“BABEŞ-BOLYAI” UNIVERSITY CLUJ-NAPOCA
FACULTY OF MATHEMATICS AND COMPUTER SCIENCE

Vasile Dincuță

Continuation methods in the study of periodic solutions for nonlinear functional-differential equations

Ph.D. Thesis

Supervisor:
Prof. dr. Radu Precup

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Introduction

One of the most used techniques to study the existence of solutions for nonlinear equations is given by continuation methods. These methods are based on Leray-Schauder theorems, also named continuation theorems, and represents one of the most powerful methods in study of operatorial equations, in particular of nonlinear functional-differential equations.

Shortly speaking, the continuation methods guarantees the existence of one solution for an equation starting from the solution of another simpler equation. If Λ and Δ are two sets, such that $\Lambda \subseteq \Delta$, and $F : \Lambda \to \Delta$ is an application, in order to solve the fixed point equation

$$F(x) = x, \tag{*}$$

we will associate to this equation another one, a "simpler" one

$$G(x) = x. \tag{**}$$

Using an homotopy, namely an application $H : \Lambda \times [0,1] \to \Delta$ which makes the connection between F and G by equalities

$$H(\cdot, 0) = G \text{ and } H(\cdot, 1) = F,$$

the continuation theorems contains conditions which guarantees that a solution of the simpler equation (**) can be "continued" to a solution of the initial equation (*).

The first approaches to continuation methods where made by H. Poincaré [64],[65] at the start of XX century for the study of existence of periodic solutions for dynamical systems and in the same time, by S. Bernstein[3] in the study of existence of solutions for second order differential equations using "a priori" boundedness methods. The first abstract formulation of the continuation principle was made by J. Leray and J. Schauder[37] in terms of topological degree theory.

Theorem 1 *[37] Let $(X, |\cdot|)$ a Banach space, $U \subset X$ an open, bounded subset, with $0 \in U$ and $H : \overline{U} \times [0,1] \to X$ a completely continuous map. Suppose that the following conditions are fulfilled:*

(a) $H(x, \lambda) \neq x$ for any $x \in \partial U$ and any $\lambda \in [0,1]$;

(b) $\nu_{LS}(J - H(\cdot, 0), U, 0) \neq 0$.

Then, there exists at least one $x \in U$ such that $H(x, 1) = x$. Moreover,

$$\nu_{LS}(J - H(\cdot, 0), U, 0) = \nu_{LS}(J - H(\cdot, 1), U, 0).$$

Here, we denote by $J : X \to X$ the identity map and by $\nu_{LS}(F, U, 0)$ we understand the Leray-Schauder degree of map F relative to set U and the origin 0.

Later A. Granas[21], stated a new version of this principle without the topological degree, known as Topological Transversality Principle. Instead of condition (b) we ask for $H(\cdot, 0)$ to be an *essential map*. We say that an application $F : \overline{U} \to C$ is essential if is fixed point free on the boundary ∂U and any other completely continuous map $G : \overline{U} \to C$ equal to F on ∂U has at least one fixed point in U.

Theorem 2 *[21] Let $(X, |\cdot|)$ a Banach space, $U \subset X$ an open, bounded subset, with $0 \in U$ and $H : \overline{U} \times [0, 1] \to X$ a completely continuous map. Suppose that the following conditions are fulfilled:*

(a) $H(x, \lambda) \neq x$ for any $x \in \partial U$ and any $t \in [0, 1]$;

(b) $H(\cdot, 0)$ is essential.

Then, there exists at least one $x \in U$ such that $H(x, 1) = x$. Moreover, $H(\cdot, 1)$ is also essential.

The Leray-Schauder Principle and the Topological Transversality Principle are two powerful tools in case we want the localization of the solution in a convex and closed set (usually a closed ball of a given R radius).

In case we want a better localization of the solution in a given shell, or we want to prove the existence of multiple solutions, we can use another tool, namely the Krasnoselskii Type Theorems on cones. Introduced for the first time on 1960 by M. Krasnoselskii[35], these results guarantees the existence of solution in a given shell for a wide number of nonlinear equations when K is a cone of a normed linear space and the involved map $F : K \to K$ is compressive

$$\begin{cases} ||F(x)|| \geq ||x|| \text{ for } ||x|| = r, \\ ||F(x)|| \leq ||x|| \text{ for } ||x|| = R; \end{cases}$$

or expansive type

$$\begin{cases} ||F(x)|| \leq ||x|| \text{ for } ||x|| = r, \\ ||F(x)|| \geq ||x|| \text{ for } ||x|| = R. \end{cases}$$

As we can see above, the compressive-expansive type conditions are requested for the map F only on the boundary of shell $K_{r,R}$, while for the interior points of shell we have only an invariance condition by map F.

These results where extended by R. Precup[66](see also R. Precup[67]) to a vectorial version of Krasnoselskii's Theorem for systems of equations. The use of this extension makes possible that the nonlinear therm of a system may have different and independent behaviors both in components and variables.

The conventional technique, if we want to apply the Leray-Schauder type principles, the Topological Transversality Theorem or the Krasnoselskii type theorems for localization of "a priori" bounded solutions, is to rewrite the problem as an integral equation, usually with the help of a Green function. Despite the fact that Green functions are specific to second order equations, is possible to build such functions in case of first order equations of this form

$$x^{'}(t) = a(t)x(t) - f(t).$$

This type of equations is very common (see [2], [7], [8], [19], [24], [25], [30], [31], [32], [34], [38], [39], [43], [44], [45], [49], [50], [51], [52], [53], [63], [76], [77], [79], [80], [81], [82], [85], [87], [89], [91], [90], [92]), different particular forms of them modeling phenomenons from populations dynamic. The simplest example in this sense is given by the logistic equation

$$x^{'}(t) = r(t)x(t)\left(1 - \frac{x(t)}{K(t)}\right),$$

equation which represents a common model for evolution in time of populations. Here

$x(t)$ represents the number of persons from population at time t,

$r(t)$ represents births rate(the number of newborns) at time t,

$K(t)$ represents the carrying capacity.

The same equation can be used for modeling the growth of tumors in medicine, evolution of neuronal networks, evolution of autocatalytical reactions and many other phenomenons.

In case of systems of equations, the logistic equation is involved in the Lotka-Volterra model

$$\begin{cases} x^{'}(t) = r_1(t)x(t)\left[1 - \dfrac{x(t) + \alpha_{12}(t)}{K_1(t)}\right] \\ y^{'}(t) = r_2(t)y(t)\left[1 - \dfrac{y(t) + \alpha_{21}(t)}{K_2(t)}\right] \end{cases}.$$

Here

$x(t), y(t)$ represents the number of persons from each population at time t,

$r_i(t)$ represents the growth rate of species i at time t,

$K_i(t)$ represents the carrying capacity of the environment for species i,

$\alpha_{i,j}$ represents the effect of species j over species i.

According to the sign of coefficients α_{ij} we have two cases:

$\alpha_{ij} \geq 0$, case in which we say we have a competition model(of pray-predator type),

$\alpha_{i,j} \leq 0$, case in which we say we have a mutualism model.

The purpose of this thesis is to study the existence of positive periodic solutions for nonlinear first order functional-differential equations of type

$$x^{'}(t) = a(t)x(t) - F(x)(t),$$

for second order equations like

$$x^{''}(t) = a(t)x(t) - F(x)(t);$$

and also, for their corresponding systems of equations. We will rewrite these equations, and systems, as fixed point problems, or as coincidence problems. To these problems we will apply the fixed point theorems of Leray-Schauder and Krasnoselskii or variants of these abstract results for coincidences.

This thesis is structured in 3 chapters and each chapter contains more sections.

Chapter 1: Preliminaries.
In this chapter we present the two abstract results from fixed point theory which are used in proofs from this thesis: the Leray-Schauder Principle and the Krasnoselskii's Theorem in cones. We also present different approaches to these two results and some extensions.

Chapter 2: Periodic solutions for functional-differential equations.
In this chapter we present an unitary theory over the problem of existence of positive periodic solutions for first order functional-differential equations. On first section we present some existence results of positive solutions obtained using the Leray-Schauder Principle; results taken from paper [58]. These results will be the starting point for those presented in section 3.1
In the next two section we introduce the Green's function and build the equivalent fixed point problem. In the next section we localize the periodic solutions by using the Krasnoselskii's Theorem in cones. Also, we give some

applications of these results, including to logistic equation. The personal contributions in these sections are given in 4 lemmas and 5 theorems. These results are also contained in paper V. Dincuţă [14].

