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CONTRIBUTIONS TO THE APPROXIMATION OF FUNCTIONS

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Introduction

The Theory of approximation is an area of mathematical analysis, which, at its core, is concerned with the approximation of functions by simpler and more easily calculated functions. As A. F. Timan remarked in [136, p. 1], the basis of the theory of approximation of functions of a real variable is the theorem discovered by K. Weierstrass [154] in 1885, which asserts that *for any continuous function f on the finite interval $[a, b]$, there exists a sequence of polynomials which converges uniformly to f on* $[a, b]$. In 1912, S. N. Bernstein [21] gives a simple and elegant proof of Weierstrass theorem, constructing, by probabilistic methods, a sequence of polynomials that converges uniformly to the function to be approximated. Thus were introduced the Bernstein operators (the applications that associate the function to be approximated with the polynomials that approximate the function). These operators belong to the class of positive linear operators.

In the '50s, the theory of approximation of functions by positive linear operators developed a lot, when T. Popoviciu [110], H. Bohman [23] and P. P. Korovkin [90, 91], discovered, independently, a simple and easily applicable criterion to check if a sequence of positive linear operators converges uniformly to the function to be approximated. This criterion says that *the necessary and sufficient condition for the uniform convergence of the sequence A_n of positive linear operators to the continuous function f on the compact interval $[a, b]$, is the uniform convergence of the sequence $A_n f$ to f for the only three functions $e_k(x) = x^k$, $k = 0, 1, 2$.* If the domain of definition of f is unbounded (for example $[0, \infty)$), then the result remains valid only for the continuous functions having a finite limit at infinity. In this case, the test functions, x^k, $k = 0, 1, 2$ are replaced by other three functions (e^{-kx}, $k = 0, 1, 2$ are an example).

To extend the theorem of Popoviciu-Bohman-Korovkin to continuous and unbounded functions defined on $[0, \infty)$, some bounds on the functions must be required, a fapt which was first noted by Z. Ditzian in [45]. In 1974, A. D. Gadjiev [60, 61] introduced the weighted space $C_\rho(I)$, which is the set of all continuous functions f on the interval $I \subseteq \mathbb{R}$ for which there exists a constant $M > 0$ such that $|f(x)| \leq M \cdot \rho(x)$, for every $x \in I$, where ρ is a positive continuous function called weight. This space is a Banach space, endowed with the norm

$$\|f\|_\rho = \sup_{x \in I} \frac{|f(x)|}{\rho(x)}.$$

The Korovkin type theorem found by Gadjiev is the following: *let $\varphi\colon [0, \infty) \to [0, \infty)$ be a strictly increasing, continuous and unbounded function and set $\rho(x) = 1 + \varphi^2(x)$; the sequence of positive linear operators $A_n : C_\rho[0, \infty) \to C_\rho[0, \infty)$ verifies*

$$\lim_{n \to \infty} \left\| A_n \varphi^i - \varphi^i \right\|_\rho = 0, \quad i = 0, 1, 2.$$

Then

$$\lim_{n \to \infty} \|A_n f - f\|_\rho = 0,$$

for every function $f \in C_\rho[0, \infty)$, for which the limit $\lim_{x \to \infty} \frac{f(x)}{\rho(x)}$ exists and is finite.

We have two basic problems in approximation theory. The first is qualitative – under what conditions will a sequence of operators approximate the identity operator. The second problem is quantitative, namely, how quickly do the operators approximate the identity operator. *The aim* *of the present thesis* is a study of quantitative results related to the above mentioned qualitative results. For the functions defined on a compact interval, the evaluation of the remainder $A_n f - f$ is done using moduli of smoothness. The simplest and best-known is the modulus of continuity, defined by the relation

$$\omega(f,\delta) = \sup\{\, |f(t) - f(x)| \,:\, t, x \in [a,b]\,,\ |t-x| \le \delta\,\}\,, \quad \delta \ge 0.$$

A first result for the estimation of the remainder using the modulus of continuity is due to Shisha and Mond [128] from 1968. The estimation is of the following form:

$$|A_n(f,x) - f(x)| \le |f(x)| \cdot |A_n(1,x) - 1| + (1 + A_n(1,x)) \cdot \omega\left(f, \sqrt{A_n((t-x)^2, x)}\right).$$

In order to obtain similar results for bounded functions defined on a noncompact interval or for unbounded functions, we use the following weighted modulus of continuity

$$\omega_\varphi(f,\delta) = \sup_{\substack{t,x \in I \\ |\varphi(t)-\varphi(x)| \le \delta}} |f(t) - f(x)|.$$

Using this modulus we obtain estimations of the remainder and characterizations of functions, which can be uniformly approximated using a given sequence of positive linear operators. The thesis is structured in four chapters.

The first chapter contains preliminary notions and results related to positive linear operators. Thus, we introduce the notion of positive linear operator and we give some properties of these operators and some examples. Then we define the moduli of smoothness of order one and two, the usual ones and those of Ditzian and Totik and we present known estimations of the remainder using these moduli.

In the first section of *the second chapter*, it is presented the weighted modulus of continuity with its properties. This modulus is close connected with the usual modulus of continuity and will play an important role in all the estimations made throughout the second and third chapters of the thesis. In the second section, we obtain quantitative results for the approximation of continuous and bounded functions on the positive semiaxis which have a finite limit at infinity. The general results from Theorem 2.10 and Theorem 2.12 and the corollaries that follow are personal contributions of the author. In the next section, we introduce a new technique to characterize the functions which can be uniformly approximated using a given sequence of positive linear operators. Old and new results are thus obtained.

Chapter three is the largest and contains results for the approximation of continuous and unbounded functions. In the first two sections, are given some known results related to the approximation on compact subsets and local approximation of unbounded functions using positive linear operators. In the third section, we give global results (quantitative results for the whole domain of definition of the functions) for the approximation of functions belonging to weighted spaces. The majority of the results from the literature related to weighted spaces are for the particular cases of polynomial and exponential weighted spaces. A first contribution in this chapter is the quantitative version of the theorem of Gadjiev. Another contribution, maybe the most important, which solves some open problems, is the extension of the technique introduced in the second chapter for bounded functions to this more general setting of weighted spaces. These results, which are presented in subsection 3.3.3, are submitted and will be published soon. In the last part of this chapter, we present different moduli of continuity used for weighted spaces. A particular interest is shown to a modulus introduced by the author and properties and estimations of the remainder using this modulus are given.

The fourth chapter contains two contributions, which use different inequalities for linear operators. The first part concerns the approximation of functions, using rational expressions with prescribed numerator, which are constructed using positive linear operators. The second part presents necessary and sufficient conditions in order that an inequality for a linear discrete functional holds for some classes of convex functions.

The bibliography is extensive, includind 157 references. The index is very useful to find some results and the list from the end with the notations used along this thesis gives the page where the definition of each symbol is located.

This thesis contains original results of the author, which have been obtained in a period of time of four years. They can be found in the six papers mentioned in the bibliography, five already published and one submitted, but there are some results and remarks which appear for the first time in this work.

Chapter 1

Preliminary concepts and results

1.1 Positive linear operators

1.1.1 Definition and properties of positive linear operators

Let M be a nonempty set and let

$$\mathcal{F}(M,\mathbb{R}) = \{\, f\colon M \to \mathbb{R}\,\},$$

be the linear space over $\mathbb{R}$ of real functions defined on M, endowed with the usual operations of addition and scalar multiplication.

