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Summary of the PhD thesis

The Abstract Gronwall Lemma and Applications

by

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Contents

Abstract

In this thesis we derive optimal bounds (explicitly or in a generic form) for the solutions to integral or differential inequalities. We re-write the equations and inequalities in terms of integral operators. Applying the Abstract Gronwall Lemma to these operators gives the optimal bounds. We also present some applications of the Abstract Gronwall Lemma to pseudoparabolic and second and third-order hyperbolic inequalities, as well as to the study of Ulam stabilities for the second-order differential equations of hyperbolic type.

Keywords

Fixed Point Theory, Picard Operators, Abstract Gronwall Lemma, Comparison Lemma, Integral Equations, Integral Inequalities, Volterra Integral Inequalities, Fredholm Integral Inequalities, Triangular Operators, Hyperbolic Inequalities, Pseudoparabolic Inequalities, Ulam Stabilities.

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Chapter 1

Introduction

Integral equations are an important part of Pure and Applied Mathematics, with applications in differential equations, mechanical vibrations, engineering, physics, numerical computations and others (see, for instance, [66] and [99]). The beginning of the theory of Integral Equations can be attributed to N. H. Abel who formulated an integral equation in 1812 when studying a problem in Mechanics. Since then many other great mathematicians including T. Lalescu (who wrote the first dissertation on Integral Equations in the world in 1911, see [42]), J. Liouville, J. Hadamard, V. Volterra, I. Fredholm, E. Goursat, D. Hilbert, E Picard, H. Poincare have contributed to the development of Integral Equations.

Gronwall-type lemmas play an important role in the area of Integral (and Differential) Equations, as technical tools used to prove existence and uniqueness of a solution and to obtain various estimates for the solutions. They can be viewed as a type of result which gives *a priori* bounds for the function which satisfies an integral or differential inequality.

In this thesis we use Gronwall-type lemmas to obtain bounds of the functions that satisfy a certain differential or integral inequality, and express these bounds as fixed points of the corresponding integral operators.

In a recent paper [86], I.A. Rus has formulated ten problems of interest in the theory of Gronwall lemmas. One of them concerns finding examples of Gronwall-type lemmas in which the upper bounds are fixed points of the corresponding operator A (**Problem 5**).

Another problem is identifying which of these Gronwall-type lemmas can be obtained as consequences of the Abstract Gronwall Lemma (**Problem 6**). Abstract Gronwall Lemma gives the lowest majorant among all possible upper bounds (see [84]). There is no general methodology to answer these questions, so they have to be obtained on case-by-case basis. Over the years many mathematicians have obtained such examples (see, for instance, [8], [22], [25], [55], [57], [83], [84], [86]), and this work presents more such results.

The second chapter is dedicated to notations, definitions, lemmas and theorems, which will be used in the subsequent chapters.

In the third chapter we study some consequences of the Abstract Gronwall Lemma in the cases of Volterra and Fredholm integral inequalities in Banach and non-Banach spaces. In general, it is difficult to find the exact solutions to the integral equations or bounds for the functions which satisfy integral inequalities; therefore we need to apply results from fixed point theory and the Picard operators technique to prove existence of solutions and to find some bounds for these solutions. We also present some particular examples of integral inequalities in the Banach spaces $C([\alpha,\beta],\mathbb{R})$ and $C([\alpha,\beta],\mathbb{R}^p)$, and study the case of infinite systems of integral equations in the space $C([\alpha,\beta],s(\mathbb{R}))$ and the Banach space $C([\alpha,\beta],l^2(\mathbb{R}))$, respectively. The results of this chapter have been published in [22] and [25].

In the fourth chapter we investigate some integral inequalities studied in [8], [41], [45], [57], for which the authors obtained bounds for the solutions using various methods. Applying the Abstract Gronwall Lemma we are able to show that their bounds are optimal or, when that is not the case, we obtain ourselves the optimal bounds. Furthermore, we prove that our bounds are lowest ones among all possible upper bounds, hence they are optimal. Some optimal bounds are expressed explicitly and others just in a generic form. The results of this chapter have been published in [21], [22] and [25].

In the last chapter we apply the Abstract Gronwall Lemma to second and third-order hyperbolic inequalities and pseudoparabolic inequalities. Adding boundary conditions to hyperbolic and pseudoparabolic equations results in Darboux problems. We use the Picard operators technique to prove the existence and the uniqueness of the solution to these Darboux problems, and we apply the Abstract Gronwall Lemma to functions that satisfy the corresponding inequalities. We also use the Riemann function to represent the solutions to these Darboux problems in the case of specific pseudoparabolic inequalities. We conclude this chapter with a study of the Ulam stabilities for the second-order differential equations of hyperbolic type. The articles [23] and [24] contain the main results of this last chapter.

Chapter 2

Preliminaries

The aim of this chapter is to present some notions and results which are necessary in the derivation of the original results of this thesis. These results can be found in the following references: [20], [31], [55], [56], [66], [76], [79], [81], [83], [86], [91], [92], [99].

The metric fixed point theorems guarantee the existence and uniqueness of fixed points of certain well defined operators in metric spaces, and usually provide a constructive method to find those fixed points. The results presented in this section can be found in: [12], [26], [37], [39], [74], [78], [81], [82], [84], [85], [91] and others.

We consider the Chebyshev and Bielecki norms, which are metrically equivalent:

$$\|x\|_C := \max_{t\in[\alpha,\beta]} |x(t)|, \tag{2.0.1}$$

$$\|x\|_\tau := \max_{t\in[\alpha,\beta]} (|x(t)| \exp(-\tau(t-\alpha))), \tau \in \mathbb{R}_+^* \tag{2.0.2}$$

With respect to these norms $C([\alpha,\beta], \mathbb{B})$ is a Banach space.

The standard tool for proving the existence and uniqueness, data dependance and comparison results for the solution to integral equations is the Picard operators technique.

Following [83] we present the basic notions and results from the Picard and Weakly Picard Operators theory (see also [22], [23], [25], [33], [64], [78], [81], [82], [84], [91]).

Gronwall lemmas are the subject of this thesis, and we use them to bound a function that satisfies a certain differential or integral inequality by the solution of the corresponding differential or integral equation. The following abstract lemmas are the main tools for the results of this thesis.

Lemma 2.0.1. *(I.A. Rus [83]) Let $(X, \to, \leq)$ be an ordered L-space and $A : X \to X$ be an operator. We assume that:*

(i) A is an Weakly Picard operator;

(ii) A is an increasing operator.

Then:

(a) $x \leq A(x) \;\Rightarrow\; x \leq A^\infty(x)$;

(b) $x \geq A(x) \;\Rightarrow\; x \geq A^\infty(x)$.

Lemma 2.0.2. *(Abstract Gronwall Lemma, [83]) Let $(X, \to, \leq)$ be an ordered L-space and $A : X \to X$ be an operator. We assume that:*

(i) A is a Picard operator ($F_A = \{x_A^*\}$);

(ii) A is an increasing operator.

Then:

(a) $x \leq A(x) \;\Rightarrow\; x \leq x_A^*$;

(b) $x \geq A(x) \;\Rightarrow\; x \geq x_A^*$.

Remark 2.0.1. *Abstract Gronwall Lemma 2.0.2 provides bounds for the solution to the integral inequality $x \leq A(x)$ in terms of the fixed point of the operator A.*

Remark 2.0.2. *In the conditions of the Abstract Gronwall Lemma 2.0.2 we have:*

$$\forall y \in (LF)_A : y \leq x_A^* \text{ and } \forall z \in (UF)_A : x_A^* \leq z. \tag{2.0.3}$$

Chapter 3

Consequences of the Abstract Gronwall Lemma

The Picard operators and Gronwall type lemmas play a significant role in the qualitative theory of integral equations. In this chapter we study some consequences of Abstract Gronwall Lemma 2.0.2 in the cases of Volterra and Fredholm integral inequalities in Banach spaces. In general it is difficult to find the exact solutions of the integral equations and inequalities. The fixed point theory and the Picard operators technique allow us to prove the existence and, furthermore, to find some bounds of these solutions. We also present some particular examples of integral inequalities in $C([\alpha,\beta], s(\mathbb{R}))$ and in the Banach spaces $C([\alpha,\beta],\mathbb{R})$, $C([\alpha,\beta],\mathbb{R}^p)$ and $C([\alpha,\beta], l^2(\mathbb{R}))$ respectively.

The original results of this chapter have been published in [22] and [25].

3.1 Volterra Integral Inequalities

Volterra integral equations and inequalities have been studied for many years in Applied Mathematics using both, the classical and the fixed-point technique methods.

In this section we present a general result for Volterra operators in Banach spaces, and then particular results for the cases of Banach spaces $C([\alpha,\beta],\mathbb{R})$, $C([\alpha,\beta],\mathbb{R}^p)$ and

$C([\alpha, \beta], l^2(\mathbb{R}))$, and also for triangular operators in the non-Banach space $C([\alpha, \beta], s(\mathbb{R}))$. In some cases, the fixed points of the corresponding operators can be determined.

3.1.1 The Space $C([\alpha, \beta], \mathbb{B})$

Let $(\mathbb{B}, +, \mathbb{R}, |\cdot|, \leq)$ be an ordered Banach space, let $K : [\alpha, \beta] \times [\alpha, \beta] \times \mathbb{B} \to \mathbb{B}$ be a functional and let $g : [\alpha, \beta] \to \mathbb{B}$ be a function.

We consider the following Volterra-type equations and inequalities:

$$x(t) = g(t) + \int_\alpha^t K(t, s, x(s))ds, \text{ for all } t \in [\alpha, \beta], \tag{3.1.1}$$

$$x(t) \leq g(t) + \int_\alpha^t K(t, s, x(s))ds, \text{ for all } t \in [\alpha, \beta], \tag{3.1.2}$$

$$x(t) \geq g(t) + \int_\alpha^t K(t, s, x(s))ds, \text{ for all } t \in [\alpha, \beta]. \tag{3.1.3}$$

Using Abstract Gronwall Lemma 2.0.2 we have the following general result (see, for instance, [78], [79], [83]):

Theorem 3.1.1. *We assume that:*

(i) $K \in C([\alpha, \beta] \times [\alpha, \beta] \times \mathbb{B}, \mathbb{B})$, $g \in C([\alpha, \beta], \mathbb{B})$;

(ii) there exists $L_K > 0$ *such that:*

$$|K(t, s, u) - K(t, s, v)| \leq L_K |u - v|,$$

for all $s, t \in [\alpha, \beta]$, *and* $u, v \in \mathbb{B}$;

We have:

(a) the equation (3.1.1) has in $C([\alpha, \beta], \mathbb{B})$ *a unique solution* x^*;

(b) the operator $A : C([\alpha, \beta], \mathbb{B}) \to C([\alpha, \beta], \mathbb{B})$ *defined by:*

$$A(x)(t) := g(t) + \int_\alpha^t K(t, s, x(s))ds, \text{ for all } t \in [\alpha, \beta] \tag{3.1.4}$$

is a PO in $\left(C([\alpha, \beta], \mathbb{B}), \overset{unif}{\to}\right)$, *and* $F_A = \{x^*\}$.

If in addition we have the hypothesis:

(iii) $K(t, s, \cdot) : \mathbb{B} \to \mathbb{B}$ *is increasing, for all* $t, s \in [\alpha, \beta]$*,*

then:

(c) if $x \in C([\alpha, \beta], \mathbb{B})$ *satisfies the inequality (3.1.2), then* $x(t) \leq x^*(t)$*, for all* $t \in [\alpha, \beta]$*;*

(d) if $x \in C([\alpha, \beta], \mathbb{B})$ *satisfies the inequality (3.1.3), then* $x(t) \geq x^*(t)$*, for all* $t \in [\alpha, \beta]$*.*

Remark 3.1.1. *The above theorem is well known, but it is very useful for understanding our results.*

3.1.2 The Space $C([\alpha, \beta], \mathbb{R})$

In this subsection we start by considering the particular case of Theorem 3.1.1 when $\mathbb{B} := \mathbb{R}$. Then we apply this theorem to the case when the functional K is linear, i.e. $K(t, s, u) = k(t, s)u$, for all $t, s \in [\alpha, \beta]$, and $u \in \mathbb{R}$. In this case the solution x^* is given in terms of the resolvent kernel.