In section 4 we prove a version of Krasnoselskii's Theorem for coincidences. In the next section we apply this abstract result to the first order equations which we studied in the previous sections. The conclusion is that the results are similar no matter what method is used, which gives consistency to our results. Our personal contributions in this section are given by 6 lemmas and Theorems 2.5.1, 2.6.1. The results are also contained in paper V. Dincuţă [11].

Capitolul 3: Periodic solutions for functional-differential systems of equations. In this chapter we extend the results from chapter 2 to systems of equations.

First, we study the existence of solutions using the Leray-Schauder Principle. The results extends those obtained by D. O'Regan and M. Meehan in [58]. We give a general existence principle and we apply this result to an integral operator with delays. Finally, we present some particular cases of this operator and also we give an application to high order equations which can be reduced to systems of first order equations. Our contributions in this section are Theorems 3.1.1, 3.1.2 and other 5 forms of these results to some particular cases. These results are contained in paper V. Dincuţă [13].

In section 2 of this chapter we study the systems of first order equations using the vectorial form of Krasnoselskii's Theorem. The obtained results naturally extend those presented in section 3 of chapter 2, and allows to nonlinear therm to have different sublinear and superlinear behaviors. In the end, we give some applications of these results, including to Lotka-Volterra systems. Our contributions in this section are given by 8 lemmas and 3 theorems.

In section 3 we give some results which guarantee the existence of solutions for systems of second order equations using the vectorial form of Krasnoselskii's Theorem. First, we seek for solutions in a given shell and next we obtain an existence result under asymptotical conditions. These results extend those obtained by D. O'Regan and H. Wang [61]. Our contributions in this section are given in 13 lemmas and Theorems 3.3.1, 3.3.2, 3.3.3. These results are contained in paper V. Dincuţă [16].

In section 4, we prove an abstract version of the vectorial version of Krasnoselskii's Theorem for coincidences, and in the next section we apply this result to the systems of first order equations studied in the previous sections. These results extend those obtained in sections 4 and 5 from chapter 2, and we conclude again that no matter that we apply Krasnoselskii's Theorem or we see the problem like a coincidence problem, the obtained results are similar. Our contributions in this section are given in 10 lemmas and Theorems 3.4.1, 3.5.1. These results are contained in paper V. Dincuţă [12].

1 Preliminaries

This chapter contains the abstract results used in proofs from this thesis.

2 Periodic solutions for functional-differential equations

The purpose of this chapter is to develop an unitary theory on the existence of periodic solutions for functional-differential equations like

$$x^{'}(t) = a(t)x(t) - F(x)(t), \tag{2.1}$$

where $a \in C_T(\mathbb{R}, \mathbb{R}_+)$ is nonidentical zero and $F : C_T(\mathbb{R}) \to C_T(\mathbb{R})$ is a continuous operator.

2.1 Periodic solutions via Leray-Schauder Principle

The results from this section are contained in paper[58] of M. Meehan and D. O'Regan and represents the inspiration source for our results from section 3.1.

Consider the equation

$$x^{'}(t) = a(t)y(t) + N(y)(t), \text{a.p.t. } t \in [0, T]. \tag{2.2}$$

By a solution of this equation we understand a function $y \in AC[0, T]$ with $y(0) = y(T)$ and which satisfies the equation (2.2) almost everywhere on $[0, T]$.

Theorem 2.1.1 *[58]] Suppose that*

$$N : C[0, T] \to L^1[0, T] \textit{ is a continuous operator,} \tag{2.3}$$

$$\begin{cases} \textit{for any constant } A \geq 0 \textit{ there exists } h_A \in L^1[0, T] \textit{ such that} \\ \textit{for any } y \in C[0, T] \textit{ with } ||y||_0 = \sup\limits_{t \in [0,T]} ||y(t)|| \leq A \\ \textit{we have } ||N(y)(t)|| \leq h_A(t) \textit{ a.p.t. } t \in [0, T], \end{cases} \tag{2.4}$$

and

$$a \in L^1[0, T] \textit{ such that } e^{-\int_0^T a(s)ds} \neq 1. \tag{2.5}$$

Moreover, suppose that there exists a constant M independent of λ with $||y||_0 \neq M$ for any solution $y \in AC[0,T]$ of the problem

$$\begin{cases} y^{'}(t) - a(t)y(t) = \lambda N(y)(t) \ \ a.p.t. \ t \in [0,T] \\ y(0) = y(T) \end{cases}$$

and any $\lambda \in (0,1)$.

Then the equation (2.2) has at least one solution $y \in AC[0,T]$ such that $||y||_0 \leq M$.

Using this principle, we can obtain the following general existence result.

Theorem 2.1.2 *[58]] Suppose that (2.3), (2.4) and (2.5) are fulfilled. Moreover, suppose that the following conditions are satisfied*

$$\begin{cases} \text{there exists a continuous function } \psi : [0,\infty) \to (0,\infty) \text{ şi } \phi \in L^1[0,T] \text{ with} \\ ||N(y)(x)|| \leq \phi(x)\psi(|y|_0) \ \ a.p.t. \ x \in [0,T] \text{ and any } y \in C[0,T], \end{cases}$$

and

$$\sup_{c \in (0,\infty)} \frac{c}{\psi(c)} > k_0;$$

where

$$k_0 = \frac{1}{|1-b(T)|} \sup_{t \in [0,T]} \left\{ \frac{b(T)}{b(t)} \int_0^t b(s)\phi(s)ds + \frac{1}{b(t)} \int_t^T b(s)\phi(s)ds \right\}$$

and

$$b(t) = e^{-\int_0^t a(x)dx}.$$

Then equation (2.2) has at least one solution in $AC[0,T]$.

Next, we give an existence result of the solutions of equation

$$\begin{cases} y^{'}(t) = N(y)(t) \text{ a.p.t. } t \in [0,T] \\ y(0) = y(T), \end{cases} \tag{2.6}$$

where the operator N is given by

$$\begin{aligned} N(y)(t) \ &= \ r(t) + y(t)g(t,y(t)) + h(t,y(t)) \\ &+ \int_0^t k_1(t,s)f_1(s,y(s))ds \\ &+ \int_0^T k_2(t,s)f_2(s,y(s))ds, \ \text{ a.p.t. } t \in [0,T]. \end{aligned} \tag{2.7}$$

2.2 Green's function for the periodic problem

Despite the fact that generally the Green's function is specific to second order differential equations, a such function can be build in case of the first order equation (2.1).

In what follows, consider the Green's function given by

$$G(t,s) = \frac{e^{-\int\limits_t^s a(\tau)d\tau}}{1 - e^{-\int\limits_0^T a(\tau)d\tau}}.$$

If we take

$$\delta = e^{-\int\limits_0^T a(\tau)d\tau} < 1,$$

it is obvious that the Green's function is bounded and satisfies the inequalities

$$0 < \frac{\delta}{1-\delta} \leq G(t,s) \leq \frac{1}{1-\delta}, \text{ for any } s \in [t, t+T]. \tag{2.8}$$

2.3 Reduction of the periodic problem to a fixed point problem

Lemma 2.3.1 *A function $x \in C_T(\mathbb{R})$ is solution for the problem*

$$\begin{cases} x^{'}(t) = a(t)x(t) - f(t) \\ x(0) = x(T) \end{cases} \tag{2.9}$$

if and only if

$$x(t) = \int\limits_t^{t+T} G(t,s)f(s)ds, \tag{2.10}$$

where $f \in C_T(\mathbb{R})$ is an arbitrary function.

Remark 2.3.1 *Using the above lema is obvious that a function $x \in C_T(\mathbb{R})$ is solution for equation (2.1) if and only if is a solution of the integral equation*

$$x(t) = \int_t^{t+T} G(t,s)F(x)(s)ds. \tag{2.11}$$

2.4 Localization of positive periodic solutions using the Krasnoselskii's principle

In this section, we study the existence of positive periodic solutions for equation (2.1), solutions which satisfies $r \leq ||x|| \leq R$, where $0 < r < R$ are two given numbers. The main result of this section is the following.

