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Improved Optimality Conditions for Scalar, Vector and Set-Valued Optimization Problems

Doctoral Thesis Summary

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Contents

Introduction

The optimization theory occupies a well-deserved high rank among mathematical areas, mainly due to its countless applications in almost all thinkable practical areas. The opportunity of conducting investigations in such an exciting research field represents a real privilege. This thesis contains the author's own results, contained in ten single authored or joint authored articles (among which three in ISI journals, six in journals refereed in other international data bases, and one still under review), addressing different problems in scalar, vector and set-valued optimization. The entire content is split into four chapters, whose detailed description is presented in the following.

Chapter 1, entitled Preliminaries, contains a listing of the notions and results taken from the specialized literature, that are used at times in the author's own proofs. It also familiarizes the reader with the notations adopted. The author's achievements at this point are Proposition 1.1.7, which provides a new useful relation between the quasi-relative interior and the quasi interior of a convex set, and Proposition 1.1.8, which, in connection to a convex cone C, assures strict positivity for the value of a non-zero linear continuous functional taken from the dual cone of C at a point form the quasi interior of C.

Chapter 2 deals with Scalar Optimization. The results therein can be considered as belonging to two main categories: those concerning sequential optimality conditions and those with regard to quasi-relative interior type constraint qualifications for strong Lagrange duality. The authors's contributions with respect to these topics have been published in Grad A. [60], [62], [64], as well as in the joint works Boţ R. I., Grad A. and Wanka G. [21], [22].

First of all, in Subsection 2.1.1, we analyze the scalar convex optimization problem with cone constraints

$$(P_C) \qquad \inf_{g(x)\in -C} f(x),$$

under the hypotheses (2.1). For a perturbation function Φ_C, defined by (2.2), Lemma 2.1.3 reveals some properties, while Lemma 2.1.4 characterizes its subdifferential. The fist sequential optimality conditions with respect to (P_C) are stated in Theorem 2.1.5. They are improved, in terms of better separation of the occurring sequences, in Theorem 2.1.7, with the help of Lemma 2.1.6.

Next we analyze the convex optimization problem with geometric and cone constraints

$$(P_{CM}) \qquad \inf_{\substack{x\in M\\ g(x)\in -C}} f(x),$$

under the hypotheses (2.1) and (2.9). The first sequential optimality conditions for (P_{CM}) can be found in Theorem 2.1.9, which uses Lemma 2.1.8. In contrast to Theorems 4.10 and 4.11 in Boţ R. I., Csetnek E. R. and Wanka G. [17], our Theorem 2.1.9 has a smaller number of sequences involved in the optimality conditions, while the function g is C-epi closed, rather than simply continuous. By using Theorem 2.1.7 we obtain an equivalent formulation of Theorem 2.1.9, presented in Theorem 2.1.10.

Within Subsection 2.1.2 we focus our attention on the convex composed optimization problem with geometric and cone constraints

$$(P_{CM}^{s\circ f}) \qquad \inf_{\substack{x\in M\\ g(x)\in -C}} (s\circ f)(x),$$

under the hypotheses (2.13). We fist consider the case when f is star K-lower semicontinuous. A perturbation function $\Phi_{s\circ f}$, given by (2.15), is chosen and characterized in Lemmas 2.1.11 and 2.1.12. Theorems 2.1.13 and 2.1.15 contain equivalent formulations for sequential optimality conditions with regard to $(P_{CM}^{s\circ f})$. Their equivalence is assured by Lemma 2.1.14. The case when K is closed and f is K-epi closed is further analyzed in Theorems 2.1.18 and 2.1.20, in whose proofs we use Lemmas 2.1.16 and 2.1.17.

By considering $\operatorname{dom} f = \operatorname{dom} g = X$, and requiring that f and g are continuous functions, we obtain in Theorem 2.1.21 optimality conditions for $(P_{CM}^{s\circ f})$, with a considerable less number of sequences. This number can be dropped even further, by taking $g \equiv 0$, as it is the case in Corollary 2.1.23. We present next a sequential Lagrange multiplier condition, when $f(x) = x$ for all $x \in X$ and $K := \{0\}$, in Theorem 2.1.24, which turns out to be a refinement of a result by Boţ R. I., Csetnek E. R. and Wanka G. [17, Theorem 4.10]. Moreover, Theorem 2.1.28, a sequential generalization of the well-known Pshenichnyi-Rockafellar Lemma, is an extension of Corollary 4.8 in Boţ R. I., Csetnek E. R. and Wanka G. [17] and, consequently, a generalization of Corollary 3.5 in Jeyakumar V. and Wu Z. Y. [83].

Example 2.1.27 comes to validate the search for sequential optimality conditions, as it presents an optimization problem for which the classical Karush-Kuhn-Tucker conditions fail, whereas the sequential ones do not.

Section 2.2 deals with sufficient strong duality conditions for a convex optimization problem with geometric, cone and affine constraints, stated in infinite dimensional spaces, and its Lagrange dual problem. The optimality conditions are specified by means of the quasi-relative interior and the quasi interior of convex sets. The author's own contributions to this field have been published in Grad A. [64].

The primal-dual pair under investigation is

$$(P_{CMA}) \qquad \inf_{\substack{x\in M\\ g(x)\in -C\\ h(x)=0}} f(x),$$

and

$$(D_{CMA}^{L}) \qquad \sup_{z^*\in C^+, w^*\in W^*} \inf_{x\in M} \left\{ f(x) + \langle z^*, g(x)\rangle + \langle w^*, h(x)\rangle \right\},$$

under the hypotheses (2.33). Theorem 2.2.2 presents a weak duality statement. In Remark 2.2.3 we compare the primal-dual pair (P_{CMA}, D_{CMA}^{L}) to the pair treated by Daniele P. and Giuffré S. [45, Theorem 3.1], emphasize their failures and argument our approach.

Theorem 2.2.6 is the main result of Section 2.2, and presents sufficient conditions in order to achieve strong duality for (P_{CMA}, D^L_{CMA}). Stronger, but in the same time easier to verify in practice versions of Theorem 2.2.6 can be encountered in Theorems 2.2.13 and 2.2.19. All the strong duality results of this section, in the particular case when the primal optimization problem (P_{CMA}) admits an optimal solution, are dealt with in the Corollaries 2.2.7, 2.2.14 and 2.2.20. One should notice that the conditions in Corollary 2.2.20 resamble those in Theorem 3.1 established by DANIELE P. and GIUFFRÉ S. [45], but they are obviously less restrictive and, furthermore, correct.

In the proof of Theorem 2.2.6 we use Lemmas 2.2.4 and 2.2.5 which characterize the set $\mathcal{E}$. Proposition 2.2.9 gives an equivalent formulation for (2.35), whilst Lemma 2.2.11 reveals sufficient conditions which ensure that (2.34) holds. Lemma 2.2.16 approaches the case when M is an affine set, and Lemma 2.2.18 deals with the situation when the function h is continuous.

We then present necessary and sufficient optimality conditions for the existence of saddle points associated with (P_{CMA}), in Theorem 2.2.21. Finally, Chapter 2 ends with a subsection devoted to an optimization problems in $L^2([0,T])$, for which we apply the strong duality results in Corollary 2.2.14.

Chapter 3 reveals aspects regarding sequential optimality conditions for vector problems, along with a new Fenchel dual problem in vector optimization. The author's contributions with respect to these topics have been published in GRAD A. [60], [61], [63], [65] and in the joint works BOŢ R. I., DUMITRU (GRAD) A. and WANKA G. [20], BOŢ R. I., GRAD A. and WANKA G. [21], [22].

To our best knowledge, sequential optimality conditions for vector optimization problems have been given in the literature for the first time in BOŢ R. I., GRAD A. and WANKA G. [21]. These results were then improved in BOŢ R. I., GRAD A. and WANKA G. [22], and GRAD A. [63].

In Section 3.1 we consider the general vector optimization problem with geometric and cone constraints

$$(P^v_{CM}) \qquad \underset{\substack{x\in M\\ g(x)\in -C}}{\text{v-min}}\, f(x),$$

under the hypotheses (3.1). Subsection 3.1.1 addresses the lower semicontinuous case. Theorem 3.1.7 presents sufficient sequential optimality conditions for S-properly efficient solutions to (P^v_{CM}), when f is star K-lower semicontinuous, while Theorem 3.1.8 contains sufficient sequential optimality conditions for T-weakly efficient solutions to (P^v_{CM}), when f is K-epi closed.

When considering f continuous, as it is done in Subsection 3.1.2, not only sufficient, but also necessary and sufficient sequential optimality conditions for S-properly efficient solutions, and for T-weakly efficient solutions to (P^v_{CM}) can be given. We actually treat two particular types of problems, as they are fairly representative and at the same time suffice as examples. Linear scalarization is the first to be considered. We give in Theorem 3.1.9 necessary and sufficient sequential optimality conditions for $S_{K^{+0}}$-properly efficient solutions to (P^v_{CM}). The second scalarization we deal with is attributed to GERSTEWITZ C. and IWANOW [54]. Theorem 3.1.12 states necessary and sufficient sequential optimality conditions for $T_{\operatorname{int} K}$-weakly efficient solutions to (P^v_{CM}).

Remark 3.1.10 and Example 3.1.11 come as a reinforcements of the research undertaken in Section 3.1 with respect to sequential optimality conditions, as they present a vector optimization problem for which the classical Karush-Kuhn-Tucker optimality conditions fail, while the sequential ones do not.

New aspects related to Fenchel-type vector dual problems can be found Section 3.2. They have been published in GRAD A. [61]. The primal problem is

$$(P_A^v) \qquad \operatorname*{v-min}_{x\in X}(f+g\circ A)(x),$$

investigated under hypotheses (3.8). The first Fenchel-type vector dual problem associated with (P_A^v) and treated in Subsection 3.2.1 is

$$(D_A^{v\leq}) \qquad \operatorname*{v-max}_{(y^*,z^*,y)\in\mathcal{A}_{D_A^{v\leq}}} h^{\leq}(y^*,z^*,y),$$

A weak duality statement for the primal-dual pair $(P_A^v, D_A^{v\leq})$ is given in Theorem 3.2.3. The proof of the strong duality theorem, i.e. Theorem 3.2.5, relies on imposing regularity conditions which actually ensures strong duality for the scalarized versions of $(P_A^v, D_A^{v\leq})$. We make a recollection of the most important regularity conditions which have been given so far in the literature, with respect to the particular situation studied in this section. Theorem 3.2.6 helps in the proof of a converse duality statement, in Theorem 3.2.7.

In Subsection 3.2.2 we start by comparing $(D_A^{v\leq})$ with another Fenchel-type vector dual problem associated with (P_A^v), whose formulation was inspired from BRECKNER W. W. and KOLUMBÁN I. [36], defined by

$$(D_A^{vBK}) \qquad \operatorname*{v-max}_{(y^*,z^*,y)\in\mathcal{A}_{D_A^{vBK}}} h^{BK}(y^*,z^*,y),$$

From Remark 3.2.9 we know that $h^{BK}(\mathcal{A}_{D_A^{vBK}}) \subseteq h^{\leq}(\mathcal{A}_{D_A^{v\leq}})$, while Theorem 3.2.10 reveals that $\text{v-max}\, h^{BK}(\mathcal{A}_{D_A^{vBK}}) = \text{v-max}\, h^{\leq}(\mathcal{A}_{D_A^{v\leq}})$. Using the weak, strong and converse duality theorems established for the primal-dual pair $(P_A^v, D_A^{v\leq})$ of vector optimization problems, we provide the same type of results for the primal-dual pair (P_A^v, D_A^{vBK}), in Theorem 3.2.12. When particularizing the spaces X, Y and Z, the Fenchel-type vector dual problems $(D_A^{v\leq})$ and (D_A^{vBK}) become exactly the classical Fenchel dual problem in scalar optimization from ROCKAFELLAR R. T. [103], as it is detailed in Remark 3.2.13.

When considering $Y := \mathbb{R}^m$ and $K := \mathbb{R}^m_+$, in addition to the vector dual problems $(D_A^{v\leq})$ and (D_A^{vBK}), we introduce a new one, whose formulation was inspired from BOŢ R. I., DUMITRU(GRAD) A. and WANKA G. [20], defined by

$$(D_A^{vBGW}) \qquad \operatorname*{v-max}_{(p,q,\lambda,t)\in\mathcal{A}_{D_A^{vBGW}}} h^{BGW}(p,q,\lambda,t),$$

Under an appropriate regularity condition, not only $h^{BK}\left(\mathcal{A}_{D_A^{vBK}}\right) \subseteq h^{BGW}\left(\mathcal{A}_{D_A^{vBGW}}\right)\cap\mathbb{R}^m$, but also $h^{BGW}\left(\mathcal{A}_{D_A^{vBGW}}\right)\cap\mathbb{R}^m \subseteq h^{\leq}\left(\mathcal{A}_{D_A^{v\leq}}\right)$ hold (see Proposition 3.2.15). As it is seen from the Examples 3.2.17 and 3.2.18, the inclusions above are strict. However, in Theorem 3.2.19 we prove that the sets of optimal solutions to (D_A^{vBGW}) and $(D_A^{v\leq})$ coincide, i.e. $\text{v-max}\left[h^{BGW}\left(\mathcal{A}_{D_A^{vBGW}}\right)\cap\mathbb{R}^m\right] = \text{v-max}\, h^{\leq}\left(\mathcal{A}_{D_A^{v\leq}}\right)$. Weak, strong and converse duality statements for the prima-dual pair (P_A^v, D_A^{vBGW}) are given in Theorem 3.2.21. Using Example 3.2.22 we reach the conclusion that a direct converse duality proof for the case of problem (D_A^{vBGW}) would be more difficult, unless using its connection to $(D_A^{v\leq})$ from Theorem 3.2.19.

Section 3.3 contains results published in Boţ R. I., Dumitru(Grad) A. and Wanka G. [20], and Grad A. [65]. It proposes a direct approach to proving weak, strong and converse duality for a Fenchel-type vector dual problem resembling (D_A^{vBGW}) from Subsection 3.2.2, this time in a finite dimensional setting. First, in Subsection 3.3.1 we analyze the scalarized problem associated with $(P_A^{!v})$ and its corresponding Fenchel scalar dual problem. The main results within are Theorem 3.3.4 (weak duality), Theorem 3.3.6 (strong duality) and Theorem 3.3.7 (optimality conditions). Then, in Subsection 3.3.2 we extend the scalar results to the vector case by defining the finite dimension Fenchel-type vector dual problem associated with $(P_A^{!v})$ as

$$(D_A^{!vBGW}) \qquad \underset{(p,q,\lambda,t)\in\mathcal{A}_{D_A^{!vBGW}}}{\text{v-max}} h^{!BGW}(p,q,\lambda,t),$$

Theorem 3.3.9 targets weak duality, while Theorem 3.3.12 aims strong duality. Propositions 3.3.15 and 3.3.16 help in proving the direct converse duality assertion for the primal-dual pair $(P_A^{!v}, D_A^{!vBGW})$ from Theorem 3.3.17, in Subsection 3.3.3.

Chapter 4 addresses a new approach to duality in set-valued optimization, by means of the quasi interior of a convex cone, and contains the author's results from Grad A. [66].

Section 4.1 is centered on the definition and some properties of two new set relations. Given a pointed, convex cone K, with nonempty quasi interior, as a strict subset of a separated locally convex space Y, we introduce two new set relations ($\trianglelefteq^l_{\mathrm{qi}\,K}$ and $\trianglelefteq^u_{\mathrm{qi}\,K}$) in Definition 4.1.3. They turn out to be transitive, as proved in Remark 4.1.4. Proposition 4.1.5 contains some of their properties. Definition 4.1.7 provides four new efficiency notions for a given $\mathcal{S} \subseteq \mathcal{P}_0(Y)$. The notations adopted for the sets of the new efficient solutions are $\text{l-Min}_{\text{qi}}\,\mathcal{S}$, $\text{l-Max}_{\text{qi}}\,\mathcal{S}$, $\text{u-Min}_{\text{qi}}\,\mathcal{S}$ and $\text{u-Max}_{\text{qi}}\,\mathcal{S}$. Proposition 4.1.10 states that $\text{l-Min}_{\text{qi}}(-\mathcal{S}) = -\,\text{u-Max}_{\text{qi}}\,\mathcal{S}$.

For a given set-valued function $F : X \to \mathcal{P}(Y)$, in Section 4.2, Definition 4.2.1 introduces the qi-conjugate of F, while Definition 4.2.4 presents the qi-subgradient of F at $\overline{x} \in X$, such that $F(\overline{x}) \neq \emptyset$. Theorem 4.2.3 may be regarded as an extension of the Fenchel-Young inequality for scalar functions, to set-valued ones.

Section 4.3, as its title suggests, deals with a perturbation approach in set-valued optimization. First of all, in Subsection 4.3.1, we tackle the unconstrained case. Given the set-valued optimization problem

$$(P^{sv}_{\text{qi}}) \qquad \underset{x\in X}{\text{l-Min}_{\text{qi}}}\, F(x),$$

we attach to it a general perturbation function Φ, and, by making use of the qi-conjugate set-valued function of Φ, we prove that

$$(D^{sv}_{\text{qi}}) \qquad \underset{T\in\mathcal{L}(W,Y)}{\text{l-Max}_{\text{qi}}} -\Phi^*_{\text{qi}\,K}(0,T).$$

is a valid dual to (P^{sv}_{qi}), where W is a topological vector space. The weak duality theorem, i.e. Theorem 4.3.4, is accompanied by Theorem 4.3.5 which provides some optimality conditions for the primal-dual pair $(P^{sv}_{\text{qi}}, D^{sv}_{\text{qi}})$. Moreover, Theorem 4.3.6 presents optimality conditions for (D^{sv}_{qi}).

Next, in Subsection 4.3.2, we focus on the constrained set-valued optimization problem

$$(CP^{sv}_{\text{qi}}) \qquad \underset{G(x)\cap(-C)\neq\emptyset}{\text{l-Min}_{\text{qi}}}\, F(x),$$

where Z is a separated locally convex space and $G : X \to \mathcal{P}(Z)$ is a proper set-valued function. Theorem 4.3.10 contains a weak duality result, whereas Theorems 4.3.11 and 4.3.12 present optimality conditions. This constrained case is derived from the unconstrained one, with the help of the set-valued indicator function.

By choosing, in Subsection 4.3.3, a suitable Lagrange-type perturbation function we are able to prove a Lagrange-type strong duality theorem, i.e. Theorem 4.3.15. Let us further notice that this result generalizes Corollary 4.7 in BOŢ R. I., CSETNEK E. R. and MOLDOVAN A. [16] from the scalar case. In its proof we use a quasi-relative interior separation theorem from BOŢ R. I., CSETNEK E. R. and WANKA G. [19].

Chapter 4 ends with Section 4.4, which provides an example formulated in $\ell^2(\mathbb{R})$, on which the strong duality Theorem 4.3.15 can be successfully applied.

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Keywords

Scalar optimization, vector optimization, set-valued optimization, Lagrange duality, Fenchel duality, sequential optimality conditions, generalized interiors of sets, constraint qualifications.

Chapter 1

Preliminaries

For the easier lecturing of this thesis we start by presenting basic notions, results and notations with respect to real-valued and vector-valued functions, as well as some elements of convex analysis.

1.1 Some Special Subsets of a Vector Space

Let X be a vector space, and let $M \subseteq X$ be a set. We recall de definitions of the linear hull ($\operatorname{lin} M$), affine hull ($\operatorname{aff} M$), convex hull ($\operatorname{co} M$), conical hull ($\operatorname{cone} M$), and convex conical hull ($\operatorname{cone}\operatorname{co} M$) of M. We also mention the definition of the normal cone associated with M, denoted by N_M, as well as the definitions of the dual cone of a nonempty cone $C \subseteq X$, denoted by C^+, and of the quasi interior of the cone C, denoted by C^{+0}.

1.1.1 Generalized Interiors of Sets

Let X be a nontrivial vector space, and let $M \subseteq X$ be a set. We recall de definitions of the algebraic interior ($\operatorname{core} M$), intrinsic core ($\operatorname{icr} M$) of M, and mention some of their properties when M is convex. For the case when X is a topological vector space we remind the interior ($\operatorname{int} M$), and the closure ($\operatorname{cl} M$) of M.

Consider now a separated topological vector space X, and let $M \subseteq X$ be a set. We mention the strong quasi-relative interior of M ($\operatorname{sqri} M$), and when M is convex, the quasi-relative interior ($\operatorname{qri} M$), and the quasi interior ($\operatorname{qi} M$).

The author's results are the following two.

Proposition 1.1.7 (Grad A. [64]) *Let M be a convex subset of a separated locally convex space X, and let $x \in M$. Then*

$$x \in \operatorname{qi} M \Longleftrightarrow \begin{cases} 0 \in \operatorname{qi}(M - M) \\ x \in \operatorname{qri} M. \end{cases}$$

Proposition 1.1.8 *Let C be a nonempty convex cone of a separated locally convex space X. Then, for all $x^* \in C^+ \backslash \{0\}$ and for all $x \in \operatorname{qi} C$, the following inequality holds:*

$$\langle x^*, x \rangle > 0. \tag{1.1}$$

1.1.2 Separation Theorems

When proving strong duality results, separation theorems are always involved. We mention not only some well-known separation theorems from the specialized literature (given by EIDELHEIT M., TUKEY J. W.), but also some newly published ones, that use the quasi-relative interior and quasi interior of a convex set (given by BOŢ R. I., CSETNEK E. R. and WANKA G. [19], CAMMAROTO F. and DI BELLA B. [38]).

1.2 Notions and Results Concerning Functions

1.2.1 Extended Real-Valued Functions

Let X be a locally convex space, and let X^* be its topological dual. We denote $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$. Given a set $M \subseteq X$ its indicator function is $\delta_M : X \to \overline{\mathbb{R}}$, and its support functional is $\sigma_M : X^* \to \overline{\mathbb{R}}$.

Let $f : X \to \overline{\mathbb{R}}$ be a function. We recall its domain $(\operatorname{dom} f)$ and its epigraph $(\operatorname{epi} f)$. Let $\overline{x} \in X$. We denote by $\partial f(\overline{x})$ the subdifferential of f at $\overline{x}$. The conjugate function of f regarding a set $M \subseteq X$ is $f_M^* : X^* \to \overline{\mathbb{R}}$, when $M = X$, we get the classical Fenchel-Moreau conjugate of f denoted by f^*. The so-called Fenchel-Young inequality proves to be extremely useful in applications is also mentioned. The conjugate function $f^{**} : X \to \overline{\mathbb{R}}$ of f^* is called the biconjugate of f.

Let $f_1 : X \to \overline{\mathbb{R}}$,..., $f_m : X \to \overline{\mathbb{R}}$ be proper functions, where $m \in \mathbb{N}$. The infimal convolution of $f_1, ..., f_m$ is the function $f_1 \Box ... \Box f_m : X \to \overline{\mathbb{R}}$. We list some of its properties.

Let X and Y be topological vector spaces. Given a linear continuous operator $T \in \mathcal{L}(X, Y)$, its adjoint operator is denoted by T^*, while the infimal function of a function $f : X \to \overline{\mathbb{R}}$ through T is $Tf : Y \to \overline{\mathbb{R}}$.

1.2.2 Extended Vector-Valued Functions

Let us consider a locally convex space Y partially ordered by a nonempty convex cone $C \subseteq Y$. To Y we attach a greatest element with respect to $\leq_C$, which does not belong to Y and which is denoted by ∞_Y. Put $Y^\bullet := Y \cup \{\infty_Y\}$. Then, for each $y \in Y^\bullet$, we have $y \leq_K \infty_Y$. Besides, we define on $Y^\bullet$ the following operations:

$$y + \infty_Y := \infty_Y, \infty_Y + y := \infty_Y, \lambda \cdot \infty_Y := \infty_Y \text{ and } \langle y^*, \infty_Y \rangle := +\infty$$

for all $y \in Y$, all $\lambda \geq 0$ and all $y^* \in C^+$.