In general it is difficult to work out an explicit expression of the resolvent kernel. For some particular cases, the integral equations can be solved and explicit forms of the kernel can be obtained. Such examples are represented, for instance, by Gronwall's Lemma and Filatov's Theorem (see [8], [22], [51] and others)

As in the linear case of the functional K, in the nonlinear one it is difficult to solve the corresponding integral equations in order to find bounds of the solutions. There are some specific Gronwall lemmas in which the fixed point of the corresponding operator can be determined. In what follows we present an example of such lemma.

Theorem 3.1.2. *(Bihari-type inequality)([22], see also [56], [96]) We assume that:*

(i) $c \in \mathbb{R}$, $p \in C([\alpha, \beta], \mathbb{R}_+)$

(ii) V *is a continuous, positive, increasing, and Lipschitz function.*

$We\ have:$

(a) *if* $x \in C[\alpha, \beta]$ *is a solution of the inequality:*

$$x(t) \leq c + \int_{\alpha}^{t} p(s)\ V(x(s))ds, \text{ for all } t \in [\alpha, \beta] \tag{3.1.5}$$

then:

$$x(t) \leq x^*(t), \ \textit{for all } t \in [\alpha, \beta]$$

where $x^*(t) = F^{-1}(\phi(t) + F(c))$.

(b) *if* $x \in C[\alpha, \beta]$ *is a solution of the inequality:*

$$x(t) \geq c + \int_{\alpha}^{t} p(s)\ V(x(s))ds, \text{ for all } t \in [\alpha, \beta] \tag{3.1.6}$$

then:

$$x(t) \geq x^*(t), \ \textit{for all } t \in [\alpha, \beta]$$

where $x^*(t) = F^{-1}(\phi(t) + F(c))$.

Here we have:

$$F(y) = \int_{\alpha}^{y} \frac{dy}{V(y)}, \quad \phi(x) = \int_{\alpha}^{x} p(s)ds,$$

and F^{-1} *is the inverse of* F.

3.1.3 The Space $C\left([\alpha, \beta], \mathbb{R}^p\right)$

In this subsection we present Theorem 3.1.1 in the case of $\mathbb{B} := \mathbb{R}^p$. In this case we have a system of equations.

3.1.4 The Spaces $C([\alpha, \beta], l^2(\mathbb{R}))$ and $C([\alpha, \beta], s(\mathbb{R}))$

The version of Theorem 3.1.1 in the particular case

$$\mathbb{B} := l^2(\mathbb{R}) = \left\{ (u_n)_{n \in N}, u_n \in \mathbb{R}, \sum_{n \in N} u_n^2 < +\infty \right\}$$

is presented at the beginning of this subsection. In this case we have an infinite system of equations.

In what follows we consider the case in which $x(t) \in s(\mathbb{R})$.

We obtain an infinite system of integral equations:

$$\begin{cases} x_0(t) = g_0(t) + \int_\alpha^t K_0(t, s, x_0(s))ds \\ x_1(t) = g_1(t) + \int_\alpha^t K_1(t, s, x_0(s), x_1(s))ds \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ x_p(t) = g_p(t) + \int_\alpha^t K_p(t, s, x_0(s), x_1(s), ..., x_p(s))ds \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots, \end{cases} \tag{3.1.7}$$

for all $t \in [\alpha, \beta]$, where $K_n : [\alpha, \beta] \times [\alpha, \beta] \times \mathbb{R}^{n+1} \to \mathbb{R}, \forall n \in \mathbb{N}$ and $g : [\alpha, \beta] \to s(\mathbb{R})$.

We consider $X_n := C[\alpha, \beta]$, $n \in \mathbb{N}$ and the operators

$$A_n : \prod_{i=0}^{n} X_i \to X_n, \text{ defined by:}$$

$$\begin{cases} A_0(x_0)(t) := g_0(t) + \int_\alpha^t K_0(t, s, x_0(s))ds, \\ A_n(x_0, x_1, ..., x_n)(t) := g_n(t) + \int_\alpha^t K_n(t, s, x_0(s), x_1(s), ..., x_n(s))ds. \end{cases} \tag{3.1.8}$$

We denote by $X := \prod_{i \in \mathbb{N}} X_i$ and

$$A : X \to X, \ A = (A_0, A_1, ..., A_p, ...), \tag{3.1.9}$$

where $A_0, ..., A_p, ...$ are given by (3.1.8).

The existence and the uniqueness of the solution to the infinite system (3.1.7) have been obtained by I. A. Rus and M. A. Şerban in [93]. We add the monotony condition for the functionals K_n, $n \in \mathbb{N}$, and applying the Abstract Gronwall Lemma 2.0.2 gives us the following theorem:

Theorem 3.1.3. *We assume that:*

(i) $g_n \in C[\alpha, \beta]$, $K_n \in C([\alpha, \beta] \times [\alpha, \beta] \times \mathbb{R}^{n+1})$, $n \in \mathbb{N}$;

(ii) there exists $L_{K_0} > 0$ *such that:*

$$|K_0(t, s, \xi_1) - K_0(t, s, \xi_2)| \leq L_{K_0} |\xi_1 - \xi_2|, \tag{3.1.10}$$

for all $t, s \in [\alpha, \beta], \xi_1, \xi_2 \in \mathbb{R}$;

(iii) there exists $L_{K_n} > 0$ *such that:*

$$|K_n(t,s,u_0,u_1,...,u_{n-1},\xi_1) - K_n(t,s,u_0,u_1,...,u_{n-1},\xi_2)| \leq L_{K_n} |\xi_1 - \xi_2| , \tag{3.1.11}$$

for all $t, s \in [\alpha,\beta], u_0, u_1, ..., u_{n-1}, \xi_1, \xi_2 \in \mathbb{R}$, $p \in \mathbb{N}^*$.

Then:

(a) the infinite system (3.1.7) has a unique solution $x^* \in C([\alpha,\beta], s(\mathbb{R}))$;

(b) the corresponding operator A *defined by (3.1.9) is a PO in* $(C([\alpha,\beta], s(\mathbb{R})), \xrightarrow{t})$.

If in addition we have the hypothesis:

(iv) $K_n(t,s,\cdot,...,\cdot) : \mathbb{R}^{n+1} \to \mathbb{R}$ *is increasing for all* $t, s \in [\alpha,\beta]$ *and* $\forall n \in \mathbb{N}$,

then:

(c) if $x \in C([\alpha,\beta], s(\mathbb{R}))$ *satisfies the corresponding inequality (3.1.2), then* $x(t) \leq x^*(t)$, *for all* $t \in [\alpha,\beta]$*;*

(d) if $x \in C([\alpha,\beta], s(\mathbb{R}))$ *satisfies the corresponding inequality (3.1.3), then* $x(t) \geq x^*(t)$, *for all* $t \in [\alpha,\beta]$.

3.2 Fredholm Integral Inequalities

Along with Volterra integral equations and inequalities, the integral inequalities of Fredholm type play an essential role in the nonlinear analysis. The Fredholm equations and inequalities have been studied by many mathematicians over the years. Some monographs on Fredholm equations have been written by D. Bainov and P. Simeonov (see [8]), C. Corduneanu (see [20]), D.S. Mitrinović, J.E. Pečarić, A.M. Fink (see [51]). See also: Sz. András ([1], [2]), F. Caliò, E. Marcchetti and V. Mureşan ([14]), C. Crăciun and N. Lungu ([22]), V. Mureşan ([52]), B.G. Pachpatte ([56], [55]), I.A. Rus ([76], [78], [81], [84], [86],[85], [83]), N. Taghizadeh and V. Khanbabai ([98]).

In this section we present a Gronwall-type lemma for Fredholm operators in Banach spaces, and then particular results for the cases of Banach spaces $C([\alpha,\beta],\mathbb{R})$, $C([\alpha,\beta],\mathbb{R}^p)$ and $C([\alpha,\beta],l^2(\mathbb{R}))$, and also for triangular operators in the non-Banach space $C([\alpha,\beta],s(\mathbb{R}))$.

3.2.1 The Space $C([\alpha, \beta], \mathbb{B})$

Let $(\mathbb{B}, +, \mathbb{R}, |\cdot|, \leq)$ be an ordered Banach space, let $H : [\alpha, \beta] \times [\alpha, \beta] \times \mathbb{B} \to \mathbb{B}$ be a functional, and let $g : [\alpha, \beta] \to \mathbb{B}$ be a function.

We consider the following Fredholm-type equation and inequalities:

$$x(t) = g(t) + \int_\alpha^\beta H(t, s, x(s))ds, \text{ for all } t \in [\alpha, \beta], \tag{3.2.1}$$

$$x(t) \leq g(t) + \int_\alpha^\beta H(t, s, x(s))ds, \text{ for all } t \in [\alpha, \beta], \tag{3.2.2}$$

$$x(t) \geq g(t) + \int_\alpha^\beta H(t, s, x(s))ds, \text{ for all } t \in [\alpha, \beta]. \tag{3.2.3}$$

Theorem 3.2.1. *We assume that:*

(i) $H \in C([\alpha, \beta] \times [\alpha, \beta] \times \mathbb{B}, \mathbb{B})$, $g \in C([\alpha, \beta], \mathbb{B})$*;*

(ii) there exists $L_H > 0$ *such that:*

$$|H(t, s, u) - H(t, s, v)| \leq L_H |u - v|,$$

for all $s, t \in [\alpha, \beta]$*, and* $u, v \in \mathbb{B}$*;*

(iii) $L_H(\beta - \alpha) < 1$*;*

We have:

(a) the equation (3.2.1) has in $C([\alpha, \beta], \mathbb{B})$ *a unique solution* x^**;*

(b) the operator $B : C([\alpha, \beta], \mathbb{B}) \to C([\alpha, \beta], \mathbb{B})$*, defined by:*

$$B(x)(t) := g(t) + \int_\alpha^\beta H(t, s, x(s))ds, \text{ for all } t \in [\alpha, \beta] \tag{3.2.4}$$

is a PO in $\left(C([\alpha, \beta], \mathbb{B}), \overset{unif}{\to}\right)$*, and* $F_B = \{x^*\}$*.*

If in addition we have the hypothesis:

(iv) $H(t, s, \cdot) : \mathbb{B} \to \mathbb{B}$ *is increasing for all* $t, s \in [\alpha, \beta]$*,*

then:

(c) if $x \in C([\alpha, \beta], \mathbb{B})$ *satisfies the inequality (3.2.2), then:*

$$x(t) \leq x^*(t), \text{ for all } t \in [\alpha, \beta]; \tag{3.2.5}$$

(d) if $x \in C([a, b], \mathbb{B})$ *satisfies the inequality (3.2.3), then:*

$$x(t) \geq x^*(t), \text{ for all } t \in [\alpha, \beta]. \tag{3.2.6}$$

3.2.2 The Space $C([\alpha, \beta], \mathbb{R})$

We start this case by considering the linear form of the functional H, i.e. $H(t, s, u) := h(t, s)u$, case for which we present a Gronwall-type result.

Remark 3.2.1. *If the integral equation has a degenerate kernel, i.e.* $h(t, s) = \sum_{i=1}^{p} a_i(t) b_i(s)$*, where a and b are continuous functions on* $[\alpha, \beta]$*, then the solution to the integral equation is, in fact, the solution of an algebraic system, which can be found explicitly.*

3.2.3 The Space $C([\alpha, \beta], \mathbb{R}^p)$

If $\mathbb{B} := \mathbb{R}^p$ the equation (3.2.1) is, in fact, a system of equations. A Gronwall-type result for this system is presented in this subsection.

3.2.4 The Spaces $C([\alpha, \beta], l^2(\mathbb{R}))$ and $C([\alpha, \beta], s(\mathbb{R}))$

The first part of this subsection consists of a Gronwall-type result for Fredholm equation and inequalities in the particular case of $C([\alpha, \beta], l^2(\mathbb{R}))$.