Theorem 2.4.1 [**V. Dincuţă [14]**] *Suppose that for two numbers r and R with $0 < r < R$, one of the next conditions is fulfilled:*

$$\text{(a.1)} \begin{cases} F \text{ is nondecreasing,} \\ \max\limits_{t\in[0,T]} F(r)(t) \leq \dfrac{1-\delta}{T} r, \\ \min\limits_{t\in[0,T]} F(\delta R)(t) \geq \dfrac{1-\delta}{\delta T} R; \end{cases}$$

$$\text{(a.2)} \begin{cases} F \text{ is nondecreasing,} \\ \min\limits_{t\in[0,T]} F(\delta r)(t) \geq \dfrac{1-\delta}{\delta T} r, \\ \max\limits_{t\in[0,T]} F(R)(t) \leq \dfrac{1-\delta}{T} R; \end{cases}$$

$$\text{(a.3)} \begin{cases} F \text{ is nonincreasing,} \\ \max\limits_{t\in[0,T]} F(\delta r)(t) \leq \dfrac{1-\delta}{T} r, \\ \min\limits_{t\in[0,T]} F(R)(t) \geq \dfrac{1-\delta}{\delta T} R; \end{cases}$$

$$\text{(a.4)} \begin{cases} F \text{ is nonincreasing,} \\ \min\limits_{t\in[0,T]} F(r)(t) \geq \dfrac{1-\delta}{\delta T} r, \\ \max\limits_{t\in[0,T]} F(\delta R)(t) \leq \dfrac{1-\delta}{T} R. \end{cases}$$

The there exists at least one positive periodic solution x of equation (2.1) such that $r \leq ||x|| \leq R$.

Next we give 4 examples, two of them relative to the logistic model.

Example 2.4.1

Consider the classic logistic model

$$\begin{cases} x^{'}(t) = a(t)x(t)\left[1 - \dfrac{x(t)}{K(t)}\right] \\ x(0) = x(T) \end{cases} \tag{2.12}$$

where a, K are positive periodic functions, nonidentical to zero and $T > 0$.

Using Theorem 2.4.1 we obtain the following result.

Theorem 2.4.2 *If there exists two numbers r and R such that*

$$r \leq \frac{1-\delta}{T \max\limits_{t\in[0,T]} \dfrac{a(t)}{K(t)}} < \frac{1-\delta}{\delta^3 T \min\limits_{t\in[0,T]} \dfrac{a(t)}{K(t)}} \leq R$$

then there exists a positive periodic solution x of problem (2.12) which satisfies $r \leq ||x|| \leq R$.

Example 2.4.2

Consider the generalized logistic model for a single species

$$\begin{cases} x^{'}(t) = x(t)[a(t) - b(t)x(t) - c(t)x(t - \tau(t))] \\ x(0) = x(T) \end{cases} \tag{2.13}$$

where a, b, c, τ are positive continuous functions and $T > 0$.

Using Theorem 2.4.1 we obtain the following result.

Theorem 2.4.3 *If there exists two numbers r and R such that*

$$r \leq \frac{1-\delta}{T \max\limits_{t\in[0,T]}[b(t) + c(t)]} < \frac{1-\delta}{\delta^3 T \min\limits_{t\in[0,T]}[b(t) + c(t)]} \leq R$$

then there exists a positive periodic solution x of problem (2.13) which satisfies $r \leq ||x|| \leq R$.

2.5 Krasnoselskii's theorem for coincidences

In this section we prove a version of Krasnoselskii's theorem for coincidences. We study the existence of positive periodic solutions in a cone for the equation

$$Lx = T(x), \tag{2.14}$$

where L is a linear application and T is a nonlinear operator. Here $L, T : X \to Y$, where X is a Banach space and Y is a normed space.

Theorem 2.5.1 [V. Dincuţă [11]] *Let $K \subset X$ be a cone, $r, R \in \mathbb{R}_+$, $0 < r < R$, $T : K \to Y$ a completely continuous map and $J : X \to Y$ a linear map such that*

(a) $L + J : X \to Y$ is invertible

(b) $(T + J)(K) \subseteq (L + J)(K) := \widetilde{K}$.

Suppose that one of the following conditions is fulfilled:

$$(c.1) \begin{cases} Lx - T(x) \notin \widetilde{K} \text{ for } ||x|| = r, \\ T(x) - Lx \notin \widetilde{K} \text{ for } ||x|| = R; \end{cases}$$

$$(c.2) \begin{cases} T(x) - Lx \notin \widetilde{K} \text{ for } ||x|| = r, \\ Lx - T(x) \notin \widetilde{K} \text{ for } ||x|| = R. \end{cases}$$

Then there exists $x \in K_{r,R}$ such that $Lx = T(x)$.

2.6 Applications of Krasnoselskii's theorem for coincidences to the periodic problem

In what follows we will apply the Theorem 2.5.1 to obtain an existence result for the periodic solutions of equation (2.1).

If we take

$$X = Y = C_T(\mathbb{R}),$$

$$Lx(t) = x(t) - x(0),$$

$$T(x)(t) = \int_0^t [a(s)x(s) - F(x)(s)]ds;$$

then equation (2.1) is equivalent with equation

$$Lx = T(x).$$

The main result of this section is the following.

Theorem 2.6.1 **[V. Dincuţă [11]]** *Suppose that for two numbers r and R with $0 < r < R$, one of the next conditions is fulfilled:*

$$\text{(a.1)} \begin{cases} F \text{ is nondecreasing}, \\ \max\limits_{t\in[0,T]} F(r)(t) < \dfrac{1-\delta}{T} r, \\ \min\limits_{t\in[0,T]} F(\delta R)(t) > \dfrac{1-\delta}{\delta T} R; \end{cases}$$

$$\text{(a.2)} \begin{cases} F \text{ is nondecreasing}, \\ \min\limits_{t\in[0,T]} F(\delta r)(t) > \dfrac{1-\delta}{\delta T} r, \\ \max\limits_{t\in[0,T]} F(R)(t) < \dfrac{1-\delta}{T} R; \end{cases}$$

$$\text{(a.3)} \begin{cases} F \text{ is nonincreasing}, \\ \max\limits_{t\in[0,T]} F(\delta r)(t) < \dfrac{1-\delta}{T} r, \\ \min\limits_{t\in[0,T]} F(R)(t) > \dfrac{1-\delta}{\delta T} R; \end{cases}$$

$$\text{(a.4)} \begin{cases} F \text{ is nonincreasing}, \\ \min\limits_{t\in[0,T]} F(r)(t) > \dfrac{1-\delta}{\delta T} r, \\ \max\limits_{t\in[0,T]} F(\delta R)(t) < \dfrac{1-\delta}{T} R. \end{cases}$$

Then there exists at least one solution x of equation (2.1) such that $r \leq ||x|| \leq R$.

3 Periodic solutions for functional-differential systems

3.1 Periodic solutions via Leray-Schauder Principle

In this section, motivated by chapter 12 from [58], we give an existence result for systems of equations like:

$$\begin{cases} y'(t) - A(t)y(t) = Ny(t) \ a.p.t. \ t \in [0, T] \\ y(0) = y(T). \end{cases} \tag{3.1}$$

Here $N : C([0, T], \mathbb{R}^n) \to C([0, T], \mathbb{R}^n)$, $N = (N_1, N_2, ..., N_n)$ is a continuous operator.

The results will extend those from [58] in two directions: to systems of equations, and to delay equations. Moreover, these results can be applied to high order equations, by reducing them to first order equations.

3.1.1 A general existence principle

First, we give a general existence principle for the solutions of system (3.1) which in particular, for $n = 1$, can be reduced to Theorem 12.1.1 from [58].