In the following, we denote by X a linear subspace of $\mathcal{F}(M,\mathbb{R})$ and by Y a linear subspace of $\mathcal{F}(N,\mathbb{R})$, where M and N are nonempty sets.

Definition 1.1. The application $A\colon X \to Y$ is called an operator. The operator A is linear, if

$$A(\alpha f + \beta g) = \alpha A f + \beta A g, \quad \text{for every } f, g \in X,\ \alpha, \beta \in \mathbb{R},$$

and is positive, if

$$Af \geq 0, \quad \text{for every } f \in X \text{ with the property } f \geq 0.$$

Proposition 1.2.

- (i) A positive linear operator is monotone.
- (ii) If A is a positive linear operator, then for every $f \in X$ we have $|Af| \leq A(|f|)$.

Proposition 1.3 (Hölder inequality for positive linear operators)**.** Let $A\colon X \to Y$ be a positive linear operator and let $p, q > 1$ be real numbers such that $1/p + 1/q = 1$. Then

$$A(|f \cdot g|) \leq (A(|f|^p))^{\frac{1}{p}} \cdot (A(|g|^q))^{\frac{1}{q}}, \quad \text{for every } f, g \in X.$$

Remark 1.4. The idea of proof is from [70] and even older (see the references of the article cited). This result extends the result presented in [69].

Remark 1.5. An important particular case is the Cauchy-Schwarz inequality for positive linear operators, which is obtained from Hölder inequality for $p = q = 2$, by using Proposition 1.2 (ii):

$$|A(f \cdot g, x)| \leq \sqrt{A(f^2, x)} \cdot \sqrt{A(g^2, x)}.$$

Proposition 1.6. Let $A\colon B(I) \to B(I)$ be a positive linear operator. Then A is bounded and has the norm $\|A\| = \|Ae_0\|$.

1.1.2 Examples of positive linear operators

Example 1.7. For a positive integer $n \geq 1$ and for a function f defined on $[0,1]$, the Bernstein operators $B_n\colon C[0,1] \to C[0,1]$ are defined by

$$B_n(f,x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right), \quad x \in [0,1].$$

They were introduced by S.N. Bernstein [21] in 1912.

Example 1.8. The Stancu operators $P_n^{\langle\alpha\rangle}\colon C[0,1] \to C[0,1]$ are defined by

$$P_n^{\langle\alpha\rangle}(f,x) = \sum_{k=0}^{n} \binom{n}{k} \frac{\prod_{i=0}^{k-1}(x+i\alpha)\prod_{j=0}^{n-k-1}(1-x+j\alpha)}{(1+\alpha)(1+2\alpha)\dots(1+(n-1)\alpha)} \cdot f\left(\frac{k}{n}\right), \qquad n \geq 1,$$

where α is a parameter which may depend only on n. They were introduced by D.D. Stancu [131] in 1968.

Example 1.9. Let $0 \leq \alpha \leq \beta$ be real numbers. For $n \geq 1$, the relation

$$P_n^{(\alpha,\beta)}(f,x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \cdot f\left(\frac{k+\alpha}{n+\beta}\right),$$

defines the Bernstein-Stancu operators $P_n^{(\alpha,\beta)}\colon C[0,1] \to C[0,1]$, introduced by D.D. Stancu [132] in 1969.

Example 1.10. For $n \geq 1$, let x_n be the greatest root of Jacobi's polynomial $J_n^{(1,0)}$ of degree n related to the interval $[0,1]$ and

$$P_{2n-1}(x) = \lambda_n \int_0^x \left(\frac{J_n^{(1,0)}(t)}{t-x_n}\right)^2 dt, \quad \text{where } \lambda_n = \frac{1}{\int_0^1 (1-x)\left(\frac{J_n^{(1,0)}(x)}{x-x_n}\right)^2 dx}.$$

Using the representation $P_{2n-1}(x) = \sum_{k=0}^{2n-1} a_k x^k$, I. Gavrea [65] introduced in 1996 the operators $H_{2n+1}\colon C[0,1] \to \Pi_{2n+1}$ defined by

$$H_{2n+1}(f,x) = \sum_{k=0}^{2n-1} \frac{a_k}{k+1} L_{k+2}(f,x),$$

where $L_n\colon C[0,1] \to \Pi_n$ are given by

$$L_n(f,x) = f(0)(1-x)^n + f(1)x^n + (n-1)\sum_{k=1}^{n-1} p_{n,k}(x) \int_0^1 p_{n-2,k-1}(t) \cdot f(t)\, dt,$$

where $p_{n,k}$ are defined by

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}.$$

The operators H_{2n+1} are linear and positive, preserving the affine functions and having the property that

$$\begin{aligned} H_{2n+1}(e_2,x) - x^2 &= x(1-x)\left(1 - \int_0^1 x^2 P_{2n-1}(x)\, dx\right) \\ &\leq x(1-x)(1-x_n) \leq \frac{Cx(1-x)}{n^2}. \end{aligned}$$

Example 1.11. The Hermite-Fejér interpolatory polynomial $\mathcal{H}_n(f,x)$ of degree $2n-1$ is defined by

$$\mathcal{H}_n(f,x) = \sum_{k=1}^{n} f(x_{k,n})(1 - x x_{k,n}) \left(\frac{T_n(x)}{n(x - x_{k,n})} \right)^2,$$

where $x_{k,n} = \cos \frac{(2k-1)\pi}{2n}$, $k = 1, 2, \ldots, n$, are the roots of Chebyshev polynomial of the first kind $T_n(x) = \cos(n \arccos x)$ and $f \in C[-1,1]$. These operators were introduced and studied by Fejér [56] in 1916. They are called Hermite, also, because they verify the interpolatory conditions of Hermite [72]

$$\mathcal{H}_n(f, x_{k,n}) = f(x_{k,n}) \text{ and } \mathcal{H}'_n(f, x_{k,n}) = 0, \quad \text{for every } k = 1, 2, \ldots, n.$$

Example 1.12. The operators $\mathcal{L}_n \colon C[0,1] \to C[0,1]$ defined by

$$\mathcal{L}_n(f,x) = \frac{\int_0^1 (1 - (u-x)^2)^n f(u)\, du}{2 \int_0^1 (1 - u^2)^n\, du}$$

are called Landau operators and were introduced by E. Landau [92] in 1908 to give another proof to the theorem of Weierstrass (see Theorem 1.30).

For a noncompact interval $I \subseteq \mathbb{R}$, let $\mathcal{D} \subset C(I)$ be a linear subspace of continuous real functions defined on I. In the following, we give some examples of positive linear operators defined on a such subspace, which will be mentioned for every particular case.

Example 1.13. For $I = [0, \infty)$, the operators $S_n \colon \mathcal{D} \to C(I)$ defined by

$$S_n(f,x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right), \quad x \in [0,\infty),\ n \geq 1,$$

are called Szász-Mirakjan operators. They were introduced by G. Mirakjan [105] in 1941 (some authors spell this name: Mirakyan) and were studied by J. Favard [55] in 1944 and by O. Szász [134] in 1950. The domain of definition of S_n is the set of all functions $f(x) = \mathcal{O}(e^{\alpha x \ln x})$, $\alpha > 0$, this fact being proved by T. Hermann [71]. Concerning the uniform approximation of these operators see Corollary 2.19 and Corollary 3.14.