The case of $C([\alpha, \beta], s(\mathbb{R}))$ follows. Here we have the following infinite system of integral equations:

$$\begin{cases} x_0(t) = g_0(t) + \int_\alpha^\beta H_0(t, s, x_0(s))ds \\ x_1(t) = g_1(t) + \int_\alpha^\beta H_1(t, s, x_0(s), x_1(s))ds \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \qquad \text{for all } t \in [\alpha, \beta] \\ x_p(t) = g_p(t) + \int_\alpha^\beta H_p(t, s, x_0(s), x_1(s), ..., x_p(s))ds \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots, \end{cases} \tag{3.2.7}$$

where $H_n : [\alpha, \beta] \times [\alpha, \beta] \times \mathbb{R}^{n+1} \to \mathbb{R}$, $\forall n \in \mathbb{N}$ and $g : [\alpha, \beta] \to s(\mathbb{R})$.

For any $n \in \mathbb{N}$ we consider $X_n := C[\alpha, \beta]$ and the operators

$$B_n : \prod_{i=0}^{n} X_i \to X_n, \text{defined by:}$$

$$\begin{cases} B_0(x_0)(t) := g_0(t) + \int_\alpha^\beta H_0(t,s,x_0(s))ds, \\ B_n(x_0,x_1,...,x_n)(t) := g_n(t) + \int_\alpha^\beta H_n(t,s,x_0(s),x_1(s),...,x_n(s))ds. \end{cases} \tag{3.2.8}$$

We denote by $X := \prod_{i\in\mathbb{N}} X_i$ and

$$B : X \to X,\ \ B = (B_0, B_1, ..., B_p, ...), \tag{3.2.9}$$

where $B_0, ..., B_p, ...$ are given by (3.2.8).

From Theorem 3.2.1 we obtain theoretical bounds for functions which satisfy the corresponding inequalities (3.2.2) and (3.2.3) in the case of triangular operators. These functions are dominated by the fixed point of the corresponding operator.

Theorem 3.2.2. *We assume that:*

(i) $g_n \in C[\alpha,\beta],\ H_n \in C([\alpha,\beta]\times[\alpha,\beta]\times\mathbb{R}^{n+1}),\ \forall n\in\mathbb{N}$*;*

(ii) there exists $L_{H_0} > 0$ *such that:*

$$|H_0(t,s,\xi_1) - H_0(t,s,\xi_2)| \le L_{H_0}\,|\xi_1 - \xi_2|\,, \tag{3.2.10}$$

for all $t,s \in [\alpha,\beta], \xi_1, \xi_2 \in \mathbb{R}$;

(iii) $L_{H_0}(\beta-\alpha) < 1$;

(iv) there exists $L_{H_n} > 0$ *such that:*

$$|H_n(t,s,u_0,u_1,...,u_{n-1},\xi_1) - H_n(t,s,u_0,u_1,...,u_{n-1},\xi_2)| \le L_{H_n}\,|\xi_1 - \xi_2|\,, \tag{3.2.11}$$

for all $t,s \in [\alpha,\beta], u_0,u_1,...,u_{n-1},\xi_1,\xi_2 \in \mathbb{R},\ n\in\mathbb{N}^*$;

(v) $L_{H_n}(\beta-\alpha) < 1,\ \forall n\in\mathbb{N}^*$;

We have:

(a) the infinite system (3.2.7) has a unique solution $x^* \in C([\alpha,\beta]\,, s(\mathbb{R}))$;

(b) the corresponding operator B *defined by (3.2.9) is a PO in* $(C([\alpha,\beta]\,,\ s(\mathbb{R})), \xrightarrow{t})$ *and* $F_B = \{x^*\}$.

If in addition we have the hypothesis:

(vi) $H_n(t,s,\cdot,...,\cdot) : \mathbb{R}^{n+1} \to \mathbb{R}$ *is increasing for all* $t,s\in[\alpha,\beta]$ *and* $\forall n\in\mathbb{N}$,

then:

(c) if $x \in C([\alpha, \beta], s(\mathbb{R}))$ *satisfies the corresponding inequality (3.1.2), then* $x(t) \leq x^*(t)$*, for all* $t \in [\alpha, \beta]$*;*

(d) if $x \in C([\alpha, \beta], s(\mathbb{R}))$ *satisfies the corresponding inequality (3.1.3), then* $x(t) \geq x^*(t)$*, for all* $t \in [\alpha, \beta]$.

3.3 Integral Inequalities with Modified Argument

Integral inequalities with modified argument are used to describe many phenomena in science, economics, population dynamics, physics, medicine etc. They have been studied by many researchers: D. Bainov and P. Simeonov [8], T.A. Burton [13], M. Dobriţoiu [27], [28], M. Dobriţoiu, I.A. Rus and M.A. Şerban [29], D. Guo, V. Lakshmikantham and X. Liu [33], N. Lungu [46], D.S. Mitrinović, J.E. Pečarić and A.M. Fink [51], V. Mureşan [53], B.G. Pachpatte [56], R. Precup and E. Kirr [71], and others.

In this section are presented Gronwall-type results, in both cases, Volterra and Fredholm-type equations and inequalities.

3.4 Multi-dimensional Integral Inequalities

Similar results to those presented in sections 3.1 and 3.2 can be obtained in the case of multi-dimensional integral inequalities (see, for instance, [8], [9], [56]). These inequalities can be used in the study of various problems in the theory of certain partial differential, integral and integro-differential equations, especially in the study of the qualitative behaviour of the solutions.

3.5 Volterra-Fredholm Integral Inequalities

In this section we consider the following mixed type Volterra-Fredholm nonlinear integral equation and inequalities:

$$x(t) = F\left(t, x(t), \int_{\alpha_1}^{t_1} \dots \int_{\alpha_m}^{t_m} K(t,s,x(s))ds, \int_{\alpha_1}^{\beta_1} \dots \int_{\alpha_m}^{\beta_m} H(t,s,x(s))ds\right), \qquad (3.5.1)$$

$$x(t) \leq F\left(t, x(t), \int_{\alpha_1}^{t_1} \dots \int_{\alpha_m}^{t_m} K(t,s,x(s))ds, \int_{\alpha_1}^{\beta_1} \dots \int_{\alpha_m}^{\beta_m} H(t,s,x(s))ds\right), \qquad (3.5.2)$$

$$x(t) \geq F\left(t, x(t), \int_{\alpha_1}^{t_1} \dots \int_{\alpha_m}^{t_m} K(t,s,x(s))ds, \int_{\alpha_1}^{\beta_1} \dots \int_{\alpha_m}^{\beta_m} H(t,s,x(s))ds\right), \qquad (3.5.3)$$

where $[\alpha_1;\beta_1]\times\ldots\times[\alpha_m;\beta_m]$ is an interval in $\mathbb{R}^m$, $K, H : [\alpha_1;\beta_1]\times\ldots\times[\alpha_m;\beta_m]\times[\alpha_1;\beta_1]\times\ldots\times[\alpha_m;\beta_m]\times\mathbb{R}\to\mathbb{R}$ are continuous functions and $F : [\alpha_1;\beta_1]\times\ldots\times[\alpha_m;\beta_m]\times\mathbb{R}^3\to\mathbb{R}$ is a functional. The mixed type Volterra-Fredholm integral equations have been studied by many authors (see [2], [55], [63], [94], [95] and others).

We prove the existence and uniqueness for the solution of integral equation (3.5.1) by standard techniques as in [2], [14], [22], [47], our integral equation (3.5.1) being more general than integral equations considered in above mentioned papers. We also establish a Gronwall-type lemma for these inequalities. The results of this section have been published in [25].

Theorem 3.5.1. *([25]) We assume that:*

(i) $K,\ H \in C([\alpha_1,\beta_1]\times\cdots\times[\alpha_m,\beta_m]\times[\alpha_1,\beta_1]\times\cdots\times[\alpha_m,\beta_m]\times\mathbb{R})$;

(ii) $F \in C([\alpha_1,\beta_1]\times\cdots\times[\alpha_m,\beta_m]\times\mathbb{R}^3)$;

(iii) there exist a, b, c *nonnegative constants such that:*

$$|F(t,u_1,v_1,w_1) - F(t,u_2,v_2,w_2)| \leq a|u_1-u_2| + b|v_1-v_2| + c|w_1-w_2|,$$

for all $t \in [\alpha_1,\beta_1]\times\cdots\times[\alpha_m,\beta_m], u_1,u_2,v_1,v_2,w_1,w_2 \in \mathbb{R}$;

(iv) there exist L_K and L_H nonnegative constants such that:

$$\begin{aligned} |K(t,s,u) - K(t,s,v)| &\leq L_K|u-v|, \\ |H(t,s,u) - H(t,s,v)| &\leq L_H|u-v|, \end{aligned}$$

for all $t, s \in [\alpha_1, \beta_1] \times \cdots \times [\alpha_m, \beta_m]$, and $u, v \in \mathbb{R}$;

(v) $a + (bL_K + cL_H)(\beta_1 - \alpha_1) \ldots (\beta_m - \alpha_m) < 1$.

Then:

(a) the equation (3.5.1) has a unique solution $x^* \in C([\alpha_1, \beta_1] \times \cdots \times [\alpha_m, \beta_m])$;

(b) the operator

$$A : C([\alpha_1, \beta_1] \times \cdots \times [\alpha_m, \beta_m]) \to C([\alpha_1, \beta_1] \times \cdots \times [\alpha_m, \beta_m])$$

defined by:

$$A(x)(t) = F\left(t, x(t), \int_{\alpha_1}^{t_1} \ldots \int_{\alpha_m}^{t_m} K(t,s,x(s))ds, \int_{\alpha_1}^{\beta_1} \ldots \int_{\alpha_m}^{\beta_m} H(t,s,x(s))ds\right) \quad (3.5.4)$$

is a Picard operator, and $F_A = \{x^*\}$;

If in addition we have the following hypotheses:

(vi) $K(t,s,\cdot)$, $H(t,s,\cdot) : \mathbb{R} \to \mathbb{R}$ are increasing for all $t, s \in [\alpha_1, \beta_1] \times \cdots \times [\alpha_m, \beta_m]$;

(vii) $F(t,\cdot,\cdot,\cdot) : \mathbb{R}^3 \to \mathbb{R}$ is increasing, for all $t \in [\alpha_1, \beta_1] \times \cdots \times [\alpha_m, \beta_m]$,

then, we have:

(c) if $x \in C([\alpha_1, \beta_1] \times \cdots \times [\alpha_m, \beta_m])$ satisfies the inequality (3.5.2), then $x(t) \leq x^*(t)$, for all $t \in [\alpha_1, \beta_1] \times \cdots \times [\alpha_m, \beta_m]$;

(d) if $x \in C([\alpha_1, \beta_1] \times \cdots \times [\alpha_m, \beta_m])$ satisfies the inequality (3.5.3), then $x(t) \geq x^*(t)$, for all $t \in [\alpha_1, \beta_1] \times \cdots \times [\alpha_m, \beta_m]$.

Proposition 3.5.1. *The conclusions of Theorem 3.5.1 remain true if instead of condition (v) we put the condition*

(v') there exists $\tau > 0$ such that:

$$a + \frac{bL_K}{\tau^m} + \frac{cL_H}{\tau^m} \cdot \prod_{i=1}^{m} e^{\tau(\beta_i - \alpha_i)} < 1.$$

In the special case when F is linear with respect to the last two variables we get the following result.