Theorem 3.1.1 [V. Dincuţă [13]] *Suppose that*

$$N : C([0,T],\mathbb{R}^n) \to L^1([0,T],\mathbb{R}^n) \text{ is a continuous operator,} \tag{3.2}$$

$$\begin{cases} \text{for any constant } B \geq 0 \text{ there exists } h_B \in L^1[0,T] \text{ such that} \\ \text{for any } y \in C([0,T],\mathbb{R}^n) \text{ with } ||y||_0 = \sup\limits_{t\in[0,T]} ||y(t)||_{\mathbb{R}^n} \leq B \\ \text{we have } ||Ny(t)||_{\mathbb{R}^n} \leq h_B(t) \text{ a.p.t. } t \in [0,T], \end{cases} \tag{3.3}$$

and

$$A \in L^1([0,T], M_{nn}(\mathbb{R})) \text{ such that } I_n - e^{-\int_0^T A(s)ds} \text{ is invertible.} \tag{3.4}$$

Here I_n is the unity matrix from $M_{nn}(\mathbb{R})$, and for a matrix $D \in M_{nn}(\mathbb{R})$ by e^D we understand the sum $\sum\limits_{k=0}^{\infty} \frac{1}{k!}D^k$.

Moreover, suppose that there exists a constant M independent of λ with $||y||_0 \neq M$ for any solution $y \in AC([0,T],\mathbb{R}^n)$ of problem

$$\begin{cases} y'(t) - A(t)y(t) = \lambda Ny(t) \text{ a.p.t. } t \in [0,T] \\ y(0) = y(T) \end{cases} \tag{3.5}$$

and any $\lambda \in (0,1)$.

Then the system (3.1) has at least one solution $y \in AC([0,T],\mathbb{R}^n)$ such that $||y||_0 \leq M$.

3.1.2 Existence of positive periodic solutions

We consider the problem

$$\begin{cases} y'(t) = Ny(t) \text{ a.p.t. } t \in [0,T] \\ y(0) = y(T). \end{cases} \tag{3.6}$$

We will discuss the particular case when the operator N is given by

$$\begin{aligned} Ny(t) &= r(t)+g(t,y(t-\theta_1))y(t-\theta_1)+h(t,y(t-\theta_2)) \\ &+\int_0^t k_1(t,s)f_1(s,y(s))ds+\int_0^T k_2(t,s)f_2(s,y(s))ds. \end{aligned} \tag{3.7}$$

.

Theorem 3.1.2 **[V. Dincuţă [13]]** *Suppose that conditions (3.2) and (3.3) are fulfilled for N given by (3.7).*

Moreover, suppose that:

$$r(t)+h(t,0)\leq 0 \ \ a.p.t. \ \ t\in[0,T]; \tag{3.8}$$

$$\begin{cases} \|h(t,y)\|_{\mathbb{R}^n}\leq \Phi_1(t)\,\|y\|_{\mathbb{R}^n}^{\alpha}+\Phi_2(t) \ \ a.p.t. \ \ t\in[0,T] \ \ and \ y\geq 0, \\ where \ 0\leq\alpha<1 and \ \Phi_1,\Phi_2\in L^1[0,T]; \end{cases} \tag{3.9}$$

$$\begin{cases} \text{there exists } \beta\in L^1([0,T],\mathbb{R}^n) \text{and } \tau\in L^1([0,T],\mathbb{R}_+) \text{ with } \beta(t)\leq g(t,y)y \\ \text{and } \|g(t,y)y\|_{\mathbb{R}^n}\leq\tau(t)\,\|y\|_{\mathbb{R}^n} \ \ a.p.t. \ \ t\in[0,T] \text{and any } y\geq 0; \\ \text{where } \tau(t)>0 \text{ on a subset of positive measure of } [0,T]; \end{cases} \tag{3.10}$$

$$\text{there exists } \rho\in L^1([0,T],\mathbb{R}^n) \text{ with } h(t,y)\geq\rho(t) \ \ a.p.t. \ \ t\in[0,T] \text{and } y\geq 0; \tag{3.11}$$

$$\begin{cases} \int\limits_0^t k_1(t,s)f_1(s,y(s))ds+\int\limits_0^T k_2(t,s)f_2(s,y(s))ds\leq 0 \\ a.p.t. \ \ t\in[0,T] \text{for any } y\in C([0,T],\mathbb{R}^n); \end{cases} \tag{3.12}$$

$$\begin{cases} \text{there exists } \rho_1\in L^1[0,T] \text{and } \rho_2\in L^1([0,T],\mathbb{R}^n) \text{ such that} \\ k_1(t,s)f_1(s,y)\geq\rho_1(s)\rho_2(t) \ \ a.p.t. \ \ t\in[0,T], \ a.p.t. \ \ s\in[0,t] \\ \text{for any } y\geq 0; \end{cases} \tag{3.13}$$

$$\begin{cases} \text{there exists } \rho_3\in L^1[0,T] \text{and } \rho_4\in L^1([0,T],\mathbb{R}^n) \text{ such that} \\ k_2(t,s)f_2(s,y)\geq\rho_3(s)\rho_4(t) \ \ a.p.t. \ \ t\in[0,T], \ a.p.t. \ \ s\in[0,T] \\ \text{and any } y\geq 0; \end{cases} \tag{3.14}$$

$$\begin{cases} \left\|\int\limits_0^t k_1(t,s)f_1(s,y(s))ds\right\|_{\mathbb{R}^n}\leq\Phi_3(t)\,\|y\|_0^{\gamma}+\Phi_4(t) \\ a.p.t. \ \ t\in[0,T] \text{and for any } y\in C([0,T],\mathbb{R}^n_+); \\ \text{where } \Phi_3,\Phi_4\in L^1([0,T],\mathbb{R}) \text{and } 0\leq\gamma<1; \end{cases} \tag{3.15}$$

$$\begin{cases} \left\| \int_0^T k_2(t,s) f_2(s,y(s))ds \right\|_{\mathbb{R}^n} \leq \Phi_5(t) \, ||y||_0^\omega + \Phi_6(t) \\ a.p.t. \;\; t \in [0,T] \text{ and for any } y \in C([0,T], \mathbb{R}_+^n); \\ \text{where } \Phi_5, \Phi_6 \in L^1([0,T], \mathbb{R}) \text{ and } 0 \leq \omega < 1; \end{cases} \tag{3.16}$$

and

$$\int_0^T [-r(t)]dt < \int_0^T \liminf_{x\to\infty}[f(t,x)x]dt + \int_0^T \liminf_{x\to\infty}[h(t,s)]dt+$$
$$+\int_0^T \int_0^t \liminf_{x\to\infty}[k_1(t,s)f_1(s,x)]dsdt + \int_0^T \int_0^T \liminf_{x\to\infty}[k_2(t,s)f_2(s,x)]dsdt. \tag{3.17}$$

Then the system (3.6) has at least one solution $y \in AC([0,T], \mathbb{R}^n)$ *such that* $y(x) \geq 0$ *for any* $x \in [0,T]$.

3.2 Krasnoselskii's vectorial theorem and periodic solutions for systems of functional-differential equations

In this section, inspired by [61] we study the existence of positive, T periodic solutions for the functional-differential system

$$\begin{cases} x^{'}(t) = a_1(t)x(t) - F_1(x,y)(t) \\ y^{'}(t) = a_2(t)y(t) - F_2(x,y)(t) \\ x(0) = x(T) \\ y(0) = y(T). \end{cases} \tag{3.18}$$

Then $a_1, a_2 \in C_T(\mathbb{R}, \mathbb{R}_+)$ and $F_1, F_2 : C_T^2(\mathbb{R}, \mathbb{R}_+) \to C_T(\mathbb{R}, \mathbb{R}_+)$ are continuous operators.

Using Lemma 2.3.1, this system is equivalent with the system

$$\begin{cases} x(t) = N_1(x,y)(t) \\ y(t) = N_2(x,y)(t) \end{cases}$$

where operators $N_1, N_2 : C_T^2(\mathbb{R}, \mathbb{R}_+) \to C_T(\mathbb{R}, \mathbb{R}_+)$ are given by

$$\begin{aligned} N_1(x,y)(t) &= \int_t^{t+T} G_1(t,s) F_1(x,y)(t) ds, \\ N_2(x,y)(t) &= \int_t^{t+T} G_2(t,s) F_2(x,y)(t) ds; \end{aligned}$$

the Green's functions are given by

$$G_i(t,s) = \frac{e^{-\int_t^s a_i(\tau) d\tau}}{1 - e^{-\int_0^T a_i(\tau) d\tau}}, \ i = 1, 2;$$

and

$$\delta_i = e^{-\int_0^T a_i(\tau) d\tau}, \ i = 1, 2.$$

The main result of this section is the following existence theorem.