Example 1.14. For $I = [0, \infty)$, the operators $V_n \colon \mathcal{D} \to C(I)$ defined by

$$V_n f(x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} f\left(\frac{k}{n}\right), \quad x \geq 0,\ n \geq 1,$$

are called Baskakov operators and were introduced by V.A. Baskakov [15] in 1957. The domain of definition $\mathcal{D}$, is the set of all functions f which have the growth $f(x) = \mathcal{O}(e^{\alpha x})$, $\alpha > 0$, this fact being proved by T. Hermann [71]. Concerning the uniform approximation of these operators see Corollary 2.21 and Corollary 3.17.

Example 1.15. For $I = [0, 1)$, the operators $M_n \colon \mathcal{D} \to C(I)$ defined by

$$M_n(f,x) = \sum_{k=0}^{\infty} \binom{n+k}{k} x^k (1-x)^{n+1} f\left(\frac{k}{n+k}\right), \quad 0 \leq x < 1,$$

are called Meyer-König and Zeller operators and were introduced in the present form by E.W. Cheney and A. Sharma [30] in 1964. Initially, they were introduced by W. Meyer-König and K. Zeller [104] in 1960, having $f(k/n+1+k)$ instead of $f(k/n+k)$. The set $\mathcal{D}$, for which it was proved in [104] the convergence of the series defining the operators, is the set of all functions f having the growth $f(x) = \mathcal{O}((1-x)^{-\alpha})$, $\alpha > 0$.

Example 1.16. For $I = [0, \infty)$ and for $n \geq 1$, the operators $L_n : \mathcal{D} \to C(I)$ defined by

$$L_n(f, x) = \frac{1}{(1+x)^n} \sum_{k=0}^{n} \binom{n}{k} x^k f\left(\frac{k}{n-k+1}\right)$$

are called Bleimann-Butzer-Hahn operators. They were introduced by G. Bleimann, P.L. Butzer and L. Hahn [22] in 1980 and were studied for the set of all functions which are bounded and uniformly continuous on $[0, \infty)$.

Example 1.17. For $I = [0, \infty)$, the operators $C_n : \mathcal{D} \to C(I)$ defined by

$$C_n f(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\beta_n\right) \binom{n}{k} \left(\frac{x}{\beta_n}\right)^k \left(1 - \frac{x}{\beta_n}\right)^{n-k},$$

for $0 \leq x \leq \beta_n$ and $C_n f(x) = f(x)$, for $x > \beta_n$, where $(\beta_n)_{n \in \mathbb{N}}$ is a sequence of positive real numbers having the properties

$$\lim_{n \to \infty} \beta_n = \infty \quad \text{and} \quad \lim_{n \to \infty} \frac{\beta_n}{n} = 0,$$

are called Bernstein-Chlodovsky operators and were introduced by I. Chlodovsky [39] in 1937. In the same paper, the author proves that $C_n(f, x)$ converges punctually to $f(x)$, if

$$\max_{x \in [0, \beta_n]} |f(x)| \cdot e^{-\alpha^2 \frac{n}{\beta_n}} \to 0, \quad n \to \infty,$$

for every $\alpha \neq 0$.

Example 1.18. For $I = \mathbb{R}$, the operators $W_n : \mathcal{D} \to C(I)$ defined by

$$W_n(f, x) = \frac{\sqrt{n}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-n\frac{(u-x)^2}{2}} f(u)\, du, \quad x \in (-\infty, \infty),$$

are called Gauss-Weierstrass operators. In 1885, K. Weierstrass [154] proves that $(W_n(f, x))_n$ converges punctually to $f(x)$, if f is continuous and bounded on $\mathbb{R}$. Using this result, he proves Theorem 1.30. In 1944, J. Favard [55] proves the convergence for the set of functions $f(x) = \mathcal{O}(e^{\alpha x^2})$, $\alpha > 0$.

Example 1.19. For $I = (0, \infty)$, the operators $P_n : \mathcal{D} \to C(I)$ defined by

$$P_n(f, x) = \frac{1}{(n-1)!} \left(\frac{n}{x}\right)^n \int_0^{\infty} e^{-\frac{nu}{x}} u^{n-1} f(u)\, du, \quad x > 0,$$

are called Post-Widder operators. They were introduced by E.L. Post [111] in 1930 and studied by D.V. Widder [152] in 1934. They are the real inversion formula for the Laplace transform (see [153], for details). In [153, p. 283-287] are given the conditions for the simple and uniform convergence of $P_n f$ toward f. R.A. Khan [88] and M.K. Khan, B. Della Vecchia and A. Fassih [89] called these operators the Gamma operators.

Example 1.20. For $I = (0, \infty)$, the operators $G_n : \mathcal{D} \to C(I)$ defined by

$$G_n(f, x) = \frac{x^{n+1}}{n!} \int_0^{\infty} e^{-xu} u^n f\left(\frac{n}{u}\right) du, \quad x > 0,\ n \geq 1,$$

are called Gamma operators. They were studied by M. Müller and A. Lupa's [99] in 1967.

Example 1.21. For $I = \mathbb{R}$, the operators $\mathcal{P}_n : \mathcal{D} \to C(I)$ defined by

$$\mathcal{P}_n(f, x) = \frac{n}{2} \int_{-\infty}^{\infty} e^{-n|u-x|} f(u)\, du$$

are called Picard operators.

1.2 The approximation of functions using positive linear operators

1.2.1 The modulus of continuity

Definition 1.22. Let $f \in C(I)$ be a continuous function defined on an interval $I \subseteq \mathbb{R}$. The function $\omega\colon C(I) \times [0,\infty) \to \mathbb{R} \cup \{\infty\}$, defined by:

$$\omega(f,\delta) = \sup\{\, |f(t) - f(x)| \;:\; t, x \in I\,,\ |t-x| \le \delta \,\}$$

is called *modulus of continuity* of the function f.

Proposition 1.23. The modulus of continuity has the following properties:

1. For $f \in B(I)$, the function $\omega(f,\cdot)$ is nonnegative, increasing, subadditive and bounded, and for $\delta \ge 0$, the function $\omega(\cdot,\delta)$ is a seminorm on $B(I)$ (subadditive and positive homogeneous).