Proposition 3.5.2. *We consider the integral equation and inequalities:*

$$x(t) = f(t, x(t)) + \int_{\alpha_1}^{t_1} \ldots \int_{\alpha_m}^{t_m} K(t,s,x(s))ds + \int_{\alpha_1}^{\beta_1} \ldots \int_{\alpha_m}^{\beta_m} H(t,s,x(s))ds, \quad (3.5.5)$$

$$x(t) \leq f(t, x(t)) + \int_{\alpha_1}^{t_1} \ldots \int_{\alpha_m}^{t_m} K(t,s,x(s))ds + \int_{\alpha_1}^{\beta_1} \ldots \int_{\alpha_m}^{\beta_m} H(t,s,x(s))ds, \quad (3.5.6)$$

$$x(t) \geq f(t, x(t)) + \int_{\alpha_1}^{t_1} \ldots \int_{\alpha_m}^{t_m} K(t,s,x(s))ds + \int_{\alpha_1}^{\beta_1} \ldots \int_{\alpha_m}^{\beta_m} H(t,s,x(s))ds. \quad (3.5.7)$$

We assume that:

(i) $K,\ H \in C([\alpha_1,\beta_1] \times \cdots \times [\alpha_m,\beta_m] \times [\alpha_1,\beta_1] \times \cdots \times [\alpha_m,\beta_m] \times \mathbb{R})$;

(ii) $f \in C([\alpha_1,\beta_1] \times \cdots \times [\alpha_m,\beta_m] \times \mathbb{R})$;

(iii) there exists $a > 0$ such that:

$$|f(t,u_1) - f(t,u_2)| \leq a|u_1 - u_2|,$$

for all $t \in [\alpha_1,\beta_1] \times \cdots \times [\alpha_m,\beta_m]$, $u_1, u_2 \in \mathbb{R}$;

(iv) there exist L_K and L_H nonnegative constants such that:

$$\begin{aligned} |K(t,s,u) - K(t,s,v)| &\leq L_K|u-v|, \\ |H(t,s,u) - H(t,s,v)| &\leq L_H|u-v|, \end{aligned}$$

for all $t,s \in [\alpha,\beta]$, and $u,v \in \mathbb{R}$;

(v) $a + (L_K + L_H)(\beta_1 - \alpha_1)\ldots(\beta_m - \alpha_m) < 1$ or there exists $\tau > 0$ such that: $a + \frac{L_K}{\tau^m} + \frac{L_H}{\tau^m} \cdot \prod_{i=1}^{m} e^{\tau(\beta_i - \alpha_i)} < 1$.

Then:

(a) the equation (3.5.5) has a unique solution $x^* \in C([\alpha_1,\beta_1] \times \cdots \times [\alpha_m,\beta_m])$;

(b) the operator $A : C([\alpha_1,\beta_1] \times \cdots \times [\alpha_m,\beta_m]) \to C([\alpha_1,\beta_1] \times \cdots \times [\alpha_m,\beta_m])$ defined by:

$$A(x)(t) = f(t, x(t)) + \int_{\alpha_1}^{t_1} \ldots \int_{\alpha_m}^{t_m} K(t,s,x(s))ds + \int_{\alpha_1}^{\beta_1} \ldots \int_{\alpha_m}^{\beta_m} H(t,s,x(s))ds$$

is a PO, and $F_A = \{x^*\}$.

If in addition we have the following hypotheses:

(vi) $f(t, \cdot) : \mathbb{R} \to \mathbb{R}$ *is increasing, for all* $t \in [\alpha_1, \beta_1] \times \cdots \times [\alpha_m, \beta_m]$;

(vii) $K(t, s, \cdot)$, $H(t, s, \cdot) : \mathbb{R} \to \mathbb{R}$ *are increasing, for all* $t, s \in [\alpha, \beta]$,

then:

(c) if $x \in C([\alpha_1, \beta_1] \times \cdots \times [\alpha_m, \beta_m])$ *satisfies the inequality (3.5.6), then* $x(t) \leq x^*(t)$, *for all* $t \in [\alpha_1, \beta_1] \times \cdots \times [\alpha_m, \beta_m]$;

(d) if $x \in C([\alpha_1, \beta_1] \times \cdots \times [\alpha_m, \beta_m])$ *satisfies the inequality (3.5.7), then* $x(t) \geq x^*(t)$, *for all* $t \in [\alpha_1, \beta_1] \times \cdots \times [\alpha_m, \beta_m]$;

Example 3.5.1. *Let us consider the Darboux problem:*

$$\begin{cases} \frac{\partial^2 x}{\partial t_1 \partial t_2}(t_1, t_2) & = f\left(t_1, t_2, x\left(t_1, t_2\right)\right), \quad (t_1, t_2) \in [\alpha_1; \beta_1] \times [\alpha_2; \beta_2] \\ \quad x\left(t_1, \alpha_2\right) & = \varphi\left(t_1\right), \qquad\qquad\qquad t_1 \in [\alpha_1; \beta_1] \\ \quad x\left(\alpha_1, t_2\right) & = \psi\left(t_2\right), \qquad\qquad\qquad t_2 \in [\alpha_2; \beta_2], \ \varphi\left(\alpha_1\right) = \psi\left(\alpha_2\right) \end{cases} \tag{3.5.8}$$

under the following hypothesis:

(i) $f \in C([\alpha_1; \beta_1] \times [\alpha_2; \beta_2] \times \mathbb{R})$, $\varphi \in C\left([\alpha_1; \beta_1]\right)$, $\psi \in C\left([\alpha_2; \beta_2]\right)$;

(ii) there exists $L_f > 0$ *such that:*

$$|f(t_1, t_2, u_1) - f(t_1, t_2, u_2)| \leq L_f \cdot |u_1 - u_2|,$$

for all $(t_1, t_2) \in [\alpha_1; \beta_1] \times [\alpha_2; \beta_2]$, $u_1, u_2 \in \mathbb{R}$.

The function $x \in C([\alpha_1, \beta_1] \times [\alpha_2, \beta_2]$ *is a solution to the Darboux problem (3.5.8) if, and only if, it is a solution to the integral equation:*

$$x\left(t_1, t_2\right) = \varphi\left(t_1\right) + \psi\left(t_2\right) - \varphi\left(\alpha_1\right) + \int_{\alpha_1}^{t_1} \int_{\alpha_2}^{t_2} f\left(\xi_1, \xi_2, x\left(\xi_1, \xi_2\right)\right) d\xi_1 d\xi_2. \tag{3.5.9}$$

So, we apply Theorem 3.5.1 (a) and (b) in the particular case of $m = 2$ *and*

$$F : [\alpha_1, \beta_1] \times [\alpha_2, \beta_2] \times \mathbb{R}^3 \to \mathbb{R}, \text{ defined by:}$$

$$F\left(t_1, t_2, u, v, w\right) = \varphi\left(t_1\right) + \psi\left(t_2\right) - \varphi\left(\alpha_1\right) + v.$$

In this case we have $a = 0$, $b = 1$, $c = 0$, $L_K = L_f$ and $L_H = 0$. Also, the condition (v') from Proposition 3.5.1 is satisfied: there exists $\tau > 0$ such that $\frac{L_f}{\tau^2} < 1$ (for example we can choose $\tau = L_f + 1$).

Then, the Darboux problem (3.5.8) has a unique solution $x^* \in C([\alpha_1; \beta_1] \times [\alpha_2; \beta_2])$, and the corresponding operator A defined by the right hand side of (3.5.9) is a PO.

Remark 3.5.1. *Theorem 3.5.1 remains true if we consider the mixed type Volterra-Fredholm functional nonlinear integral equation (3.5.1) in a general Banach space $\mathbb{B}$ instead of the Banach space $\mathbb{R}$.*

Next we consider an example of an infinite system of integral equations. We apply the version of Theorem 3.5.1 in a Banach space to obtain the existence and the uniqueness of the solution.

Example 3.5.2. *We consider the following infinite system of integral equations:*

$$x_n(t) = f_n(t) + \int_{\alpha_1}^{t_1} \cdots \int_{\alpha_m}^{t_m} k(t,s) x_{n+1}(s) ds + \int_{\alpha_1}^{\beta_1} \cdots \int_{\alpha_m}^{\beta_m} h(t,s) x_{n+2}(s) ds, \quad (3.5.10)$$

for any $n \in \mathbb{N}$, under the following hypotheses:

(i) $f_n \in C([\alpha_1, \beta_1] \times \cdots \times [\alpha_m, \beta_m])$, $n \in \mathbb{N}$, $f_n(t) \to 0$, $n \to +\infty$, for every $t \in [\alpha_1, \beta_1] \times \cdots \times [\alpha_m, \beta_m]$;

(ii) $k, h \in C([\alpha_1, \beta_1] \times \cdots \times [\alpha_m, \beta_m] \times [\alpha_1, \beta_1] \times \cdots \times [\alpha_m, \beta_m])$;

(iii) $(m_k + m_h)(\beta_1 - \alpha_1) \dots (\beta_m - \alpha_m) < 1$ or there exists $\tau > 0$ such that: $\frac{m_k}{\tau^m} + \frac{m_h}{\tau^m} \cdot \prod_{i=1}^{m} e^{\tau(\beta_i - \alpha_i)} < 1$, where

$$\begin{aligned} m_k &= \max_{t,s \in [\alpha_1,\beta_1] \times \cdots \times [\alpha_m,\beta_m]} |k(t,s)|, \\ m_h &= \max_{t,s \in [\alpha_1,\beta_1] \times \cdots \times [\alpha_m,\beta_m]} |h(t,s)|. \end{aligned}$$

Let $(\mathbb{B}, \|\cdot\|)$ be the Banach space, where

$$\mathbb{B} = c_0 = \{\mathbf{u} = (u_0, u_1, \dots, u_n, \dots) \in s(\mathbb{R}) : u_n \to 0\}$$

and

$$\|\mathbf{u}\| = \max_{n \in \mathbb{N}} |u_n|.$$

Let $\mathbf{u} = (u_0, u_1, \ldots, u_n, \ldots) \in \mathbb{B}$. *We denote by*

$$
\begin{aligned}
\mathbf{f} &= (f_0, f_1, \ldots, f_n, \ldots), \\
\mathbf{K} &= (K_0, K_1, \ldots, K_n, \ldots), \\
\mathbf{H} &= (H_0, H_1, \ldots, H_n, \ldots),
\end{aligned}
$$

where

$$
\begin{aligned}
K_n(t, s, \mathbf{u}) &= k(t, s) u_{n+1}, \\
H_n(t, s, \mathbf{u}) &= h(t, s) u_{n+2},
\end{aligned}
$$

for all $t, s \in [\alpha_1, \beta_1] \times \cdots \times [\alpha_m, \beta_m]$. *From* (i) *and* (ii) *we have that* $\mathbf{f} \in C([\alpha_1, \beta_1] \times \cdots \times [\alpha_m, \beta_m], \mathbb{B})$ *and* $\mathbf{K}, \mathbf{H} \in C([\alpha_1, \beta_1] \times \cdots \times [\alpha_m, \beta_m] \times [\alpha_1, \beta_1] \times \cdots \times [\alpha_m, \beta_m], \mathbb{B})$. *Also,*

$$
\begin{aligned}
\|\mathbf{K}(t, s, \mathbf{u}) - \mathbf{K}(t, s, \mathbf{v})\| &\leq m_k \|\mathbf{u} - \mathbf{v}\|, \\
\|\mathbf{H}(t, s, \mathbf{u}) - \mathbf{H}(t, s, \mathbf{v})\| &\leq m_h \|\mathbf{u} - \mathbf{v}\|,
\end{aligned}
$$

for all $t, s \in [\alpha_1, \beta_1] \times \cdots \times [\alpha_m, \beta_m]$ *and* $\mathbf{u}, \mathbf{v} \in \mathbb{B}$. *Conditions (i)-(v) of Theorem 3.5.1 (the case of a Banach space) are satisfied, therefore we get that the equation (3.5.10) has a unique solution* $\mathbf{x}^* = (x_0^*, x_1^*, \ldots, x_n^*, \ldots) \in C([\alpha_1, \beta_1] \times \cdots \times [\alpha_m, \beta_m], \mathbb{B})$.

In this chapter we have reduced the study of various integral equations and inequalities to fixed point problems. We have studied the existence and uniqueness of the solutions to integral equations as fixed points of the corresponding operators. We have also derived some general Gronwall-type results for the integral inequalities and have given examples of such results in the particular cases of Banach and non-Banach spaces.

Chapter 4

Applications to Optimality Results

For some specific operators A (see [8], [41], [57]) a priori bounds y^* of the solutions to integral inequalities of the type $x \leq A(x)$ have been obtained:

$$x \leq y^*.$$

Using fixed point theory and the Abstract Gronwall Lemma 2.0.2 we show that their bounds are optimal (see Remark 2.0.2), or obtain ourselves the optimal ones. Hence, in these cases the fixed point x_A^* of the operator A is the optimal bound in the sense:

$$x \leq x_A^* \text{ and } \forall y : x \leq y \Rightarrow x_A^* \leq y.$$

The results of this chapter have been published in [21], [22] and [25].