Theorem 3.2.1 *Suppose that we have two pairs of numbers* (r_1, R_1) *and* (r_2, R_2) *with* $0 < r_1 < R_1$ *and* $0 < r_2 < R_2$*, such that:*

(a) for any $y \in \mathbb{R}_+$ *one of the following conditions is fulfilled:*

$$\text{(a.1)} \begin{cases} F_1(\cdot, y) \ \text{is nondecreasing}, \\ \max\limits_{t \in [0,T]} F_1(r_1, y)(t) \le \dfrac{1-\delta_1}{T} r_1, \\ \min\limits_{t \in [0,T]} F_1(\delta_1 R_1, y)(t) \ge \dfrac{1-\delta_1}{\delta_1 T} R_1; \end{cases}$$

$$\text{(a.2)} \begin{cases} F_1(\cdot, y) \ \text{is nondecreasing}, \\ \min\limits_{t \in [0,T]} F_1(\delta_1 r_1, y)(t) \ge \dfrac{1-\delta_1}{\delta_1 T} r_1, \\ \max\limits_{t \in [0,T]} F_1(R_1, y)(t) \le \dfrac{1-\delta_1}{T} R_1; \end{cases}$$

$$\text{(a.3)} \begin{cases} F_1(\cdot, y) \ \text{is nonincreasing}, \\ \max\limits_{t \in [0,T]} F_1(\delta_1 r_1, y)(t) \le \dfrac{1-\delta_1}{T} r_1, \\ \min\limits_{t \in [0,T]} F_1(R_1, y)(t) \ge \dfrac{1-\delta_1}{\delta_1 T} R_1; \end{cases}$$

$$
(a.4)\ \begin{cases} F_1(\cdot,y) \text{ is nonincreasing,} \\ \min\limits_{t\in[0,T]} F_1(r_1,y)(t) \geq \dfrac{1-\delta_1}{\delta_1 T} r_1, \\ \max\limits_{t\in[0,T]} F_1(\delta_1 R_1,y)(t) \leq \dfrac{1-\delta_1}{T} R_1; \end{cases}
$$

(b) for any $x \in \mathbb{R}_+$ *one of the following conditions is fulfilled:*

$$
(b.1)\ \begin{cases} F_2(x,\cdot) \text{ is nondecreasing,} \\ \max\limits_{t\in[0,T]} F_2(x,r_2)(t) \leq \dfrac{1-\delta_2}{T} r_2, \\ \min\limits_{t\in[0,T]} F_2(x,\delta_2 R_2)(t) \geq \dfrac{1-\delta_2}{\delta_2 T} R_2; \end{cases}
$$

$$
(b.2)\ \begin{cases} F_2(x,\cdot) \text{ is nondecreasing,} \\ \min\limits_{t\in[0,T]} F_2(x,\delta_2 r_2)(t) \geq \dfrac{1-\delta_2}{\delta_2 T} r_2, \\ \max\limits_{t\in[0,T]} F_2(x,R_2)(t) \leq \dfrac{1-\delta_2}{T} R_2; \end{cases}
$$

$$
(b.3)\ \begin{cases} F_2(x,\cdot) \text{ is nonincreasing,} \\ \max\limits_{t\in[0,T]} F_2(x,\delta_2 r_2)(t) \leq \dfrac{1-\delta_2}{T} r_2, \\ \min\limits_{t\in[0,T]} F_2(x,R_2)(t) \geq \dfrac{1-\delta_2}{\delta_2 T} R_2; \end{cases}
$$

$$
(b.4)\ \begin{cases} F_2(x,\cdot) \text{ is nonincreasing,} \\ \min\limits_{t\in[0,T]} F_2(x,r_2)(t) \geq \dfrac{1-\delta_2}{\delta_2 T} r_2, \\ \max\limits_{t\in[0,T]} F_2(x,\delta_2 R_2)(t) \leq \dfrac{1-\delta_2}{T} R_2. \end{cases}
$$

Then there exists a solution (x^*, y^*) *of system (3.18) such that* $r_1 \leq ||x^*|| \leq R_1$ *and* $r_2 \leq ||y^*|| \leq R_2$.

Remark 3.2.1 *In the previous theorem we can have 16 cases for functions* F_1 *and* F_2. *This allows to the nonlinearities* $F_1(x,y)$ *and* $F_2(x,y)$ *to have different and independent sublinear and superlinear behaviors in* x *and* y.

Next, we give two examples, one to the Lotka-Volterra model

$$\begin{cases} x^{'}(t) = a_1(t)x(t)\left[1 - \alpha_{11}\dfrac{x(t)}{K_1(t)} - \alpha_{12}\dfrac{f(y(t))}{K_1(t)}\right] \\ y^{'}(t) = a_2(t)y(t)\left[1 - \alpha_{21}\dfrac{g(x(t))}{K_2(t)} - \alpha_{22}\dfrac{y(t)}{K_2(t)}\right] \\ x(0) = x(T) \\ y(0) = y(T) \end{cases} \tag{3.19}$$

where a_1, a_2, K_1, K_2 are positive, continuous functions, nonidentical zero, T-periodic and $T > 0$.

Using Theorem 3.2.1 we obtain the following result.

Theorem 3.2.2 *Suppose that functions $f, g : \mathbb{R} \to [0, \infty)$ are continuous and such that*

$$0 < \min_{u\in\mathbb{R}} f(u) < \max_{u\in\mathbb{R}} f(u) < \infty,$$
$$0 < \min_{u\in\mathbb{R}} g(u) < \max_{u\in\mathbb{R}} g(u) < \infty.$$

Moreover, suppose that there exists the numbers r_1, r_2, R_1 and R_2 such that

$$r_1 \le \frac{1}{\alpha_{11}}\left(\frac{1-\delta_1}{T\max\limits_{t\in[0,T]}\dfrac{a_1(t)}{K_1(t)}} - \alpha_{12}\max_{u\in\mathbb{R}} f(u)\right),$$

$$R_1 \ge \frac{1}{\alpha_{11}\delta_1^2}\left(\frac{1-\delta_1}{\delta_1 T\min\limits_{t\in[0,T]}\dfrac{a_1(t)}{K_1(t)}} - \alpha_{12}\min_{u\in\mathbb{R}} f(u)\right),$$

$$r_2 \le \frac{1}{\alpha_{22}}\left(\frac{1-\delta_2}{T\max\limits_{t\in[0,T]}\dfrac{a_2(t)}{K_2(t)}} - \alpha_{21}\max_{u\in\mathbb{R}} g(u)\right),$$

$$R_2 \ge \frac{1}{\alpha_{22}\delta_2^2}\left(\frac{1-\delta_2}{\delta_2 T\min\limits_{t\in[0,T]}\dfrac{a_2(t)}{K_2(t)}} - \alpha_{21}\min_{u\in\mathbb{R}} g(u)\right).$$

Then there exists a solution (x^*, y^*) of system (3.19) which satisfy $r_1 \le ||x^*|| \le R_1$ and $r_2 \le ||y^*|| \le R_2$.

3.3 The Krasnoselskii's vectorial theorem and periodic solutions for systems of second order differential equations.

The purpose of this section is to study the existence of periodic positive solutions for the problem

$$\begin{cases} x''(t) + m_1^2 x(t) = \lambda_1 G_1(t) F_1(x, y)(t) \\ y''(t) + m_2^2 y(t) = \lambda_2 G_2(t) F_2(x, y)(t) \\ x(0) = x(2\pi) \\ y(0) = y(2\pi) \\ x'(0) = x'(2\pi) \\ y'(0) = y'(2\pi) \end{cases} \tag{3.20}$$

where $m_1, m_2 \in (0, \frac{1}{2})$ are constants, $\lambda_1, \lambda_2 > 0$ positive parameters, and

$$\begin{matrix} G_i(t) = diag[g_1^i(t), g_2^i(t), ..., g_n^i(t)], \\ F_i(x, y) = [f_1^i(x, y), f_2^i(x, y), ..., f_n^i(x, y)]^T \end{matrix}, i \in \{1, 2\}.$$

The case of a single equation was studied in [61], our results extends them and can be found in paper [16].
In this section, we will consider fulfilled the following conditions:

(H1) $f_j^i : \mathbb{R}_+^{2n} \to [0, \infty)$ is continuous, for $j = 1, ..., n$ and $i = 1, 2$;

(H2) $g_j^i : [0, 2\pi] \to [0, \infty)$ is continuous and nonidentical zero, for $j = 1, ..., n$ and $i = 1, 2$.