2. The function f is uniformly continuous on I if and only if

$$\lim_{\delta \searrow 0} \omega(f,\delta) = 0.$$

3. For every $\delta, \lambda \ge 0$, it is true the inequality:

$$\omega(f,\lambda\delta) \le (1+\lambda) \cdot \omega(f,\delta)\,.$$

4. For every $\delta > 0$ we have

$$|f(y) - f(x)| \le \left(1 + \frac{|y-x|}{\delta}\right) \cdot \omega(f,\delta)\,.$$

5. For every $\delta > 0$ we have

$$|f(y) - f(x)| \le \left(1 + \frac{(y-x)^2}{\delta^2}\right) \cdot \omega(f,\delta)\,.$$

Proposition 1.24. For a compact interval I, the modulus of continuity is equivalent with the K-functional

$$K_1(f,t) = \inf_{g' \in C(I)} (\|f-g\| + t\,\|g'\|), \quad t > 0,$$

i.e., there are the constants $C_1, C_2 > 0$ and $\delta_0 > 0$ such that

$$C_1 \cdot \omega(f,\delta) \le K_1(f,\delta) \le C_2 \cdot \omega(f,\delta)\,, \quad \text{for } \delta < \delta_0.$$

Theorem 1.25. *Let $A\colon C(I) \to B(I)$ be a positive linear operator. Then*
(i) if $f \in C(I) \cap B(I)$, then we have

$$\begin{aligned}|A(f,x) - f(x)| &\le |f(x)| \cdot |A(e_0,x) - 1| \\ &\quad + \left(A(e_0,x) + \frac{A((t-x)^2,x)}{\delta^2}\right) \omega(f,\delta)\,.\end{aligned}$$

(ii) if f is differentiable on I and $f' \in C(I) \cap B(I)$, then

$$\begin{aligned}|A(f,x) - f(x)| &\le |f(x)| \cdot |A(e_0,x) - 1| + |f'(x)| \cdot |A(e_1,x) - xA(e_0,x))| \\ &\quad + \left(\sqrt{A(e_0,x)A((t-x)^2,x)} + \frac{A((t-x)^2,x)}{\delta}\right) \omega(f',\delta)\,.\end{aligned}$$

Remark 1.26. The estimations from Theorem 1.25 are based on the ideas of O. Shisha and B. Mond [128] from 1968.

Theorem 1.27 (Popoviciu-Bohman-Korovkin). *Let $A_n\colon C[a,b] \to C[a,b]$ be a sequence of positive linear operators. If*

$$\lim_{n\to\infty} A_n(e_k, x) = e_k(x), \quad k = 0, 1, 2,$$

uniformly on $[a,b]$, then

$$\lim_{n\to\infty} A_n(f, x) = f(x),$$

uniformly on $[a,b]$, for every continuous function f defined on $[a,b]$.

Remark 1.28. The theorem 1.27 was discovered by H. Bohman [23] in 1952 and by P.P. Korovkin [90, 91] in 1953. T. Popoviciu [110] obtained in 1950 this result for the polynomial operators. For other details related to the history of this theorem and its generalizations, you can consult the survey article [51], for its 102 referencies and the monografy [7], for its teoretical notions and examples. This result shows that a necessary and sufficient condition for the uniform convergence of a sequence of positive linear operators A_n toward a continuous function f on a compact interval $[a,b]$, is the convergence of $A_n f$ toward f for only three functions, $e_k(x) = x^k$, $k = 0, 1, 2$.

Remark 1.29. Let $I \subseteq \mathbb{R}$ be a noncompact interval, and let $\mathcal{D} \subset C(I)$ be a subspace on which is defined a sequence of positive linear operators $A_n\colon \mathcal{D} \to C(I)$. If we have the relations $T(A_n(e_k)) \to T(e_k)$, $k = 0, 1, 2$ uniformly on $[a,b] \subset I$, where $T\colon C(I) \to C[a,b]$ is defined by $T(f) = f|_{[a,b]}$, then, using Theorem 1.27 we have

$$\lim_{n\to\infty} A_n f = f, \quad \text{uniformly on } [a,b],$$

for every continuous function $f \in \mathcal{D}$.

Theorem 1.30 (Weierstrass [154]). *Let $f \in C[a,b]$ be a continuous function. Then, for every $\varepsilon > 0$, there is a polynomial $P(x)$ with real coefficients, such that*

$$|f(x) - P(x)| < \varepsilon, \quad \text{for every } x \in [a,b].$$

Theorem 1.31 (Jackson [85]). *Let $f \in C[a,b]$ be a continuous function. Then*

$$\inf_{p\in\Pi_n} \|f - p\| \le C \cdot \omega\left(f, \frac{b-a}{n}\right),$$

where C is a constant not depending on n and f, and Π_n is the set of polynomials with real coefficients having the degree less or equal with n.

Theorem 1.32. *Let $A_n\colon C[a,b] \to \Pi_n$ be a sequence of polynomial positive linear operators. Then, at least one of the functions $e_k = x^k$, $k = 0, 1, 2$ cannot be approximated by $A_n e_k$ with an order better than n^{-2}.*

1.2.2 The modulus of smoothness of order two

1.2.3 The moduli of smoothness of Ditzian and Totik

Chapter 2

Uniform approximation of continuous and bounded functions

2.1 The weighted modulus of continuity

Definition 2.1. Let $\varphi\colon I \to J$ be a strictly increasing and continuous one-to-one map and $f \in B(I)$. Then, for $\delta \geq 0$ we define by

$$\omega_\varphi(f,\delta) = \sup_{\substack{t,x\in I\\ |\varphi(t)-\varphi(x)|\leq\delta}} |f(t)-f(x)|. \tag{2.1}$$

the *weighted modulus of continuity* of f.

Remark 2.2. For $\varphi(x) = x$ we obtain the usual modulus of continuity. The weighted modulus is a particular case of a more general modulus (see [68], for example)

$$\omega_d(f,\delta) = \sup\{\, |f(t)-f(x)| \,:\, t,x \in X,\ d(t,x) \leq \delta\,\},$$

where f is a bounded function defined on X and (X,d) is a compact metric space.

Proposition 2.3. The following relation is true:

$$\omega_\varphi(f,\delta) = \omega\left(f \circ \varphi^{-1}, \delta\right). \tag{2.2}$$

Proposition 2.4. The weighted modulus of continuity has the properties:

1. For $f \in B(I)$, the function $\omega_\varphi(f,\cdot)$ is nonnegative, increasing, subadditive and bounded, and for $\delta \geq 0$, the function $\omega_\varphi(\cdot,\delta)$ is a seminorm on $B(I)$ (subadditive and positive homogeneous).

2. (i) If the function $f\circ\varphi^{-1}$ is uniformly continuous on J, then, for every sequence $(\delta_n)_{n\geq1}$ of real numbers having the property $\lim_{n\to\infty}\delta_n = 0$, we have $\lim_{n\to\infty}\omega_\varphi(f,\delta_n) = 0$.

(ii) If $(\delta_n)_{n\geq1}$ is a sequence of positive real numbers such that $\lim_{n\to\infty}\omega_\varphi(f,\delta_n) = 0$, then $f\circ\varphi^{-1}$ is uniformly continuous on J.

3. For every $\delta, \lambda \geq 0$ we have

$$\omega_\varphi(f,\lambda\delta) \leq (1+\lambda)\cdot\omega_\varphi(f,\delta).$$

4. For every $\delta > 0$ the following inequality is true:

$$|f(y)-f(x)| \leq \left(1 + \frac{|\varphi(y)-\varphi(x)|}{\delta}\right)\cdot\omega_\varphi(f,\delta).$$

2.2 The approximation of continuous and bounded functions on the positive half-line

Remark 2.5. To extend the Theorem of Popoviciu-Bohman-Korovkin for the space $C[0,\infty)$, we consider first, those functions which are continuous and bounded on $[0,\infty)$. To be more precise, let $C^*[0,\infty)$ be the set of all continuous functions which have finite limit at infinity, which is a normed space, endowed with the uniform norm $\|f\| = \sup_{x\geq 0} |f(x)|$,

$$C^*[0,\infty) = \left\{ f\colon [0,\infty) \to \mathbb{R} \,|\, f \text{ continuous on } [0,\infty) \text{ and } \lim_{x\to\infty} f(x) = L \in \mathbb{R} \right\}.$$

In [139] and [1] it is used the notation $C[0,\infty]$, but we prefer $C^*[0,\infty)$, which is used in [7], [51] and [24].