Given a function $f : X \to Y^\bullet$, we recall its domain $(\operatorname{dom} f)$ and its cone-epigraph $(\operatorname{epi}_C f)$. Moreover, we mention different generalized convexity notions for vector-valued functions. A review of the main extensions of lower semicontinuity (cone-lower semicontinuity, star cone-lower semicontinuity, cone-epi closedness) to the vector case ends the chapter.

Chapter 2

Scalar Optimization

Several original results concerning optimality conditions and duality for different types of scalar optimization problems are contained within the present chapter. They were published by the author alone in Grad A. [60], [62], [64], or in the joint works Boţ R. I., Grad A. and Wanka G. [21], [22]. Most of the theorems and corollaries represent generalizations and/or improvements of results previously given by other authors.

From the historical point of view, the first major achievements obtained in the optimization theory were given for scalar problems. Despite being an essential subject of interest for the scientific community during the last decades, this topic still presents some challenges, focused mainly on the widening of the set of problems for which less and less restrictive optimality conditions can be imposed. Several monographs were written on the subject, starting mainly from the middle of the last century. We mention among them the books by Barbu V. and Precupanu T. [5], Blaga L. and Lupşa L. [6], Boţ R. I., Grad S. M. and Wanka G. [26], Breckner W. W. [33], Rockafellar R. T. [103] and Zălinescu C. [121].

2.1 Sequential Optimality Conditions for Convex Optimization Problems with Geometric and Cone Constraints

This section contains sequential optimality conditions for several types of simple and composed scalar convex optimization problems, conditions that are, to our best knowledge, the most general given so far in the literature.

Boţ R. I., Csetnek E. R. and Wanka G. [17], [18], have recently given sequential optimality conditions in convex optimization via a perturbation approach improving several previously given results. We extend the results given in [17], [18], and in Grad A. [60], for the case of scalar convex composed optimization problems with geometric and cone constraints defined with cone-epi closed vector-valued functions. Moreover, we rediscover as particular cases some of the results in the above mentioned papers.

Let $(X, \|\cdot\|)$ be a reflexive Banach space, and let $(X^*, \|\cdot\|_*)$ be its topological dual space. Let $(x_n^*)_{n\in\mathbb{N}}$ be a sequence in X^*. We write $x_n^* \xrightarrow{\omega^*} 0$ ($x_n^* \xrightarrow{\|\cdot\|_*} 0$) for the case when $(x_n^*)_{n\in\mathbb{N}}$ converges to 0 in the weak* (strong) topology.

Theorem 2.1.1 (Boţ R. I., Csetnek E. R., Wanka G. [17]) *Let X be a reflexive Banach space, and let Y be a Banach space. Let $\Phi : X \times Y \to \overline{\mathbb{R}}$ be a proper, convex and lower semicontinuous function such that* $\inf_{x\in X} \Phi(x, 0) < +\infty$*. Further, let* $\overline{x} \in \operatorname{dom} \Phi(\cdot, 0)$*. Then the following statements are equivalent:*

(a) *$\overline{x}$ is a minimizer of $\Phi(\cdot, 0)$ on X.*
(b) *There exist two sequences $((x_n, y_n))_{n\in\mathbb{N}}$ in $\operatorname{dom} \Phi$ and $((x_n^*, y_n^*))_{n\in\mathbb{N}}$ in $X^* \times Y^*$, respectively, such that*

$$\begin{cases} \forall n \in \mathbb{N} : (x_n^*, y_n^*) \in \partial\Phi(x_n, y_n); \\ x_n \to \overline{x}, y_n \to 0, x_n^* \to 0, \langle y_n^*, y_n\rangle \to 0, \Phi(x_n, y_n) - \Phi(\overline{x}, 0) \to 0. \end{cases}$$

Remark 2.1.2 Theorem 2.1.1 can also be obtained from Zălinescu C. [121, Theorem 3.1.6] where one should particularize the following elements: define the linear continuous operator $A : X \to X \times Y$ by $Ax := (x, 0)$, and take the function $f := \Phi \circ A$. ∎

2.1.1 Convex Optimization Problems

Within this subsection there are provided improvements of some results established by Boţ R. I., Csetnek E. R. and Wanka G. [18] for quite a general class of scalar convex optimization problems, where the constraint defining function is cone-convex and cone-epi closed. We start by considering the general scalar convex optimization problem with cone constraints

$$(P_C) \qquad \inf_{g(x)\in -C} f(x),$$

stated under the following hypotheses:

$$(2.1) \qquad \begin{cases} X \text{ is a reflexive Banach space}, Z \text{ is a Banach space;} \\ C \subseteq Z \text{ is a nonempty, closed and convex cone;} \\ f : X \to \overline{\mathbb{R}} \text{ is a proper, convex and lower semicontinuous function;} \\ g : X \to Z^\bullet \text{ is a proper, } C\text{-convex and } C\text{-epi closed function;} \\ g^{-1}(-C) \cap \operatorname{dom} f \neq \emptyset. \end{cases}$$

We denote the set of feasible solutions to (P_C) by $\mathcal{A}_{P_C} := g^{-1}(-C) \cap \operatorname{dom} f$.

We consider the perturbation function $\Phi_C : X \times X \times Z \to \overline{\mathbb{R}}$, defined by

$$(2.2) \qquad \Phi_C(x, p, q) := \begin{cases} f(x + p) & \text{if } g(x) - q \in -C \\ +\infty & \text{otherwise.} \end{cases}$$

Lemma 2.1.3 (Grad A. [62]) *Let the hypotheses* (2.1) *be satified. Then the function Φ_C is proper, convex and lower semicontinuous. Furthermore, it holds*

$$(2.3) \qquad \operatorname{dom} \Phi_C(\cdot, 0, 0) = \mathcal{A}_{P_C},$$

and therefore the following inequality is valid:

$$(2.4) \qquad \inf_{x\in X} \Phi_C(x, 0, 0) < +\infty.$$

Lemma 2.1.4 (Grad A. [62]) *Let the hypotheses* (2.1) *be satisfied. If* $(\overline{x}, \overline{p}, \overline{q}) \in \operatorname{dom} \Phi_C$, *then*

$$\partial\Phi_C(\overline{x}, \overline{p}, \overline{q}) = \left\{ \begin{array}{l} (x^*, p^*, -q^*) \in X^* \times X^* \times (-C^+) : p^* \in \partial f(\overline{x} + \overline{p}), \\ x^* - p^* \in \partial(q^* \circ g)(\overline{x}), \langle q^*, g(\overline{x}) - \overline{q}\rangle = 0 \end{array} \right\}. \tag{2.5}$$

The following theorem contains the previously announced necessary and sufficient sequential optimality conditions for problem (P_C).

Theorem 2.1.5 (Grad A. [62]) *Let the hypotheses* (2.1) *be satisfied. An element* $\overline{x} \in \mathcal{A}_{P_C}$ *is an optimal solution to problem* (P_C) *if and only if there exists two sequences*

$$((x_n, p_n, q_n))_{n\in\mathbb{N}} \text{ in } X \times (\operatorname{dom} f) \times C \text{ and } ((x_n^*, p_n^*, q_n^*))_{n\in\mathbb{N}} \text{ in } X^* \times X^* \times C^+, \tag{2.6}$$

respectively, such that

$$\left\{ \begin{array}{l} \forall n \in \mathbb{N} : p_n^* \in \partial f(p_n), x_n^* \in \partial(q_n^* \circ g)(x_n), \langle q_n^*, q_n\rangle = 0; \\ x_n \to \overline{x}, p_n \to \overline{x}, g(x_n) + q_n \to 0, x_n^* + p_n^* \to 0; \\ \langle p_n^*, p_n - x_n\rangle - \langle q_n^*, g(x_n)\rangle \to 0, f(p_n) - f(\overline{x}) \to 0. \end{array} \right. \tag{2.7}$$

A sequential optimality condition quite similar to that in Theorem 2.1.5 was obtained by Boţ R. I., Csetnek E. R. and Wanka G. [18, Corollary 3.5]. However, our condition may be easier applied in practice, since the sequences involved are better separated.

Lemma 2.1.6 *Let the hypotheses* (2.1) *be satisfied, and let* $\overline{x} \in \mathcal{A}_{P_C}$. *Moreover, consider the sequences* $((x_n, p_n, q_n))_{n\in\mathbb{N}}$ *in* $X \times (\operatorname{dom} f) \times C$ *and* $((x_n^*, p_n^*, q_n^*))_{n\in\mathbb{N}}$ *in* $X^* \times X^* \times C^+$ *such that*

$$\left\{ \begin{array}{l} \forall n \in \mathbb{N} : p_n^* \in \partial f(p_n), x_n^* \in \partial(q_n^* \circ g)(x_n); \\ x_n \to \overline{x}, p_n \to \overline{x}, x_n^* + p_n^* \to 0. \end{array} \right.$$

Suppose that the real sequences $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ *and* $(c_n)_{n\in\mathbb{N}}$ *are defined by:*

$$a_n := \langle p_n^*, p_n - x_n\rangle - \langle q_n^*, g(x_n)\rangle, b_n := \langle p_n^*, p_n - \overline{x}\rangle - \langle q_n^*, g(\overline{x})\rangle \text{ and}$$

$$c_n := -\langle p_n^*, x_n - \overline{x}\rangle - \langle q_n^*, g(x_n) - g(\overline{x})\rangle.$$

Then

$$a_n \to 0 \text{ if and only if } b_n \to 0 \text{ and } c_n \to 0.$$

Applying Lemma 2.1.6, we obtain the following equivalent formulation of Theorem 2.1.5.

Theorem 2.1.7 *Let the hypotheses* (2.1) *be satisfied. An element* $\overline{x} \in \mathcal{A}_{P_C}$ *is an optimal solution to problem* (P_C) *if and only if there exist two sequences*

$$((x_n, p_n, q_n))_{n\in\mathbb{N}} \text{ in } X \times \operatorname{dom} f \times C \text{ and } ((x_n^*, p_n^*, q_n^*))_{n\in\mathbb{N}} \text{ in } X^* \times X^* \times C^+,$$

respectively, such that

$$\left\{ \begin{array}{l} \forall n \in \mathbb{N} : p_n^* \in \partial f(p_n), x_n^* \in \partial(q_n^* \circ g)(x_n), \langle q_n^*, q_n\rangle = 0; \\ x_n \to \overline{x}, p_n \to \overline{x}, g(x_n) + q_n \to 0, x_n^* + p_n^* \to 0; \\ \langle p_n^*, p_n - \overline{x}\rangle - \langle q_n^*, g(\overline{x})\rangle \to 0, \langle p_n^*, x_n - \overline{x}\rangle + \langle q_n^*, g(x_n) - g(\overline{x})\rangle \to 0; \\ f(p_n) - f(\overline{x}) \to 0. \end{array} \right. \tag{2.8}$$

Next we focus on the convex optimization problem with geometric and cone constraints

$$(P_{CM}) \qquad \inf_{\substack{x\in M\\ g(x)\in -C}} f(x).$$

To this end, we suppose that the hypotheses (2.1), and also the following additional assumptions, are satisfied:

$$(2.9) \qquad \begin{cases} M\subseteq X \text{ is a nonempty, closed and convex set;}\\ M\cap(\operatorname{dom} f)\cap g^{-1}(-C)\neq\emptyset. \end{cases}$$

We denote the set of feasible solutions to (P_{CM}) by $\mathcal{A}_{P_{CM}} := M\cap(\operatorname{dom} f)\cap g^{-1}(-C)$.

The problem (P_{CM}) can be considered as a (P_C)-type problem. The argument for this statement is given in the following.

First of all, we consider the function $\tilde f : X\to\overline{\mathbb{R}}$ defined by $\tilde f := f+\delta_M$.

Lemma 2.1.8 *Let the hypotheses* (2.1) *and* (2.9) *be satisfied. Then the function $\tilde f$ is proper, convex and lower semicontinuous.*

From the considerations stated above, the optimization problem (P_{CM}) can be rewritten as:

$$(2.10) \qquad \inf_{g(x)\in -C} \tilde f(x).$$

By applying Theorem 2.1.5 to this problem we obtain sequential optimality conditions for problem (P_{CM}).

Theorem 2.1.9 (Grad A. [62]) *Let the hypotheses* (2.1) *and* (2.9) *be satisfied. An element $\overline{x}\in\mathcal{A}_{P_{CM}}$ is an optimal solution to problem (P_{CM}) if and only if there exist two sequences*

$$((x_n,p_n,q_n))_{n\in\mathbb{N}} \text{ in } X\times(\operatorname{dom} f\cap M)\times C \text{ and } ((x_n^*,p_n^*,q_n^*))_{n\in\mathbb{N}} \text{ in } X^*\times X^*\times C^+,$$

respectively, such that

$$(2.11) \qquad \begin{cases} \forall n\in\mathbb{N}: p_n^*\in\partial(f+\delta_M)(p_n), x_n^*\in\partial(q_n^*\circ g)(x_n), \langle q_n^*,q_n\rangle=0;\\ x_n\to\overline{x}, p_n\to\overline{x}, g(x_n)+q_n\to 0, x_n^*+p_n^*\to 0;\\ \langle p_n^*,p_n-x_n\rangle-\langle q_n^*,g(x_n)\rangle\to 0, f(p_n)-f(\overline{x})\to 0. \end{cases}$$

Theorem 2.1.9 has a a smaller number of sequences involved in the optimality conditions in comparison to Theorems 4.10 and 4.11 in Boţ R. I., Csetnek E. R. and Wanka G. [17], while the function g is C-epi-closed rather than simply continuous. When $\operatorname{dom} f = X$ and f is continuous, then the condition $f(p_n)-f(\overline{x})\to 0$ in (2.11) is superfluous since it can be deduced from $p_n\to\overline{x}$. When g is continuous, the system (2.11) gives an improved sequential Lagrange multiplier condition for (P_{CM}). In this particular case, we obtain an improvement of Theorem 4.1 in Thibault L. [114], as it will be seen in Theorem 2.1.21, stated in Subsection 2.1.3.

By using Theorem 2.1.7 we obtain the following equivalent formulation of Theorem 2.1.9.

Theorem 2.1.10 *Let the hypotheses* (2.1) *and* (2.9) *be satisfied. An element* $\overline{x} \in \mathcal{A}_{P_{CM}}$ *is an optimal solution to problem* (P_{CM}) *if and only if there exists two sequences*

$$((x_n, p_n, q_n))_{n\in\mathbb{N}} \text{ in } X \times (\operatorname{dom} f \cap M) \times C \text{ and } ((x_n^*, p_n^*, q_n^*))_{n\in\mathbb{N}} \text{ in } X^* \times X^* \times C^+,$$

respectively, such that

$$\left\{\begin{array}{l} \forall n \in \mathbb{N} : p_n^* \in \partial(f + \delta_M)(p_n), x_n^* \in \partial(q_n^* \circ g)(x_n), \langle q_n^*, q_n\rangle = 0; \\ x_n \to \overline{x}, p_n \to \overline{x}, g(x_n) + q_n \to 0, x_n^* + p_n^* \to 0; \\ \langle p_n^*, p_n - \overline{x}\rangle - \langle q_n^*, g(\overline{x})\rangle \to 0, \langle p_n^*, x_n - \overline{x}\rangle + \langle q_n^*, g(x_n) - g(\overline{x})\rangle \to 0 \\ f(p_n) - f(\overline{x}) \to 0. \end{array}\right. \tag{2.12}$$

2.1.2 Convex Composed Optimization Problems

Within this subsection there are presented results concerning sequential optimality conditions for the following scalar convex composed optimization problem with geometric and cone constraints:

$$(P_{CM}^{s\circ f}) \qquad \inf_{\substack{x\in M,\\ g(x)\in -C}} (s \circ f)(x),$$

stated under the following hypotheses:

$$\left\{\begin{array}{l} X \text{ is a reflexive Banach space}, Y \text{ and } Z \text{ are Banach spaces;} \\ K \subseteq Y \text{ and } C \subseteq Z \text{ are nonempty convex cones;} \\ M \subseteq X \text{ is a nonempty, closed and convex set;} \\ f : X \to Y^\bullet \text{ is a proper, } K\text{-convex function;} \\ g : X \to Z^\bullet \text{ is a proper, } C\text{-convex function;} \\ s : Y^\bullet \to \overline{\mathbb{R}} \text{ is a proper, convex, lower semicontinuous} \\ \qquad\qquad \text{and } K\text{-increasing function with } s(\infty_Y) = +\infty; \\ \{x \in M \cap (\operatorname{dom} f) \cap g^{-1}(-C) : f(x) \in \operatorname{dom} s\} \neq \emptyset. \end{array}\right. \tag{2.13}$$

For the sake of simplicity for writing, in the following we will use the set notation

$$\mathcal{A}_{s\circ f} := \{x \in M \cap (\operatorname{dom} f) \cap g^{-1}(-C) : f(x) \in \operatorname{dom} s\}.$$

The Case When f is Star K-Lower Semicontinuous

We formulate sequential optimality conditions for $(P_{CM}^{s\circ f})$ supposing that not only the hypotheses (2.13) are satisfied, but also the following assumptions:

$$f \text{ is star } K\text{-lower semicontinuous}, C \text{ is closed}, g \text{ is } C\text{-epi closed.} \tag{2.14}$$

We consider the perturbation function $\Phi_{s\circ f} : X \times Y \times Z \to \overline{\mathbb{R}}$ defined by

$$\Phi_{s\circ f}(x, y, z) := \left\{\begin{array}{ll} s(f(x) + y) & \text{if } x \in M, g(x) - z \in -C \\ +\infty & \text{otherwise.} \end{array}\right. \tag{2.15}$$

Lemma 2.1.11 (Boţ R. I., Grad A., Wanka G. [22]) *Let the hypotheses* (2.13) *and* (2.14) *be satisfied. Then the function* $\Phi_{s\circ f}$ *is proper, convex and lower semicontinuous. Furthermore, it holds*

$$\operatorname{dom}\Phi_{s\circ f}(\cdot,0,0)=\mathcal{A}_{s\circ f}, \tag{2.16}$$

and the following inequality is satisfied:

$$\inf_{x\in X}\Phi_{s\circ f}(x,0,0)<+\infty. \tag{2.17}$$

Lemma 2.1.12 *Let the hypotheses* (2.13) *and* (2.14) *be satisfied. If* $(\overline{x},\overline{p},\overline{q})\in\operatorname{dom}\Phi_{s\circ f}$, *then*

$$\partial\Phi_{s\circ f}(\overline{x},\overline{p},\overline{q})=\left\{\begin{array}{l}(x^*,p^*,-q^*)\in X^*\times K^+\times(-C^+):\\ x^*\in\partial\left((p^*\circ f)+(q^*\circ g)+\delta_M\right)(\overline{x}),\\ p^*\in\partial s\left(f(\overline{x})+\overline{p}\right),\langle q^*,g(\overline{x})-\overline{q}\rangle=0\end{array}\right\}. \tag{2.18}$$

The following theorem provides sequential optimality conditions for problem $(P_{CM}^{s\circ f})$.

Theorem 2.1.13 *Let the hypotheses* (2.13) *and* (2.14) *be satisfied. An element* $\overline{x}\in\mathcal{A}_{s\circ f}$ *is an optimal solution to problem* $(P_{CM}^{s\circ f})$ *if and only if there exist two sequences*

$$((x_n,y_n,z_n))_{n\in\mathbb{N}} \text{ in } (M\cap\operatorname{dom}f)\times(\operatorname{dom}s)\times(-C) \text{ and } ((x_n^*,y_n^*,z_n^*))_{n\in\mathbb{N}} \text{ in } X^*\times K^+\times C^+,$$

respectively, such that

$$\left\{\begin{array}{l}\forall n\in\mathbb{N}: x_n^*\in\partial\left((y_n^*\circ f)+(z_n^*\circ g)+\delta_M\right)(x_n), y_n^*\in\partial s(y_n),\langle z_n^*,z_n\rangle=0;\\ x_n\to\overline{x}, y_n-f(x_n)\to 0, z_n-g(x_n)\to 0, x_n^*\to 0;\\ \langle y_n^*,y_n-f(x_n)\rangle-\langle z_n^*,g(x_n)\rangle\to 0, s(y_n)-s(f(\overline{x}))\to 0.\end{array}\right. \tag{2.19}$$

Lemma 2.1.14 *Let the hypotheses* (2.13) *and* (2.14) *be satisfied, and let* $\overline{x}\in\mathcal{A}_{s\circ f}$. *Moreover, let us consider two sequences* $((x_n,y_n,z_n))_{n\in\mathbb{N}}$ *in* $(M\cap\operatorname{dom}f)\times(\operatorname{dom}s)\times(-C)$ *and* $((x_n^*,y_n^*,z_n^*))_{n\in\mathbb{N}}$ *in* $X^*\times K^+\times C^+$, *respectively, such that*

$$\left\{\begin{array}{l}\forall n\in\mathbb{N}: x_n^*\in\partial\left((y_n^*\circ f)+(z_n^*\circ g)+\delta_M\right)(x_n), y_n^*\in\partial s(y_n);\\ x_n\to\overline{x}, x_n^*\to 0, s(y_n)-s(f(\overline{x}))\to 0.\end{array}\right. \tag{2.20}$$

Suppose that the real sequences $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ *and* $(c_n)_{n\in\mathbb{N}}$ *are defined by:*

$$a_n:=\langle y_n^*,y_n-f(x_n)\rangle-\langle z_n^*,g(x_n)\rangle, b_n:=\langle y_n^*,y_n-f(\overline{x})\rangle+\langle z_n^*,-g(\overline{x})\rangle, \text{ and}$$
$$c_n:=\langle y_n^*,f(x_n)-f(\overline{x})\rangle+\langle z_n^*,g(x_n)-g(\overline{x})\rangle.$$

Then

$$a_n\to 0 \text{ if and only if } b_n\to 0 \text{ and } c_n\to 0.$$

By considering Lemma 2.1.14 we give the following equivalent formulation of Theorem 2.1.13.