3.3.1 Positive periodic solutions in a given shell

We consider the following Green's functions:

$$G_i(t, s) = \begin{cases} \dfrac{\sin m_i(t - s) + \sin m_i(2\pi - t + s)}{2m_i(1 - \cos 2m_i\pi)}, & \text{if } 0 \leq s \leq t \leq 2\pi \\ \dfrac{\sin m_i(s - t) + \sin m_i(2\pi - s + t)}{2m_i(1 - \cos 2m_i\pi)}, & \text{if } 0 \leq t \leq s \leq 2\pi \end{cases}, i = 1, 2.$$

Denote by

$$\widetilde{G}_i(x) = \frac{\sin(m_i x) + \sin m_i(2\pi - x)}{2m_i(1 - \cos 2m_i\pi)}, x \in [0, 2\pi], i = 1, 2.$$

and let $\sigma_i = \cos m_i\pi, i = 1, 2.$

Also, we consider the following notations:

$$N_i = \lambda_i \widetilde{G}_i(\pi) \sum_{j=1}^{n} \int_0^{2\pi} g_j^i(s)ds,$$
$$M_i = \lambda_i \sigma_i \widetilde{G}_i(0) \min_{j=\overline{1,n}} \int_0^{2\pi} g_j^i(s)ds,$$

for $i = 1, 2$.

The main result of this section is the following.

Theorem 3.3.1 [V. Dincuţă [16]] *Let $0 < r_1 < R_1$ and $0 < r_2 < R_2$. Suppose that (H1) and (H2) takes place and that one of the following conditions is fulfilled:*

$$\text{(H3.1)} \begin{cases} \textit{for } x, y \in \mathbb{R}_+^n \textit{ with } r_2 \le |y| \le R_2 \textit{ we have:} \\ (1) f_j^1(x,y) < \dfrac{r_1}{N_1}, j = 1, ..., n \textit{ if } \sigma_1 r_1 \le \sum\limits_{k=1}^{n} x_k \le r_1, \\ (2) \exists j^* \in \{1, ..., n\} \textit{ such that } f_{j^*}^1(x,y) > \sigma_1 \dfrac{R_1}{M_1} \textit{ if } \sigma_1 R_1 \le \sum\limits_{k=1}^{n} x_k \le R_1. \\ \textit{for } x, y \in \mathbb{R}_+^n \textit{ with } r_1 \le |x| \le R_1 \textit{ we have:} \\ (3) f_j^2(x,y) < \dfrac{r_2}{N_2}, j = 1, ..., n \textit{ if } \sigma_2 r_2 \le \sum\limits_{k=1}^{n} y_k \le r_2, \\ (4) \exists j^* \in \{1, ..., n\} \textit{ such that } f_{j^*}^2(x,y) > \sigma_2 \dfrac{R_2}{M_2} \textit{ if } \sigma_2 R_2 \le \sum\limits_{k=1}^{n} y_k \le R_2. \end{cases}$$

$$\text{(H3.2)} \begin{cases} \textit{for } x, y \in \mathbb{R}_+^n \textit{ with } r_2 \le |y| \le R_2 \textit{ we have:} \\ (1) f_j^1(x,y) < \dfrac{r_1}{N_1}, j = 1, ..., n \textit{ if } \sigma_1 r_1 \le \sum\limits_{k=1}^{n} x_k \le r_1, \\ (2) \exists j^* \in \{1, ..., n\} \textit{ such that } f_{j^*}^1(x,y) > \sigma_1 \dfrac{R_1}{M_1} \textit{ if } \sigma_1 R_1 \le \sum\limits_{k=1}^{n} x_k \le R_1. \\ \textit{for } x, y \in \mathbb{R}_+^n \textit{ with } r_1 \le |x| \le R_1 \textit{ we have:} \\ (3) \exists j^* \in \{1, ..., n\} \textit{ such that } f_{j^*}^2(x,y) > \sigma_2 \dfrac{r_2}{M_2} \textit{ if } \sigma_2 r_2 \le \sum\limits_{k=1}^{n} y_k \le r_2, \\ (4) f_j^2(x,y) < \dfrac{R_2}{N_2}, j = 1, ..., n \textit{ if } \sigma_2 R_2 \le \sum\limits_{k=1}^{n} y_k \le R_2. \end{cases}$$

$$(H3.3)\begin{cases} \textit{for } x,y\in\mathbb{R}^n_+ \textit{ with } r_2\le |y|\le R_2 \textit{ we have:}\\ (1)\exists j^*\in\{1,...,n\} \textit{ such that } f^1_{j^*}(x,y)>\sigma_1\dfrac{r_1}{M_1} \textit{ if } \sigma_1 r_1\le \sum\limits_{k=1}^n x_k\le r_1,\\ (2) f^1_j(x,y)<\dfrac{R_1}{N_1}, j=1,...,n \textit{ if } \sigma_1R_1\le \sum\limits_{k=1}^n x_k\le R_1.\\ \textit{for } x,y\in\mathbb{R}^n_+ \textit{ with } r_1\le |x|\le R_1 \textit{ we have:}\\ (3) f^2_j(x,y)<\dfrac{r_2}{N_2}, j=1,...,n \textit{ if } \sigma_2 r_2\le \sum\limits_{k=1}^n y_k\le r_2,\\ (4)\exists j^*\in\{1,...,n\} \textit{ such that } f^2_{j^*}(x,y)>\sigma_2\dfrac{R_2}{M_2} \textit{ if } \sigma_2 R_2\le \sum\limits_{k=1}^n y_k\le R_2. \end{cases}$$

$$(H3.4)\begin{cases} \textit{for } x,y\in\mathbb{R}^n_+ \textit{ with } r_2\le |y|\le R_2 \textit{ we have:}\\ (1)\exists j^*\in\{1,...,n\} \textit{ such that } f^1_{j^*}(x,y)>\sigma_1\dfrac{r_1}{M_1} \textit{ if } \sigma_1 r_1\le \sum\limits_{k=1}^n x_k\le r_1,\\ (2) f^1_j(x,y)<\dfrac{R_1}{N_1}, j=1,...,n \textit{ if } \sigma_1R_1\le \sum\limits_{k=1}^n x_k\le R_1.\\ \textit{for } x,y\in\mathbb{R}^n_+ \textit{ with } r_1\le |x|\le R_1 \textit{ we have:}\\ (3)\exists j^*\in\{1,...,n\} \textit{ such that } f^2_{j^*}(x,y)>\sigma_2\dfrac{r_2}{M_2} \textit{ if } \sigma_2 r_2\le \sum\limits_{k=1}^n y_k\le r_2,\\ (4) f^2_j(x,y)<\dfrac{R_2}{N_2}, j=1,...,n \textit{ if } \sigma_2 R_2\le \sum\limits_{k=1}^n y_k\le R_2. \end{cases}$$

Then there exists a solution (x^*,y^*) of problem (3.20) such that $r_1\le ||x^*||\le R_1$ and $r_2\le ||y^*||\le R_2$.