Theorem 2.6. *Let $A_n\colon C^*[0,\infty) \to C^*[0,\infty)$ be a sequence of positive linear operators with the property that*

$$\lim_{n\to\infty} A_n(e^{-kt}, x) = e^{-kx}, \quad k = 0, 1, 2,$$

hold uniformly on $[0,\infty)$. Then

$$\lim_{n\to\infty} A_n f(x) = f(x),$$

holds uniformly on $[0,\infty)$, for every function $f \in C^*[0,\infty)$.

Remark 2.7. The proof of the above theorem is for the first time mentioned in the paper [24]. Using a remark from [26], we construct another compactification of the positive half-line $[0,\infty)$ to obtain another theorem of Korovkin type.

Theorem 2.8. *Let $A_n\colon C^*[0,\infty) \to C^*[0,\infty)$ be a sequence of positive linear operators such that the relations*

$$\lim_{n\to\infty} A_n\left(\left(\frac{t}{1+t}\right)^k, x\right) = \left(\frac{x}{1+x}\right)^k, \quad k = 0, 1, 2,$$

hold uniformly on $[0,\infty)$. Then

$$\lim_{n\to\infty} A_n f(x) = f(x),$$

holds uniformly on $[0,\infty)$, for every $f \in C^*[0,\infty)$.

Remark 2.9. The condition that f has a finite limit at infinity is essential, as the following examples show (the second example is from [63]). The sequence of positive linear operators A_n, defined by

$$A_n(f,x) = \begin{cases} f(x) + e^{x-n}[f(x+1) - f(x)], & x \in [0,n] \\ f(x), & x > n, \end{cases}$$

which transforms every function $f \in C^*[0,\infty)$ into a function from $C^*[0,\infty)$, verifies the conditions $\sup_{x\geq 0} \left|A_n(e^{-kt}, x) - e^{-kx}\right| \to 0$, for $k = 0, 1, 2$. But, for $f^*(x) = \cos \pi x$ which is continuous and bounded on $[0,\infty)$, even uniformly continuous on $[0,\infty)$, we have

$$\|A_n f^* - f^*\| = \sup_{x\in[0,n]} \left|2e^{x-n} \cos \pi x\right| = 2.$$

The sequence B_n defined by

$$B_n(f,x) = \begin{cases} f(x) + \frac{1-\frac{x}{n}}{n+1} \left[\left(x+\frac{3}{2}\right) f\left(x+\frac{1}{2}\right) - (x+1)f(x)\right], & x \in [0,n] \\ f(x), & x > n, \end{cases}$$

is a sequence of positive linear operators with $\left\|B_n(t^k/(1+t)^k, x) - x^k/(1+x)^k\right\| \to 0$, for $k = 0, 1, 2$ and $\|B_n f^* - f^*\| \geq 2$.

Next, we present some quantitative results, estimating the error of approximation, using the weighted modulus of continuity mentioned in Definition 2.1. We denote as in Holhoş [77]

$$\omega^*(f,\delta) = \sup_{\substack{x,t\geq 0 \\ |e^{-x}-e^{-t}|\leq\delta}} |f(x)-f(t)|,$$

and

$$\omega^{\#}(f,\delta) = \sup_{\substack{x,t\geq 0 \\ \left|\frac{x}{1+x}-\frac{t}{1+t}\right|\leq\delta}} |f(x)-f(t)|,$$

for every $\delta \geq 0$ and every $f \in C^*[0,\infty)$.

Theorem 2.10 (Holhoş [77])**.** *Let $A_n : C^*[0,\infty) \to C^*[0,\infty)$ be a sequence of positive linear operators with*

$$\begin{aligned} \|A_n 1 - 1\| &= a_n, \\ \|A_n(e^{-t},x) - e^{-x}\| &= b_n, \\ \left\|A_n(e^{-2t},x) - e^{-2x}\right\| &= c_n, \end{aligned}$$

where a_n, b_n and c_n converge to 0, when n tends to infinity. Then

$$\|A_n f - f\| \leq \|f\| \cdot a_n + (2+a_n)\cdot\omega^*\left(f, \sqrt{a_n+2b_n+c_n}\right),$$

for every $f \in C^*[0,\infty)$.

Remark 2.11. Similarly, if we replace e^{-t} and e^{-2t} with $t/(1+t)$ and $t^2/(1+t)^2$ and the modulus $\omega^*(f,\delta)$ with $\omega^{\#}(f,\delta)$ we obtain another estimation of the error of approximation.

Theorem 2.12 (Holhoş [77])**.** *Let $A_n : C[0,\infty) \to C[0,\infty)$ be a sequence of positive linear operators, preserving the affine functions and*

$$\sup_{x\geq 0} \frac{|A_n(t^2,x)-x^2|}{(1+x)^2} = d_n \to 0, \quad (n\to\infty)$$

then

$$\|A_n f - f\| \leq 2\cdot\omega^{\#}\left(f, \sqrt{d_n}\right),$$

for every $f \in C^*[0,\infty)$.

Remark 2.13. Because the inequality $|e^{-t}-e^{-x}| \leq |t-x|$ is true for every $t,x\geq 0$, we deduce that

$$\omega(f,\delta) \leq \omega^*(f,\delta), \quad \text{for every } \delta \geq 0,$$

and because $|e^{-t}-e^{-x}| = e^{-\theta}|t-x| \geq e^{-M}|t-x|$ is true only for $t,x\in[0,M]$, we obtain

$$\omega^*(f,\delta) \leq \omega(f,e^M\delta) \leq (1+e^M)\cdot\omega(f,\delta).$$

The inequality $\left|\frac{x}{1+x}-\frac{t}{1+t}\right| \leq |x-t|$, for $x,t\geq 0$, shows us that

$$\omega(f,\delta) \leq \omega^{\#}(f,\delta),$$

and $\left|\frac{x}{1+x}-\frac{t}{1+t}\right| \geq \frac{|x-t|}{(1+M)^2}$, for $x,t\in[0,M]$ proves that

$$\omega^{\#}(f,\delta) \leq \omega(f,(1+M)^2\delta) \leq (1+M)^2\cdot\omega(f,\delta),$$

where $M>0$, is an integer.

Because of these inequalities we cannot replace the weighted moduli from Theorem 2.10 and Theorem 2.12 with the usual modulus of continuity unless we approximate the functions on $[0,M]$.