Theorem 2.1.15 *Let the hypotheses* (2.13) *and* (2.14) *be satisfied. An element* $\overline{x}\in\mathcal{A}_{s\circ f}$ *is an optimal solution to problem* $(P_{CM}^{s\circ f})$ *if and only if there exist two sequences*

$$((x_n,y_n,z_n))_{n\in\mathbb{N}} \text{ in } (M\cap\operatorname{dom}f)\times(\operatorname{dom}s)\times(-C) \text{ and } ((x_n^*,y_n^*,z_n^*))_{n\in\mathbb{N}} \text{ in } X^*\times K^+\times C^+,$$

respectively, such that

$$\left\{\begin{array}{l}\forall n\in\mathbb{N}: x_n^*\in\partial\left((y_n^*\circ f)+(z_n^*\circ g)+\delta_M\right)(x_n), y_n^*\in\partial s(y_n),\langle z_n^*,z_n\rangle=0;\\ x_n\to\overline{x}, y_n-f(x_n)\to 0, z_n-g(x_n)\to 0, x_n^*\to 0;\\ \langle y_n^*,y_n-f(\overline{x})\rangle-\langle z_n^*,g(\overline{x})\rangle\to 0,\\ \langle y_n^*,f(x_n)-f(\overline{x})\rangle+\langle z_n^*,g(x_n)-g(\overline{x})\rangle\to 0, s(y_n)-s(f(\overline{x}))\to 0.\end{array}\right. \tag{2.21}$$

The Case When f is K-epi Closed

We provide sequential optimality conditions for $(P_{CM}^{s\circ f})$ when not only the hypotheses (2.13) are satisfied, but also the following assumptions:

$$C \text{ and } K \text{ are closed}, f \text{ is } K\text{-epi closed}, g \text{ is } C\text{-epi closed}. \tag{2.22}$$

In order to obtain the desired sequential optimality conditions for problem $(P_{CM}^{s\circ f})$ under the hypotheses (2.13) and (2.22) we use the following intermediate problem:

$$(P_{\widetilde{s}}) \qquad \inf_{H(x,y)\leq_{K\times C}(0,0)} \widetilde{s}(x,y),$$

where the function $\widetilde{s} : X \times Y \to \overline{\mathbb{R}}$ is defined by $\widetilde{s}(x,y) := s(y) + \delta_M(x)$ for all (x,y) in $X \times Y$. The function $H : X \times Y \to (Y \times Z)^\bullet$ generating the constraints of $(P_{\widetilde{s}})$ is defined by

$$H(x,y) := (f(x) - y, g(x)) \text{ for all } (x,y) \in X \times Y.$$

Lemma 2.1.16 (Grad A. [62]) *Let the hypotheses* (2.13) *and* (2.22) *be satisfied. Then the following statements are true:*

(a) *The function $\widetilde{s}$ is proper, convex and lower semicontinuous.*
(b) *The function H is proper, $K \times C$-convex and $K \times C$-epi closed.*
(c) *$(P_{\widetilde{s}})$ is an optimization problem of type (P_C).*

Lemma 2.1.17 (Grad A. [62]) *Let the hypotheses* (2.13) *and* (2.22) *be satisfied. Then the following statements are true:*

(a) *If $(\overline{x},\overline{y}) \in \operatorname{dom}\widetilde{s}$ is an optimal solution to $(P_{\widetilde{s}})$, then $(\overline{x}, f(\overline{x}))$ is also an optimal solution to $(P_{\widetilde{s}})$.*
(b) *$\overline{x} \in \mathcal{A}_{s\circ f}$ is an optimal solution to $(P_{CM}^{s\circ f})$ if and only if $(\overline{x}, f(\overline{x}))$ is an optimal solution to $(P_{\widetilde{s}})$.*

Theorem 2.1.18 (Grad A. [62]) *Let the hypotheses* (2.13) *and* (2.22) *be satisfied, and let Y be a reflexive space. An element $\overline{x} \in \mathcal{A}_{s\circ f}$ is an optimal solution to $(P_{CM}^{s\circ f})$ if and only if there exist two sequences*

$$((x_n, y_n, u_n, v_n, t_n, q_n))_{n\in\mathbb{N}} \text{ in } X \times Y \times M \times (\operatorname{dom} s) \times K \times C$$

and

$$((x_n^*, u_n^*, v_n^*, t_n^*, q_n^*))_{n\in\mathbb{N}} \text{ in } X^* \times X^* \times Y^* \times K^+ \times C^+,$$

respectively, such that

$$\begin{cases} \forall n \in \mathbb{N} : u_n^* \in N_M(u_n), v_n^* \in \partial s(v_n), x_n^* \in \partial(t_n^* \circ f + q_n^* \circ g)(x_n), \\ \langle t_n^*, t_n\rangle = 0, \langle q_n^*, q_n\rangle = 0; \\ x_n \to \overline{x}, u_n \to \overline{x}, y_n \to f(\overline{x}), v_n \to f(\overline{x}), f(x_n) + t_n \to f(\overline{x}), \\ g(x_n) + q_n \to 0, x_n^* + u_n^* \to 0, -t_n^* + v_n^* \to 0; \\ \langle u_n^*, u_n - x_n\rangle + \langle v_n^*, v_n - y_n\rangle - \langle t_n^*, f(x_n) - y_n\rangle - \langle q_n^*, g(x_n)\rangle \to 0, \\ s(v_n) - s(f(\overline{x})) \to 0. \end{cases} \tag{2.23}$$

Remark 2.1.19 The system (2.23) contains actually the set of solution for another convex optimization problem. The condition $u_n^* \in N_M(u_n)$ means actually that u_n is the optimal solution of the optimization problem $\sup_{x\in M}\langle u_n^*, x\rangle$, therefore $\langle u_n^*, u_n\rangle = \max_{x\in M}\langle u_n^*, x\rangle$. ∎

In terms of the separation of the sequences, an improved version of Theorem 2.1.18 can be obtained by applying in its proof Theorem 2.1.7 instead of Theorem 2.1.5.

Theorem 2.1.20 *Let the hypotheses* (2.13) *and* (2.22) *be satisfied, and let Y be a reflexive space. An element $\overline{x} \in \mathcal{A}_{s\circ f}$ is an optimal solution to $(P_{CM}^{s\circ f})$ if and only if there exist two sequences*

$$((x_n, y_n, u_n, v_n, t_n, q_n))_{n\in\mathbb{N}} \text{ in } X \times Y \times M \times (\operatorname{dom} s) \times K \times C$$

and

$$((x_n^*, u_n^*, v_n^*, t_n^*, q_n^*))_{n\in\mathbb{N}} \text{ in } X^* \times X^* \times Y^* \times K^+ \times C^+,$$

respectively, such that

$$(2.24)\qquad \begin{cases} \forall n \in \mathbb{N} : u_n^* \in N_M(u_n), v_n^* \in \partial s(v_n), x_n^* \in \partial(t_n^* \circ f + q_n^* \circ g)(x_n), \\ \langle t_n^*, t_n\rangle = 0, \langle q_n^*, q_n\rangle = 0; \\ x_n \to \overline{x}, u_n \to \overline{x}, y_n \to f(\overline{x}), v_n \to f(\overline{x}), f(x_n) + t_n \to f(\overline{x}), g(x_n) + q_n \to 0, \\ x_n^* + u_n^* \to 0, -t_n^* + v_n^* \to 0; \\ \langle u_n^*, x_n - \overline{x}\rangle + \langle v_n^*, y_n - f(\overline{x})\rangle + \langle t_n^*, f(x_n) - y_n\rangle + +\langle q_n^*, g(x_n) - g(\overline{x})\rangle \to 0, \\ \langle u_n^*, u_n - \overline{x}\rangle + \langle v_n^*, v_n - f(\overline{x})\rangle - \langle q_n^*, g(\overline{x})\rangle \to 0, s(v_n) - s(f(\overline{x})) \to 0. \end{cases}$$

The Case When f and g Are Continuous

We deal with $(P_{CM}^{s\circ f})$ under the hypotheses (2.13) with the additional assumptions:

$$(2.25)\qquad C \text{ is closed}, \operatorname{dom} f = \operatorname{dom} g = X, f \text{ and } g \text{ are continuous functions.}$$

Theorem 2.1.21 (Boţ R.I., Grad A., Wanka G. [22]) *Let the hypotheses* (2.13) *and* (2.25) *be satisfied. An element $\overline{x} \in \mathcal{A}_{s\circ f}$ is an optimal solution to problem $(P_{CM}^{s\circ f})$ if and only if there exist two sequences*

$$((x_n, y_n, z_n))_{n\in\mathbb{N}} \text{ in } M \times \operatorname{dom} s \times (-C) \text{ and } ((u_n^*, v_n^*, t_n^*, y_n^*, z_n^*))_{n\in\mathbb{N}} \text{ in } X^* \times X^* \times X^* \times K^+ \times C^+,$$

respectively, such that

$$(2.26)\qquad \begin{cases} \forall n \in \mathbb{N} : u_n^* \in \partial(y_n^* \circ f)(x_n), v_n^* \in \partial(z_n^* \circ g)(x_n), \\ t_n^* \in N_M(x_n), y_n^* \in \partial s(y_n), \langle z_n^*, z_n\rangle = 0; \\ x_n \to \overline{x}, y_n \to f(\overline{x}), z_n \to g(\overline{x}), u_n^* + v_n^* + t_n^* \to 0; \langle y_n^*, y_n - f(\overline{x})\rangle - \langle z_n^*, g(\overline{x})\rangle \to 0, \\ \langle y_n^*, f(x_n) - f(\overline{x})\rangle + \langle z_n^*, g(x_n) - g(\overline{x})\rangle \to 0, s(y_n) - s(f(\overline{x})) \to 0. \end{cases}$$

Remark 2.1.22 We analyze at this point the conclusions in Theorem 2.1.20 when working under the added assumptions (2.25). First we notice that

$$\partial(t_n^* \circ f + q_n^* \circ g)(x_n) = \partial(t_n^* \circ f)(x_n) + \partial(q_n^* \circ g)(x_n) \text{ for all } n \in \mathbb{N}.$$

Therefore, the conditions from (2.24) can be modified according to the details listed below. Fix an arbitrary $n \in \mathbb{N}$. We have that $x_n^* \in \partial(t_n^* \circ f + q_n^* \circ g)(x_n)$ is equivalent to the existence of two functionals a_n^* and b_n^* in X^* such that $x_n^* = a_n^* + b_n^*$ along with $a_n^* \in \partial(t_n^* \circ f)(x_n)$ and $b_n^* \in \partial(q_n^* \circ g)(x_n)$. Moreover, as $x_n \to \overline{x}$ it holds $f(x_n) \to f(\overline{x})$ and $g(x_n) \to g(\overline{x})$. Furthermore we notice that $t_n \to 0$ and $q_n \to -g(\overline{x})$. ∎

Corollary 2.1.23 (Boţ R.I., Grad A., Wanka G. [22]) *Let the hypotheses* (2.13) *and* (2.25) *be satisfied, and let $g \equiv 0$. An element $\overline{x} \in M$ is an optimal solution to the optimization problem*

$$\inf_{x \in M} (s \circ f)(x) \tag{2.27}$$

if and only if there exist two sequences

$$((x_n, y_n))_{n \in \mathbb{N}} \text{ in } M \times \operatorname{dom} s \text{ and } ((u_n^*, t_n^*, y_n^*))_{n \in \mathbb{N}} \text{ in } X^* \times X^* \times K^+,$$

respectively, such that

$$\begin{cases} \forall n \in \mathbb{N} : u_n^* \in \partial(y_n^* \circ f)(x_n), t_n^* \in N_M(x_n), y_n^* \in \partial s(y_n); \\ x_n \to \overline{x}, y_n \to f(\overline{x}), u_n^* + t_n^* \to 0; \\ \langle y_n^*, y_n - f(\overline{x}) \rangle \to 0, \langle y_n^*, f(x_n) - f(\overline{x}) \rangle \to 0, s(y_n) - s(f(\overline{x})) \to 0. \end{cases} \tag{2.28}$$

2.1.3 A Sequential Lagrange Multipliers Condition

Using again Theorem 2.1.21 we are able to derive sequential optimality conditions for the convex optimization problem with geometric and cone constraints

$$(P_{s\circ id}) \qquad \inf_{\substack{x \in M \\ g(x) \in -C}} s(x),$$

under the following hypotheses:

$$\begin{cases} X \text{ is a reflexive Banach space, } Z \text{ is a Banach space;} \\ \emptyset \neq C \subseteq Z \text{ is a closed convex cone}, \emptyset \neq M \subseteq X \text{ is a closed convex set;} \\ g : X \to Z \text{ is a continuous, } C\text{-convex function;} \\ s : X \to \overline{\mathbb{R}} \text{ is a proper, convex and lower semicontinuous function;} \\ M \cap g^{-1}(-C) \cap \operatorname{dom} s \neq \emptyset. \end{cases} \tag{2.29}$$

We denote the set of feasible solutions to $(P^{s\circ id})$ by $\mathcal{A}_{s\circ id} := M \cap g^{-1}(-C) \cap \operatorname{dom} s$.

Next we present a sequential Lagrange multipliers condition which turns out to be a refinement of a result by Boţ R. I., Csetnek E. R. and Wanka G. [17, Theorem 4.10].

Theorem 2.1.24 (Boţ R. I., Grad A., Wanka G. [22]) *Let the hypotheses* (2.29) *be satisfied. An element $\overline{x} \in \mathcal{A}_{s\circ id}$ is an optimal solution to problem $(P_{s\circ id})$ if and only if there exist two sequences*

$$((x_n, y_n, z_n))_{n \in \mathbb{N}} \text{ in } M \times (\operatorname{dom} s) \times (-C) \text{ and } ((v_n^*, t_n^*, y_n^*, z_n^*))_{n \in \mathbb{N}} \text{ in } X^* \times X^* \times X^* \times C^+,$$

respectively, such that

$$\begin{cases} \forall n \in \mathbb{N} : v_n^* \in \partial(z_n^* \circ g)(x_n), t_n^* \in N_M(x_n), y_n^* \in \partial s(y_n), \langle z_n^*, z_n \rangle = 0; \\ x_n \to \overline{x}, y_n \to \overline{x}, z_n \to g(\overline{x}), y_n^* + v_n^* + t_n^* \to 0, \langle y_n^*, y_n - \overline{x} \rangle - \langle z_n^*, g(\overline{x}) \rangle \to 0, \\ \langle y_n^*, x_n - \overline{x} \rangle + \langle z_n^*, g(x_n) - g(\overline{x}) \rangle \to 0, s(y_n) - s(\overline{x}) \to 0. \end{cases} \tag{2.30}$$

Remark 2.1.25 A similar Lagrange multiplier rule may be derived by particularizing the functions involved in Theorem 2.1.10. ∎

Remark 2.1.26 According to ROCKAFELLAR R. T. [103, Theorem 20], for the scalar optimization problem (P_{soid}) the Karush-Kuhn-Tucker condition, hereinafter named KKT condition, is:

$$\overline{x} \in \mathcal{A}_{soid} \text{ is a solution to } (P_{soid}) \text{ if and only if } 0 \in \left(s + \delta_{\{u \in M: g(u) \in -C\}}\right)(\overline{x}).$$

Considering an arbitrary $x \in \mathcal{A}_{soid}$, the following relations hold:

$$\begin{aligned} \partial \left(s + \delta_{\{u \in M: g(u) \in -C\}}\right)(x) &\supseteq \bigcup_{\substack{z^* \in C^+, \\ (z^* \circ g)(x) = 0}} \partial(s + (z^* \circ g) + \delta_M)(x) \\ &\supseteq \partial s(x) + \bigcup_{\substack{z^* \in C^+, \\ (z^* \circ g)(x) = 0}} \partial((z^* \circ g) + \delta_M)(x). \end{aligned}$$

Thus, given an element $\overline{x}$ in $\mathcal{A}_{soid}$ such that

$$0 \in \bigcup_{\substack{z^* \in C^+, \\ (z^* \circ g)(\overline{x}) = 0}} \partial(s + (z^* \circ g) + \delta_M)(\overline{x}) \text{ or } 0 \in \partial s(\overline{x}) + \bigcup_{\substack{z^* \in C^+, \\ (z^* \circ g)(\overline{x}) = 0}} \partial((z^* \circ g) + \delta_M)(\overline{x}),$$

it results that $\overline{x}$ is an optimal solution to (P_{soid}). ∎

Example 2.1.27 (BOŢ R. I., GRAD A., WANKA G. [22]) Let us consider the spaces $X := \mathbb{R}$, $Z := \mathbb{R}^2$, and the sets $C := \mathbb{R}^2_+$, $M := \mathbb{R}$. Moreover, we define the functions $s : \mathbb{R} \to \overline{\mathbb{R}}$ and $g : \mathbb{R} \to \mathbb{R}^2$ by

$$s(x) := -\sqrt{x} + \delta_{\mathbb{R}_+}(x) \text{ and } g(x) := (-1 - x, x) \text{ for all } x \in \mathbb{R},$$

respectively. Then the function s is proper, convex and lower semicontinuous. Moreover, the function g is $\mathbb{R}^2_+$-convex and continuous, and it holds $M \cap g^{-1}(-C) \cap \operatorname{dom} s \neq \emptyset$. The element $\overline{x} := 0$ is the (unique) optimal solution of problem (P_{soid}). Since

$$\bigcup_{\substack{z^* \in C^+, \\ (z^* \circ g)(0) = 0}} \partial(s + (z^* \circ g) + \delta_M)(0) = \partial s(0) + \bigcup_{\substack{z^* \in C^+, \\ (z^* \circ g)(0) = 0}} \partial((z^* \circ g) + \delta_M)(0) = \emptyset,$$

the classical KKT optimality conditions (stated in Remark 2.1.26) fail. Nevertheless, the sequential optimality conditions in (2.30) are satisfied. ∎

A sequential generalization of the well-known Pshenichnyi-Rockafellar Lemma can be given by taking in Theorem 2.1.24 only geometric constraints. It is stated below. (see also PSHENICHNYI B. N. [102] and ROCKAFELLAR R. T. [103])

Theorem 2.1.28 (BOŢ R. I., GRAD A., WANKA G. [22]) *Let the hypotheses* (2.29) *be satisfed. An element* $\overline{x} \in \mathcal{A}_{soid}$ *is an optimal solution to the problem*

$$\inf_{x \in M} s(x) \tag{2.31}$$

if and only if there exist two sequences $((x_n, y_n))_{n \in \mathbb{N}}$ *in* $M \times \operatorname{dom} s$ *and* $((t_n^*, y_n^*,))_{n \in \mathbb{N}}$ *in* $X^* \times X^*$, *respectively, such that*

$$\begin{cases} \forall n \in \mathbb{N} : y_n^* \in \partial s(y_n), t_n^* \in N_M(x_n); \\ x_n \to \overline{x}, y_n \to \overline{x}, y_n^* + t_n^* \to 0; \\ \langle y_n^*, y_n - \overline{x} \rangle \to 0, \langle y_n^*, x_n - \overline{x} \rangle \to 0, s(y_n) - s(\overline{x}) \to 0. \end{cases} \tag{2.32}$$

Theorem 2.1.28 is also a refinement of Corollary 4.8 established by Boţ R. I., Csetnek E. R. and Wanka G. [17] and, consequently, a generalization of Corollary 3.5 in Jeyakumar V. and Wu Z. Y. [83].

2.2 Quasi-Relative Interior Constraint Qualifications for Convex Optimization Problems

As pointed out by Boţ R. I., Csetnek E. R. and Moldovan A. [16], the main duality theorems in [45] and [46] contain too many assumptions, along with a mistake in the proof. Having this in mind, we set forth to give a general and valid strong duality theorem for a convex optimization problem with geometric, cone and affine constraints stated in infinite dimensional spaces and its Lagrange dual problem, achieving two goals: firstly it corrects the results in [45], and secondly it completes some statements in [16]. Our results come as corrections and improvements to Daniele P. and Giuffré S. [45] and have been published in Grad A. [64].

2.2.1 Strong Lagrange Duality Theorems

Our goal is to give a rather weak sufficient condition which ensures strong duality between the convex optimization problem with geometric, cone and affine constraints

$$(P_{CMA}) \qquad \inf_{\substack{x\in M\\ g(x)\in -C\\ h(x)=0}} f(x),$$

on the one hand, and its Lagrange dual problem

$$(D^L_{CMA}) \qquad \sup_{z^*\in C^+, w^*\in W^*} \inf_{x\in M} \big\{f(x)+\langle z^*, g(x)\rangle + \langle w^*, h(x)\rangle\big\},$$

on the other hand.

The general hypotheses we work under are described below:

$$(2.33) \qquad \begin{cases} X \text{ is a vector space, } Z \text{ and } W \text{ are separated locally convex spaces;}\\ M\subseteq X \text{ is a nonempty convex set}, C\subseteq Z \text{ is nonempty convex cone;}\\ f: M\to\mathbb{R} \text{ is a convex function;}\\ g: M\to Z \text{ is a } C\text{-convex function;}\\ h: X\to W \text{ is an affine function;}\\ \{x\in M: g(x)\in -C, h(x)=0\}\neq\emptyset. \end{cases}$$

In the following we use the notation

$$\mathcal{A}_{CMA} := \{x\in M: g(x)\in -C, h(x)=0\}.$$

Remark 2.2.1 (a) We recall that the function $g: M\to Z$ is said to be C-convex if

$$g(\lambda x_1 + (1-\lambda)x_2) \leq_C \lambda g(x_1) + (1-\lambda) g(x_2)$$

for all $x_1, x_2 \in M$ and all $\lambda \in [0, 1]$, which means that

$$\lambda g(x_1) + (1 - \lambda)g(x_2) - g(\lambda x_1 + (1 - \lambda)x_2) \in C$$

for all $x_1, x_2 \in M$ and all $\lambda \in [0, 1]$. We further notice that $g(M) + C$ and $h(M)$ are convex sets. Moreover,

$$(g, h)(M) + C \times \{0\} \text{ and } (f, g, h)(M) + \mathbb{R}_+ \times C \times \{0\}$$

are also convex sets.

(b) In imposing the hypotheses (2.33) we took into consideration not only the resemblance with the problem encountered in [45], but also the fact that we intend to apply the separation theorem for the quasi-relative interior. ∎

The **weak duality theorem** always holds.

Theorem 2.2.2 *The objective values $v(P_{CMA})$ and $v(D^L_{CMA})$ of the optimization problems (P_{CMA}) and (D^L_{CMA}), respectively, satisfy the inequality*

$$v(D^L_{CMA}) \leq v(P_{CMA}).$$

Remark 2.2.3 In the particular case when X, Z and W are normed spaces, Daniele P. and Giuffré S. [45, Theorem 3.1] gave a strong duality theorem for the dual-pair (P_{CM}, D^L_{CMA}). As pointed out by Boţ R. I., Csetnek E. R. and Moldovan A. [16], Theorem 3.1 in [45] presents two major issues: it has a mistake in the proof; it has too many hypotheses. ∎

When $v(P_{CMA}) = -\infty$, strong duality obviously holds. Therefore, for the rest of this subsection we consider that $v(P_{CMA}) \in \mathbb{R}$.

Let us define the following set:

$$\mathcal{E} := (v(P_{CMA}), 0, 0) - (f, g, h)(M) - \mathbb{R}_+ \times C \times \{0\},$$

where $\mathbb{R}_+ := [0, +\infty)$. Notice that the set $-\mathcal{E}$ is analogous to the conical extension used by Giannessi F. [55] in the theory of image space analysis.

Lemma 2.2.4 *Let the hypotheses* (2.33) *be satisfied. Then the following statements are true:*

(a) *$\mathcal{E}$ is a convex set.*

(b) *(P_{CMA}) has an optimal solution if and only if $(0, 0, 0) \in \mathcal{E}$.*

(c) *If (P_{CMA}) has an optimal solution, then $\operatorname{co}(\mathcal{E} \cup \{(0, 0, 0)\}) = \operatorname{co} \mathcal{E} = \mathcal{E}$.* ∎

The following lemma will be used in the proof of the strong Lagrange duality theorem within this section.

Lemma 2.2.5 (Grad A. [64]) *Let the hypotheses* (2.33) *be satisfied. Then the following statements are true:*

(a) *If $x \in M$ and $c \in C$ satisfy $(g(x) + c, h(x)) \in \operatorname{qri}((g, h)(M) + C \times \{0\})$, then*

$$(v(P_{CMA}) - f(x) - t, -g(x) - c, -h(x)) \in \operatorname{qri} \mathcal{E} \textit{ for all } t > 0.$$

(b) *If* $(r_0, z_0, w_0) \in \operatorname{qri} \mathcal{E}$*, then* $(-z_0, -w_0) \in \operatorname{qri}((g,h)(M) + C \times \{0\})$.

(c) $\operatorname{qri} \mathcal{E} \neq \emptyset$ *if and only if* $\operatorname{qri}((g,h)(M) + C \times \{0\}) \neq \emptyset$.

The main result within this section, which provides a sufficient condition ensuring **strong duality** between (P_{CMA}) and (D^L_{CMA}), is stated and proved below.

Theorem 2.2.6 (Grad A. [64]) *Let the hypotheses* (2.33) *be satisfied, and let us assume that*

$$(0,0) \in \operatorname{qi}((g,h)(M) + C \times \{0\}) \tag{2.34}$$

and that

$$(0,0,0) \notin \operatorname{qri} \operatorname{co}(\mathcal{E} \cup \{(0,0,0)\}). \tag{2.35}$$

Then strong duality holds between (P_{CMA}) *and* (D^L_{CMA})*, i.e.* $v(P_{CMA}) = v(D^L_{CMA})$ *and* (D^L_{CMA}) *has an optimal solution.*

When the primal optimization problem (P_{CMA}) admits an optimal solution, we obtain the following results.