4.1 Optimality Results in Explicit Form

In this section we consider some integral inequalities of the type

$$x \leq A(x)$$

for which we derive explicit expressions of the optimal bound x_A^*.

In [57] B.G. Pachpatte has considered the following inequality:

$$x(t) \leq f(t) + \int_\alpha^t g(s)x(s)ds + \int_\alpha^\beta h(s)x(s)ds, \text{ for all } t \in [\alpha, \beta]. \tag{4.1.1}$$

We also consider the inequality:

$$x(t) \geq f(t) + \int_{\alpha}^{t} g(s)x(s)ds + \int_{\alpha}^{\beta} h(s)x(s)ds, \text{ for all } t \in [\alpha, \beta] . \tag{4.1.2}$$

Remark 4.1.1. *Using direct methods B. G. Pachpatte has obtained a bound for the functions satisfying the inequality (4.1.1). We prove in the next theorem that the bound found by Pachpatte is a fixed point of the corresponding operator*

$$A : (C([\alpha, \beta], \mathbb{R}_+), \overset{\|\cdot\|_C}{\to}, \leq) \to (C([\alpha, \beta], \mathbb{R}_+), \overset{\|\cdot\|_C}{\to}, \leq),$$

$$A(x)(t) = f(t) + \int_{\alpha}^{t} g(s)x(s)ds + \int_{\alpha}^{\beta} h(s)x(s)ds, \textit{ for all } t \in [\alpha, \beta], \tag{4.1.3}$$

where $\|\cdot\|_C$ is the Chebyshev's norm defined by (2.0.1), hence it is optimal. We also derive optimal bounds for the functions satisfying the inequality (4.1.2).

Theorem 4.1.1. *We assume that:*

(i) $f, g, h \in C([\alpha, \beta], \mathbb{R}_+)$;

(ii) f is a continuously differentiable function on $[\alpha, \beta]$, and $f(t) \geq 0$, for all $t \in [\alpha, \beta]$;

(iii)

$$(\beta - \alpha)(M_g + M_h) < 1,$$

where $M_g = \max_{t \in [\alpha, \beta]} |g(t)|$ and $M_h = \max_{t \in [\alpha, \beta]} |h(t)|$;

(iv) $p_1 = \int_{\alpha}^{\beta} h(s) \exp(\int_{\alpha}^{s} g(r)dr)ds < 1$.

We have:

(a) if $x \in C([\alpha, \beta], \mathbb{R}_+)$ satisfies the inequality (4.1.1), then:

$$x(t) \leq M_1 \exp(\int_{\alpha}^{t} g(s)ds) + \int_{\alpha}^{t} f'(s) \exp(\int_{s}^{t} g(r)dr)ds, \textit{ for all } t \in [\alpha, \beta], \tag{4.1.4}$$

where

$$M_1 = \frac{1}{1 - p_1} \left[f(\alpha) + \int_{\alpha}^{\beta} h(s)(\int_{\alpha}^{s} f'(\tau) \exp(\int_{\tau}^{s} g(r)dr)d\tau)ds \right];$$

(b) if $x \in C([\alpha,\beta], \mathbb{R}_+)$ satisfies the inequality (4.1.2), then:

$$x(t) \geq M_1 \exp(\int_\alpha^t g(s)ds) + \int_\alpha^t f'(s) \exp(\int_s^t g(r)dr)ds, \text{ for all } t \in [\alpha,\beta], \tag{4.1.5}$$

where

$$M_1 = \frac{1}{1-p_1}\left[f(\alpha) + \int_\alpha^\beta h(s)(\int_\alpha^s f'(\tau)\exp(\int_\tau^s g(r)dr)d\tau)ds\right]. \tag{4.1.6}$$

Next we present an optimal bound of the solutions to an inequality considered by Chandirov (see [8]). The bound obtained by him is not a fixed point of the corresponding operator A.

Theorem 4.1.2. *We assume that:*

(i) f, g, h are continuous functions on $J = [\alpha,\beta]$;

(ii) $g(t) \geq 0$, for all $t \in J$.

We have:

(a) if $x \in C[\alpha,\beta]$ satisfies the inequality:

$$x(t) \leq f(t) + \int_\alpha^t [g(s)x(s) + h(s)]\, ds, \text{ for all} t \in J, \tag{4.1.7}$$

then:

$$x(t) \leq f(\alpha)\exp(\int_\alpha^t g(s)ds) + \int_\alpha^t [f'(r) + h(r)]\exp(\int_r^t g(s)ds)dr; \tag{4.1.8}$$

(b) if $x \in C[\alpha,\beta]$ satisfies the inequality:

$$x(t) \geq f(t) + \int_\alpha^t [g(s)x(s) + h(s)]\, ds, \text{ for all} t \in J, \tag{4.1.9}$$

then:

$$x(t) \geq f(\alpha)\exp(\int_\alpha^t g(s)ds) + \int_\alpha^t [f'(r) + h(r)]\exp(\int_r^t g(s)ds)dr. \tag{4.1.10}$$

Remark 4.1.2. *Theorem 4.1.2 gives optimal bounds for $x(t)$, as the right hand side of the inequality (4.1.8) is the unique fixed point of the corresponding operator A.*

Another example is the Bernoulli-type integral inequality. An upper bound of the solutions of this integral inequality has been obtained by N. Lungu (see [44]). We obtain a similar result by using the Picard operators' technique.

Theorem 4.1.3. *([21]) We assume that:*

(i) $x \in C([\alpha, \beta], \mathbb{R}_+),\ f, g \in C[\alpha, \beta]:$

(ii) $f(t) \geq 0,\ g(t) \geq 0,$ *for all* $t \in [\alpha, \beta];$

(iii) $[(R+c)M_1 + (R+c)^p M_2](\beta - \alpha) \leq R$, *where* M_1, M_2, R *are such that:*

$$|f(t)| \leq M_1, |g(t)| \leq M_2, \forall t \in [\alpha, \beta]$$

and

$$x \in \overline{B}(c, R) \subset (C([\alpha, \beta], \mathbb{R}_+), \|\cdot\|_\tau) \implies x(t) \in \mathbb{R}_+, \forall t \in [\alpha, \beta], \forall p \in \mathbb{R}^* \backslash \{1\}.$$

Then:

(a) there exists a unique solution $x^* \in \overline{B}(c, \mathbb{R})$ *to the equation:*

$$x(t) = c + \int_\alpha^t [f(s)x(s) + g(s)x^p(s)]\, ds, \text{ for all } t \in [\alpha, \beta]; \tag{4.1.11}$$

(b) if $x \in \overline{B}(c, \mathbb{R})$ *satisfies the inequality:*

$$x(t) \leq c + \int_\alpha^t [f(s)x(s) + g(s)x^p(s)]\, ds, \text{ for all } t \in [\alpha, \beta]; \tag{4.1.12}$$

then:

$$x(t) \leq \exp(\int_\alpha^t f(s)ds) \left[c^{1-p} + (1-p)\int_\alpha^t g(s)\exp\left[(p-1)\int_\alpha^s f(r)dr\right]ds\right]^{\frac{1}{1-p}}, \tag{4.1.13}$$

for all $t \in [\alpha, \beta]$.

(c) if $x \in \overline{B}(c, \mathbb{R})$ *satisfies the inequality:*

$$x(t) \geq c + \int_\alpha^t [f(s)x(s) + g(s)x^p(s)]\, ds, \text{ for all } t \in [\alpha, \beta]; \tag{4.1.14}$$

then:

$$x(t) \geq \exp(\int_\alpha^t f(s)ds) \left[c^{1-p} + (1-p)\int_\alpha^t g(s)\exp\left[(p-1)\int_\alpha^s f(r)dr\right]ds\right]^{\frac{1}{1-p}}, \tag{4.1.15}$$

for all $t \in [\alpha, \beta]$.

4.2 Optimality Results in Generic Form

In this section we consider an inequality of the type $x \leq A(x)$ for which the optimal bound x_A^* cannot be derived explicitly and will only be presented in the generic form. In some particular cases explicit expressions for x_A^* exist (see [22]), and we present their derivation.

Theorem 4.2.1. *(Wendroff-type, [41], see also [8], [46], [51], [56]) We assume that:*

(i) $\varphi \in C([0,\alpha] \times [0,\beta], \mathbb{R}_+)$, $a \in \mathbb{R}_+$;

(ii) φ *is increasing.*

If $x \in C([0,\alpha] \times [0,\beta])$ *is a solution to the inequality:*

$$x(t_1,t_2) \leq a + \int_0^{t_1}\int_0^{t_2} \varphi(s,t)x(s,t)dsdt, \quad t_1 \in [0,\alpha],\ t_2 \in [0,\beta], \tag{4.2.1}$$

then:

$$x(t_1,t_2) \leq a \exp\left(\int_0^{t_1}\int_0^{t_2} \varphi(s,t)dsdt\right). \tag{4.2.2}$$

We consider $(X, \to, \leq) := C(D, \overset{\|\cdot\|_\tau}{\to}, \leq)$, where $D = [0,\alpha] \times [0,\beta]$, and $\|\cdot\|_\tau$ is the Bielecki norm on $C(D)$:

$$\|x\|_\tau := \max_D(|x(t_1,t_2)| \exp(-\tau(t_1+t_2))), \quad \tau \in \mathbb{R}_+^*. \tag{4.2.3}$$

The corresponding operator $A : X \to X$ is defined by:

$$A(x)(t_1,t_2) := a + \int_0^{t_1}\int_0^{t_2} \varphi(s,t)x(s,t)dsdt, \quad (t_1,t_2) \in D. \tag{4.2.4}$$

This operator is an increasing Picard operator, but the function:

$$(t_1,t_2) \mapsto a \exp\left(\int_0^{t_1}\int_0^{t_2} \varphi(s,t)dsdt\right) \tag{4.2.5}$$

is not a fixed point of the operator A. Therefore we have proved the following remark:

Remark 4.2.1. *The right hand side of (4.2.2) is not a fixed point of the operator* A*, so Theorem 4.2.1 is not a consequence of the Abstract Gronwall Lemma 2.0.2.*

On the other hand, the Abstract Gronwall Lemma 2.0.2 allows us to obtain a theoretical upper bound of the type of the right-hand side of (4.2.2) without finding an explicit form of this bound.

Theorem 4.2.2. *(see [22])We assume that:*

(i) $\varphi \in C([0,\alpha] \times [0,\beta], \mathbb{R}_+)$, $a \in \mathbb{R}_+$;

(ii) φ *is increasing.*

If $x \in C([0,\alpha]\times[0,\beta])$ *is a solution to the inequality (4.2.1), then* $x(t_1,t_2) \leq x_A^*(t_1,t_2)$, *where* $x_A^*(t_1,t_2)$ *is the unique fixed point of the corresponding operator* A *defined by (4.2.4).*

Next we present particular cases of the inequality (4.2.1), for which we derive the optimal bounds of the solutions. One of them is the following:

Example 4.2.1. *(Wendroff's inequality (4.2.1) for* $\varphi(t_1,t_2) \equiv 1$*, see [22])*

Let $a \in \mathbb{R}_+$ *and* $c \in \mathbb{R}$ *be given. If* $x \in C(D,\mathbb{R}_+)$ *is a solution to the inequality:*

$$x(t_1,t_2) \leq a + c^2 \int_0^{t_1} \int_0^{t_2} x(s,t)dsdt, \tag{4.2.6}$$

with conditions:

$$x(t_1,0) = x(0,t_2) = a, \quad t_1 \in [0,\alpha],\ t_2 \in [0,\beta],$$

then:

$$x(t_1,t_2) \leq a\exp(c^2 t_1 t_2),\ \forall\ t_1 \in [0,\alpha], t_2 \in [0,\beta]. \tag{4.2.7}$$

In this case the corresponding operator $A : X \to X$ *is given by:*

$$A(x)(t_1,t_2) := \alpha + c^2 \int_0^{t_1} \int_0^{t_2} x(s,t)dsdt, \quad t_1 \in [0,\alpha],\ t_2 \in [0,\beta]. \tag{4.2.8}$$

This operator is an increasing Picard operator, but the function:

$$(t_1,t_2) \mapsto a\exp(c^2 t_1 t_2)$$

is not a fixed point of the operator A.