Corollary 2.14. For the Szász-Mirakjan operators $S_n : C^*[0,\infty) \to C^*[0,\infty)$ defined in Example 1.13 and for $f \in C^*[0,\infty)$, we have the estimations

$$\|S_n f - f\| \le 2 \cdot \omega^* \left(f, \frac{1}{\sqrt{n}} \right), \quad n \ge 1,$$

and

$$\|S_n f - f\| \le 2 \cdot \omega^\# \left(f, \frac{1}{2\sqrt{n}} \right), \quad n \ge 1.$$

Corollary 2.15. For the Baskakov operators$V_n : C^*[0,\infty) \to C^*[0,\infty)$ defined in Example 1.14 and for $f \in C^*[0,\infty)$, we have

$$\|V_n f - f\| \le 2 \cdot \omega^* \left(f, \frac{5}{2\sqrt{n}} \right), \quad n \ge 2,$$

and

$$\|V_n f - f\| \le 2 \cdot \omega^\# \left(f, \frac{1}{\sqrt{n}} \right), \quad n \ge 1.$$

Corollary 2.16. For the Bernstein-Chlodovsky operators defined in Example 1.17 and for a function $f \in C^*[0,\infty)$, we have the estimations

$$\|C_n f - f\| \le 2 \cdot \omega^* \left(f, \sqrt{\frac{\beta_n}{n}} \right), \quad n \ge 1,$$

and

$$\|C_n f - f\| \le 2 \cdot \omega^\# \left(f, \sqrt{\frac{\beta_n}{4n}} \right), \quad n \ge 1.$$

Corollary 2.17. The Bleimann-Butzer-Hahn operators $L_n \colon C^*[0,\infty) \to C^*[0,\infty)$ defined in Example 1.16, verify for every $f \in C^*[0,\infty)$

$$\|L_n f - f\| \le 2 \cdot \omega^\# \left(f, \frac{2}{\sqrt{n+1}} \right), \quad n \ge 1.$$

2.3 Uniform approximation of functions on noncompact intervals

The next results give a characterization of the functions which can be uniformly approximated by a sequence of positive linear operators. The results are published in [76] and they present another approach to this problem already solved (see [140] and [40]).

Theorem 2.18 (Holhoş [76])**.** *Let $A_n \colon C(I) \to C(I)$ be a sequence of positive linear operators preserving the constant functions. Then, the following statements are true:*
a) if $\sup_{x\in I} A_n(|\varphi(t) - \varphi(x)|, x) = a_n \to 0$ and $f \circ \varphi^{-1}$ is uniformly continuous on J, then $\|A_n f - f\| \to 0$ and moreover

$$\|A_n f - f\| \le 2 \cdot \omega_\varphi (f, a_n).$$

b) if $\|A_n f - f\| \to 0$ and $(A_n f) \circ \varphi^{-1}$ is uniformly continuous on J, then $f \circ \varphi^{-1}$ is uniformly continuous on J.

Corollary 2.19. For the Szász-Mirakjan operators

$$S_n f(x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right),$$

we have $\|S_n f - f\| \to 0$ if $f(x^2)$ is uniformly continuous on $[0,\infty)$. If f is bounded and continuous on $[0,\infty)$ and $\|S_n f - f\| \to 0$, then $f(x^2)$ is uniformly continuous on $[0,\infty)$ and

$$\|S_n f - f\| \le 2 \cdot \omega\left(f(t^2), \frac{1}{\sqrt{n}}\right), \quad n \ge 1. \tag{2.3}$$

Remark 2.20. In [137, 140] V. Totik proves that $S_n f$ converge uniformly to f for $f \in C[0,\infty) \cap B[0,\infty)$, if and only if $f(x^2)$ is uniformly continuous on $[0,\infty)$. The estimation (2.3) is mentioned for the first time in the article of J. de la Cal and J. Cárcamo [40, Theorem 1].

Corollary 2.21. For the Baskakov operators

$$V_n f(x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} f\left(\frac{k}{n}\right),$$

the convergence $\|V_n f - f\| \to 0$ is true, if $f(e^x - 1)$ is uniformly continuous on $[0,\infty)$. If f is bounded and continuous on $[0,\infty)$ and $V_n f$ converges uniformly to f on $[0,\infty)$, then $f(e^x-1)$ is uniformly continuous on $[0,\infty)$. Furthermore,

$$\|V_n f - f\| \le 2 \cdot \omega\left(f(e^t - 1), \frac{1}{\sqrt{n-1}}\right), \quad n \ge 2. \tag{2.4}$$

Remark 2.22. Totik [138, 140] proves that for the function f continuous and bounded on $[0,\infty)$, $V_n f$ converges uniformly to f, if and only if $f(e^x)$ is uniformly continuous on $[0,\infty)$. The estimation (2.4) is similar to that from [40, Theorem 7].

Corollary 2.23. The Meyer-König and Zeller operators

$$M_n(f,x) = \sum_{k=0}^{\infty} \binom{n+k}{k} x^k (1-x)^{n+1} f\left(\frac{k}{n+k}\right)$$

have the property that $\|M_n f - f\| \to 0$, if $f(1-e^{-x})$ is uniformly continuous on $[0,\infty)$. If f is bounded and continuous on $[0,1)$ and $M_n f$ converges uniformly on $[0,1)$ to f, then $f(1-e^{-x})$ is uniformly continuous on $[0,\infty)$. Moreover,

$$\|M_n f - f\| \le 2 \cdot \omega\left(f(1-e^{-t}), \frac{1}{\sqrt{n}}\right), \quad \text{for } n \ge 1. \tag{2.5}$$

Remark 2.24. In [138] Totik proves that $M_n f$ converges to f uniformly, for the functions f which are continuous and bounded on $[0,\infty)$, if and only if $f\left(\frac{e^x}{1+e^x}\right)$ is uniformly continuous on $[0,\infty)$. In [140], the condition for the uniform continuity of $f\left(\frac{e^x}{1+e^x}\right)$ is proved to be equivalent to the uniform continuity of $f(1-e^{-x})$ on $[0,\infty)$. The estimation (2.5) is similar to that from [40, Theorem 8].

Corollary 2.25. For the Gauss-Weierstrass operators

$$W_n(f,x) = \frac{\sqrt{n}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-n\frac{(u-x)^2}{2}} f(u)\, du,$$

we have $\|W_n f - f\| \to 0$, if f is uniformly continuous on $\mathbb{R}$. If f is bounded and continuous on $\mathbb{R}$ and $W_n f \to f$ uniformly on $\mathbb{R}$, then f is uniformly continuous on $\mathbb{R}$. Moreover,

$$\|W_n f - f\| \le 2 \cdot \omega\left(f, \frac{1}{\sqrt{n}}\right), \quad \text{for } n \ge 1. \tag{2.6}$$

Remark 2.26. Stancu [133] proved that $\|W_n f - f\| \to 0$, if f is uniformly continuous on $\mathbb{R}$. Holhoş [76] proved that if f is bounded and continuous on $\mathbb{R}$ and $W_n f$ converges uniformly to f on $\mathbb{R}$, then f is uniformly continuous on $\mathbb{R}$. The estimation (2.6) is similar to that from [133].

Corollary 2.27. The Bleimann-Butzer-Hahn operatorsL_n have the property that $\|L_n f - f\| \to 0$, if $f(x^{-2}-1)$ is uniformly continuous on $(0,1]$. If f is bounded and continuous on $[0,\infty)$ and $L_n f$ converges uniformly on $[0,\infty)$ to f, then $f(x^{-2}-1)$ is uniformly continuous on $(0,1]$. Furthemore, we have the estimation

$$\|L_n f - f\| \leq 2 \cdot \omega \left(f(t^{-2}-1), \frac{1}{\sqrt{n+1}} \right), \quad \text{for } n \geq 1. \tag{2.7}$$

Remark 2.28. The function $f(x^{-2}-1)$ is uniformly continuous on $(0,1]$ if and only if $f \in C^*[0,\infty)$. Totik [139] proved that if $f \in C^*[0,\infty)$, then $L_n f$ converges uniformly to f. The estimation (2.7) is mentioned for the first time in Holhoş [76].