Corollary 2.2.7 (Grad A. [64]) *Let the hypotheses* (2.33) *be satisfied, and let* $\overline{x} \in \mathcal{A}_{CMA}$ *be an optimal solution to* (P_{CMA})*. Moreover, let* (2.34) *be satisfied and* $(0,0,0) \notin \operatorname{qri}(\mathcal{E})$*. Then the following statements are true:*

(a) $v(P_{CMA}) = v(D^L_{CMA})$ *and problem* (D^L_{CMA}) *has an optimal solution.*

(b) *For each optimal solution* $(z^*, w^*) \in C^+ \times W^*$ *to* (D^L_{CMA})*, the equality* $\langle z^*, g(\overline{x}) \rangle = 0$ *holds.*

Remark 2.2.8 We underline that

$$(0,0,0) \in \operatorname{qri} \mathcal{E} \Longrightarrow (0,0) \in \operatorname{qri}((g,h)(M) + C \times \{0\}),$$

which is a particular case of Lemma 2.2.5 (b), and that

$$(0,0,0) \in \operatorname{qi} \mathcal{E} \Longrightarrow (0,0) \in \operatorname{qi}((g,h)(M) + C \times \{0\}),$$

whose proof is a simple application of the definition of the quasi-interior. However, it is possible to have

$$(0,0) \in \operatorname{qi}((g,h)(M) + C \times \{0\}) \text{ and } (0,0,0) \notin \operatorname{qi} \mathcal{E},$$

as it will be seen in an example from Subsection 2.2.3. Hence there exist practical situations in which Corollary 2.2.7 can be applied. ∎

The following proposition allows us to give an equivalent formulation for condition (2.35) occurring in Theorem 2.2.6.

Proposition 2.2.9 (Grad A. [64]) *Let the hypotheses* (2.33) *and* (2.34) *be satisfied. Then*

$$(0,0,0) \in \operatorname{qri} \operatorname{co}(\mathcal{E} \cup \{(0,0,0)\}) \text{ if and only if } (0,0,0) \in \operatorname{qi} \operatorname{co}(\mathcal{E} \cup \{(0,0,0)\}).$$

Remark 2.2.10 In view of Proposition 2.2.9, if the hypotheses (2.33) and (2.34) are satisfied, we notice that condition (2.35) in Theorem 2.2.6 can be replaced with the equivalent condition $(0,0,0) \notin \operatorname{qi}\operatorname{co}(\mathcal{E} \cup \{(0,0,0)\})$, while the condition $(0,0,0) \notin \operatorname{qri}\mathcal{E}$ in Corollary 2.2.7 can be replaced with $(0,0,0) \notin \operatorname{qi}\mathcal{E}$. ∎

In the following we reveal some sufficient conditions which ensure that (2.34) holds.

Lemma 2.2.11 (Grad A. [64]) *Let the hypotheses* (2.33) *be satisfied, and let us assume that there exists an element* $x_0 \in M$ *such that*

$$g(x_0) \in -\operatorname{qri}(C) \text{ and } h(x_0) = 0. \tag{2.36}$$

Moreover, let the conditions

$$0 \in \operatorname{qi}(C - C) \text{ (or equivalently } \operatorname{cl}(C - C) = Z). \tag{2.37}$$

and

$$0 \in \operatorname{qi}(h(M)) \tag{2.38}$$

be satisfied. Then $(0,0) \in \operatorname{qi}((g,h)(M) + C \times \{0\})$.

Remark 2.2.12 The condition $0 \in \operatorname{qi} h(M)$ in Lemma 2.2.11 can be replaced with $0 \in \operatorname{qri} h(M)$ and $0 \in \operatorname{qi}(h(M) - h(M))$. ∎

Using Lemma 2.2.11, we deduce from Theorem 2.2.6 and Corollary 2.2.7 the following strong duality theorem and its corresponding corollary, which have stronger hypotheses. Nevertheless, in some situations, their hypotheses are easier to be verified.

Theorem 2.2.13 (Grad A. [64]) *Let the hypotheses* (2.33) *be satisfied, and assume that there exists an* $x_0 \in M$ *such that* (2.36) *is satisfied. Furthermore, let* (2.35), (2.37) *and* (2.38) *be valid. Then strong duality holds between* (P_{CMA}) *and* (D^L_{CMA}), *i.e.* $v(P_{CMA}) = v(D^L_{CMA})$ *and* (D^L_{CMA}) *has an optimal solution.*

Corollary 2.2.14 (Grad A. [64]) *Let the hypotheses* (2.33) *be satisfied, let* $\overline{x} \in \mathcal{A}_{CMA}$ *be an optimal solution to problem* (P_{CMA}), *and let* $x_0 \in M$ *be such that* (2.36) *is satisfied. Moreover, let* (2.37) *and* (2.38) *be valid, and let* $(0,0,0) \notin \operatorname{qri}(\mathcal{E})$. *Then the following statements are true:*

(a) $v(P_{CMA}) = v(D^L_{CMA})$ *and* (D^L_{CMA}) *has at least one optimal solution.*
(b) *For each optimal solution* $(z^*, w^*) \in C^+ \times W^*$ *to* (D^L_{CMA}), *the equality* $\langle z^*, g(\overline{x})\rangle = 0$ *holds.*

Remark 2.2.15 When M is an affine set, then $h(M)$ is also an affine set, thus $h(M) = \operatorname{qri} h(M)$. Therefore, (2.38) can be modified to $\operatorname{qi} h(M) \neq \emptyset$.

On the other hand, the hypothesis (2.38) can be replaced with $0 \in \operatorname{qi}(h(M) - h(M))$. ∎

We state now sufficient conditions for (2.38).

Lemma 2.2.16 (Grad A. [64]) *Let the hypotheses* (2.33) *be satisfied, and let* M *be an affine set. Then*

$$\operatorname{cl} h(M - M) = W \text{ and } 0 \in h(M), \tag{2.39}$$

if and only if $0 \in \operatorname{qi} h(M)$.

Remark 2.2.17 In the hypotheses of Lemma 2.2.16 one could obtain from Theorem 2.2.13 and Corollary 2.2.14 corresponding strong duality theorems. They will be weaker results, but under their particular assumptions, most likely easier to be verified in practice. ∎

Lemma 2.2.18 (Grad A. [64]) *In addition to the hypotheses* (2.33), *suppose that X is a separated locally convex space and that h is continuous. If there exists $x_0 \in \operatorname{qri} M$ such that $h(x_0) = 0$ and $\operatorname{cl} h(M - M) = W$, then $0 \in \operatorname{qi} h(M)$.*

Theorem 2.2.19 (Grad A. [64]) *In addition to the hypotheses* (2.33) *suppose that X is a separated locally convex space and that h is continuous. Assume that there exists $x_0 \in \operatorname{qri} M$ such that* (2.36) *is satisfied. Moreover, let* (2.35) *and* (2.37) *be valid and $\operatorname{cl} h(M - M) = W$. Then strong duality holds between (P_{CMA}) and (D^L_{CMA}), i.e. $v(P_{CMA}) = v(D^L_{CMA})$ and (D^L_{CMA}) has an optimal solution.*

Corollary 2.2.20 (Grad A. [64]) *In addition the hypotheses of Theorem 2.2.19 suppose that $\overline{x} \in \mathcal{A}_{CMA}$ is an optimal solution to (P_{CMA}). Then the following statements are true:*

(a) *$v(P_{CMA}) = v(D^L_{CMA})$ and (D^L_{CMA}) has at least one optimal solution.*

(b) *For each optimal solution $(z^*, w^*) \in C^+ \times W^*$ to (D^L_{CMA}), the equality $\langle z^*, g(\overline{x})\rangle = 0$ holds.*

One should notice that the conditions in Corollary 2.2.20 are similar to those in Theorem 3.1 established by Daniele P. and Giuffré S. [45], but they are obviously less restrictive and, furthermore, correct.

2.2.2 Saddle Points

The function $L : M \times C^+ \times W^* \to \mathbb{R}$, defined by

$$L(x, z^*, w^*) := f(x) + \langle z^*, g(x)\rangle + \langle w^*, h(x)\rangle \text{ for all } (x, z^*, w^*) \in M \times C^+ \times W^*,$$

is called **the Lagrange function** associated with (P_{CMA}).

An element $(\overline{x}, \overline{z}^*, \overline{w}^*) \in M \times C^+ \times W^*$ is said to be a **saddle point** of the Lagrange function associated with (P_{CMA}) if

$$\begin{aligned} f(\overline{x}) + \langle z^*, g(\overline{x})\rangle + \langle w^*, h(\overline{x})\rangle &\leq f(\overline{x}) + \langle \overline{z}^*, g(\overline{x})\rangle + \langle \overline{w}^*, h(\overline{x})\rangle \leq \\ &\leq f(x) + \langle \overline{z}^*, g(x)\rangle + \langle \overline{w}^*, h(x)\rangle \end{aligned}$$

for all $(x, z^*, w^*) \in M \times C^+ \times W^*$.

Theorem 2.2.21 *Let the hypotheses of Theorem 2.2.6 (or of Theorem 2.2.13, or of Theorem 2.2.19, respectively) be satisfied, and let $\overline{x} \in \mathcal{A}_{CMA}$. Then $\overline{x}$ is an optimal solution to problem (P_{CMA}) if and only if there exists $(\overline{z}^*, \overline{w}^*) \in C^+ \times W^*$ such that $(\overline{x}, \overline{z}^*, \overline{w}^*)$ is a saddle point of the Lagrange function associated with (P_{CMA}) and that the equality $\langle \overline{z}^*, g(\overline{x})\rangle = 0$ holds.*

2.2.3 An Application to an Optimization Problems in $L^2([0,T],\mathbb{R}^2)$

The theory developed within Section 2.2 has a wide application area. We are going to present an application in the reflexive Banach space $L^2([0,T],\mathbb{R}^m)$, where $T>0$ is a real constant and $m \geq 1$ is a natural number. First of all let us remind that one can consider the convex ordering cone

$$C_m := \{w \in L^2([0,T],\mathbb{R}^m) : w(t) \geq 0 \text{ a.e. in } [0,T]\},$$

for which $\operatorname{int} C_m, \operatorname{core} C_m$ and $\operatorname{sqri} C_m$ are empty sets, but

$$\operatorname{qri} C_m = \{w \in L^2([0,T],\mathbb{R}^m) : w(t) > 0 \text{ a.e. in } [0,T]\}$$

(see Borwein J. M. and Lewis A. S. [9]). Moreover, the equality $C_m - C_m = L^2([0,T],\mathbb{R}^m)$ holds, and the dual cone of C_m is actually equal to C_m.

For the easier lecturing of the example, we use the notation $\ll \eta, u \gg_m$ for the value of a linear continuous functional $\eta \in (L^2([0,T],\mathbb{R}^m))^* = L^2([0,T],\mathbb{R}^m)$ at $u \in L^2([0,T],\mathbb{R}^m)$, which is calculated as

$$\ll \eta, u \gg_m = \int_0^T \langle \eta(t), u(t)\rangle dt = \int_0^T \sum_{i=1}^m \eta_i(t)u_i(t)dt.$$

The framework we work under, which is actually a particularization of the general framework (2.33), is listed in the following:

$$(2.40)\qquad \begin{cases} X = M = W := L^2([0,T],\mathbb{R}^2), Z := L^2([0,T],\mathbb{R}); \\ f : L^2([0,T],\mathbb{R}^2) \to \mathbb{R} \text{ is defined by } f(u) := \ll \beta, u_1^2 \gg_1; \\ g : L^2([0,T],\mathbb{R}^2) \to L^2([0,T],\mathbb{R}) \text{ is defined by } g(u) := u_2; \\ h : L^2([0,T],\mathbb{R}^2) \to L^2([0,T],\mathbb{R}^2) \text{ is defined by } h(u) := \Phi(u) - \rho; \\ \qquad u = (u_1,u_2) \in L^2([0,T],\mathbb{R}^2), \rho = (-1,1); \\ \beta \in L^2([0,T],\mathbb{R}) \text{ with } \beta(t) \geq 0 \text{ a.e. in } [0,T]; \\ \Phi := \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}. \end{cases}$$

Under the given assumptions, f is a convex function, g is a C_1-convex one and h is affine. Let us consider the following set:

$$\mathcal{A}_{eq} := \left\{ u \in L^2([0,p],\mathbb{R}^2) : u_2 \in -C_1, \Phi u(t) = \rho(t) \text{ a.e. in } [0,T] \right\}.$$

We show that strong duality holds between the problem

$$(P_{eq}) \qquad \min_{u \in \mathcal{A}_{eq}} \ll \beta, u_1^2 \gg_1$$

and its Lagrange dual problem

$$(D_{eq}^L) \qquad \sup_{\substack{z^* \in C_1, \\ w^* \in W^*}} \inf_{u \in L^2([0,T],\mathbb{R}^2)} \left\{ \ll \beta, u_1^2 \gg_1 + \ll z^*, g(u) \gg_1 + \ll w^*, \Phi u - \rho \gg_2 \right\}.$$

The hypotheses of Corollary 2.2.14 are satisfied. Thus strong duality holds between (P_{eq}) and (D_{eq}^L), i.e. $v(P_{eq}) = v(D_{eq}^L)$ and (D_{eq}^L) has an optimal solution. Furthermore, for each optimal solution $(\overline{z}^*, \overline{w}^*) \in C_1 \times L^2([0,T],\mathbb{R}^2)$ to (D_{eq}^L) the equality $\ll \overline{z}^*, g(\overline{x}) \gg_1 = 0$ holds.

Chapter 3

Vector Optimization

Several original results concerning vector optimization problems are presented in this chapter. They were published either by the author alone, in Grad A. [60], [61], [63], [65], or in the joint works Boţ R. I., Dumitru(Grad) A. and Wanka G. [20], Boţ R. I., Grad A. and Wanka G. [21], [22].

3.1 Sequential Optimality Conditions

To our best knowledge, sequential optimality conditions for vector optimization problems have been given in the literature for the first time in Boţ R. I., Grad A. and Wanka G. [21]. These results were then improved in Boţ R. I., Grad A. and Wanka G. [22], and Grad A. [63], and are presented in this section.

We provide two types of sequential optimality conditions for the general convex vector optimization problem with geometric and cone constraints

$$(P^v_{CM}) \qquad \operatorname*{v-min}_{\substack{x\in M\\ g(x)\in -C}} f(x),$$

stated under the following hypotheses:

$$(3.1) \qquad \begin{cases} X \text{ is a reflexive Banach space}, Y \text{ and } Z \text{ are Banach spaces;}\\ K\subseteq Y \text{ is a nonempty, pointed and convex cone;}\\ C\subseteq Z \text{ is a nonempty, closed and convex cone;}\\ M\subseteq X \text{ is a nonempty, closed and convex set;}\\ f:X\to Y^\bullet \text{ is a proper } K\text{-convex function;}\\ g:X\to Z^\bullet \text{ is a proper, } C\text{-convex and } C\text{-epi closed function;}\\ (\operatorname{dom} f)\cap(\operatorname{dom} g)\cap M\cap g^{-1}(-C)\neq\emptyset. \end{cases}$$

We denote the set of feasible solutions to (P^v_{CM}) by

$$\mathcal{A}_{P^v_{CM}} := (\operatorname{dom} f)\cap(\operatorname{dom} g)\cap M\cap g^{-1}(-C).$$

For the particular case of our primal vector optimization problem (P^v_{CM}), we work with four optimality notions, defined in the following.

Definition 3.1.1 (Jahn J. [78]) *Let the hypotheses* (3.1) *be satisfied. Then an element* $\overline{x} \in \mathcal{A}_{P^v_{CM}}$ *is called a* ***Pareto-efficient solution*** *to* (P^v_{CM}) *if, for each* $x \in \mathcal{A}_{P^v_{CM}}$ *satisfying*

$$f(x) \leq_K f(\overline{x}), \text{ the equality } f(x) = f(\overline{x}) \text{ holds.}$$

Let us consider a nonempty set S of convex, K-strongly increasing real-valued functions on Y, i.e.

$$S \subseteq \{s : Y \to \mathbb{R} : s \text{ is convex and } K\text{-strongly increasing}\}.$$

Definition 3.1.2 (Gerstewitz C. [52], Göpfert A., Gerth C. [56]) *Let the hypotheses* (3.1) *be satisfied. Then an element* $\overline{x} \in \mathcal{A}_{P^v_{CM}}$ *is said to be an* S-***properly efficient solution*** *to* (P^v_{CM}) *if there exists a function* $s \in S$ *such that*

$$s(f(\overline{x})) \leq s(f(x)) \text{ for all } x \in \mathcal{A}_{P^v_{CM}}.$$

Remark 3.1.3 (a) Each S-properly efficient solution to the vector optimization problem (P^v_{CM}) is actually an optimal solution to the following scalar, convex and composed optimization problem with geometric and cone constraints:

$$\inf_{\substack{x \in M \\ g(x) \in -C}} (s \circ f)(x),$$

which is exactly problem $(P^{s \circ f}_{CM})$ studied in Section 2.1, with the convention $s(\infty_Y) := +\infty$.

(b) Each S-properly efficient solution to (P^v_{CM}) is also a Pareto-efficient one. ∎

If $\operatorname{int} K \neq \emptyset$, one can also introduce other efficiency notions, among which, we mention at this point only the following two.

Definition 3.1.4 (Jahn J. [78]) *Let the hypotheses* (3.1) *be satisfied, and let* $\operatorname{int} K \neq \emptyset$. *Then an element* $\overline{x} \in \mathcal{A}_{P^v_{CM}}$ *is said to be a* ***weakly efficient solution*** *to* (P^v_{CM}) *if there exists no* $x \in \mathcal{A}_{P^v_{CM}}$ *such that*

$$f(x) - f(\overline{x}) \in - \operatorname{int} K.$$

Let us consider a nonempty set T of convex, K-strictly increasing real-valued functions on Y, i.e.

$$T \subseteq \{t : Y \to \mathbb{R} : t \text{ is convex and } K\text{-strictly increasing}\}.$$

With its help one can define the following new class of vector optimal solutions.

Definition 3.1.5 *Let the hypotheses* (3.1) *be satisfied, and let* $\operatorname{int} K \neq \emptyset$. *Then an element* $\overline{x} \in \mathcal{A}_{P^v_{CM}}$ *is said to be a* T-***weakly efficient solution*** *to* (P^v_{CM}) *if there exists a function* $t \in T$ *such that*

$$t(f(\overline{x})) \leq t(f(x)) \text{ for all } x \in \mathcal{A}_{P^v_{CM}}.$$

Remark 3.1.6 (a) Each T-weakly efficient solution to the vector optimization problem (P^v_{CM}) is actually an optimal solution to the following scalar, convex and composed optimization problem with geometric and cone constraints:

$$\inf_{\substack{x \in M \\ g(x) \in -C}} (t \circ f)(x),$$

which is a $(P^{s \circ f}_{CM})$-type problem, with the convention that $t(\infty_Y) = +\infty$.

(b) Each T-weakly efficient solution to (P^v_{CM}) is also a weakly efficient one. ∎

3.1.1 Sufficient Sequential Optimality Conditions

Using some results from Section 2.1 we can give sufficient sequential optimality conditions for S-properly efficient solutions and for T-weakly efficient solutions to (P^v_{CM}).

We first analyze the case when f is a star K-lower semicontinuous function.

Theorem 3.1.7 *Let us consider the vector optimization problem* (P^v_{CM}) *under the hypotheses* (3.1) *and suppose that f is star K-lower semicontinuous. Moreover, let $\overline{x} \in \mathcal{A}_{P^v_{CM}}$. If there exist a lower semicontinuous function $s \in S$ and two sequences*

$$((x_n, y_n, z_n))_{n\in\mathbb{N}} \text{ in } (M \cap \operatorname{dom}(f)) \times Y \times -C$$

and

$$((x_n^*, y_n^*, z_n^*))_{n\in\mathbb{N}} \text{ in } X^* \times K^+ \times C^+,$$

respectively, such that

$$(3.2)\qquad \begin{cases} \forall n \in \mathbb{N} : x_n^* \in \partial\left((y_n^* \circ f) + (z_n^* \circ g) + \delta_M\right)(x_n), y_n^* \in \partial s(y_n), \langle z_n^*, z_n\rangle = 0; \\ x_n \to \overline{x}, y_n - f(x_n) \to 0, z_n - g(x_n) \to 0, x_n^* \to 0; \\ \langle y_n^*, y_n - f(\overline{x})\rangle - \langle z_n^*, g(\overline{x})\rangle \to 0, \\ \langle y_n^*, f(x_n) - f(\overline{x})\rangle + \langle z_n^*, g(x_n) - g(\overline{x})\rangle \to 0, s(y_n) - s(f(\overline{x})) \to 0, \end{cases}$$

then $\overline{x}$ is a S-properly efficient solution to problem (P^v_{CM}) .

We continue with giving sufficient sequential optimality conditions for the case when f is K-epi closed.

Theorem 3.1.8 (Grad A. [63]) *Let us consider the vector optimization problem* (P^v_{CM}) *under the hypotheses* (3.1)*. Assume further that Y is reflexive, K is closed, and f is K-epi closed. Moreover, let $\overline{x} \in \mathcal{A}_{P^v_{CM}}$. If there exist a lower semicontinuous function $t \in T$ and two sequences*

$$((x_n, y_n, u_n, v_n, t_n, q_n))_{n\in\mathbb{N}} \text{ in } X \times Y \times M \times Y \times K \times C$$

and

$$((x_n^*, u_n^*, v_n^*, t_n^*, q_n^*))_{n\in\mathbb{N}} \text{ in } X^* \times X^* \times Y^* \times K^+ \times C^+,$$

respectively, such that

$$(3.3)\qquad \begin{cases} \forall n \in \mathbb{N} : u_n^* \in N_M(u_n), v_n^* \in \partial t(v_n), x_n^* \in \partial((t_n^* \circ f) + (q_n^* \circ g))(x_n), \\ \langle t_n^*, t_n\rangle = 0, \langle q_n^*, q_n\rangle = 0; \\ x_n \to \overline{x}, u_n \to \overline{x}, y_n \to f(\overline{x}), v_n \to f(\overline{x}), \\ f(x_n) + t_n \to f(\overline{x}), g(x_n) + q_n \to 0, x_n^* + u_n^* \to 0, -t_n^* + v_n^* \to 0; \\ \langle u_n^*, u_n - \overline{x}\rangle + \langle v_n^*, v_n - f(\overline{x})\rangle - \langle q_n^*, g(\overline{x})\rangle \to 0, \\ \langle u_n^*, x_n - \overline{x}\rangle + \langle v_n^*, y_n - f(\overline{x})\rangle + \langle t_n^*, f(x_n) - y_n\rangle \\ +\langle q_n^*, g(x_n) - g(\overline{x})\rangle \to 0, t(v_n) - t(f(\overline{x})) \to 0, \end{cases}$$

then $\overline{x}$ is a T-weakly efficient solution to problem (P^v_{CM}) .

3.1.2 Necessary and Sufficient Optimality Conditions

Imposing further conditions on the functions involved in the definition of the vector optimization problem (P^v_{CM}), we can give not only sufficient, but also necessary and sufficient optimality conditions.