Using Theorem 4.2.2 we obtain:

$$x(t_1,t_2) \leq x_A^*(t_1,t_2), \tag{4.2.9}$$

where the fixed point $x_A^*(t_1, t_2)$ is (see [46], [49]):

$$x_A^*(t_1, t_2) = a\ J_0(2c\sqrt{t_1 t_2}). \tag{4.2.10}$$

Here $J_0(2c\sqrt{t_1 t_2})$ is the Bessel function.

In this chapter we used the Abstract Gronwall Lemma to improve the bounds of the solutions to specific inequalities. For some solutions we derived the bounds explicitly and for the solutions of Wendroff-type inequalities we found theoretical bounds only. However, we have given some examples of Wendroff-type inequalities for which the bounds of the solutions can be derived explicitly.

Chapter 5

Applications to the Study of Solutions and to Stability

In this chapter we present some applications of the Abstract Gronwall Lemma to various differential inequalities. We also apply a specific Gronwall-type lemma to the study of different types of Ulam stabilities for the second-order differential equations of hyperbolic type. The results of this chapter can be found in [23] and [24].

5.1 Applications to the Study of Solutions to Various Inequalities

In this section we consider different types of differential equation to which we add boundary conditions. This results in Darboux problems. We prove the existence and uniqueness of the solutions to these problems. Using Abstract Gronwall Lemma we also prove that the solutions to these Darboux problem are the optimal bounds for the functions which satisfy the corresponding differential inequalities.

5.1.1 Second-order Hyperbolic Inequalities

Hyperbolic equations and inequalities represent a vast domain of Partial Differential Equations, with many practical applications. They have been studied by many authors (see [26], [35], [46], [49], [75] and others).

In this subsection we study the qualitative behaviour of the solutions to some second-order differential equations of hyperbolic type.

Let $\alpha > 0$, $\beta > 0$, $\overline{D} := [0,\alpha] \times [0,\beta]$ and $\mathbb{B}$ be a real or complex Banach space. We consider the following hyperbolic inequality:

$$\frac{\partial^2 x}{\partial t_1 \partial t_2}(t_1,t_2) \leq f(t_1,t_2,x(t_1,t_2),\frac{\partial x}{\partial t_1}(t_1,t_2),\frac{\partial x}{\partial t_2}(t_1,t_2)),\ \text{for all } (t_1,t_2) \in \overline{D} \tag{5.1.1}$$

and the Darboux problem:

$$\frac{\partial^2 x}{\partial t_1 \partial t_2}(t_1,t_2) = f(t_1,t_2,x(t_1,t_2),\frac{\partial x}{\partial t_1}(t_1,t_2),\frac{\partial x}{\partial t_2}(t_1,t_2)),\ \text{for all } (t_1,t_2) \in \overline{D} \tag{5.1.2}$$

$$\begin{cases} x(t_1,0) = \varphi(t_1),\ \text{for all } t_1 \in [0,\alpha]\,; \\ x(0,t_2) = \psi(t_2),\ \text{for all } t_2 \in [0,\beta]\,; \\ \varphi(0) = \psi(0), \end{cases} \tag{5.1.3}$$

where $f : \overline{D} \times \mathbb{B}^3 \to \mathbb{B}$, $\varphi \in C^1([0,\alpha]\,,\mathbb{B})$, $\psi \in C^1([0,\beta]\,,\mathbb{B})$, $x \in C^1(\overline{D},\mathbb{B})$ and $\frac{\partial^2 x}{\partial t_1 \partial t_2} \in C(\overline{D},\mathbb{B})$.

We prove the existence and uniqueness for the solution of the Darboux problem (5.1.2) + (5.1.3) by Picard operators technique, and also give a Gronwall type result for the inequality (5.1.1). In the case of $\mathbb{B}$:=$\mathbb{R}$ this inequality has been studied in [49], [46].... We present the results in the general case of a Banach space $\mathbb{B}$.

5.1.2 Third-order Hyperbolic Inequalities

In this subsection we generalise the results obtained in the previous section for second-order differential equations to the case of third-order differential equations of hyperbolic type.

5.1.3 Pseudoparabolic Inequalities

Pseudoparabolic equations and inequalities represent an important chapter of Partial Differential Equations, with many practical applications in electromagnetism, heat conduction etc (see [43], [46], [49], [73] etc).

In this subsection we consider the following pseudoparabolic inequality:

$$\frac{\partial^3 x}{\partial t_1^2 \partial t_2}(t_1,t_2) \leq F(t_1,t_2,x(t_1,t_2),\frac{\partial x}{\partial t_2}(t_1,t_2),\frac{\partial^2 x}{\partial t_1^2}(t_1,t_2)),\ (t_1,t_2)\in\overline{D}, \tag{5.1.4}$$

and the corresponding Darboux problem:

$$\frac{\partial^3 x}{\partial t_1^2 \partial t_2}(t_1,t_2) = F(t_1,t_2,x(t_1,t_2),\frac{\partial x}{\partial t_2}(t_1,t_2),\frac{\partial^2 x}{\partial t_1^2}(t_1,t_2)),\ (t_1,t_2)\in\overline{D}, \tag{5.1.5}$$

$$\begin{cases} x(t_1,0)=h(t_1),\ \text{for all } t_1\in[0,\alpha] \\ x(0,t_2)=g_1(t_2), \text{ for all } t_2\in[0,\beta] \\ \frac{\partial x}{\partial t_1}(0,t_2)=g_2(t_2), \text{ for all } t_2\in[0,\beta]\,, \end{cases} \tag{5.1.6}$$

where $\overline{D}:=[0,\alpha]\times[0,\beta]\,,\ F\in C(\overline{D}\times\mathbb{R}^3,\mathbb{R}),\ h\in C^2\,[0,\alpha]\,,\ g_1,\ g_2\in C^1\,[0,\beta],\ x\in C^1(\overline{D})$, $\frac{\partial^2 x}{\partial t_1^2},\ \frac{\partial^3 x}{\partial t_1^2\partial t_2}\in C(\overline{D}),\ h(0)=h'(0)=g_1(0)=g_2(0)=0.$

We apply the Picard operators technique to prove the existence and the uniqueness of the solution to the Darboux problem (5.1.5)+(5.1.6), and we give a Gronwall-type result for the inequality (5.1.4). We also use the Riemann function to represent the solution to the Darboux problem (5.1.5)+(5.1.6) for specific pseudoparabolic inequalities. The results are presented in [23].

Throughout this subsection we assume that:

$\alpha>0,\ \beta>0$ and $\overline{D}:=[0,\alpha]\times[0,\beta]\,.$

Theorem 5.1.1. *(see [23]) We assume that:*

(i) $F\in C(\overline{D}\times\mathbb{R}^3,\mathbb{R})$*;*

(ii) there exists $L_F>0$ *such that:*

$$|F(t_1,t_2,u_1,v_1,w_1)-F(t_1,t_2,u_2,v_2,w_2)|\leq L_F\max(|u_1-u_2|\,,|v_1-v_2|\,,|w_1-w_2|),$$

for all $(t_1, t_2) \in \overline{D}$ *and* $u_i, v_i, w_i \in \mathbb{R}$, $i \in \{1, 2\}$;

(iii) $h \in C^2\left[0, \alpha\right],\ g_1, g_2 \in C^1\left[0, \beta\right]$;

(iv) $F(t_1, t_2, \cdot, \cdot, \cdot) : \mathbb{R}^3 \to \mathbb{R}$ *is increasing, for all* $(t_1, t_2) \in \overline{D}$.

Then:

(a) the Darboux problem (5.1.5)+(5.1.6) has a unique solution $x^*(t_1, t_2)$;

(b) if $x(t_1, t_2)$ *satisfies the inequality (5.1.4) with the boundary conditions (5.1.6), then:*

$$x(t_1, t_2) \le x^*(t_1, t_2).$$

In the remainder of this subsection we consider some particular examples of inequality (5.1.4). By Theorem 5.1.1 any solution of such inequality is dominated by the solution of the corresponding Darboux problem, which we solve explicitly in terms of the Riemann function. Here we present one of them.

Example 5.1.1. *Let us consider the inequality* (5.1.4) *in the case*

$F = \frac{\partial x}{\partial t_2} - \frac{\partial^2 x}{\partial t_1^2}$ *(see [23], [19], also [73]):*

$$\frac{\partial^3 x}{\partial t_1^2 \partial t_2}(t_1, t_2) \le \frac{\partial x}{\partial t_2}(t_1, t_2) - \frac{\partial^2 x}{\partial t_1^2}(t_1, t_2), \tag{5.1.7}$$

and the corresponding Darboux problem:

$$\frac{\partial^3 x}{\partial t_1^2 \partial t_2}(t_1, t_2) - \frac{\partial x}{\partial t_2}(t_1, t_2) + \frac{\partial^2 x}{\partial t_1^2}(t_1, t_2) = 0, \tag{5.1.8}$$

with the boundary conditions (5.1.6).

With the notation

$$(Lx)(t_1, t_2) := \frac{\partial^3 x}{\partial t_1^2 \partial t_2}(t_1, t_2) - \frac{\partial x}{\partial t_2}(t_1, t_2) + \frac{\partial^2 x}{\partial t_1^2}(t_1, t_2)$$

the equation (5.1.8) becomes:

$$Lx = 0.$$

From Theorem 5.1.1 it follows that the problem (5.1.8) + (5.1.6) has a unique solution $x^*(t_1, t_2)$, *and if* $x(t_1, t_2)$ *is a solution to the problem (5.1.7) + (5.1.6), then:*

$$x(t_1, t_2) \le x^*(t_1, t_2). \tag{5.1.9}$$

In this case the solution $x^*(t_1, t_2)$ can be represented using the Riemann function $v(t_1, t_2)$, which is solution of the adjoint equation (see [49], [73], [46]) :

$$(L^*v)(t_1, t_2) := \frac{\partial^3 v}{\partial t_1^2 \partial t_2}(t_1, t_2) - \frac{\partial v}{\partial t_2}(t_1, t_2) - \frac{\partial^2 v}{\partial t_1^2}(t_1, t_2) = 0.$$

The solution $x^*(t_1, t_2)$ is:

$$\begin{aligned} x^*(t_1, t_2) &= h(t_1) - \int_0^{t_1} h'(s) v_s(s, 0; t_1, t_2) ds \\ &+ \int_0^{t_2} \left[g_2'(t) v_t(0, t; t_1, t_2) - g_1'(t) v_{st}(0, t; t_1, t_2) \right] dt \\ &+ \int_0^{t_2} \left[g_2(t) v_t(0, t; t_1, t_2) - g_1'(t) v_s(0, t; t_1, t_2) \right] dt, \end{aligned}$$

where $v(t_1, t_2)$ is the Riemann function corresponding to $L^*v = 0$.

5.2 Applications to Ulam Stabilities of Hyperbolic Equations

In this section we use the following Gronwall Lemma to give some results on Ulam-Hyers stability and generalised Ulam-Hyers-Rassias stability of the hyperbolic partial differential equation studied in subsection 5.1.1 of this chapter.

Lemma 5.2.1. *(see [41], also [56]) We assume that:*

(i) $x, \varphi, g \in C(\mathbb{R}_+^n, \mathbb{R}_+)$;

(ii) for any $t \geq t_0$ we have:

$$x(t) \leq g(t) + \int_{t_0}^{t} \varphi(s) x(s) ds;$$

(iii) $g(t)$ is positive and increasing.

Then:

$$x(t) \leq g(t) \exp \int_s^t \varphi(r) dr, \text{ for any } t \geq t_0.$$

Results on Ulam stability for the functional equations are well known (see, for instance, [15], [34], [38], [67], [68]).

We start by presenting some definitions and results on different types of Ulam stability for a hyperbolic partial differential equation.