Chapter 3

The approximation of continuous and unbounded functions

3.1 The local approximation

3.2 The approximation on compact subsets

3.3 The approximation on weighted spaces

3.3.1 Weighted spaces: definition and examples

In the previous sections we have mentioned some results for the local approximation and approximation on compact subsets of an unbounded function using positive linear operators. The function f to be approximated must satisfy some growth condition

$$f(x) = \mathcal{O}(\rho(x)). \tag{3.1}$$

In this section we want to obtain global results (on the entire domain of definition of the functions) for the approximation of unbounded functions using positive linear operators.

Definition 3.1. For the interval $I \subseteq \mathbb{R}$ the continuous function $\rho\colon I \to (0, \infty)$is called weight. We call weighted space, the set $B_\rho(I)$, which represents the space of all functions $f\colon I \to \mathbb{R}$, for which there exists $M > 0$, such that $|f(x)| \leq M \cdot \rho(x)$, for every $x \in I$. This space can be endowed with the ρ-norm

$$\|f\|_\rho = \sup_{x \in I} \frac{|f(x)|}{\rho(x)}.$$

We define the subspace $C_\rho(I) = C(I) \cap B_\rho(I)$, and for $I = [0, \infty)$

$$C_\rho^*[0, \infty) = \left\{ f \in C_\rho[0, \infty), \lim_{x \to +\infty} \frac{f(x)}{\rho(x)} = K < +\infty \right\}.$$

Example 3.2. The polynomial weighted spaces $C_N[0, \infty)$ are obtained from the weight

$$\rho(x) = 1 + x^N, \quad x \geq 0,\ N > 0.$$

Some authors use the weight $w_N(x) = (1+x)^N$. The weight space associated with this weight is the same with $C_N[0, \infty)$ with equivalent norms. There are many who have studied the approximation of functions on these weighted spaces using positive linear operators: A. Aral [10, 11], F. Altomare [8], A. Attalienti [12], M. Becker [16], J. Bustamante [28], M. Campiti [12], A. Ciupa [34, 35, 36, 38],

O. Doğru [49], A. Erençýn [53], A.D. Gadjiev [61], S. Graczyg [123], V. Gupta [11, 151], E. Ibikli [81], N. Ispir [83, 156], A.-J. López-Moreno [97], E. Mangino [8], L. Morales de la Cruz [28], J.M. Quesada [28], L. Rempulska [114, 119, 120, 121, 122, 123, 124], M. Skorupka [114, 119, 121, 122], F. Taşdelen [53], K. Tomczak [124], Z. Walczak [120, 144, 145, 146, 149, 150, 151], I. Yuksel [156].

Example 3.3. The exponential weight spaces $C_p[0,\infty)$ are defined by means of the weight

$$\rho(x) = e^{px}, \quad x \geq 0,\ p > 0.$$

From those who studied the problem of approximation of functions on such spaces using positive linear operators, we mention: M. Becker [17], A. Ciupa [31, 32, 33, 37], Z. Ditzian [46], M. Leśniewich [93], L. Rempulska [93, 114, 115, 116, 117, 118], M. Skorupka [114, 115, 116], Z. Walczak [117, 118, 147, 148].

Example 3.4. D.-X. Zhou [157] uses the weight

$$w(x) = x^{-a}(1+x)^b, \quad 1 > a > 0,\, b > 0,$$

to obtain a similar result to that of Becker [16], for the Szász-Mirakjan operators. In [41, 42], B. Della Vecchia, G. Mastroianni and J. Szabados considered a general class of weights including

$$w(x) = e^{x^\beta}, \quad x \geq 0,\, \beta > 0.$$

Remark 3.5. The spaces $B_\rho(I)$, $C_\rho(I)$ and $C_\rho^*[0,\infty)$ are Banach spaces. The completeness of these spaces comes from the completeness of the spaces $B(I)$, $C(I) \cap B(I)$ and $C^*[0,\infty)$, by the relation between ρ-norm and uniform norm: $\|f\|_{\rho=} \|f/\rho\|$.

Remark 3.6. Let A be a positive linear operator defined on $C_\rho(I)$. A necessary and sufficient condition for the operator A to map $C_\rho(I)$ into $B_\rho(I)$ is

$$A(\rho, x) \leq M \cdot \rho(x), \text{ for every } x \in I,$$

where $M > 0$ is a constant independent o x. This condition assures the boundedness (the continuity) of the operator A. The norm of the operator A is given by

$$\|A\|_{C_\rho \to B_\rho} = \|A\rho\|_\rho .$$

3.3.2 Theorems of Korovkin type

The following result is the first extension of the Popoviciu-Bohman-Korovkin Theorem 1.27 for the weighted spaces. A similar result, but for the more general case of weighted spaces of functions defined on locally compact Hausdorff spaces see [125] and [7].

Theorem 3.7 (Gadjiev)**.** *Let $\varphi\colon [0,\infty) \to [0,\infty)$ be a strictly increasing continuous function with the property that $\lim_{x\to\infty} \varphi(x) = \infty$ and let $\rho(x) = 1+\varphi^2(x)$ be a weight. If the sequence of positive linear operators $A_n : C_\rho[0,\infty) \to B_\rho[0,\infty)$, satisfies*

$$\lim_{n\to\infty} \left\|A_n\varphi^i - \varphi^i\right\|_\rho = 0, \quad i = 0,1,2,$$

then, for every $f \in C_\rho^*[0,\infty)$, we have

$$\lim_{n\to\infty} \|A_n f - f\|_\rho = 0.$$

Remark 3.8. This theorem of A.D. Gadjiev is stated in [60] in 1974 and proved in 1976 (see [61]), for the weighted space of functions defined on $\mathbb{R}$.

In the following, we present a quantitative version of Theorem 3.7.

Theorem 3.9 (Holhoş [74])**.** *Let $\varphi\colon [0,\infty) \to [0,\infty)$ be a strictly increasing continuous function with the properties that $\varphi(0) = 0$ and $\lim_{x\to\infty} \varphi(x) = \infty$. Let $\rho(x) = 1 + \varphi^2(x)$ be the weight and let $A_n : C_\rho[0,\infty) \to B_\rho[0,\infty)$ be a sequence of positive linear operators for which*

$$\|A_n 1 - 1\| = a_n,$$
$$\|A_n\varphi - \varphi\|_{\rho^{\frac{1}{2}}} = b_n,$$
$$\left\|A_n\varphi^2 - \varphi^2\right\|_{\rho} = c_n,$$

where $(a_n)_{n\geq 1}, (b_n)_{n\geq 1}, (c_n)_{n\geq 1}$ are sequences converging to 0. Using the following notation $\delta_n = \sqrt{a_n + 2b_n + c_n}$ and considering the sequence of real numbers $(\eta_n)_{n\geq 1}$ with the properties

$$\lim_{n\to\infty} \eta_n = \infty \quad \text{and} \quad \lim_{n\to\infty} \rho^{\frac{1}{2}}(\eta_n)\delta_n = 0,$$

we have for every $f \in C^*_\rho[0,\infty)$ the estimation

$$\begin{aligned}
\|A_n f - f\|_\rho &\leq K_f \cdot (a_n + c_n)\\
&+ \left(\|f\|_\rho + K_f\right)\left[\rho^{\frac{1}{2}}(\eta_n)\delta_n\sqrt{1+a_n} + a_n + \delta_n\sqrt{\delta_n^2 + 4}\right]\\
&+ (2 + a_n)\,\omega_\varphi\left(\frac{f}{\rho}, \rho^{\frac{1}{2}}(\eta_n)\delta_n\right)\\
&+ 2r_n(3 + 2a_n + 2c_n),
\end{aligned}$$

where $K_f = \lim\limits_{x\to\infty} \dfrac{f(x)}{\rho(x)}$ and $r_n = \sup\limits_{x\geq \eta_n} \left|\dfrac{f(x)}{\rho(x)} - K_f\right|$.