We consider the vector optimization problem (P^v_{CM}) under the hypotheses (3.1) and add the following assumptions:

$$\left\{ \begin{array}{l} \operatorname{dom} f = \operatorname{dom} g = X; \\ f \text{ and } g \text{ are continuous.} \end{array} \right. \tag{3.4}$$

Linear Scalarization

The most famous scalarization in vector optimization involves K-strongly increasing linear functionals. Let us start by noticing that, for each $k^* \in K^{+0}$, the function $s_{k^*} : Y \to \mathbb{R}$ defined by

$$s_{k^*}(y) := \langle k^*, y \rangle \text{ for all } y \in Y$$

is continuous, convex and K-strongly increasing. Then, considering the set

$$S_{K^{+0}} := \{s_{k^*} : k^* \in K^{+0}\},$$

an element $\overline{x} \in \mathcal{A}_{P^v_{CM}}$ is an $S_{K^{+0}}$**-properly efficient solution** to (P^v_{CM}) if there exists a $\overline{k}^* \in K^{+0}$ such that

$$\langle \overline{k}^*, f(\overline{x}) \rangle \leq \langle \overline{k}^*, f(x) \rangle \text{ for all } x \in \mathcal{A}_{P^v_{CM}}.$$

The following necessary and sufficient sequential optimality condition can be given.

Theorem 3.1.9 (Boţ R.I., Grad A., Wanka G. [22]) *Let the hypotheses* (3.1) *and* (3.4) *be satisfied. Then an element $\overline{x} \in \mathcal{A}_{P^v_{CM}}$ is an $S_{K^{+0}}$-properly efficient solution to (P^v_{CM}) if and only if there exist a function $\overline{k}^* \in K^{+0}$ and two sequences*

$$((x_n, z_n))_{n\in\mathbb{N}} \text{ in } M \times (-C) \text{ and } ((u_n^*, v_n^*, t_n^*, z_n^*))_{n\in\mathbb{N}} \text{ in } X^* \times X^* \times X^* \times C^+,$$

respectively, such that

$$\left\{ \begin{array}{l} \forall n \in \mathbb{N} : u_n^* \in \partial(\overline{k}^* \circ f)(x_n), v_n^* \in \partial(z_n^* \circ g)(x_n), t_n^* \in N_M(x_n), \langle z_n^*, z_n \rangle = 0; \\ x_n \to \overline{x}, z_n \to g(\overline{x}), u_n^* + v_n^* + t_n^* \to 0; \\ \langle z_n^*, g(\overline{x}) \rangle \to 0, \langle z_n^*, g(x_n) \rangle \to 0. \end{array} \right. \tag{3.5}$$

Remark 3.1.10 (Boţ R. I., Grad A., Wanka G. [22]) By taking into consideration Remark 2.1.26 one can easily see that if, for a fixed $\overline{x} \in \mathcal{A}_{P^v_{CM}}$, there exists an element $\overline{k}^* \in K^{+0}$ such that

$$0 \in \bigcup_{\substack{z^* \in C^+, \\ (z^* \circ g)(\overline{x}) = 0}} \partial((\overline{k}^* \circ f) + (z^* \circ g) + \delta_M)(\overline{x})$$

or

$$0 \in \partial(\overline{k}^* \circ f)(\overline{x}) + \bigcup_{\substack{z^* \in C^+, \\ (z^* \circ g)(\overline{x}) = 0}} \partial((z^* \circ g) + \delta_M)(\overline{x}),$$

then $\overline{x}$ is a $S_{K^{+0}}$-properly efficient solution to (P^v_{CM}). ∎

Example 3.1.11 (Boţ R. I., Grad A., Wanka G. [22]) Let us consider the spaces $X := \mathbb{R}$, $Y := \mathbb{R}^2$, $Z := \mathbb{R}$, and the sets $K := \mathbb{R}^2_+$, $C := \mathbb{R}_+$, $M := \mathbb{R}$. Moreover, we define the functions $f : \mathbb{R} \to \mathbb{R}^2$ and $g : \mathbb{R} \to \mathbb{R}$ by

$$f(x) := (x, x^2) \text{ and } g(x) := x^2 \text{ for all } x \in \mathbb{R},$$

respectively. Then f is $\mathbb{R}^2_+$-convex and continuous, while g is $\mathbb{R}_+$-convex and continuous. Furthermore, the feasibility condition $\mathcal{A}_{P^v_{CMA}} \neq \emptyset$ is satisfied. Obviously, $\overline{x} := 0$ is an $S_{K^{+0}}$-properly efficient solution to (P^v_{CM}), but there is no $k^* \in \operatorname{int}(\mathbb{R}^2_+)$ such that one of the optimality conditions in Remark 3.1.10 is fulfilled. This is not the case for the sequential optimality conditions given in (3.5). ∎

Set Scalarization

Some quite recent scalarization methods are based on already given or constructed sets which have to satisfy some conditions. The scalarization function we use in the following is attributed to Gerstewitz C. and Iwanow [54]. It was thoroughly investigated by Gerth C. and Weidner P. [51] and Zălinescu C. [119]. However, in the context of vector optimization, it was used before, for example, by Rubinov A. [105] and by Pascoletti A. and Serafini P. [99].

We work under the added assumptions that

$$\begin{cases} K \subset Y \text{ is closed;} \\ \operatorname{int} K \neq \emptyset. \end{cases} \tag{3.6}$$

For each $\mu \in \operatorname{int} K$ we consider the function $t_\mu : Y \to \mathbb{R}$ defined by

$$t_\mu(y) := \inf\{r \in \mathbb{R} : y \in r\mu - K\} \text{ for all } y \in Y.$$

According to Göpfert A., Riahi H., Tammer C. and Zălinescu C. [57, Corollary 2.3.5], this function is K-strictly increasing, convex and continuous. Moreover, let us consider the following set of K-strictly increasing, convex and continuous functions:

$$T_{\operatorname{int} K} := \{t_\mu : \mu \in \operatorname{int} K\}.$$

Then we can give the following sequential characterization of $T_{\operatorname{int} K}$-weakly efficient solutions.

Theorem 3.1.12 (Boţ R.I., Grad A., Wanka G. [22]) *Let the hypotheses* (3.1), (3.4) *and* (3.6) *be satisfied. Then an element $\overline{x} \in \mathcal{A}_{P^v_{CM}}$ is a $T_{\operatorname{int} K}$-weakly efficient solution to (P^v_{CM}) if and only if there exist a $\overline{\mu} \in \operatorname{int} K$ and two sequences*

$$((x_n, y_n, z_n))_{n\in\mathbb{N}} \text{ in } M \times Y \times (-C) \text{ and } ((u^*_n, v^*_n, t^*_n, y^*_n, z^*_n))_{n\in\mathbb{N}} \text{ in } X^* \times X^* \times X^* \times K^+ \times C^+,$$

respectively, such that

$$\begin{cases} \forall n \in \mathbb{N} : u^*_n \in \partial(y^*_n \circ f)(x_n), v^*_n \in \partial(z^*_n \circ g)(x_n), t^*_n \in N_M(x_n), \\ \langle y^*_n, \overline{\mu}\rangle = 1, \sigma_{\{k^* \in K^+ : \langle k^*, \overline{\mu}\rangle = 1\}}(y_n) = \langle y^*_n, y_n\rangle, \langle z^*_n, z_n\rangle = 0; \\ u^*_n + v^*_n + t^*_n \to 0, x_n \to \overline{x}, y_n \to f(\overline{x}), z_n \to g(\overline{x}), \\ \langle y^*_n, y_n - f(\overline{x})\rangle - \langle z^*_n, g(\overline{x})\rangle \to 0, \langle y^*_n, f(x_n) - f(\overline{x})\rangle + \langle z^*_n, g(x_n) - g(\overline{x})\rangle \to 0. \end{cases} \tag{3.7}$$

3.2 Fenchel-Type Vector Duality in Locally Convex Spaces

Recent advances with respect to Fenchel duality in vector optimization are presented within this section, results that have been published in Grad A. [61]. It is important to mention that all three Fenchel-type vector dual problems under consideration in this part of the thesis prove to be natural extensions of the classical Fenchel dual problem from Rockafellar R. T. [103] in scalar optimization.

3.2.1 A General Fenchel-Type Vector Dual Probelm

The primal vector optimization problem we deal with is defined by

$$(P_A^v) \qquad \underset{x\in X}{\text{v-min}}(f+g\circ A)(x).$$

We tackle it under the following hypotheses:

$$(3.8) \qquad \begin{cases} X, Y \text{ and } Z \text{ are separated locally convex spaces;} \\ K \subseteq Y \text{ is a nontrivial, pointed and convex cone;} \\ f : X \to Y^\bullet \text{ and } g : Z \to Y^\bullet \text{ are proper } K\text{-convex functions;} \\ A : X \to Z \text{ is a linear continuous operator;} \\ (\operatorname{dom} f) \cap A^{-1}(\operatorname{dom} g) \neq \emptyset. \end{cases}$$

The set of feasible solutions to (P_A^v) is denoted by $\mathcal{A}_{P_A^v} := (\operatorname{dom} f) \cap A^{-1}(\operatorname{dom} g)$.

Related to problem (P_A^v) we investigate properly efficient solutions.

Definition 3.2.1 *Let the hypotheses* (3.8) *be satisfied. Then an element* $\overline{x} \in \mathcal{A}_{P_A^v}$ *is said to be a* ***properly efficient*** *solution to problem* (P_A^v) *if there exists an* $y^* \in K^{+0}$ *such that*

$$\langle y^*, (f+g\circ A)(\overline{x})\rangle \leq \langle y^*, (f+g\circ A)(x)\rangle \textit{ for all } x \in \mathcal{A}_{P_A^v}.$$

The first Fenchel-type vector dual problem associated with (P_A^v), and analyzed in this section, is formulated as

$$(D_A^{v\leq}) \qquad \underset{(y^*,z^*,y)\in\mathcal{A}_{D_A^{v\leq}}}{\text{v-max}} h^{\leq}(y^*,z^*,y),$$

where

$$\mathcal{A}_{D_A^{v\leq}} := \{(y^*,z^*,y) \in K^{+0} \times Z^* \times Y : \langle y^*, y\rangle \leq -(y^*\circ f)^*(-A^*z^*) - (y^*\circ g)^*(z^*)\}.$$

The objective function $h^{\leq} : \mathcal{A}_{D_A^{v\leq}} \to Y$ is defined by

$$h^{\leq}(y^*,z^*,y) := y \text{ for all } (y^*,z^*,y) \in \mathcal{A}_{D_A^{v\leq}}.$$

Related to the vector dual problem $(D_A^{v\leq})$ we investigate Pareto-efficient solutions.

Definition 3.2.2 *Let the hypotheses* (3.8) *be satisfied. Then an element* $(\overline{y}^*, \overline{z}^*, \overline{y}) \in \mathcal{A}_{D_A^{v\leqq}}$ *is said to be a* ***Pareto-efficient solution*** *to* $(D_A^{v\leqq})$ *if, for each element* $(y^*, z^*, y) \in \mathcal{A}_{D_A^{v\leqq}}$ *satisfying*

$$h^{\leqq}(\overline{y}^*, \overline{z}^*, \overline{y}) \leq_K h^{\leqq}(y^*, z^*, y),$$

the equality

$$h^{\leqq}(\overline{y}^*, \overline{z}^*, \overline{y}) = h^{\leqq}(y^*, z^*, y)$$

holds.

We denote by v-max $h^{\leqq}(\mathcal{A}_{D_A^{v\leqq}})$ the set of all Pareto-efficient solution to $(D_A^{v\leqq})$.

We start by stating and proving the **weak duality theorem** for the prima-dual pair $(P_A^v, D_A^{v\leqq})$ of optimization problems.

Theorem 3.2.3 (Grad A. [61]) *There exist no* $x \in X$ *and no* $(y^*, z^*, y) \in \mathcal{A}_{D_A^{v\leqq}}$ *such that*

$$(f + g \circ A)(x) \leq_K h^{\leqq}(y^*, z^*, y) \text{ and } (f + g \circ A)(x) \neq h^{\leqq}(y^*, z^*, y).$$

Remark 3.2.4 In order to ensure strong duality between (P_A^v) and $(D_A^{v\leqq})$ a regularity condition has to be fulfilled. As it will be seen in the proof of the forthcoming Theorem 3.2.5, the required regularity condition actually ensures the existence of strong duality for the scalar optimization problem

$$(\overline{y}^* P_A^v) \qquad \inf_{x \in X} \{(\overline{y}^* \circ f)(x) + (\overline{y}^* \circ g)(Ax)\}$$

and its Fenchel dual problem

$$(\overline{y}^* D_A^v) \qquad \sup_{z^* \in Z^*} \{-(\overline{y}^* \circ f)^*(-A^* z^*) - (\overline{y}^* \circ g)^*(z^*)\}$$

for all $\overline{y}^* \in K^{+0}$. ∎

The first regularity condition which we mention for the primal-dual pair $(\overline{y}^* P_A^v, \overline{y}^* D_A^v)$, condition derived from Ekeland I. and Temam R. [49], is:

$$(RC_A^{v1}) \qquad \exists x_0 \in \operatorname{dom} f \cap A^{-1}(\operatorname{dom} g) \text{ such that } g \text{ is continuous at } A(x_0).$$

In Fréchet spaces one can state the following regularity conditions for the primal-dual pair $(\overline{y}^* P_A^v, \overline{y}^* D_A^v)$:

$$(RC_A^{v2}) \qquad \begin{cases} X \text{ and } Z \text{ are Fréchet spaces;} \\ f \text{ and } g \text{ are star } K\text{-lower semicontinuous functions;} \\ 0 \in \operatorname{sqri}\left[(\operatorname{dom} g) - A(\operatorname{dom} f)\right]; \end{cases}$$

$$(RC_A^{v2'}) \qquad \begin{cases} X \text{ and } Z \text{ are Fréchet spaces;} \\ f \text{ and } g \text{ are star } K\text{-lower semicontinuous functions;} \\ 0 \in \operatorname{core}\left[(\operatorname{dom} g) - A(\operatorname{dom} f)\right]; \end{cases}$$

$$(RC_A^{v2''}) \qquad \begin{cases} X \text{ and } Z \text{ are Fréchet spaces;} \\ f \text{ and } g \text{ are star } K\text{-lower semicontinuous functions;} \\ 0 \in \operatorname{int}\left[(\operatorname{dom} g) - A(\operatorname{dom} f)\right]. \end{cases}$$

In the finite dimensional setting one can use the following regularity condition:

$$(RC_A^{v3}) \qquad \begin{cases} \dim\{\mathrm{lin}[(\mathrm{dom}\, g) - A(\mathrm{dom}\, f)]\} < +\infty; \\ \mathrm{ri}(\mathrm{dom}\, g) \cap \mathrm{ri}(A(\mathrm{dom}\, f)) \neq \emptyset. \end{cases}$$

When $X := \mathbb{R}^n$ and $Z := \mathbb{R}^m$ the condition (RC^{v3}) becomes

$$(RC_A^{v4}) \qquad \exists x' \in \mathrm{ri}(\mathrm{dom}\, f) \text{ such that } Ax' \in \mathrm{ri}(\mathrm{dom}\, g).$$

The condition (RC_A^{v4}) is the classical regularity condition for the scalar Fenchel duality in finite dimensional spaces and has been stated by Rockafellar R. T. [103].

In the following we state the **strong duality theorem** for the primal-dual pair $(P_A^v, D_A^{v\leq})$ of vector optimization problems.

Theorem 3.2.5 (Grad A. [61], Boţ R. I., Grad S. M., Wanka G. [26]) *Let the hypotheses* (3.8)*, and one of the regularity conditions from* (RC_A^{v1}) *to* (RC_A^{v3})*, be satisfied. If* $\overline{x} \in \mathcal{A}_{P_A^v}$ *is a properly efficient solution to* (P_A^v)*, then there exists a Pareto-efficient solution* $(\overline{y}^*, \overline{z}^*, \overline{y}) \in \mathcal{A}_{D_A^{v\leq}}$ *to* $(D_A^{v\leq})$ *such that*

$$(f + g \circ A)(\overline{x}) = h^{\leq}(\overline{y}^*, \overline{z}^*, \overline{y}).$$

The forthcoming result plays a crucial role in proving a converse duality theorem.

Theorem 3.2.6 (Grad A. [61], Boţ R. I., Grad S. M., Wanka G. [26]) *Let the hypotheses* (3.8)*, and one of the regularity conditions from* (RC_A^{v1}) *to* (RC_A^{v3})*, be satisfied, and let* $\mathcal{A}_{D_A^{v\leq}} \neq \emptyset$*. Then the following inclusion holds:*

$$Y \setminus \mathrm{cl}\left\{(f + g \circ A)\left(\mathcal{A}_{P_A^v}\right) + K\right\} \subseteq \mathrm{core}\, h^{\leq}(\mathcal{A}_{D_A^{v\leq}}).$$

We are now able to state and prove the **converse duality theorem** for the primal-dual pair $(P_A^v, D_A^{v\leq})$.

Theorem 3.2.7 (Grad A. [61], Boţ R. I., Grad S. M., Wanka G. [26]) *Let the hypotheses* (3.8)*, and one of the regularity conditions from* (RC_A^{v1}) *to* (RC_A^{v3})*, be satisfied. Moreover, let the set* $(f + g \circ A)(\mathcal{A}_{P_A^v}) + K$ *be closed. Then for each Pareto-efficient solution* $(\overline{y}^*, \overline{z}^*, \overline{y}) \in \mathcal{A}_{D_A^{v\leq}}$ *to* $(D_A^{v\leq})$ *there exists a properly efficient solution* $\overline{x} \in \mathcal{A}_{P_A^v}$ *to* (P_A^v) *such that*

$$(f + g \circ A)(\overline{x}) = h^{\leq}(\overline{y}^*, \overline{z}^*, \overline{y}).$$

3.2.2 A Comparison Among three Fenchel-Type Vector Dual Problems

The scalar Fenchel duality was involved for the first time in the definition of a vector dual problem by Breckner W. W. and Kolumbán I. [36] in a very general framework. Inspired by the approach introduced in this work, one gets the following dual vector optimization problem associated with (P_A^v):

$$(D_A^{vBK}) \qquad \operatorname*{v\text{-}max}_{(y^*, z^*, y) \in \mathcal{A}_{D_A^{vBK}}} h^{BK}(y^*, z^*, y),$$

where

$$\mathcal{A}_{D_A^{vBK}} := \{(y^*, z^*, y) \in K^{+0} \times Z^* \times Y : \langle y^*, y\rangle = -(y^* \circ f)^*(-A^* z^*) - (y^* \circ g)^*(z^*)\}.$$

The objective function $h^{BK} : \mathcal{A}_{D_A^{vBK}} \to Y$ is defined by

$$h^{BK}(y^*, z^*, y) := y \text{ for all } (y^*, z^*, y) \in \mathcal{A}_{D_A^{vBK}}.$$

Definition 3.2.8 *Let the hypotheses* (3.8) *be satisfied. Then an element* $(\overline{y}^*, \overline{z}^*, \overline{y}) \in \mathcal{A}_{D_A^{vBK}}$ *is said to be a* ***Pareto-efficient*** *solution to* (D_A^{vBK}) *if for each element* $(y^*, z^*, y) \in \mathcal{A}_{D_A^{vBK}}$ *satisfying*

$$h^{BK}(\overline{y}^*, \overline{z}^*, \overline{y}) \leq_K h^{BK}(y^*, z^*, y),$$

the equality

$$h^{BK}(\overline{y}^*, \overline{z}^*, \overline{y}) = h^{BK}(y^*, z^*, y)$$

holds.

We denote by v-max $h^{BK}(\mathcal{A}_{D_A^{vBK}})$ the set of all Pareto-efficient solutions to (D_A^{vBK}).

Remark 3.2.9 It can be easily remarked from the definition, without any other additional assumptions, that the following inclusion:

$$h^{BK}(\mathcal{A}_{D_A^{vBK}}) \subseteq h^{\leq}(\mathcal{A}_{D_A^{v\leq}}).$$

holds. ∎

Theorem 3.2.10 (Grad A. [61]) *Let the hypotheses* (3.8) *be satisfied. Then the following equality holds:*

$$\text{v-max}\, h^{BK}(\mathcal{A}_{D_A^{vBK}}) = \text{v-max}\, h^{\leq}(\mathcal{A}_{D_A^{v\leq}}).$$

Remark 3.2.11 In the proof of the previous theorem, no assumptions regarding the nature of the functions and sets involved in the formulation of (P_A^v) were made. This means that the sets of efficient elements of $h^{\leq}(\mathcal{A}_{D_A^{v\leq}})$ and $h^{BK}(\mathcal{A}_{D_A^{vBK}})$ are always identical. ∎

Using the weak, strong and converse duality theorems established for the primal-dual pair $(P_A^v, D_A^{v\leq})$ of vector optimization problems, we can prove similar results for the primal-dual pair (P_A^v, D_A^{vBK}).

Theorem 3.2.12 (Grad A. [61]) *The following statements are true:*

(a) (Weak duality) *There exist no* $x \in X$ *and no* $(y^*, z^*, y) \in \mathcal{A}_{D_A^{vBK}}$ *such that*

$$(f + g \circ A)(x) \leq_K h^{BK}(y^*, z^*, y) \text{ and } (f + g \circ A)(x) \neq h^{BK}(y^*, z^*, y).$$

(b) (Strong duality) *Let the hypotheses* (3.8)*, and one of the regularity conditions from* (RC_A^{v1}) *to* (RC_A^{v3})*, be satisfied. If* $\overline{x} \in \mathcal{A}_{P_A^v}$ *is a properly efficient solution to* (P_A^v)*, then there exists a Pareto-efficient solution* $(\overline{y}^*, \overline{z}^*, \overline{y}) \in \mathcal{A}_{D_A^{vBK}}$ *to* (D_A^{vBK}) *such that*

$$(f + g \circ A)(\overline{x}) = h^{BK}(\overline{y}^*, \overline{z}^*, \overline{y}).$$

(c) (Converse duality) *Let the hypotheses* (3.8)*, and one of the regularity conditions from* (RC_A^{v1}) *to* (RC_A^{v3})*, be satisfied. Moreover, let* $(f + g \circ A)(\mathcal{A}_{P_A^v}) + K$ *be a closed set. Then for each Pareto-efficient solution* $(\overline{y}^*, \overline{z}^*, \overline{y}) \in \mathcal{A}_{D_A^{vBK}}$ *to* (D_A^{vBK}) *there exists a properly efficient solution* $\overline{x} \in \mathcal{A}_{P_A^v}$ *to* (P_A^v) *such that* $(f + g \circ A)(\overline{x}) = h^{BK}(\overline{y}^*, \overline{z}^*, \overline{y})$.

Remark 3.2.13 When particularizing the spaces X, Y and Z, the Fenchel-type vector dual problems $(D_A^{v\leq})$ and (D_A^{vBK}) are exactly the classical Fenchel dual problem in scalar optimization. ∎

The Case When $Y := \mathbb{R}^m$

In the following we focus our attention on the special case when $Y := \mathbb{R}^m$ and $K := \mathbb{R}^m_+$. In addition to the two vector dual problems $(D_A^{v\leqq})$ and (D_A^{vBK}) studied so far in this section, we introduce a new one, whose formulation was inspired from Boţ R. I., Dumitru (Grad) A. and Wanka G. [20]. Nevertheless, a more particular case was treated there, namely when $X := \mathbb{R}^n$ and $Z := \mathbb{R}^k$, while in this section X and Z are separated locally convex spaces.