Throughout this section we consider

$$\varepsilon > 0,\ \alpha, \beta \in (0, \infty],\ \varphi \in C\left([0, \alpha) \times [0, \beta), \mathbb{R}_+\right),$$

where $(\mathbb{B}, |\cdot|)$ is a real or complex Banach space.

We consider the following hyperbolic partial differential equation:

$$\frac{\partial^2 x}{\partial t_1 \partial t_2}(t_1, t_2) = f(t_1, t_2, x(t_1, t_2), \frac{\partial x}{\partial t_1}(t_1, t_2), \frac{\partial x}{\partial t_2}(t_1, t_2)), \tag{5.2.1}$$

for $0 \leq t_1 < \alpha$, $0 \leq t_2 < \beta$, where $f \in C\left([0, \alpha) \times [0, \beta) \times \mathbb{B}^3, \mathbb{B}\right)$.

We also consider the following inequalities:

$$\left|\frac{\partial^2 y}{\partial t_1 \partial t_2}(t_1, t_2) - f(t_1, t_2, y(t_1, t_2), \frac{\partial y}{\partial t_1}(t_1, t_2), \frac{\partial y}{\partial t_2}(t_1, t_2))\right| \leq \varepsilon, \tag{5.2.2}$$

$$\left|\frac{\partial^2 y}{\partial t_1 \partial t_2}(t_1, t_2) - f(t_1, t_2, y(t_1, t_2), \frac{\partial y}{\partial t_1}(t_1, t_2), \frac{\partial y}{\partial t_2}(t_1, t_2))\right| \leq \varphi(t_1, t_2), \tag{5.2.3}$$

$$\left|\frac{\partial^2 y}{\partial t_1 \partial t_2}(t_1, t_2) - f(t_1, t_2, y(t_1, t_2), \frac{\partial y}{\partial t_1}(t_1, t_2), \frac{\partial y}{\partial t_2}(t_1, t_2))\right| \leq \varepsilon\varphi(t_1, t_2), \tag{5.2.4}$$

for all $t_1 \in [0, \alpha)$, $t_2 \in [0, \beta)$.

We need the following definitions and results (see [87], [90] and [89])

Definition 5.2.1. *A function x is a solution to the equation (5.2.1) if $x \in C\left([0, \alpha) \times [0, \beta)\right) \cap C^1\left([0, \alpha) \times [0, \beta)\right)$, $\frac{\partial^2 x}{\partial t_1 \partial t_2} \in C\left([0, \alpha) \times [0, \beta)\right)$ and x satisfies (5.2.1).*

Definition 5.2.2. *The equation (5.2.1) is Ulam-Hyers stable if there exists the real numbers C_f^1, C_f^2 and $C_f^3 > 0$ such that for any $\varepsilon > 0$ and for any solution y to the inequality (5.2.2) there exists a solution x to the equation (5.2.1) with:*

$$\begin{cases} |y(t_1, t_2) - x(t_1, t_2)| \leq C_f^1 \varepsilon,\ \forall\ t_1 \in [0, \alpha),\ \forall\ t_2 \in [0, \beta), \\ \left|\frac{\partial y}{\partial t_1}(t_1, t_2) - \frac{\partial x}{\partial t_1}(t_1, t_2)\right| \leq C_f^2 \varepsilon,\ \forall\ t_1 \in [0, \alpha),\ \forall\ t_2 \in [0, \beta), \\ \left|\frac{\partial y}{\partial t_2}(t_1, t_2) - \frac{\partial x}{\partial t_2}(t_1, t_2)\right| \leq C_f^3 \varepsilon,\ \forall\ t_1 \in [0, \alpha),\ \forall\ t_2 \in [0, \beta). \end{cases} \tag{5.2.5}$$

Definition 5.2.3. *The equation (5.2.1) is generalised Ulam-Hyers-Rassias stable if there exists the real numbers* $C^1_{f,\varphi}, C^2_{f,\varphi}$ *and* $C^3_{f,\varphi} > 0$ *such that for any* $\varepsilon > 0$ *and for any solution* y *to the inequality (5.2.3) there exists a solution* x *to the equation (5.2.1) with:*

$$\begin{cases} |y(t_1,t_2) - x(t_1,t_2)| \le C^1_{f,\varphi}\varphi(t_1,t_2),\ \forall\ t_1 \in [0,\alpha),\ \forall\ t_2 \in [0,\beta), \\ \left|\frac{\partial y}{\partial t_1}(t_1,t_2) - \frac{\partial x}{\partial t_1}(t_1,t_2)\right| \le C^2_{f,\varphi}\varphi(t_1,t_2),\ \forall\ t_1 \in [0,\alpha),\ \forall\ t_2 \in [0,\beta), \\ \left|\frac{\partial y}{\partial t_2}(t_1,t_2) - \frac{\partial x}{\partial t_2}(t_1,t_2)\right| \le C^3_{f,\varphi}\varphi(t_1,t_2),\ \forall\ t_1 \in [0,\alpha),\ \forall\ t_2 \in [0,\beta). \end{cases} \tag{5.2.6}$$

In the following we denote:

$$\begin{aligned} x_1(t_1,t_2) &= \frac{\partial x}{\partial t_1}(t_1,t_2),\ x_2(t_1,t_2) = \frac{\partial x}{\partial t_2}(t_1,t_2), \\ y_1(t_1,t_2) &= \frac{\partial y}{\partial t_1}(t_1,t_2),\ y_2(t_1,t_2) = \frac{\partial y}{\partial t_2}(t_1,t_2). \end{aligned}$$

In what follows we give a result on the existence and uniqueness of the solution to the equation (5.2.1), and also on Ulam-Hyers stability for the same equation in the case $\alpha < \infty$ and $\beta < \infty$.

Theorem 5.2.1. *We assume that:*

(i) $\alpha < \infty,\ \beta < \infty$;

(ii) $f \in C([0,\alpha] \times [0,\beta] \times \mathbb{B}^3, \mathbb{B})$;

(iii) there exists $L_f > 0$ *such that:*

$$|f(t_1,t_2,z_1,z_2,z_3) - f(t_1,t_2,t_1,t_2,t_3)| \le L_f \max\{|z_i - t_i|, i = 1,2,3\}, \tag{5.2.7}$$

for all $t_1 \in [0,\alpha]$, $t_2 \in [0,\beta]$ *and* $z_1,z_2,z_3,t_1,t_2,t_3 \in \mathbb{B}$.

Then:

(a) for $\phi \in C^1([0,\alpha],\mathbb{B})$ *and* $\psi \in C^1([0,\beta],\mathbb{B})$ *the equation (5.2.1) has a unique solution, which satisfies:*

$$x(t_1,0) = \phi(t_1),\ \textit{for all } t_1 \in [0,\alpha], \tag{5.2.8}$$

and

$$x(0,t_2) = \psi(t_2),\ \textit{for all } t_2 \in [0,\beta]; \tag{5.2.9}$$

(b) the equation (5.2.1) is Ulam-Hyers stable.

Remark 5.2.1. *If $\alpha = \infty$ or $\beta = \infty$, then the equation (5.2.1) is not Ulam-Hyers stable.*

In the following we consider the hyperbolic partial differential equation (5.2.1) and the inequality (5.2.3) in the case $\alpha = \infty$ and $\beta = \infty$, and we prove the generalised Ulam-Hyers-Rassias stability of equation (5.2.1).

Theorem 5.2.2. *We assume that:*

(i) $f \in C([0,\infty) \times [0,\infty) \times \mathbb{B}^3, \mathbb{B})$;

(ii) there exists $l_f \in C^1([0,\infty) \times [0,\infty), \mathbb{R}_+)$ such that:

$$|f(t_1,t_2,z_1,z_2,z_3) - f(t_1,t_2,t_1,t_2,t_3)| \le l_f(t_1,t_2) \max\{|z_i - t_i|, i = 1,2,3\}, \tag{5.2.10}$$

for all $t_1, t_2 \in [0,\infty)$;

(iii) there exist $\lambda_\varphi^1, \lambda_\varphi^2, \lambda_\varphi^3 > 0$ such that:

$$\begin{cases} \int_0^{t_1}\int_0^{t_2} \varphi(s,t)dsdt \le \lambda_\varphi^1 \varphi(t_1,t_2), \text{ for all } t_1,t_2 \in [0,\infty), \\ \int_0^{t_2} \varphi(t_1,t)dt \le \lambda_\varphi^2 \varphi(t_1,t_2), \text{ for all } t_1,t_2 \in [0,\infty), \\ \int_0^{t_1} \varphi(s,t_2)ds \le \lambda_\varphi^3 \varphi(t_1,t_2), \text{ for all } t_1,t_2 \in [0,\infty); \end{cases} \tag{5.2.11}$$

(iv) $\varphi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is increasing.

Then the equation (5.2.1) with $\alpha = \infty$ and $\beta = \infty$ is generalised Ulam-Hyers-Rassias stable.

Bibliography

[1] S. András, *Fredholm-Volterra equations*, P.U.M.A., **13** (2002), no. 1-2, 21-30.

[2] S. András, *Ecuaţii integrale Fredholm-Volterra*, Ed. Didactică si Pedagogică, Bucureşti, 2005.

[3] S. András, *Gronwall type inequalities via subconvex sequences*, Seminar on Fixed Point Theory, **3** (2002), 183-188.

[4] M. Angrisani and M. Clavelli, *Synthetic approaches to problems of fixed points in metric spaces*, Ann. Mat. Pura Appl., **170** (1996), 1-12.

[5] J. Appell, A.S. Kalitvin, *Existence results for integral equations: Spectral methods vs. Fixed point theory*, Fixed Point Theory, **7** (2006), no. 2, 219-234.

[6] S. Ashirov, Ya. D. Mamedov, *A Volterra-type integral equation,* UMJ 40, **4** (1988), 510-515.

[7] M. Ashyraliyev, *Generalizations of Gronwall's integral inequality and their discrete analogies*, CWI Netherlands, no. E0520 (2005), 1-27.

[8] D. Bainov, P. Simeonov, *Integral inequalities and applications*, Kluwer Academic Publisher, Dordrecht/Boston/London, no. 57, 1992.

[9] D. Bainov, A. I. Zahariev, A. D. Myshkis, *Asymptotic properties of a class of operator-differential inequalities*, Publ. RIMS Kyoto Univ. 20, **5** (1984), 903-911.

[10] P. R. Beesack, *Comparison theorems and integral inequalities for Volterra integral equations*, Proc.Am.Math.Soc., **20** (1969), 61-66.

[11] R. Bellman, *The stability of solutions of differential equations*, Duke Math. Journal, **10** (1943), no. 4, 643-647.

[12] C. Bessaga, *On the converse of the Banach fixed point principle*, Colloq.Math., **7** (1959), 41-43.

[13] T.A. Burton, *Volterra integral and differential equations*, Academic Press, New York, 1983.

[14] F. Caliò, E. Marcchetti, V. Mureşan, *On some Volterra-Fredholm integral equations*, Int. J. Pure Appl. Math., **31** (2006), no. 2, 173-184.

[15] L. Cădariu, V. Radu, *The Fixed Points Method for the Stability of Some Functional Equations*, Carphatian J. Math., **23**(2007), no. 1-2, 63-72.

[16] C.J. Chen, W. S. Cheung, D. Zhao, *Gronwall-Bellman-type integral inequalities and applications to BVPs*, Journal of inequalities and Applications, 2009, 1-15.

[17] Y.J. Cho, S.S. Dragomir, Y.H. Kim, *On some integral inequalities with iterated integrals*, J. Korean Math. Soc., **43** (2006), no. 3, 563-578.

[18] S.C. Chu, F.T. Matcalf, *On Gronwall's inequality*, Proc. Am. Math. Soc., **18** (1967), 439-440.

[19] D.L. Colton, *Pseudoparabolic equations in one space variable*, Journal of Diff. Eq., **12** (1972), 559-565.

[20] C. Corduneanu, *Integral equations and applications*, Cambridge Univ. Press, 1991.

[21] C. Crăciun, *On some Gronwall inequalities*, Seminar on Fixed Point Theory, Cluj-Napoca, **1** (2000), 31-34.

[22] C. Crăciun, N. Lungu, *Abstract and concrete Gronwall lemmas*, Fixed Point Theory, **10** (2009), no. 2, 221-228.