Next, we give an application to the system

$$\begin{cases} x''(t)+\dfrac{1}{16}x(t)=tx^{\alpha}(t)h(y(t))\\ y''(t)+\dfrac{1}{9}y(t)=ty^{\beta}(t)k(x(t)). \end{cases} \tag{3.21}$$

Theorem 3.3.2 [V. Dincuţă [16]] *Let $0<r_1<R_1$, $0<r_2<R_2$ and suppose that functions $h,k:\mathbb{R}\to[0,\infty)$ are continuous and such that*

$$\begin{aligned} 0<\min_{r_2\le t\le R_2} h(t)<\max_{r_2\le t\le R_2} h(t)<\infty,\\ 0<\min_{r_1\le t\le R_1} k(t)<\max_{r_1\le t\le R_1} k(t)<\infty. \end{aligned}$$

Also, suppose that one of the following conditions is fulfilled

$$(a.1)\begin{cases} \max\limits_{r_2\le t\le R_2} h(t)\cdot r_1^{\alpha-1}<\dfrac{1}{4\sqrt{2}\pi^2} \textit{ and } \min\limits_{r_2\le t\le R_2} h(t)\cdot R_1^{\alpha-1}>\dfrac{1}{2\sqrt{2}\pi^2}\cdot\dfrac{1}{\sigma_1^{\alpha-1}},\\ \max\limits_{r_1\le t\le R_1} k(t)\cdot r_2^{\beta-1}<\dfrac{1}{6\sqrt{3}\pi^2} \textit{ and } \min\limits_{r_1\le t\le R_1} k(t)\cdot R_2^{\beta-1}>\dfrac{2}{\sqrt{3}\pi^2}\cdot\dfrac{1}{\sigma_2^{\beta-1}}; \end{cases}$$

$$(a.2)\begin{cases} \max\limits_{r_2\le t\le R_2} h(t)\cdot r_1^{\alpha-1} < \dfrac{1}{4\sqrt{2}\pi^2} \ \textit{and} \ \min\limits_{r_2\le t\le R_2} h(t)\cdot R_1^{\alpha-1} > \dfrac{1}{2\sqrt{2}\pi^2}\cdot\dfrac{1}{\sigma_1^{\alpha-1}}, \\ \min\limits_{r_1\le t\le R_1} k(t)\cdot r_2^{\beta-1} > \dfrac{2}{\sqrt{3}\pi^2}\cdot\dfrac{1}{\sigma_2^{\beta-1}} \ \textit{and} \ \max\limits_{r_1\le t\le R_1} k(t)\cdot R_2^{\beta-1} < \dfrac{1}{6\sqrt{3}\pi^2}; \end{cases}$$

$$(a.3)\begin{cases} \min\limits_{r_2\le t\le R_2} h(t)\cdot r_1^{\alpha-1} > \dfrac{1}{2\sqrt{2}\pi^2}\cdot\dfrac{1}{\sigma_1^{\alpha-1}} \ \textit{and} \ \max\limits_{r_2\le t\le R_2} h(t)\cdot R_1^{\alpha-1} < \dfrac{1}{4\sqrt{2}\pi^2}, \\ \max\limits_{r_1\le t\le R_1} k(t)\cdot r_2^{\beta-1} < \dfrac{1}{6\sqrt{3}\pi^2} \ \textit{and} \ \min\limits_{r_1\le t\le R_1} k(t)\cdot R_2^{\beta-1} > \dfrac{2}{\sqrt{3}\pi^2}\cdot\dfrac{1}{\sigma_2^{\beta-1}}; \end{cases}$$

$$(a.4)\begin{cases} \min\limits_{r_2\le t\le R_2} h(t)\cdot r_1^{\alpha-1} > \dfrac{1}{2\sqrt{2}\pi^2}\cdot\dfrac{1}{\sigma_1^{\alpha-1}} \ \textit{and} \ \max\limits_{r_2\le t\le R_2} h(t)\cdot R_1^{\alpha-1} < \dfrac{1}{4\sqrt{2}\pi^2}, \\ \min\limits_{r_1\le t\le R_1} k(t)\cdot r_2^{\beta-1} > \dfrac{2}{\sqrt{3}\pi^2}\cdot\dfrac{1}{\sigma_2^{\beta-1}} \ \textit{and} \ \max\limits_{r_1\le t\le R_1} k(t)\cdot R_2^{\beta-1} < \dfrac{1}{6\sqrt{3}\pi^2}. \end{cases}$$

The the system (3.21) has a solution (x^*, y^*) *such that* $r_1 \le ||x|| \le R_1$ *and* $r_2 \le ||y|| \le R_2$.

3.3.2 Positive periodic solutions in asymptotic conditions

In the previous section we discussed the existence of positive periodic solutions in a given shell. Here, we will give some sufficient conditions on the nonlinearities $f^1(x,y), f^2(x,y)$ which will guarantee the existence of a such shell.

For $y \in \mathbb{R}_+^n$ and any $j = 1, ..., n$ we consider the following notations:

$$\begin{aligned} &f_{10}^j(y) = \lim_{|x|\to 0}\frac{f_j^1(x,y)}{|x|} \text{ and } F_{10}(y) = \max_{j=1,\dots,n} f_{10}^j(y), \\ &f_{1\infty}^j(y) = \lim_{|x|\to\infty}\frac{f_j^1(x,y)}{|x|} \text{ and } F_{1\infty}(y) = \max_{j=1,\dots,n} f_{1\infty}^j(y), \\ &\widehat{f}_1^j(t,y) = \max\{f_j^1(x,y) : x\in\mathbb{R}_+^n \text{ and } |x|\le t\}, \\ &\widehat{f}_{10}^j(y) = \lim_{t\to 0}\frac{\widehat{f}_1^j(t,y)}{t} \text{ and } \widehat{f}_{1\infty}^j(y) = \lim_{t\to\infty}\frac{\widehat{f}_1^j(t,y)}{t}. \end{aligned}$$

Similarly, for $x \in \mathbb{R}_+^n$ and any $j = 1, ..., n$ we consider:

$$f_{20}^{j}(x)=\lim_{|y|\to 0}\frac{f_j^2(x,y)}{|y|} \text{ and } F_{20}(x)=\max_{j=1,\dots,n} f_{20}^{j}(x),$$
$$f_{2\infty}^{j}(x)=\lim_{|y|\to \infty}\frac{f_j^2(x,y)}{|y|} \text{ and } F_{2\infty}(x)=\max_{j=1,\dots,n} f_{2\infty}^{j}(x),$$
$$\widehat{f_2^j}(x,t)=\max\{f_j^2(x,y) : y\in\mathbb{R}_+^n \text{ and } |y|\le t\},$$
$$\widehat{f_{20}^j}(x)=\lim_{t\to 0}\frac{\widehat{f_2^j}(x,t)}{t} \text{ and } \widehat{f_{2\infty}^j}(x)=\lim_{t\to \infty}\frac{\widehat{f_2^j}(x,t)}{t}.$$

The main result of this section is the following.

Theorem 3.3.3 **[V. Dincuţă [16]]** *Suppose that (H1) and (H2) takes place. Moreover, suppose that one of the next conditions is fulfilled:*

(H4.1) $\begin{cases} F_{10}(y)=0 \text{ and } F_{1\infty}(y)=\infty \text{ for } y\in\mathbb{R}_+^n, \\ F_{20}(x)=0 \text{ and } F_{2\infty}(x)=\infty \text{ for } x\in\mathbb{R}_+^n. \end{cases}$

(H4.2) $\begin{cases} F_{10}(y)=0 \text{ and } F_{1\infty}(y)=\infty \text{ for } y\in\mathbb{R}_+^n, \\ F_{20}(x)=\infty \text{ and } F_{2\infty}(x)=0 \text{ for } x\in\mathbb{R}_+^n. \end{cases}$

(H4.3) $\begin{cases} F_{10}(y)=\infty \text{ and } F_{1\infty}(y)=0 \text{ for } y\in\mathbb{R}_+^n, \\ F_{20}(x)=0 \text{ and } F_{2\infty}(x)=\infty \text{ for } x\in\mathbb{R}_+^n. \end{cases}$

(H4.4) $\begin{cases} F_{10}(y)=\infty \text{ and } F_{1\infty}(y)=0 \text{ for } y\in\mathbb{R}_+^n, \\ F_{20}(x)=\infty \text{ and } F_{2\infty}(x)=0 \text{ for } x\in\mathbb{R}_+^n. \end{cases}$

Then there exists $0<r_1<R_1$ and $0<r_2<R_2$ such that problem (3.20) has a solution $(x^*,y^*)\in K_{r,R}$.

3.4 Krasnoselskii's vectorial theorem for coincidences

First we will give a vectorial version for coincidences of Krasnoselskii's theorem in cones. We study the existence of positive periodic solutions for the system

$$\begin{cases} L_1x=T_1(x,y) \\ L_2y=T_2(x,y) \end{cases}, \tag{3.22}$$

where $L_1,L_2 : X\to Y$ are linear maps and $T_1,T_2 : X\times X\to Y$ are two nonlinear operators. Here X is a Banach space and Y is a normed space.