Let us emphasize that the primal problem

$$(P_A^v) \qquad \operatorname*{v-min}_{x\in X}(f + g \circ A)(x)$$

is now investigated under the following hypotheses:

$$(3.9) \qquad \begin{cases} X \text{ and } Z \text{ are separated locally convex spaces;} \\ f := (f_1, f_2, ...f_m) \text{ and } g := (g_1, g_2, ..., g_m) \text{ are vector functions;} \\ \forall i \in \{1, ..., m\} : f_i : X \to \overline{\mathbb{R}} \ \text{ is a proper convex function;} \\ \forall i \in \{1, ..., m\} : g_i : Z \to \overline{\mathbb{R}} \ \text{ is a proper convex function;} \\ A : X \to Z \text{ is a linear continuous operator;} \\ \left(\bigcap\limits_{i=1}^{m} \operatorname{dom} f_i\right) \cap A^{-1}\left(\bigcap\limits_{i=1}^{m} \operatorname{dom} g_i\right) \neq \emptyset. \end{cases}$$

Let us further notice that

$$\mathcal{A}_{P_A^v} = \left(\bigcap_{i=1}^{m} \operatorname{dom} f_i\right) \cap A^{-1}\left(\bigcap_{i=1}^{m} \operatorname{dom} g_i\right).$$

Let $x \in X$ be such that there exists $i_0 \in \{1, ..., m\}$ satisfying $f_{i_0}(x) = +\infty$. Then, by convention, $f(x) = \infty_{\mathbb{R}^m}$. Similarly, if there exists $j_0 \in \{1, ..., m\}$ satisfying $g_{j_0}(x) = +\infty$, then, by convention, $g(x) = \infty_{\mathbb{R}^m}$.

Furthermore, we consider the following regularity condition:

$$(RC_A^{v_m}) \qquad \begin{cases} \exists x' \in \left(\bigcap\limits_{i=1}^{m} \operatorname{dom} f_i\right) \cap A^{-1}\left(\bigcap\limits_{i=1}^{m} \operatorname{dom} g_i\right) \text{ such that} \\ m-1 \text{ functions } f_i, \text{ with } i \in \{1, ..., m\}, \text{ are continuous at } x'; \\ g_i \text{ is continuous at } Ax' \text{ for all } i \in \{1, ..., m\}. \end{cases}$$

We associate with (P_A^v) the following dual optimization problem:

$$(D_A^{vBGW}) \qquad \operatorname*{v-max}_{(p,q,\lambda,t)\in\mathcal{A}_{D_A^{vBGW}}} h^{BGW}(p, q, \lambda, t),$$

where

$$\mathcal{A}_{D_A^{vBGW}} := \left\{ \begin{array}{ll} (p, q, \lambda, t) : & p := (p_1, ..., p_m) \in X^* \times ... \times X^*, \\ & q := (q_1, ..., q_m) \in Z^* \times ... \times Z^*, \\ & \lambda := (\lambda_1, ..., \lambda_m) \in \operatorname{int} \mathbb{R}^m_+, \\ & t := (t_1, ..., t_m) \in \mathbb{R}^m, \\ & \sum\limits_{i=1}^{m} \lambda_i \left(p_i + A^* q_i\right) = 0, \sum\limits_{i=1}^{m} \lambda_i t_i = 0 \end{array} \right\}.$$

The objective function is defined by

$$h^{BGW}(p,q,\lambda,t) := \left(h_1^{BGW}(p,q,\lambda,t),...,h_m^{BGW}(p,q,\lambda,t)\right) \text{ for all } (p,q,\lambda,t) \in \mathcal{A}_{D_A^{vBGW}},$$

with

$$h_i^{BGW}(p,q,\lambda,t) := -f_i^*(p_i) - g_i^*(q_i) + t_i \text{ for all } i \in \{1,...,m\}.$$

Definition 3.2.14 *Let the hypotheses* (3.9) *be satisfied. Then an element* $(\overline{p},\overline{q},\overline{\lambda},\overline{t}) \in \mathcal{A}_{D_A^{vBGW}}$ *is said to be a* ***Pareto-efficient*** *solution to* (D_A^{vBGW}) *if* $h^{BGW}(\overline{p},\overline{q},\overline{\lambda},\overline{t}) \in \mathbb{R}^m$*, and for each element* $(p,q,\lambda,t) \in \mathcal{A}_{D_A^{vBGK}}$ *such that* $h^{BGW}(p,q,\lambda,t) \in \mathbb{R}^m$ *satisfying the inequality*

$$h^{BGW}(\overline{y}^*,\overline{z}^*,\overline{y}) \leq_{\mathbb{R}_+^m} h^{BK}(y^*,z^*,y),$$

the equality

$$h^{BGW}(\overline{y}^*,\overline{z}^*,\overline{y}) = h^{BK}(y^*,z^*,y)$$

holds.

We denote by v-max $\left[h^{BGW}(\mathcal{A}_{D_A^{vBK}}) \cap \mathbb{R}^m\right]$ the set of all Pareto-efficient solutions to (D_A^{vBGW}).

We present next relations among the image sets of $(D_A^{v\leq})$, (D_A^{vBK}) and (D_A^{vBGW}).

Proposition 3.2.15 (Grad A. [61]) *Let the hypotheses* (3.9) *and the regularity condition* $(RC_A^{v_m})$ *be satisfied. Then the image sets of the three Fenchel-type vector dual problems* $(D_A^{\leq})$, (D_A^{vBK}) *and* (D_A^{vBGW}) *associated with* (P_A^v) *satisfy the following inclusions:*

$$h^{BK}\left(\mathcal{A}_{D_A^{vBK}}\right) \subseteq h^{BGW}\left(\mathcal{A}_{D_A^{vBGW}}\right) \cap \mathbb{R}^m; \tag{3.10}$$

$$h^{BGW}\left(\mathcal{A}_{D_A^{vBGW}}\right) \cap \mathbb{R}^m \subseteq h^{\leq}\left(\mathcal{A}_{D_A^{v\leq}}\right). \tag{3.11}$$

Remark 3.2.16 (a) The inclusions in Proposition 3.2.15 are in general strict, i.e. the following chain is valid:

$$h^{BK}\left(\mathcal{A}_{D_A^{vBK}}\right) \underset{\neq}{\subseteq} h^{BGW}\left(\mathcal{A}_{D_A^{vBGW}}\right) \cap \mathbb{R}^m \underset{\neq}{\subseteq} h^{\leq}\left(\mathcal{A}_{D_A^{v\leq}}\right), \tag{3.12}$$

as it will be proved with the help of Examples 3.2.17 and 3.2.18.

(b) The relation (3.11) remains valid even if the regularity condition $(RC_A^{v_m})$ is not satisfied, as it results from the proof of Proposition 3.2.15. ∎

Example 3.2.17 (Boţ R. I., Dumitru (Grad) A., Wanka G. [20]) Let us consider the spaces $X := \mathbb{R}$, $Z := \mathbb{R}$, the linear continuous operator $A : \mathbb{R} \to \mathbb{R}$ defined by $A(x) := x$ for all $x \in X$, and the functions $f, g : \mathbb{R} \to \mathbb{R}^2$ defined by

$$f(x) := \left(2x^2 - 1, x^2\right) \quad \text{and} \quad g(x) := (-2x, -x+1) \text{ for all } x \in \mathbb{R}.$$

We prove that

$$h^{BK}\left(\mathcal{A}_{D_A^{vBK}}\right) \subset h^{BGW}\left(\mathcal{A}_{D_A^{vBGW}}\right) \cap \mathbb{R}^m.$$

∎

Example 3.2.18 (BOŢ R. I., DUMITRU (GRAD) A., WANKA G. [20]) Let us consider the spaces $X := \mathbb{R}$, $Z := \mathbb{R}$, the linear continuous operator $A : \mathbb{R} \to \mathbb{R}$ defined by $A(x) := x$ for all $x \in \mathbb{R}$, and the functions $f, g : \mathbb{R} \to \mathbb{R}^2$ defined by

$$f\,(x) := (x-1, -x-1) \quad \text{and} \quad g\,(x) := (x, -x) \ \text{ for all } x \in \mathbb{R}.$$

We prove that

$$h^{BGW}\left(\mathcal{A}_{D_A^{vBGW}}\right) \cap \mathbb{R}^m \subset h^{\leq}\left(\mathcal{A}_{D_A^{v\leq}}\right).$$

∎

Below we prove that the sets of optimal solutions to (D_A^{vBGW}) and $(D_A^{v\leq})$ coincide.

Theorem 3.2.19 (GRAD A. [61]) *Let the hypotheses* (3.9), *and the regularity conditions* $(RC_A^{v_m})$, *be satisfied. Then the following set equality holds:*

$$\text{v-max}\left[h^{BGW}\left(\mathcal{A}_{D_A^{vBGW}}\right) \cap \mathbb{R}^m\right] = \text{v-max}\, h^{\leq}\left(\mathcal{A}_{D_A^{v\leq}}\right). \tag{3.13}$$

Remark 3.2.20 Let the hypotheses (3.9), and the regularity conditions $(RC_A^{v_m})$, be satisfied. We would like to emphasize that from Theorems 3.2.10 and 3.2.19 along with Examples 3.2.18 and 3.2.17, we obtain the following equalities for the sets of optimal solutions of the three Fenchel-type vector dual problems $(D_A^{\leq})$, (D_A^{vBK}) and (D_A^{vBGW}) associated with the primal problem (P_A^v):

$$\text{v-max}\, h^{BK}(\mathcal{A}_{D_A^{vBK}}) = \text{v-max}\left[h^{BGW}\left(\mathcal{A}_{D_A^{vBGW}}\right) \cap \mathbb{R}^m\right] = \text{v-max}\, h^{\leq}\left(\mathcal{A}_{D_A^{v\leq}}\right),$$

even though

$$h^{BK}\left(\mathcal{A}_{D_A^{vBK}}\right) \underset{\neq}{\subseteq} h^{BGW}\left(\mathcal{A}_{D_A^{vBGW}}\right) \cap \mathbb{R}^m \underset{\neq}{\subseteq} h^{\leq}\left(\mathcal{A}_{D_A^{v\leq}}\right).$$

∎

Using the weak, strong and converse duality theorems between the primal-dual pair $(P_A^v, D_A^{v\leq})$ of the vector optimization problems, similar results can be proved for the primal-dual pair (P_A^v, D_A^{vBGW}).

Theorem 3.2.21 (GRAD A. [61]) *The following statements are true:*

(a) (Weak duality) *There exist no* $x \in X$ *and no* $(p, q, \lambda, t) \in \mathcal{A}_{D_A^{vBGW}}$ *with* $h^{BGW}(p, q, \lambda, t)$ *in* $\mathbb{R}^m$, *such that*

$$(f + g \circ A)(x) \leq_{\mathbb{R}^m_+} h^{BGW}(p, q, \lambda, t) \text{ and } (f + g \circ A)(x) \neq h^{BGW}(p, q, \lambda, t).$$

(b) (Strong duality) *Let the hypotheses* (3.9) *and the regularity condition* $(RC_A^{v_m})$ *be satisfied. If* $\overline{x} \in \mathcal{A}_{P_A^v}$ *is a properly efficient solution to* (P_A^v), *then there exists a Pareto-efficient solution* $(\overline{p}, \overline{q}, \overline{\lambda}, \overline{t}) \in \mathcal{A}_{D_A^{vBGW}}$ *to* (D_A^{vBGW}) *such that*

$$(f + g \circ A)(\overline{x}) = h^{BGW}(\overline{p}, \overline{q}, \overline{\lambda}, \overline{t}).$$

(c) (Converse duality) *Let the hypotheses* (3.9) *and the regularity condition* $(RC_A^{v_m})$ *be satisfied. If the set* $(f + g \circ A)(\mathcal{A}_{P_A^v}) + K$ *is closed, then for each Pareto-efficient solution* $(\overline{p}, \overline{q}, \overline{\lambda}, \overline{t}) \in \mathcal{A}_{D_A^{vBGW}}$ *to* (D_A^{vBGW}) *there exists a properly efficient solution* $\overline{x} \in \mathcal{A}_{P_A^v}$ *to* (P_A^v) *such that*

$$(f + g \circ A)(\overline{x}) = h^{BGW}(\overline{p}, \overline{q}, \overline{\lambda}, \overline{t}).$$

As it will be seen in the following example, Theorem 3.2.6, which was important in the proof of the converse duality for the dual $(D_A^{v\leq})$, does not hold for the more particular dual problems (D_A^{vBK}) and (D_A^{vBGW}).

Example 3.2.22 (Grad A. [61]) Let us consider the spaces $X := \mathbb{R}$, $Z := \mathbb{R}$ and $Y := \mathbb{R}^2$, the linear continuous operator $A : \mathbb{R} \to \mathbb{R}$ defined by $A(x) := x$ for all $x \in \mathbb{R}$, and the functions $f, g : \mathbb{R} \to \mathbb{R}^2$ defined by

$$f(x) := (-3x + 7, 2x) \text{ and } g(x) := (3x - 7, -2x) \text{ for all } x \in \mathbb{R}.$$

We show that

$$\mathbb{R}^2 \setminus \operatorname{cl}\left((f+g)(\mathbb{R}) + \mathbb{R}^2_+\right) \not\subseteq \operatorname{core} h^{BGW}(\mathcal{A}_{D_A^{vBGW}}) \cap \mathbb{R}^2.$$

∎

Remark 3.2.23 Using Example 3.2.22, we reach the conclusion that a direct converse duality proof for the case of problem (D_A^{vBGW}) would be more difficult, unless using its connection to $(D_A^{v\leq})$ from Theorem 3.2.19. ∎

3.3 A Direct Approach to Problem (D_A^{vBGW}) in Finite Dimensional Spaces

This section contains a direct approach regarding strong and converse duality for a Fenchel-type vector dual problem resembling (D_A^{vBGW}) from Section 3.2, this time in a finite dimensional setting. The results were published in the joint-article Boţ R. I., Dumitru(Grad) A. and Wanka G. [20], and the direct proof of the converse duality theorem is published in Grad A. [65]. As in the general case, we make use of scalarization.

The primal problem under consideration is

$$(P_A^{!v}) \qquad \operatorname*{v-min}_{x \in \mathbb{R}^n} (f + g \circ A)(x),$$

and is dealt with under the following hypotheses:

$$(3.14) \qquad \begin{cases} f := (f_1, f_2, ... f_m), g := (g_1, g_2, ..., g_m) \text{ are vector functions;} \\ I, J \subseteq \{1, ..., m\} \text{ are two sets such that} \\ f_i : \mathbb{R}^n \to \overline{\mathbb{R}} \text{ and } g_j : \mathbb{R}^w \to \overline{\mathbb{R}} \text{ are proper polyhedral functions} \\ \qquad \text{for all } i \in I \text{ and all } j \in J; \\ f_k : \mathbb{R}^n \to \overline{\mathbb{R}} \text{ and } g_l : \mathbb{R}^w \to \overline{\mathbb{R}} \text{ are proper convex functions} \\ \qquad \text{for all } k \in \{1, ..., m\} \setminus I \text{ and all } l \in \{1, ..., m\} \setminus J; \\ A : \mathbb{R}^n \to \mathbb{R}^w \text{ is a linear operator;} \\ \left(\bigcap_{i=1}^m \operatorname{dom} f_i\right) \cap A^{-1}\left(\bigcap_{i=1}^m \operatorname{dom} g_i\right) \neq \emptyset. \end{cases}$$

In the following we use the notation

$$\mathcal{A}_{P_A^{!v}} := \left(\bigcap_{i=1}^m \operatorname{dom} f_i\right) \cap A^{-1}\left(\bigcap_{i=1}^m \operatorname{dom} g_i\right).$$

The regularity condition used in order to ensure strong duality, both in the scalar and vector case, is stated below:

$$(RC_A^{!v}) \qquad \exists\ x' \in \left(\bigcap_{i\in I} \operatorname{dom} f_i\right) \cap \bigcap_{k\in\{1,...,m\}\setminus I} \operatorname{ri}(\operatorname{dom} f_k) \ \text{ such that}$$

$$Ax' \in \left(\bigcap_{j\in J} \operatorname{dom} g_j\right) \cap \bigcap_{l\in\{1,...,m\}\setminus J} \operatorname{ri}(\operatorname{dom} g_l).$$

On $\mathbb{R}^m$ we consider the partial ordering induced by the non-negative orthant $\mathbb{R}^m_+$. Consequently, for $x, y \in \mathbb{R}^m$, one has

$$x \leq_{\mathbb{R}^m_+} y \quad \text{if and only if} \quad x_i \leq y_i \text{ for all } i \in \{1, ..., m\}.$$

Definition 3.3.1 *Let the hypotheses* (3.14) *be satisfied. Then an element* $\overline{x} \in \mathcal{A}_{P_A^{!v}}$ *is said to be a* ***Pareto-efficient solution*** *to problem* $(P_A^{!v})$ *if, for each* $x \in \mathcal{A}_{P_A^{!v}}$ *satisfying*

$$(f + g \circ A)(x) \leq_{\mathbb{R}^m_+} (f + g \circ A)(\overline{x}),$$

the following equality holds:

$$(f + g \circ A)(x) = (f + g \circ A)(\overline{x}).$$

Definition 3.3.2 *Let the hypotheses* (3.14) *be satisfied. Then an element* $\overline{x} \in \mathcal{A}_{P_A^{!v}}$ *is said to be a* ***properly efficient solution*** *to problem* $(P_A^{!v})$ *if there exists* $\lambda := (\lambda_1, ..., \lambda_m) \in \operatorname{int} \mathbb{R}^m_+$ *such that for all* $\mathcal{A}_{P_A^{!v}}$ *the following inequality holds:*

$$\sum_{i=1}^m \lambda_i (f_i + g_i \circ A)(\overline{x}) \leq \sum_{i=1}^m \lambda_i (f_i + g_i \circ A)(x).$$

Remark 3.3.3 Each properly efficient solution is also a Pareto-efficient one, while the reverse claim does not hold in general. ∎

3.3.1 Duality for the Scalarized Problem

In order to associate a vector dual problem with $(P_A^{!v})$, considered within the framework (3.14), we develop first a duality theory for the following scalar optimization problem (motivated by the definition of a properly efficient solution):

$$(\lambda P_A^{!v}) \qquad \inf_{x\in\mathbb{R}^n} \sum_{i=1}^m \lambda_i (f_i + g_i \circ A)(x),$$

where $\lambda := (\lambda_1, ..., \lambda_m)$ is arbitrarily chosen in $\operatorname{int} \mathbb{R}^m_+$.

According to Rockafellar R. T. [103, Corollary 31.2.1], the classical Fenchel dual problem to $(\lambda P_A^{!v})$ is

$$\sup_{q\in\mathbb{R}^w} \left[-\left(\sum_{i=1}^m \lambda_i f_i\right)^*(-A^*q) - \left(\sum_{i=1}^m \lambda_i g_i\right)^*(q)\right].$$

With regard to our purposes, this dual has the drawback that the functions involved do not appear separately. Therefore we consider as a dual problem to $(\lambda P_A^{!v})$ a refinement of it, namely

$$(\lambda D_A^{!v}) \qquad \sup_{\substack{p_i \in \mathbb{R}^n, q_i \in \mathbb{R}^w \\ i \in \{1,\ldots m\} \\ \sum_{i=1}^m \lambda_i (p_i + A^* q_i) = 0}} \sum_{i=1}^m \lambda_i \left[-f_i^* \left(p_i\right) - g_i^* \left(q_i\right) \right].$$

For the scalar problems $(\lambda P_A^{!v})$ and $(\lambda D_A^{!v})$ we denote by $v\left(\lambda P_A^{!v}\right)$ and $v\left(\lambda D_A^{!v}\right)$ their optimal objective values, respectively.

It is possible to obtain $(\lambda D_A^{!v})$ as a dual to $(\lambda P_A^{!v})$ by the perturbation approach employing an appropriate perturbation function. For more details on this subject we refer the reader to Rockafellar R. T. [104] and Boţ R. I. [13].

In the following we prove that $\left(\lambda D_A^{!v}\right)$ is indeed a dual problem to $(\lambda P_A^{!v})$. More precisely, we show that weak duality always holds, and that strong duality holds when convexity assumptions and the regularity condition $(RC_A^{!v})$ are fulfilled. We start by establishing the following **scalar weak duality theorem**.

Theorem 3.3.4 (Boţ R. I., Dumitru (Grad) A., Wanka G. [20]) *Let $\lambda := (\lambda_1, \ldots, \lambda_m)$ be in $\operatorname{int} \mathbb{R}_+^m$. For the primal-dual pair $\left(\lambda P_A^{!v}, \lambda D_A^{!v}\right)$ the following inequality holds:*

$$v\left(\lambda D_A^{!v}\right) \leq v\left(\lambda P_A^{!v}\right).$$

Remark 3.3.5 One can notice that weak duality holds without any convexity assumptions for the functions involved. However, for the strong duality one needs this assumption to be fulfilled. ∎

By imposing the regularity condition $(RC_A^{!v})$ we are able to give the following **scalar strong duality theorem**.

Theorem 3.3.6 (Boţ R. I., Dumitru (Grad) A., Wanka G. [20]) *Let the hypotheses* (3.14) *and the regularity condition $(RC_A^{!v})$ be satisfied. Moreover, let $\lambda := (\lambda_1, \ldots, \lambda_m) \in \operatorname{int} \mathbb{R}_+^m$. Then the following equality holds:*

$$v\left(\lambda P_A^{!v}\right) = v\left(\lambda D_A^{!v}\right).$$

Furthermore, the dual problem $\left(\lambda D_A^{!v}\right)$ has an optimal solution.

The next result contains necessary and sufficient **optimality conditions** that can be obtained from Theorem 3.3.6, for the primal-dual pair $(\lambda P_A^{!v}, \lambda D_A^{!v})$.

Theorem 3.3.7 (Boţ R.I., Dumitru (Grad) A., Wanka G. [20]) *Let the hypotheses* (3.14) *and the regularity condition $(RC_A^{!v})$ be satisfied. Moreover, let $\lambda := (\lambda_1, \ldots, \lambda_m) \in \operatorname{int} \mathbb{R}_+^m$. Then the following assertions are true:*

(a) *Let $\overline{x} \in \mathbb{R}^n$ be an optimal solution to $\left(\lambda P_A^{!v}\right)$. Then there exist*

$$\overline{p} := (\overline{p}_1, \ldots, \overline{p}_m) \in \mathbb{R}^n \times \ldots \times \mathbb{R}^n \text{ and } \overline{q} := (\overline{q}_1, \ldots, \overline{q}_m) \in \mathbb{R}^w \times \ldots \times \mathbb{R}^w$$

such that $(\overline{p},\overline{q})$ *is an optimal solution to* $\left(\lambda D_A^{!v}\right)$ *and the following equalities hold:*

$$\begin{array}{ll}\text{(i)} & f_i(\overline{x})+f_i^*(\overline{p}_i)=\langle\overline{p}_i,\overline{x}\rangle \textit{ for all } i\in\{1,...,m\};\\ \text{(ii)} & (g_i\circ A)(\overline{x})+g_i^*(\overline{q}_i)=\langle(A^*\overline{q}_i),\overline{x}\rangle \textit{ for all } i\in\{1,...,m\};\\ \text{(iii)} & \sum\limits_{i=1}^m\lambda_i(\overline{p}_i+A^*\overline{q}_i)=0.\end{array}$$

(b) *If* $\overline{x}\in\mathbb{R}^n$, $\overline{p}:=(\overline{p}_1,...,\overline{p}_m)\in\mathbb{R}^n\times...\times\mathbb{R}^n$ *and* $\overline{q}:=(\overline{q}_1,...,\overline{q}_m)\in\mathbb{R}^w\times...\times\mathbb{R}^w$ *are such that* (i), (ii) *and* (iii) *are fulfilled, then* $\overline{x}$ *is an optimal solution to* $\left(\lambda P_A^{!v}\right)$, $(\overline{p},\overline{q})$ *is an optimal solution to* $\left(\lambda D_A^{!v}\right)$, *and* $v(\lambda P_A^{!v})=v(\lambda D_A^{!v})$.

The optimality conditions in Theorem 3.3.7 will play a decisive role when proving the vector strong duality in Theorem 3.3.12.

3.3.2 A New Fenchel-Type Vector Dual Problem

By using the results obtained in the previous subsection, we are now able to formulate a Fenchel-type multiobjective dual $(D_A^{!vBGW})$ to $(P_A^{!v})$.