[23] C. Crăciun, N. Lungu, *Pseudoparabolic inequalities* (submitted).

[24] C. Crăciun, N. Lungu, *Ulam-Hyers-Rassias stability of a hyperbolic partial differential equation* (submitted).

[25] C. Crăciun, M.A. Şerban, *A nonlinear integral equation via Picard operators*, Fixed Point Theory (to appear).

[26] G.S. Desző, *Ecuaţii hiperbolice cu argument modificat. Tehnica punctului fix*, Presa Univ. Clujeană, Cluj-Napoca, 2003.

[27] M. Dobriţoiu, *Existence and continuous dependence on data of the solution of an integral equation*, Bulletins for applied&computer mathematics, Budapest, BAM-CVI/2005.

[28] M. Dobriţoiu, *Properties of the solution of an integral equation with modified argument*, Carpathian Journal of Mathematics, Baia Mare, **23** (2007), no. 1-2, 77-80.

[29] M. Dobriţoiu, I.A. Rus, M.A. Şerban, *An integral equation arising from infectious diseases via Picard operators*, Studia Univ. Babeş-Bolyai, Mathematica, vol LII, **3** (2007), 81-94.

[30] S.S. Dragomir, *On some Gronwall type lemmas*, Studia Univ. Babeş-Bolyai, Mathematica, XXXIII, **4** (1988), 29-36.

[31] M. Fréchet, *Les espaces abstraits*, Gauthier-Villars, Paris, 1928.

[32] H. Gronwall, *Note on the derivative with respect to a parameter of the solutions of a system of differential equations*, Ann. of Math, **20** (1919), no. 4, 292-296.

[33] D. Guo, V. Lakshmikantham, X. Liu, *Nonlinear integral equations in abstract spaces*, Kluwer Academic Publishers, Dordrecht-Boston-London, 1996.

[34] D.H. Hyers, G. Isac and Th. M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhäuser, Basel, 1998.

[35] D.V. Ionescu, *Sur une classe d'équations functionelles*, Thèse, Paris, 1927.

[36] D.V.Ionescu, *Ecuatii diferentiale si integrale*, Editura Didactică si Pedagogică, Bucureşti, 1972.

[37] L. Janos, *A converse of Banach's contraction theorem*, Proc. A.M.S., **18** (1967), 287-289.

[38] S.M. Jung, K.S. Lee, *Ulam-Hyers-Rassias stability of linear differential equations of second order*, J. Comput. Math. Optim., **3** (2007), no. 3, 193-200.

[39] W.A. Kirk, B. Sims, *Handbook of metrical fixed point theory*, Kluwer Academic Publishers, Dordrecht, 2001.

[40] M.A. Krasnoselskii, *Topological methods in the theory of nonlinear integral equations*, Pergamon Press, Oxford-London-New York-Paris, 1964.

[41] V. Lakshmikantham, S. Leela and A.A. Martynyuk, *Stability analysis of nonlinear systems*, New York, 1989.

[42] T. Lalescu, *Introducere in teoria ecuatiilor integrale*, Ed. Acad. Bucuresti, 1956 (first edition in 1911).

[43] K.B. Liaskos, I.G. Stratis, A.N. Yannacopoulos, *Pseudoparabolic equations with additive noise and applications*, Math. Methods in the Applied Sciences, **32** (2008), no. 8, 963-985.

[44] N. Lungu, *On some Gronwall-Bihari-type inequalities*, Libertas Mathematica, **20** (2000), 67-70.

[45] N. Lungu, *On some Gronwall-Bihari-Wendorff-type inequalities*, Seminar on Fixed Point Theory, **3** (2002), 249-254.

[46] N. Lungu, *Qualitative problems in the theory of hyperbolic differential equations*, Digital Data, Cluj-Napoca, 2006.

[47] N. Lungu, *On some Volterra integral inequalities*, Fixed Point Theory, **8** (2007), no. 1, 39-45.

[48] N. Lungu, D. Popa, *On some differential inequalities*, Seminar of Fixed Point Theory, **3** (2002), 323-327.

[49] N. Lungu, I.A. Rus, *Hyperbolic differential inequalities*, Libertas Mathematica, **21** (2001), 35-40.

[50] N. Lungu, I.A. Rus, *Gronwall inequalities via Picard operators* (to appear).

[51] D.S. Mitrinović, J.E. Pečarić, A.M. Fink, *Inequalities involving functions and their integrals and derivatives,* Kluwer Academic Publishers, Dordrecht-Boston-London, 1991.

[52] V. Mureşan, *A Gronwall type inequality for Fredholm Operators*, Mathematica, Tome 41(64), no. **2** (1999), 227-231.

[53] V. Mureşan, *Functional-Integral Equations*, Media-Mira, Cluj-Napoca, 2003.

[54] J. Norbury, A.M. Stuart, *Volterra integral equations and a new Gronwall inequality. Part 1: The linear case*, Proc. Royal Soc. Edinburgh, **106A** (1987), 361-373.

[55] B.G. Pachpate, *On the existence and uniqueness of solutions of Volterra-Fredholm integral equations*, Mathematics Seminar Notes, **10** (1982), 733-742.

[56] B.G. Pachpatte, *Inequalities for differential and integral equations,* Academic Press, New York-London, 1998.

[57] B.G. Pachpatte, *Explicit bounds on Gamidov type integral inequalities*, Tamkang Journal of Mathematics, **37** (2006), no. **1**, 1-9.

[58] B.G. Pachpatte, *Some basic theorems on difference-differencial equations*, Electronic Journal of Differential Equations, no.75 (2008), 1-11.

[59] B.G. Pachpatte, *On certain Volterra integral and integro-differential equations*, Facta Universitas, **23** (2008), 1-12.

[60] E. De Pascale, P.P. Zabreiko, *Fixed point theorems for operators in spaces of continuous functions,* Fixed Point Theory, 5 (2004), no. 1, 117-129.

[61] P. Pavel, *Sur un systeme d'equations aux derivees partielles*, Seminar on Differential Equations, Preprint **3** (1989), 121-138.

[62] I.G. Petrovschi, *Lectii de teoria ecuatiilor integrale*, Ed. Tehnica, 1947.

[63] A. Petruşel, *Fredholm-Volterra integral equations and Maia's theorem,* Univ. Babeş-Bolyai, Preprint **3 (**1988), 79-82.

[64] A. Petruşel, *Multivalued weakly Picard operators and applications*, Scientiae Mathematicae Japonicae, **59** (2004), 167-202.

[65] A. Petruşel, I.A. Rus, *Fixed point theorems in ordered L-spaces*, Proc. Am. Math. Soc., **134** (2005), no.**2**, 411-418.

[66] A.D. Polyanin, A.V. Manzhirov, *Handbook of integral equations*, CRC Press, London, 1998.

[67] D. Popa, *Functional equations. Set-valued solutions. Stability*, Technical Univ. Press, Cluj-Napoca, 2006.

[68] D. Popa, *On the Stability of the General Linear Equation*, Result. Math. **53** (2009), 383-389.

[69] D. Popa, N. Lungu, *On an operatorial inequality*, Demonstratio Mathematica, **38** (2005), 667-674.

[70] R. Precup, *Ecuatii integrale neliniare*, UBB Cluj-Napoca, 1993.

[71] R. Precup, E. Kirr, *Analysis of nonlinear integral equation modelling infection diseases*, Proc. of the International Conference, Univ. of West, Timisoara, 1997, 178-195.

[72] R. Precup, *Methods in nonlinear integral equations*, Kluwer Academic Publishers, Dordrecht/Boston/London, 2002.

[73] W. Rundell, M. Stecher, *Remarks concerning the support solutions of pseudoparabolic equations,* Proc. Amer. Math. Soc., **63** (1977), no.1, 77-81.

[74] I.A. Rus, *Metrical fixed point theorems*, Univ. of Cluj-Napoca, 1979.

[75] I.A. Rus, *On the problem of Darboux-Ionescu*, Babeş-Bolyai Univ., Preprint no. 1, 1981.

[76] I.A. Rus, *Generalized contractions*, Seminar on Fixed Point Theory, 1–130, Preprint, 83-3, Univ. "Babeş-Bolyai", Cluj-Napoca, 1983.

[77] I.A. Rus, *Weakly Picard mappings*, Comment. Math. Caroline, **34** (1993), no. 4, 769-773.

[78] I.A. Rus, *Ecuaţii diferenţiale, ecuaţii integrale şi sisteme dinamice*, Transilvania Press, 1996.

[79] I.A. Rus, *Picard operators and applications*, Seminar on Fixed Point Theory, Cluj-Napoca, preprint **3** (1996).

[80] I.A. Rus, *Who authored the first integral equations book in the world?*, Seminar on Fixed Point Theory Cluj-Napoca, **1** (2000), 81-86.

[81] I.A. Rus, *Generalized contractions and applications*, Cluj University Press, Cluj-Napoca, 2001.

[82] I.A. Rus, *Weakly Picard operators and applications*, Seminar on Fixed Point Theory Cluj-Napoca, **2** (2001), 41-58.

[83] I.A. Rus, *Picard operators and applications*, Scientiae Mathematicae Japonicae, **58** (2003), no. 1, 191-219.

[84] I.A. Rus, *Fixed points, upper and lower fixed points: abstract Gronwall lemmas,* Carpathian J. Math., **20** (2004), no. 1, 125-134.

[85] I.A. Rus, *The theory of a metrical fixed point theorem: theoretical and applicative relevance,* Fixed Point Theory, **9** (2008), 541-559.

[86] I.A. Rus, *Gronwall Lemmas: Ten open problems*, Scientiae Mathematicae Japonicae, **70** (2009), no. 2, 221-228.

[87] I.A. Rus, *Ulam stability of ordinary differential equations*, Studia Univ. Babeş-Bolyai, Mathematica, **54** (2009), no. 4.

[88] I.A. Rus, *Remarks on Ulam stability of the operatorial equations*, Fixed Point Theory, **10** (2009), no. 2, 305-320.

[89] I.A. Rus, *Gronwall lemma approach to the Hyers-Ulam-Rassias stability of an integral equation*, Nonlinear analysis and variational problems, Springer New York, **35** (2010).

[90] I.A. Rus, N. Lungu, *Ulam stability of a nonlinear hyperbolic partial differential equation*, Carphatian J. Math., **24** (2008), no. 3, 403-408.

[91] I.A. Rus, A. Petruşel, M.A. Şerban, *Weakly Picard operators:equivalent definitions, applications and open problems*, Fixed Point Theory, **7** (2006), No.1, 3-22.

[92] I.A. Rus, A. Petruşel, G. Petruşel, *Fixed point theory*, Cluj University Press, 2008.

[93] I.A. Rus, M.A. Şerban, *Operators on infinite dimensional cartesian product,* Analele Univ.Vest Timisoara (to appear).

[94] A. Sîncelean, *On a class of functional-integral equations,* Seminar on Fixed Point Theory, Cluj-Napoca, **1** (2000), 87-92.

[95] M.A. Şerban, *Teoria punctului fix pentru operatori definiţi pe produs cartezian*, Presa Univ. Clujeană, Cluj-Napoca, 2002.

[96] V.Ya. Stetsenko, M. Shaban, *On operatorial inequalities analogous to Gronwall-Bihari ones*, D. A. N. Tadj., **29** (1986), 393-398 (in Russian).

[97] N.E. Tatar, *An impulsive nonlinear singular version of the Gronwall-Bihari inequality*, Journal of Inequalities and Applications, 2006, 1-12.

[98] N. Taghizadeh, V. Khanbabai, *On the formalization of the solution of Fredholm integral equations with degenerate kernel*, International Mathematical Forum, **3** (2008), no.14, 695-701.

[99] F.G. Tricomi, *Integral equations,* Dover Publications, 1985.

[100] M. Zima, *The abstract Gronwall Lemma for some nonlinear operators*, Demonstratio Mathematica, **XXXI** (1998), no.2, 325-332.

[101] K. Young-Ho, *Explicit bounds on some nonlinear integral inequalities,* J. of Ineq. in Pure and Applied Math., **7** (2006), no. 2, art 48.