Theorem 3.4.1 **[V. Dincuţă [12]]** *Let K_1,K_2 two cones in X; $(r_i,R_i)\in\mathbb{R}_+^2$ such that $0<r_i<R_i$ for $i=1,2$; $T_1,T_2 : K_1\times K_2\to Y$ two completely continuous maps and $J_1,J_2 : X\to Y$ two linear maps such that*

(a) $L_i+J_i : X\to Y$ is invertible for $i=1,2$,

(b) $\begin{cases} (L_1+J_1)^{-1}[T_1(K_1,K_2)+J_1(K_1)] \subset K_1, \\ (L_2+J_2)^{-1}[T_2(K_1,K_2)+J_2(K_2)] \subset K_2. \end{cases}$

Moreover, suppose that one of the following conditions is fulfilled:

$$(c.1) \begin{cases} L_1x - T_1(x,y) \notin (L_1+J_1)(K_1) \text{ for } x \in \partial(K_1)_{r_1} \text{ and } y \in K_2, \\ T_1(x,y) - L_1x \notin (L_1+J_1)(K_1) \text{ for } x \in \partial(K_1)_{R_1} \text{ and } y \in K_2, \\ L_2y - T_2(x,y) \notin (L_2+J_2)(K_2) \text{ for } y \in \partial(K_2)_{r_2} \text{ and } x \in K_1, \\ T_2(x,y) - L_2y \notin (L_2+J_2)(K_2) \text{ for } y \in \partial(K_2)_{R_2} \text{ and } x \in K_1. \end{cases}$$

$$(c.2) \begin{cases} L_1x - T_1(x,y) \notin (L_1+J_1)(K_1) \text{ for } x \in \partial(K_1)_{r_1} \text{ and } y \in K_2, \\ T_1(x,y) - L_1x \notin (L_1+J_1)(K_1) \text{ for } x \in \partial(K_1)_{R_1} \text{ and } y \in K_2, \\ T_2(x,y) - L_2y \notin (L_2+J_2)(K_2) \text{ for } y \in \partial(K_2)_{r_2} \text{ and } x \in K_1, \\ L_2y - T_2(x,y) \notin (L_2+J_2)(K_2) \text{ for } y \in \partial(K_2)_{R_2} \text{ and } x \in K_1. \end{cases}$$

$$(c.3) \begin{cases} T_1(x,y) - L_1x \notin (L_1+J_1)(K_1) \text{ for } x \in \partial(K_1)_{r_1} \text{ and } y \in K_2, \\ L_1x - T_1(x,y) \notin (L_1+J_1)(K_1) \text{ for } x \in \partial(K_1)_{R_1} \text{ and } y \in K_2, \\ L_2y - T_2(x,y) \notin (L_2+J_2)(K_2) \text{ for } y \in \partial(K_2)_{r_2} \text{ and } x \in K_1, \\ T_2(x,y) - L_2y \notin (L_2+J_2)(K_2) \text{ for } y \in \partial(K_2)_{R_2} \text{ and } x \in K_1. \end{cases}$$

$$(c.4) \begin{cases} T_1(x,y) - L_1x \notin (L_1+J_1)(K_1) \text{ for } x \in \partial(K_1)_{r_1} \text{ and } y \in K_2, \\ L_1x - T_1(x,y) \notin (L_1+J_1)(K_1) \text{ for } x \in \partial(K_1)_{R_1} \text{ and } y \in K_2, \\ T_2(x,y) - L_2y \notin (L_2+J_2)(K_2) \text{ for } y \in \partial(K_2)_{r_2} \text{ and } x \in K_1, \\ L_2y - T_2(x,y) \notin (L_2+J_2)(K_2) \text{ for } y \in \partial(K_2)_{R_2} \text{ and } x \in K_1. \end{cases}$$

Then there exists $(x^*,y^*) \in K$ *a solution of system (3.22) such that* $r_1 \leq ||x^*|| \leq R_1$ *and* $r_2 \leq ||y^*|| \leq R_2$.

3.5 Applications of Krasnoselskii's vectorial theorem for coincidences to systems of functional-differential equations

In this section we will apply the Theorem 3.4.1 in order to obtain an existence result for the periodic solutions of problem (3.18).

In a similar way like in section 2.6, the system (3.18) is equivalent with the coincidences system:

$$\begin{cases} L_1x = T_1(x,y), \\ L_2y = T_2(x,y). \end{cases}$$

The main result of this section is the following.

Theorem 3.5.1 [V. Dincuţă [12]] *Suppose that we have two pairs of numbers* (r_1,R_1) *and* (r_2,R_2) *with* $0 < r_1 < R_1$ *and* $0 < r_2 < R_2$*, such that:*

(a) for any $y \in \mathbb{R}_+$ one of the following conditions is fulfilled:

$$\text{(a.1)} \begin{cases} F_1(\cdot, y) \text{ is nondecreasing,} \\ \max\limits_{t\in[0,T]} F_1(r_1, y)(t) < \dfrac{1-\delta_1}{T} r_1, \\ \min\limits_{t\in[0,T]} F_1(\delta_1 R_1, y)(t) > \dfrac{1-\delta_1}{\delta_1 T} R_1; \end{cases}$$

$$\text{(a.2)} \begin{cases} F_1(\cdot, y) \text{ is nondecreasing,} \\ \min\limits_{t\in[0,T]} F_1(\delta_1 r_1, y)(t) > \dfrac{1-\delta_1}{\delta_1 T} r_1, \\ \max\limits_{t\in[0,T]} F_1(R_1, y)(t) < \dfrac{1-\delta_1}{T} R_1; \end{cases}$$

$$\text{(a.3)} \begin{cases} F_1(\cdot, y) \text{ is nonincreasing,} \\ \max\limits_{t\in[0,T]} F_1(\delta_1 r_1, y)(t) < \dfrac{1-\delta_1}{T} r_1, \\ \min\limits_{t\in[0,T]} F_1(R_1, y)(t) > \dfrac{1-\delta_1}{\delta_1 T} R_1; \end{cases}$$

$$\text{(a.4)} \begin{cases} F_1(\cdot, y) \text{ is nonincreasing,} \\ \min\limits_{t\in[0,T]} F_1(r_1, y)(t) > \dfrac{1-\delta_1}{\delta_1 T} r_1, \\ \max\limits_{t\in[0,T]} F_1(\delta_1 R_1, y)(t) < \dfrac{1-\delta_1}{T} R_1. \end{cases}$$

(b) for any $x \in \mathbb{R}_+$ one of the following conditions is fulfilled:

$$\text{(b.1)} \begin{cases} F_2(x, \cdot) \text{ is nondecreasing,} \\ \max\limits_{t\in[0,T]} F_2(x, r_2)(t) < \dfrac{1-\delta_2}{T} r_2, \\ \min\limits_{t\in[0,T]} F_2(x, \delta_2 R_2)(t) > \dfrac{1-\delta_2}{\delta_2 T} R_2; \end{cases}$$

$$\text{(b.2)} \begin{cases} F_2(x, \cdot) \text{ is nondecreasing,} \\ \min\limits_{t\in[0,T]} F_2(x, \delta_2 r_2)(t) > \dfrac{1-\delta_2}{\delta_2 T} r_2, \\ \max\limits_{t\in[0,T]} F_2(x, R_2)(t) < \dfrac{1-\delta_2}{T} R_2; \end{cases}$$

$$\text{(b.3)} \begin{cases} F_2(x, \cdot) \text{ is nonincreasing,} \\ \max\limits_{t\in[0,T]} F_2(x, \delta_2 r_2)(t) < \dfrac{1-\delta_2}{T} r_2, \\ \min\limits_{t\in[0,T]} F_2(x, R_2)(t) > \dfrac{1-\delta_2}{\delta_2 T} R_2; \end{cases}$$

$$
(b.4)\begin{cases}
F_2(x,\cdot)\ \textit{is nonincreasing},\\
\min\limits_{t\in[0,T]} F_2(x,r_2)(t) > \dfrac{1-\delta_2}{\delta_2 T} r_2,\\
\max\limits_{t\in[0,T]} F_2(x,\delta_2 R_2)(t) < \dfrac{1-\delta_2}{T} R_2.
\end{cases}
$$

Then there exists a solution (x^*, y^*) *of system (3.18) such that* $r_1 \le ||x^*|| \le R_1$ *and* $r_2 \le ||y^*|| \le R_2$.

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