The Fenchel-type vector dual problem is

$$(D_A^{!vBGW})\qquad \underset{(p,q,\lambda,t)\in\mathcal{A}_{D_A^{!vBGW}}}{\text{v-max}} h^{!BGW}(p,q,\lambda,t),$$

where

$$\mathcal{A}_{D_A^{!vBGW}}:=\left\{\begin{array}{ll}(p,q,\lambda,t): & p:=(p_1,...,p_m)\in\mathbb{R}^n\times...\times\mathbb{R}^n,\\ & q:=(q_1,...,q_m)\in\mathbb{R}^w\times...\times\mathbb{R}^w,\\ & \lambda:=(\lambda_1,...,\lambda_m)\in\operatorname{int}\mathbb{R}_+^m,\\ & t:=(t_1,...,t_m)\in\mathbb{R}^m,\\ & \sum\limits_{i=1}^m\lambda_i(p_i+A^*q_i)=0,\sum\limits_{i=1}^m\lambda_it_i=0\end{array}\right\}.$$

The objective function is defined by

$$h^{!BGW}(p,q,\lambda,t):=\left(h_1^{!BGW}(p,q,\lambda,t),...,h_m^{!BGW}(p,q,\lambda,t)\right) \text{ for all } (p,q,\lambda,t)\in\mathcal{A}_{D_A^{!vBGW}},$$

with

$$h_i^{!BGW}(p,q,\lambda,t):=-f_i^*(p_i)-g_i^*(q_i)+t_i \text{ for all } i\in\{1,...,m\}.$$

One can easily notice that problem $(D_A^{!vBGW})$ is actually a particularization of problem (D_A^{vBGW}), stated this time in a finite dimensional setting.

Definition 3.3.8 *An element* $\left(\overline{p},\overline{q},\overline{\lambda},\overline{t}\right)\in\mathcal{A}_{D_A^{!vBGW}}$ *is said to a* ***Pareto-efficient solution*** *to problem* $\left(D_A^{!vBGW}\right)$ *if* $h^{!BGW}(\overline{p},\overline{q},\overline{\lambda},\overline{t})\in\mathbb{R}^m$, *and if for each* $(p,q,\lambda,t)\in\mathcal{A}_{D_A^{!vBGW}}$ *such that* $h^{!BGW}(p,q,\lambda,t)\in\mathbb{R}^m$ *satisfying*

$$h^{!BGW}\left(\overline{p},\overline{q},\overline{\lambda},\overline{t}\right)\leq_{\mathbb{R}_+^m}h^{!BGW}(p,q,\lambda,t),$$

the equality

$$h^{!BGW}\left(\overline{p},\overline{q},\overline{\lambda},\overline{t}\right)=h^{!BGW}(p,q,\lambda,t)$$

holds.

We denote by v-max $\left[h^{!BGW}(\mathcal{A}_{P_A^{!v}}) \cap \mathbb{R}^m\right]$ the set of all Pareto-efficient solutions to $(D_A^{!vBGW})$.

We present next the **weak duality theorem** for the primal-dual pair $\left(P_A^{!v}, D_A^{!vBGW}\right)$.

Theorem 3.3.9 (Boţ R. I., Dumitru (Grad) A., Wanka G. [20]) *There exist no $x \in \mathbb{R}^n$, and no $(p, q, \lambda, t) \in \mathcal{A}_{D_A^{!vBGW}}$ with $h^{!BGW}(p, q, \lambda, t) \in \mathbb{R}^m$, such that*

$$(f + g \circ A)(x) \leqq_{\mathbb{R}^m_+} h^{!BGW}(p, q, \lambda, t) \text{ and } (f + g \circ A)(x) \neq h^{!BGW}(p, q, \lambda, t).$$

Remark 3.3.10 As in the scalar case, for proving the weak duality theorem neither convexity assumptions for the functions involved, nor regularity conditions have been used. ∎

Remark 3.3.11 Theorem 3.3.9 can be seen as a particularization of Theorem 3.2.21 (a). ∎

We state now the finite dimensional **strong duality theorem**.

Theorem 3.3.12 (Boţ R. I., Dumitru (Grad) A., Wanka G. [20]) *Let the hypotheses* (3.14) *and the regularity condition $\left(RC_A^{!v}\right)$ be satisfied. Moreover, let $\overline{x} \in \mathcal{A}_{P_A^{!v}}$ be a properly efficient solution to $\left(P_A^{!v}\right)$. Then there exists a Pareto-efficient solution $\left(\overline{p}, \overline{q}, \overline{\lambda}, \overline{t}\right) \in \mathcal{A}_{D_A^{!vBGW}}$ to $\left(D_A^{!vBGW}\right)$ such that*

$$h^{!BGW}\left(\overline{p}, \overline{q}, \overline{\lambda}, \overline{t}\right) = (f + g \circ A)(\overline{x}).$$

Remark 3.3.13 Theorem 3.3.12 can be seen as a particularization of Theorem 3.2.21 (b). ∎

Remark 3.3.14 In the particular case when $n = 1$ (we denote f_1 and g_1 by f and g, respectively, for short) the dual $(D_A^{!vBGW})$ is exactly the classical Fenchel dual problem (cf. Rockafellar R.T. [103]) to the primal scalar problem

$$\inf_{x \in \mathbb{R}^n} (f + g \circ A)(x).$$

This means that $\left(D_A^{!vBGW}\right)$ is a natural extension of the classical scalar Fenchel dual problem to the finite dimensional vector case. ∎

3.3.3 Direct Converse Duality

In this subsection we present a direct proof for a converse duality theorem related to the primal-dual pair $(P_A^{!v}, D_A^{!vBGW})$. Let us recall that $(D_A^{!vBGW})$ is a particularization of (D_A^{vBGW}), and that for the later problem, the converse duality was proved indirectly, by making use of the converse duality for problem $(D_A^{v\leqq})$ (see also Remark 3.2.23).

We start by presenting two helping properties. With each $\lambda := (\lambda_1, ..., \lambda_m) \in \operatorname{int} \mathbb{R}^m_+$ we associate the following set:

$$\mathcal{A}^{\lambda}_{D_A^{!vBGW}} := \left\{ \begin{array}{ll} (p, q, t): & p := (p_1, ..., p_m) \in \mathbb{R}^n \times ... \times \mathbb{R}^n, \\ & q := (q_1, ..., q_m) \in \mathbb{R}^w \times ... \times \mathbb{R}^w, \\ & t := (t_1, ..., t_m) \in \mathbb{R}^m, \\ & \sum_{i=1}^m \lambda_i (p_i + A^* q_i) = 0, \sum_{i=1}^m \lambda_i t_i = 0 \end{array} \right\}.$$

Moreover, with each $y \in \mathbb{R}^m$ we associate the set

$$\Lambda(y) := \left\{ \lambda \in \operatorname{int} \mathbb{R}^m_+ : \exists (p, q, t) \in \mathcal{A}^{\lambda}_{D^{!vBGW}_A} \text{ such that } \langle \lambda, y \rangle = \langle \lambda, h^{!BGW}(p, q, \lambda, t) \rangle \right\}.$$

Finally we make the following notation:

$$\Pi := \{ y \in \mathbb{R}^m : \Lambda(y) \neq \emptyset \}.$$

Our first result emphasizes a connection between the image set of the dual problem $(D^{!vBGW}_A)$ and this set Π.

Proposition 3.3.15 (Grad A. [65]) *Let the hypotheses* (3.14) *be satisfied. Then the following equality holds:*

$$h^{!BGW}(\mathcal{A}_{D^{!vBGW}_A}) \cap \mathbb{R}^m = \Pi.$$

Our next result provides an interesting and useful characterization of the maximal elements of the set Π.

Proposition 3.3.16 (Grad A. [65]) *Let the hypotheses* (3.14) *be satisfied. Then an element $\overline{y} \in \Pi$ is a maximal Pareto-efficient element to the set Π if and only if for each $y \in \Pi$ and each $\lambda \in \Lambda(y)$ the following inequality holds:*

$$\langle \lambda, \overline{y} \rangle \geq \langle \lambda, y \rangle. \tag{3.15}$$

We are now able to state and prove a **converse duality theorem** for the primal-dual pair $(P^{!vBGW}_A, D^{!vBGW}_A)$, in finite dimensional spaces.

Theorem 3.3.17 (Grad A. [65]) *Let the hypotheses* (3.14) *and the regularity condition* $(RC^{!v}_A)$ *be satisfied. Assume further that, for each $\lambda := (\lambda_1, ..., \lambda_m) \in \operatorname{int} \mathbb{R}^m_+$ satisfying the inequality*

$$\inf_{x \in \mathbb{R}^n} \sum_{i=1}^m \lambda_i (f_i + g_i \circ A)(x) > -\infty,$$

there exists an $x_\lambda \in \mathcal{A}^{P^{!v}_A}$ such that

$$\sum_{i=1}^m \lambda_i (f_i + g_i \circ A)(x_\lambda) = \min_{x \in \mathbb{R}^n} \sum_{i=1}^m \lambda_i (f_i + g_i \circ A)(x). \tag{3.16}$$

Then, for each Pareto-efficient solution $(\overline{p}, \overline{q}, \overline{\lambda}, \overline{t}) \in \mathcal{A}_{D^{!vBGW}_A}$ to $(D^{!vBGW}_A)$ the following statements hold:

(a) $h^{!BGW}(\overline{p}, \overline{q}, \overline{\lambda}, \overline{t}) \in \operatorname{cl}\left[(f + g \circ A)(\mathcal{A}_{P^{!v}_A}) + \mathbb{R}^m_+ \right]$.

(b) *There exists a properly efficient solution $x_{\overline{\lambda}} \in \mathcal{A}_{P^{!v}_A}$ to $(P^{!v}_A)$ such that*

$$\sum_{i=1}^m \overline{\lambda}_i \left[(f_i + g_i \circ A)(x_{\overline{\lambda}}) - h^{!BGW}_i(\overline{p}, \overline{q}, \overline{\lambda}, \overline{t}) \right] = 0.$$

(c) *If the set $(f + g \circ A)(\mathcal{A}_{P^{!v}_A}) + \mathbb{R}^m_+$ is closed, then there exists a properly efficient solution $\overline{x} \in \mathcal{A}_{P^{!v}_A}$ to $(P^{!v}_A)$ such that*

$$(f + g \circ A)(\overline{x}) = h^{!BGW}(\overline{p}, \overline{q}, \overline{\lambda}, \overline{t}).$$

Remark 3.3.18 Theorem 3.3.17 (c) can be seen as a particular case of Theorem 3.2.21 (c). However, we were able to give a more relaxed converse duality statement in its first part, i.e. Theorem 3.3.17 (b), which is not connected to the general converse duality theorem stated in locally convex spaces.∎

Chapter 4

Set-Valued Optimization

Set-valued optimization problems represent the best choice when attempting to model real-life occurring situations. This advantage comes along with a disadvantage, i.e. the lack of a unitary approach to set-valued efficient solutions. The author's achievements connected to this topic are gathered in the article GRAD A. [66]. Our results are more general than those established by HERNÁNDEZ E. and RODRÍGUEZ-MARIN L. [71] for weak efficieny, since the quasi interior of a set is a more general notion than the interior. It is important to notice that in the particular case of a scalar optimization problem with vector constraints, our set-valued qi-efficiency conditions collapse into the classical ones, for example those in BOŢ R. I., CSETNEK E. R. and MOLDOVAN A. [16].

4.1 Two New Set Relations Defined by Means of the Quasi Interior

We consider the following framework:

$$\left\{ \begin{array}{l} Y \text{ is a separated locally convex space;} \\ K \subset Y \text{ is a pointed, convex cone with } \operatorname{qi} K \neq \emptyset. \end{array} \right. \tag{4.1}$$

Let us recall that

$$\mathcal{P}_0(Y) := \{A : A \subseteq Y \text{ and } A \neq \emptyset\}.$$

We start by presenting some set relations defined with the help of a convex cone, introduced by KUROIWA D. [84].

Definition 4.1.1 (KUROIWA D. [84]) *Let the hypotheses* (4.1) *be satisfied, and let A and B belong to $\mathcal{P}_0(Y)$. Then we write:*

(a) $A \leq^l B$ *if* $B \subseteq A + K$;

(b) $A \leq^u B$ *if* $A \subseteq B - K$;

(c) $A \sim^l B$ *if* $A \leq^l B$ *and* $B \leq^l A$.

Remark 4.1.2 KUROIWA D. [84] proved that $\sim^l$ is an equivalence relation on $\mathcal{P}_0(Y)$. ∎

Definition 4.1.3 (Grad A. [66]) *Let the framework* (4.1) *be satisfied, and let A and B belong to $\mathcal{P}_0(Y)$. Then we write:*

(a) $A \trianglelefteq^l_{\mathrm{qi}\,K} B$ *if* $B \subseteq A + \mathrm{qi}\,K$*;*

(b) $A \trianglelefteq^u_{\mathrm{qi}\,K} B$ *if* $A \subseteq B - \mathrm{qi}\,K$*.*

Remark 4.1.4 The relations $\trianglelefteq^l_{\mathrm{qi}\,K}$ and $\trianglelefteq^u_{\mathrm{qi}\,K}$ are transitive. ∎

Proposition 4.1.5 (Grad A. [66]) *Let the hypotheses* (4.1) *be satisfied, and let A and B belong to $\mathcal{P}_0(Y)$. Then the following statements are true:*

(a) $A \sim^l B$ *if and only if* $A + K = B + K$*.*

(b) *If* $A \sim^l B$*, then* $A + \mathrm{qi}\,K = B + \mathrm{qi}\,K$*.*

(c) *If* $A \trianglelefteq^l_{\mathrm{qi}\,K} B$ *and* $B \trianglelefteq^l_{\mathrm{qi}\,K} A$*, then* $A \sim^l B$*.*

(d) *If* $A \trianglelefteq^l_{\mathrm{qi}\,K} B$ *and* $B \preceq^l A$*, then* $B \trianglelefteq^l_{\mathrm{qi}\,K} A$*.*

(e) $A \trianglelefteq^l_{\mathrm{qi}\,K} B$ *if and only if* $-B \trianglelefteq^u_{\mathrm{qi}\,K} -A$*.*

(f) *If* $A \trianglelefteq^l_{\mathrm{qi}\,K} B$ *and* $y \in Y$*, then* $A + y \trianglelefteq^l_{\mathrm{qi}\,K} B + y$*.*

(g) *If* $A \sim^l B$ *and* $y \in Y$*, then* $A + y \sim^l B + y$*.*

Remark 4.1.6 From the statements (b) and (c) in Proposition 4.1.5 we obtain

$$A \trianglelefteq^l_{\mathrm{qi}\,K} B \text{ and } B \trianglelefteq^l_{\mathrm{qi}\,K} A \Longrightarrow A \sim^l B \Longrightarrow A + \mathrm{qi}\,K = B + \mathrm{qi}\,K.$$

This chain of implications cannot be reversed. ∎

With the help of the relations introduced in Definition 4.1.3 we define four new efficiency notions for sets.

Definition 4.1.7 (Grad A. [66]) *Let the hypotheses* (4.1) *be satisfied, and let $\mathcal{S} \subseteq \mathcal{P}_0(Y)$. A set $A \in \mathcal{S}$ is said to be:*

(a) *an* l-Min$_{\mathrm{qi}}$-**efficient set** *of* $\mathcal{S}$*, if for each set* $B \in \mathcal{S}$ *satisfying*

$$B \trianglelefteq^l_{\mathrm{qi}\,K} A, \text{ the relation } A \trianglelefteq^l_{\mathrm{qi}\,K} B \text{ holds.}$$

(b) *an* l-Max$_{\mathrm{qi}}$-**efficient set** *of* $\mathcal{S}$*, if for each set* $B \in \mathcal{S}$ *satisfying*

$$A \trianglelefteq^l_{\mathrm{qi}\,K} B, \text{ the relation } B \trianglelefteq^l_{\mathrm{qi}\,K} A \text{ holds.}$$

(c) *an* u-Min$_{\mathrm{qi}}$-**efficient set** *of* $\mathcal{S}$*, if for each set* $B \in \mathcal{S}$ *satisfying*

$$B \trianglelefteq^u_{\mathrm{qi}\,K} A, \text{ the relation } A \trianglelefteq^u_{\mathrm{qi}\,K} B \text{ holds.}$$

(d) *an* u-$\mathrm{Max}_{\mathrm{qi}}$**-efficient set** *of* $\mathcal{S}$*, if for each set* $B \in \mathcal{S}$ *satisfying*

$$A \trianglelefteq^u_{\mathrm{qi}\,K} B, \text{ the relation } B \trianglelefteq^u_{\mathrm{qi}\,K} A \text{ holds.}$$

The sets of all l-$\mathrm{Min}_{\mathrm{qi}}$-efficient, l-$\mathrm{Max}_{\mathrm{qi}}$-efficient, u-$\mathrm{Min}_{\mathrm{qi}}$-efficient and u-$\mathrm{Max}_{\mathrm{qi}}$-efficient sets of $\mathcal{S}$ are denoted by

$$\text{l-Min}_{\mathrm{qi}}\,\mathcal{S}, \text{l-Max}_{\mathrm{qi}}\,\mathcal{S}, \text{u-Min}_{\mathrm{qi}}\,\mathcal{S} \text{ and u-Max}_{\mathrm{qi}}\,\mathcal{S}, \text{ respectively.}$$

Remark 4.1.8 Hernández E. and Rodríguez-Marin L. [70], [71] considered some weak set-efficiency notions defined with $\leq^l$ when dealing with convex cones with nonempty topological interior. Our notions are more general. ∎

Remark 4.1.9 Let the hypotheses (4.1) be satisfied, and let $\mathcal{S} \subseteq \mathcal{P}_0(Y)$. From Definition 4.1.7 and Proposition 4.1.5 (f) it follows that

$$y + \text{l-Min}_{\mathrm{qi}}\,\mathcal{S} = \text{l-Min}_{\mathrm{qi}}(y + \mathcal{S}) \text{ for all } y \in Y.$$

Similar equalities hold for l-$\mathrm{Max}_{\mathrm{qi}}\,\mathcal{S}$, u-$\mathrm{Min}_{\mathrm{qi}}\,\mathcal{S}$ and u-$\mathrm{Max}_{\mathrm{qi}}\,\mathcal{S}$, respectively. ∎

Proposition 4.1.10 (Grad A. [66]) *Let the hypotheses* (4.1) *be satisfied, and let* $\mathcal{S} \subseteq \mathcal{P}_0(Y)$*. Then the following equality holds:*

$$\text{l-Min}_{\mathrm{qi}}(-\mathcal{S}) = -\,\text{u-Max}_{\mathrm{qi}}\,\mathcal{S}, \text{ where } -\mathcal{S} = \{-A : A \in \mathcal{S}\}. \tag{4.2}$$

4.2 qi-Conjugate Functions and qi-Subgradients

We consider the following hypotheses:

$$\left\{ \begin{array}{l} X \text{ is a topological vector space}, Y \text{ is a separated locally convex space;} \\ K \subset Y \text{ is a pointed, convex cone with } \mathrm{qi}\,K \neq \emptyset; \\ F : X \to \mathcal{P}(Y) \text{ is a proper set-valued function.} \end{array} \right. \tag{4.3}$$

4.2.1 qi-Conjugate Set-Valued Functions

Definition 4.2.1 (Grad A. [66]) *Let the hypotheses* (4.3) *be satisfied. The* qi**-conjugate** *function of* F *is the set-valued function* $F^*_{\mathrm{qi}\,K} : \mathcal{L}(X,Y) \to \mathcal{P}(\mathcal{P}(Y))$ *defined by*

$$F^*_{\mathrm{qi}\,K}(T) := \text{u-Max}_{\mathrm{qi}}\{Tx - F(x) : x \in X\} \text{ for all } T \in \mathcal{L}(X,Y).$$

Remark 4.2.2 Taking into consideration Definition 4.1.7 (d) and the hypotheses (4.3), it results that the following equality holds:

$$F^*_{\mathrm{qi}\,K}(T) = \text{u-Max}_{\mathrm{qi}}\{Tx - F(x) : x \in \mathrm{dom}\,F\}.$$

In the following we prove a result which may be regarded as an **extension of the Fenchel-Young inequality** to this set-valued setting.

Theorem 4.2.3 (Grad A. [66]) *Let the hypotheses* (4.3) *be satisfied, let* $x_0, x_1 \in \operatorname{dom} F$, *and let* $T \in \mathcal{L}(X,Y)$ *be such that*

$$F(x_1) - Tx_1 \in -F^*_{\mathrm{qi}_K}(T). \tag{4.4}$$

Then the following statements hold:

(a) *If* $F(x_0) - Tx_0 \trianglelefteq^l_{\mathrm{qi}_K} F(x_1) - Tx_1$, *then* $F(x_1) - Tx_1 \trianglelefteq^l_{\mathrm{qi}_K} F(x_0) - Tx_0$.

(b) *If* $F(x_0) - Tx_0 \trianglelefteq^l_{\mathrm{qi}_K} F(x_1) - Tx_1$, *then* $F(x_1) - Tx_1 \sim^l F(x_0) - Tx_0$.

4.2.2 qi-Subgradients of Set-Valued Functions

With the help of the qi-conjugate function we extend the notions of the subgradient and subdifferential to set-valued functions.

Definition 4.2.4 (Grad A. [66]) *Let the hypotheses* (4.3) *be satisfied, and let* $\overline{x} \in \operatorname{dom} F$.

(a) *An operator* $T \in \mathcal{L}(X,Y)$ *is said to be a* qi-**subgradient** *of the set-valued function* F *at* $\overline{x}$ *if*

$$T\overline{x} - F(\overline{x}) \in F^*_{\mathrm{qi}_K}(T).$$

(b) *The set of all* qi-*subgradients associated with the set-valued function* F *at* $\overline{x}$ *is called the* qi-**subdifferential** *of* F *at* $\overline{x}$ *and is denoted by* $\partial_{\mathrm{qi}_K} F(\overline{x})$.

By convention, if $\overline{x} \notin \operatorname{dom} F$, *then we consider by definition that* $\partial_{\mathrm{qi}\,K} F(\overline{x}) = \emptyset$.

Remark 4.2.5 In light of Definition 4.2.4, the condition (4.4) in Theorem 4.2.3 can be equivalently rewritten as $T \in \partial_{\mathrm{qi}\,K} F(x_1)$. ∎

Similar to the scalar and vector case, we prove the following property.

Proposition 4.2.6 (Grad A. [66]) *Let the hypotheses* (4.3) *be satisfied, and let* $\overline{x} \in \operatorname{dom} F$. *Then*

$$F(\overline{x}) \in \text{l-Min}_{\mathrm{qi}}\{F(x) : x \in X\} \textit{ if and only if } 0 \in \partial_{\mathrm{qi}_K} F(\overline{x}).$$

4.3 A Quasi Interior Perturbation Approach to Set-Valued Optimization

4.3.1 Unconstrained Set-Valued Optimization

In this subsection we consider the unconstrained set-valued optimization problem

$$(P^{sv}_{\mathrm{qi}}) \qquad \operatorname*{l-Min_{qi}}_{x \in X} F(x),$$

which will be studied under the hypotheses (4.3) and the additional assumption that

$$W \text{ is a topological vector space.} \tag{4.5}$$

Definition 4.3.1 (Grad A. [66]) *Let the hypotheses* (4.3) *be satisfied. An element* $\overline{x} \in \operatorname{dom} F$ *is a* qi**-efficient solution** *to* (P^{sv}_{qi}) *if*

$$F(\overline{x}) \in \text{l-Min}_{\text{qi}}\{F(x) : x \in X\} = \text{l-Min}_{\text{qi}}\{F(x) : x \in \operatorname{dom} F\}.$$

We develop a general duality theory based on a quasi interior perturbation approach.

Definition 4.3.2 *Let the hypotheses* (4.3) *and* (4.5) *be satisfied. A set-valued function* $\Phi : X \times W \to \mathcal{P}(Y)$*, which satisfies the equality*

$$\Phi(x, 0) = F(x) \text{ for all } x \in X,$$

is called a **perturbation function** *associated with* F*.*

Consider an arbitrary perturbation function Φ associated with (P^{sv}_{qi}). The qi-conjugate function of Φ is the set-valued function $\Phi^*_{\text{qi}_K} : \mathcal{L}(X,Y) \times \mathcal{L}(W,Y) \to \mathcal{P}(\mathcal{P}(Y))$ defined by

$$\Phi^*_{\text{qi}_K}(H,T) := \text{u-Max}_{\text{qi}}\{Hx + Tw - \Phi(x,w) : (x,w) \in X \times W\}$$

for all $(H,T) \in \mathcal{L}(X,Y) \times \mathcal{L}(W,Y)$.

We introduce the following new set-valued dual problem associated with (P^{sv}_{qi}):

$$(D^{sv}_{\text{qi}}) \qquad \underset{T \in \mathcal{L}(W,Y)}{\text{l-Max}_{\text{qi}}} \left[-\Phi^*_{\text{qi}\,K}(0,T)\right].$$

For the sake of simplicity we consider the notation

$$\mathcal{A}_{D^{sv}_{\text{qi}}} := \left\{(T,x,w) \in \mathcal{L}(W,Y) \times \operatorname{dom}\Phi : -Tw + \Phi(x,w) \in -\Phi^*_{\text{qi}_K}(0,T)\right\}.$$

Definition 4.3.3 (Grad A. [66]) *Let the hypotheses* (4.3) *and* (4.5) *be satisfied. An operator* $\widetilde{T} \in \mathcal{L}(W,Y)$ *is said to be a* qi**-efficient solution** *to the dual problem* (D^{sv}_{qi}) *if there exists an* $(\widetilde{x},\widetilde{w}) \in \operatorname{dom}\Phi$ *such that*

$$(\widetilde{T},\widetilde{x},\widetilde{w}) \in \mathcal{A}_{D^{sv}_{\text{qi}}} \text{ and } -\widetilde{T}\widetilde{w} + \Phi(\widetilde{x},\widetilde{w}) \in \text{l-Max}_{\text{qi}}\{-Tw + \Phi(x,w) : (T,x,w) \in \mathcal{A}_{D^{sv}_{\text{qi}}}\}.$$

With the help of the following **set-valued weak duality** theorem we certify the fact that (D^{sv}_{qi}) is actually a dual problem to (P^{sv}_{qi}).

Theorem 4.3.4 (Grad A. [66]) *Let the hypotheses* (4.3) *and* (4.5) *be satisfied, let* $x_0 \in \operatorname{dom} F$*, and let* $(T,x,w) \in \mathcal{A}_{D^{sv}_{\text{qi}}}$*. Then the following statements are true:*

(a) *If* $F(x_0) \trianglelefteq^l_{\text{qi}_K} -Tw + \Phi(x,w)$*, then* $-Tw + \Phi(x,w) \trianglelefteq^l_{\text{qi}_K} F(x_0)$*.*

(b) *If* $F(x_0) \trianglelefteq^l_{\text{qi}_K} -Tw + \Phi(x,w)$*, then* $-Tw + \Phi(x,w) \sim^l F(x_0)$*.*

The forthcoming result contains some **optimality conditions** for the primal-dual pair $(P^{sv}_{\text{qi}}, D^{sv}_{\text{qi}})$ of set-valued optimization problems.

Theorem 4.3.5 (Grad A. [66]) *Let the hypotheses* (4.3) *and* (4.5) *be satisfied, let $\overline{x} \in \operatorname{dom} F$, and let $(\widetilde{T}, \widetilde{x}, \widetilde{w}) \in \mathcal{A}_{D^{sv}_{\text{qi}}}$ be such that*

$$F(\overline{x}) \trianglelefteq^{l}_{\text{qi}\,K} -\widetilde{T}\widetilde{w} + \Phi(\widetilde{x}, \widetilde{w}). \tag{4.6}$$

Then the following statements are true:

(a) $\overline{x}$ *is a* qi-*efficient solution to* (P^{sv}_{qi}).

(b) $\widetilde{T}$ *is a* qi-*efficient solution to* (D^{sv}_{qi}).

The next theorem contains further **optimality conditions** for the dual problem (D^{sv}_{qi}).

Theorem 4.3.6 (Grad A. [66]) *Let the hypotheses* (4.3) *and* (4.5) *be satisfied, and let $\overline{x} \in \operatorname{dom} F$. It there exists an operator $\overline{T} \in \mathcal{L}(W, Y)$ such that $(\overline{T}, \overline{x}, 0) \in \mathcal{A}_{D^{sv}_{\text{qi}}}$, then $\overline{T}$ is a* qi-*efficient solution to* (D^{sv}_{qi}).

Remark 4.3.7 Let the hypotheses of Theorem 4.3.6 be satisfied. Then each operator $\widetilde{T} \in \mathcal{L}(W, Y)$ for which there exists an $(\widetilde{x}, \widetilde{w}) \in \operatorname{dom} \Phi$ such that the following conditions are valid:

$$(\widetilde{T}, \widetilde{x}, \widetilde{w}) \in \mathcal{A}_{D^{sv}_{\text{qi}}} \text{ and } F(\overline{x}) \trianglelefteq^{l}_{\text{qi}\,K} -\widetilde{T}\widetilde{w} + \Phi(\widetilde{x}, \widetilde{w}),$$

is a qi-efficient solution to (D^{sv}_{qi}). The proof follows easily by applying Theorem 4.3.5. ■

4.3.2 Constrained Set-Valued Optimization

We consider the general set-valued optimization problem with cone constraints

$$(CP^{sv}_{\text{qi}}) \qquad \underset{G(x)\cap(-C)\neq\emptyset}{\text{l-Min}_{\text{qi}}} \; F(x)$$

stated under the following hypotheses:

$$\left\{\begin{array}{l} X \text{ and } W \text{ are topological vector spaces}, Y \text{ and } Z \text{ are separated locally convex spaces;} \\ K \subset Y \text{ is a pointed, convex cone with } \operatorname{qi} K \neq \emptyset; \\ C \subset Z \text{ is a nonempty, pointed and convex cone;} \\ F : X \to \mathcal{P}(Y) \text{ and } G : X \to \mathcal{P}(Z) \text{ are proper set-valued functions;} \\ \{x \in (\operatorname{dom} F) \cap (\operatorname{dom} G) : G(x) \cap (-C) \neq \emptyset\} \neq \emptyset. \end{array}\right. \tag{4.7}$$

In the following we use the notation

$$\mathcal{A}_{CP^{sv}_{\text{qi}}} := \{x \in (\operatorname{dom} F) \cap (\operatorname{dom} G) : G(x) \cap (-C) \neq \emptyset\}.$$

Definition 4.3.8 *Let D and E be vector spaces, and let $M \subseteq D$. The* ***indicator set-valued function*** $\Delta^E_M : D \to \mathcal{P}(E)$ *associated with the set M with respect to the space E is defined by*

$$\Delta^E_M(x) := \begin{cases} \{0\} & \textit{if } x \in M \\ \emptyset & \textit{if } x \notin M. \end{cases}$$

We can now reexpress (CP^{sv}_{qi}) as an unconstrained set-valued optimization problem, with a modified objective function, as:

$$\underset{x\in X}{\text{l-Min}_{\mathrm{qi}}} \left[F(x) + \Delta^Y_{\mathcal{A}_{CP^{sv}_{\mathrm{qi}}}}(x) \right].$$

A **perturbation function** associated with $F + \Delta^Y_{\mathcal{A}_{CP^{sv}_{\mathrm{qi}}}}$ is actually a set-valued function $\Phi^{sv}_C : X \times W \to \mathcal{P}(Y)$ such that

$$\Phi^{sv}_C(x,0) = F(x) + \Delta^Y_{\mathcal{A}_{CP^{sv}_{\mathrm{qi}}}}(x) \text{ for all } x \in X.$$

A set-valued dual problem associated with (CP^{sv}_{qi}) with the help of Φ^{sv}_C is

$$(CD^{sv}_{\mathrm{qi}}) \qquad \underset{T\in\mathcal{L}(W,Y)}{\text{l-Max}_{\mathrm{qi}}} \left[-(\Phi^{sv}_C)^*_{\mathrm{qi}\,K}(0,T) \right].$$

In the following we use the notation

$$\mathcal{A}_{CD^{sv}_{\mathrm{qi}}} := \left\{ (T,x,w) \in \mathcal{L}(W,Y) \times \operatorname{dom} \Phi^{sv}_C : -Tw + \Phi^{sv}_C(x,w) \in -(\Phi^{sv}_C)^*_{\mathrm{qi}_K}(0,T) \right\}.$$

Definition 4.3.9 (Grad A. [66]) *Let the hypotheses* (4.7) *be satisfied. An operator* $\widetilde{T} \in \mathcal{L}(W,Y)$ *is said to be a* qi-**efficient solution** *to the dual problem* (CD^{sv}_{qi}) *if there exists an* $(\widetilde{x},\widetilde{w}) \in \operatorname{dom} \Phi^{sv}_C$ *such that*

$$(\widetilde{T},\widetilde{x},\widetilde{w}) \in \mathcal{A}_{CD^{sv}_{\mathrm{qi}}} \text{ and } -\widetilde{T}\widetilde{w} + \Phi^{sv}_C(\widetilde{x},\widetilde{w}) \in \text{l-Max}_{\mathrm{qi}}\{-Tw + \Phi^{sv}_C(x,w) : (T,x,w) \in \mathcal{A}_{CD^{sv}_{\mathrm{qi}}}\}.$$

We start with the **weak duality theorem**.

Theorem 4.3.10 (Grad A. [66]) *Let the hypotheses* (4.7) *be satisfied, let* $x_0 \in \mathcal{A}_{P^{sv}_C}$, *and let* $(T,x,w) \in \mathcal{A}_{CD^{sv}_{\mathrm{qi}}}$. *Then the following statements are true:*

(a) *If* $F(x_0) \trianglelefteq^l_{\mathrm{qi}_K} -Tw + \Phi^{sv}_C(x,w)$, *then* $-Tw + \Phi^{sv}_C(x,w) \trianglelefteq^l_{\mathrm{qi}_K} F(x_0)$.

(b) *If* $F(x_0) \trianglelefteq^l_{\mathrm{qi}_K} -Tw + \Phi^{sv}_C(x,w)$, *then* $-Tw + \Phi^{sv}_C(x,w) \sim^l F(x_0)$.

We continue with two theorems which provide optimality conditions.

Theorem 4.3.11 (Grad A. [66]) *Let the hypotheses* (4.7) *be satisfied, let* $\overline{x} \in \mathcal{A}_{CP^{sv}_{\mathrm{qi}}}$, *and let* $(\widetilde{T},\widetilde{x},\widetilde{w}) \in \mathcal{A}_{CD^{sv}_{\mathrm{qi}}}$ *be such that*

$$(4.8) \qquad F(\overline{x}) \trianglelefteq^l_{\mathrm{qi}\,K} -\widetilde{T}\widetilde{w} + \Phi^{sv}_C(\widetilde{x},\widetilde{w}).$$

Then the following statements are true:

(a) $\overline{x}$ *is a* qi-*efficient solution to* (CP^{sv}_{qi}).

(b) $\widetilde{T}$ *is a* qi-*efficient solution to* (CD^{sv}_{qi}).

Theorem 4.3.12 (Grad A. [66]) *Let the hypotheses* (4.7) *be satisfied, and let* $\overline{x} \in \mathcal{A}_{CP^{sv}_{\text{qi}}}$*. If there exists an operator* $\overline{T} \in \mathcal{L}(W,Y)$ *such that* $(\overline{T}, \overline{x}, 0) \in \mathcal{A}_{CD^{sv}_{\text{qi}}}$*, then* $\overline{T}$ *is a* qi*-efficient solution to* (CD^{sv}_{qi})*.*

Remark 4.3.13 Let the hypotheses of Theorem 4.3.12 be satisfied. Then each operator $\widetilde{T} \in \mathcal{L}(W,Y)$ for which there exists an $(\widetilde{x}, \widetilde{w}) \in \operatorname{dom} \Phi^{sv}_C$ such that the following conditions are valid:

$$(\widetilde{T}, \widetilde{x}, \widetilde{w}) \in \mathcal{A}_{LCD^{sv}_{\text{qi}}} \text{ and } F(\overline{x}) \trianglelefteq^{l}_{\text{qi}\,K} -\widetilde{T}\widetilde{w} + \Phi^{sv}_C(\widetilde{x}, \widetilde{w}),$$

is a qi-efficient solution to (CD^{sv}_{qi}). ∎

4.3.3 A Set-Valued Lagrange Multipliers Rule

In this subsection we associate with a general set-valued optimization problem with cone constraints of the form (CP^{sv}_{qi}) a dual problem, obtained by particularizing the perturbation function in a manner similar to the classical Lagrange approach from the scalar case.

Let the hypotheses (4.7) be satisfied. We consider the Lagrange-type perturbation function $\Phi^{sv}_L : X \times Z \to \mathcal{P}(Y)$ associated with problem (CP^{sv}_{qi}), defined by

$$\Phi^{sv}_L(x,z) := \begin{cases} F(x) & \text{if } x \in X \text{ and } (G(x) - z) \cap (-C) \neq \emptyset \\ \emptyset & \text{otherwise.} \end{cases}$$

Given an operator $T \in \mathcal{L}(Z,Y)$, the set-valued qi-conjugate function associated with the Lagrange-type perturbation function Φ^{sv}_L at $(0,T)$

$$(\Phi^{sv}_L)^*_{\text{qi}\,K}(0,T) := \text{u-Max}_{\text{qi}}\{Tz - \Phi^{sv}_L(x,z) : (x,z) \in X \times Z\}.$$

We attach to (CP^{sv}_{qi}) the following set-valued Lagrange-type dual problem:

$$(LCD^{sv}_{\text{qi}}) \qquad \underset{T \in \mathcal{L}(Z,Y)}{\text{l-Max}_{\text{qi}}} \left[- (\Phi^{sv}_L)^*_{\text{qi}\,K}(0,T) \right].$$

In the following we use the notation

$$\mathcal{A}_{LCD^{sv}_{\text{qi}}} := \left\{ \begin{array}{ll} (T,x,z): & T \in \mathcal{L}(Z,Y), x \in X, z \in G(x) + C, \\ & -Tz + F(x) \in -(\Phi^{sv}_L)^*_{\text{qi}\,K}(0,T) \end{array} \right\}.$$

Definition 4.3.14 *Let the hypotheses* (4.7) *be satisfied. An operator* $\widetilde{T} \in \mathcal{L}(Z,Y)$ *is said to be a* qi***-efficient solution*** *to the dual problem* (LCD^{sv}_{qi}) *if there exists an* $(\widetilde{x}, \widetilde{z}) \in \operatorname{dom} \Phi^{sv}_L$ *such that*

$$(\widetilde{T}, \widetilde{x}, \widetilde{z}) \in \mathcal{A}_{LCD^{sv}_{\text{qi}}} \text{ and } -\widetilde{T}\widetilde{z} + F(\widetilde{x}) \in \text{l-Max}_{\text{qi}} \left\{ -Tz + F(x) : (T,x,z) \in \mathcal{A}_{LCD^{sv}_{\text{qi}}} \right\}.$$

We next present a **strong duality theorem**.

Theorem 4.3.15 (Grad A. [66]) *Let the hypotheses* (4.7) *be satisfied, let* (F,G) *be a* $K \times C$*-convex function, and let* $\overline{x} \in \mathcal{A}_{CP^{sv}_{\text{qi}}}$ *be such that there exists* $\overline{y} \in F(\overline{x})$ *with the property*

$$(4.9) \qquad (\overline{y}, 0) \notin \operatorname{qri}\left[(F,G)(X) + K \times C\right].$$

Moreover, assume that

$$0 \in \operatorname{qi}[G(X) + C]. \tag{4.10}$$

Then there exists an operator $\overline{T} \in \mathcal{L}(Z, Y)$ *such that* $\overline{T}$ *is a* qi-*efficient solution to the dual problem* (LCD_{qi}^{sv}).

Remark 4.3.16 Let the hypotheses of Theorem 4.3.15 be satisfied. Then each operator $\widetilde{T} \in \mathcal{L}(Z, Y)$ for which there exists an $(\widetilde{x}, \widetilde{z}) \in \operatorname{dom} \Phi_L^{sv}$ such that the following conditions are valid:

$$(\widetilde{T}, \widetilde{x}, \widetilde{z}) \in \mathcal{A}_{LCD_{\mathrm{qi}}^{sv}} \text{ and } -\overline{T}0 + \Phi_L^{sv}(\overline{x}, 0) \trianglelefteq^l_{\mathrm{qi}\,K} -\widetilde{T}\widetilde{z} + \Phi_L^{sv}(\widetilde{x}, \widetilde{z}),$$

is a qi-efficient solution to (LCD_{qi}^{sv}). ∎

4.4 An Application to a Set-Valued Optimization Problem in $\ell^2(\mathbb{R})$

In this section we present an example of a set-valued optimization problem for which we can apply the strong-duality statements of Theorem 4.3.15. Our application was inspired by Example 2.10 in Csetnek E. R. [43], and is formulated under the following particular instance of the framework (4.7):

$$\left\{ \begin{array}{l} X := \ell^2(\mathbb{R}), Y := \mathbb{R}, Z := \ell^2(\mathbb{R}), K := \mathbb{R}_+, C := \ell^2_+(\mathbb{R}); \\ F : \ell^2(\mathbb{R}) \to \mathcal{P}(\mathbb{R}) \text{ is defined by } F(\mu) := \left\{ \begin{array}{ll} \{\|\mu\|_{\ell^2(\mathbb{R})}\} & \text{if } \mu \in \ell^2_+(\mathbb{R}) \\ \emptyset & \text{otherwise} \end{array} \right. ; \\ G : \ell^2(\mathbb{R}) \to \mathcal{P}(\ell^2(\mathbb{R})) \text{ is defined by } G(\mu) := \left\{ \begin{array}{ll} \{-\mu\} & \text{if } \mu \in \ell^2_+(\mathbb{R}) \\ \emptyset & \text{otherwise} \end{array} \right. . \end{array} \right. \tag{4.11}$$

We notice that $(\operatorname{dom} F) \cap (\operatorname{dom} G) = \ell^2_+(\mathbb{R})$, and $\operatorname{qi} K = \operatorname{qi} \mathbb{R}_+$. As $\mathbb{R}$ is a finite dimensional space, we have $\operatorname{qi} \mathbb{R}_+ = \operatorname{int} \mathbb{R}_+ = (0, +\infty)$.

Let us recall some important properties concerning the set

$$\ell^2(\mathbb{R}) := \{\mu : \mathbb{R} \to \mathbb{R} : \sum_{x \in \mathbb{R}} |\mu(x)|^2 < +\infty\}.$$

The function $\|\cdot\|_{\ell^2(\mathbb{R})} : \ell^2(\mathbb{R}) \to \mathbb{R}$ defined by

$$\|\mu\|_{\ell^2(\mathbb{R})} := \left(\sum_{x \in \mathbb{R}} |\mu(x)|^2 \right)^{\frac{1}{2}} = \left(\sup_{F \in \mathcal{P}_0(\mathbb{R}), F \text{ finite}} \sum_{x \in F} |\mu(x)|^2 \right)^{\frac{1}{2}} \text{ for all } \mu \in \ell^2(\mathbb{R})$$

is a norm on $\ell^2(\mathbb{R})$, and the vector space $\ell^2(\mathbb{R})$, equipped with this norm is a Banach space. The dual space $(\ell^2(\mathbb{R}))^*$ is identified with $\ell^2(\mathbb{R})$. Moreover, the set

$$\ell^2_+(\mathbb{R}) := \{\mu \in \ell^2(\mathbb{R}) : \mu(x) \geq 0 \text{ for all } x \in \mathbb{R}\}$$

is a pointed convex cone, and from Borwein J. M., Lucet Y. and Mordukhovich B. [11, Remark 2.20] we know that $\operatorname{qri}(\ell^2_+(\mathbb{R})) = \emptyset$. Furthermore, it holds

$$\ell^2_+(\mathbb{R}) - \ell^2_+(\mathbb{R}) = \ell^2(\mathbb{R}). \tag{4.12}$$

It is easy to see that $\overline{\mu} := 0 \in \ell^2(\mathbb{R})$ is a qi-efficient solution to the set-valued optimization problem

$$(P^{sv}_{\ell^2(\mathbb{R})}) \qquad \underset{\substack{\mu\in\ell^2_+(\mathbb{R})\\ G(\mu)\cap(-\ell^2_+(\mathbb{R}))\neq\emptyset}}{\text{l-Min}_{\text{qi}}} \left[\|\mu\|_{\ell^2(\mathbb{R})}\right].$$

We associate with $(P^{sv}_{\ell^2(\mathbb{R})})$ a Lagrange-type set-valued dual problem, with the help of the perturbation function $\Phi_{\ell^2(\mathbb{R})} : \ell^2(\mathbb{R}) \times \ell^2(\mathbb{R}) \to \mathbb{R}$ defined by

$$\Phi_{\ell^2(\mathbb{R})}(\mu,\zeta) := \left\{\|\mu\|_{\ell^2(\mathbb{R})} : \mu \in \ell^2_+(\mathbb{R}), \zeta \in -\mu + \ell^2_+(\mathbb{R})\right\}, \text{ for all } (\mu,\zeta) \in \ell^2(\mathbb{R}) \times \ell^2(\mathbb{R}).$$

The qi-conjugate set-valued function associated with $\Phi_{\ell^2(\mathbb{R})}$ is

$$(\Phi_{\ell^2(\mathbb{R})})^*_{\text{qi}\,\mathbb{R}_+} : \mathcal{L}(\ell^2(\mathbb{R}),\mathbb{R}) \times \mathcal{L}(\ell^2(\mathbb{R}),\mathbb{R}) \to \mathcal{P}(\mathcal{P}(\mathbb{R})),$$

defined by

$$\begin{aligned}(\Phi_{\ell^2(\mathbb{R})})^*_{\text{qi}\,\mathbb{R}_+}(0,T) :&= \text{u-Max}_{\text{qi}}\{T\zeta - \Phi(\mu,\zeta) : (\mu,\zeta) \in \ell^2(\mathbb{R}) \times \ell^2(\mathbb{R})\}\\ &= \text{u-Max}_{\text{qi}}\{T\zeta - \|\mu\|_{\ell^2(\mathbb{R})} : \mu \in \ell^2_+(\mathbb{R}), \zeta \in -\mu + \ell^2_+(\mathbb{R})\}\end{aligned}$$

for all $T \in \mathcal{L}(\ell^2(\mathbb{R}),\mathbb{R})$.

A Lagrange-type set-valued dual problem associated with $(P^{sv}_{\ell^2(\mathbb{R})})$ is

$$(LD^{sv}_{\ell^2(\mathbb{R})}) \qquad \underset{T\in\mathcal{L}(\ell^2(\mathbb{R}),\mathbb{R})}{\text{l-Max}_{\text{qi}}} \left[-(\Phi_{\ell^2(\mathbb{R})})^*_{\text{qi}\,\mathbb{R}_+}(0,T)\right].$$

We are able to prove that the hypotheses of Theorem 4.3.15 are satisfied. This means that there exists an operator $\overline{T} \in \mathcal{L}(\ell^2(\mathbb{R}),\mathbb{R})$ such that $\overline{T}$ is a qi-efficient solution to $(LD^{sv}_{\ell^2(\mathbb{R})})$.

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