3.3.3 Uniform approximation in weighted spaces

In this section, we consider as weight an increasing and differentiable function $\rho\colon I \to [1,\infty)$, where $I \subseteq \mathbb{R}$ is a noncompact interval of the real line. Let $\varphi\colon I \to J$, $(J \subset \mathbb{R})$, be a differentiable one-to-one map with the property that $\varphi'(x) > 0$, for every $x \in I$. Some of the following results are presented in [78]. They give an answer to two open problems mentioned in the survey article [27]:

Let $\mathcal{D}$ be a linear subspace of $\mathbb{R}^I$ and let $A_n\colon \mathcal{D} \to C(I)$ be a sequence of positive linear operators. For which weights ρ, does A_n map $C_\rho[0,\infty) \cap \mathcal{D}$ onto $C_\rho[0,\infty)$ with uniformly bounded norms?

2. For which functions $f \in C_\rho[0,\infty)$ do we have $\|A_n - f\|_{\rho} \to 0$, when $n \to \infty$?

A result related to the problem 2 is mentioned in the same paper [27, Theorem 3.8].

Theorem 3.10. *Let ρ be an arbitrary weight and let $G \in C[0,\infty)$ be a function with the properties: (i) G is differentiable, $G'(x) > 0$ for every $x > 0$ and G' is an increasing, continuous function on $(0,\infty)$ and (ii) there exist the constants $x_0 > 0$, $h_1 > 0$ and C such that $G' \circ G^{-1} \in \mathrm{Lip}[x_0,\infty)$ and for every $x \geq x_0$ and $h \in (0,h_1)$ we have $G'(x+h) \leq C \cdot G'(x)$. Let $A_n\colon C_\rho[0,\infty) \to C_\rho[0,\infty)$ be a sequence of positive linear operators with the property*

$$G' \circ G^{-1} \cdot \frac{A_n(f)}{\rho} \in \mathrm{Lip}[x_0,\infty), \quad f \in C_\rho[0,\infty), n \in \mathbb{N}.$$

If for some $f \in C_\rho[0,\infty)$ one has $\lim_{n\to\infty} \|A_n - f\|_\rho = 0$, then the function $(f/\rho) \circ G$ is uniformly continuous.

Theorem 3.11 (Holhoş [78])**.** *Let $A_n : C_\rho(I) \to C_\rho(I)$ be a sequence of positive linear operators preserving constant functions. Suppose the following conditions are satisfied*

$$\sup_{x\in I} A_n(|\varphi(t) - \varphi(x)|, x) = a_n \to 0, \ \ (n \to \infty) \tag{3.2}$$

$$\sup_{x\in I} \frac{A_n(|\rho(t) - \rho(x)|, x)}{\rho(x)} = b_n \to 0. \ \ (n \to \infty) \tag{3.3}$$

If $A_n(f,x)$ is differentiable and there exists a constant $K(f,n)$ such that

$$\frac{|(A_nf)'(x)|}{\varphi'(x)} \leq K(f,n)\cdot\rho(x), \quad \textit{for every } x\in I, \tag{3.4}$$

and, ρ and φ are given such that there is a constant $\alpha>0$ with the property

$$\frac{\rho'(x)}{\varphi'(x)} \leq \alpha\cdot\rho(x), \quad \textit{for every } x\in I, \tag{3.5}$$

then, the following statements are equivalent

- (i) $\|A_nf-f\|_\rho \to 0$ *as* $n\to\infty$.
- (ii) $\frac{f}{\rho}\circ\varphi^{-1}$ *is uniformly continuous on J.*

Furthermore, we have the estimation

$$\|A_nf-f\|_\rho \leq b_n\cdot\|f\|_\rho + 2\cdot\omega_\varphi\left(\frac{f}{\rho}, a_n\right), \quad \textit{for every } n\geq 1. \tag{3.6}$$

Remark 3.12. For $\rho(x)=1$, the result of Theorem 3.11 was obatined by Totik [140, 141], by J. de la Cal and Cárcamo [40] and by Holhoş [76] (see Theorem 2.18 from this thesis).

Remark 3.13. For the Bleimann-Butzer-Hahn operatorswe have found in the Corollary 2.27 the function $\varphi(x)=\frac{1}{\sqrt{1+x}}$. The maximal weight is $\rho(x)=e^{\frac{\alpha}{\sqrt{1+x}}}$, which is a bounded function on $[0,\infty)$, which shows the equivalence of this weight with $\rho(x)=1$. So, the appropriate space for global approximation of functions using the Bleimann-Butzer-Hahn operators is $C_\rho[0,\infty)=C^*[0,\infty)$.

Corollary 3.14. For $\alpha>0$ and $\rho(x)=e^{\alpha\sqrt{x}}$, the Szász-Mirakjan operators $S_n\colon C_\rho[0,\infty)\to C_\rho[0,\infty)$ have the property that

$$\|S_nf-f\|_\rho \to 0, \text{ when } n\to\infty$$

if and only if

$$f(x^2)e^{-\alpha x} \text{ is uniformly continuous on } [0,\infty).$$

Moreover, for every $f\in C_\rho[0,\infty)$ and every $n\geq 1$, one has

$$\|S_nf-f\|_\rho \leq \|f\|_\rho\cdot\frac{\alpha C}{\sqrt{n}} + 2\cdot\omega\left(f(t^2)e^{-\alpha t}, \frac{1}{\sqrt{n}}\right),$$

where $C=\sup_{n\in\mathbb{N}}\frac{1}{2}\sqrt{\|S_n\rho^2\|_{\rho^2}+2\|S_n\rho\|_\rho+1}<\infty$ is a constant depending only on α.

Remark 3.15. The result from Corollary 3.14 for the limit case, α=0, was obtained in [137], [140], [40] and [76] (see the Corollary 2.19).

Remark 3.16. In [16], Becker studied the global approximation of functions using Szász-Mirakjan operators for the polynomial weight $\rho(x)=1+x^N$, $N\in\mathbb{N}$. Becker, Kucharsky and Nessel [17] studied the global approximation for the exponential weight $\rho(x)=e^{\beta x}$. But because

$$\sup_{x\geq 0}\frac{S_n(e^{\beta t},x)}{e^{\beta x}} = \sup_{x\geq 0} e^{nx(e^{\frac{\beta}{n}}-1)-\beta x} = +\infty,$$

they obtain results only for the space $C(\eta)=\cap_{\beta>\eta}C_\beta$, where C_β is C_ρ for $\rho=e^{\beta x}$. It is also mentioned, that for any $f\in C_\beta$ we have $S_nf\in C_\gamma$, for $\gamma>\beta$ and for $n>\beta/\ln(\gamma/\beta)$. Ditzian [46], also, give some inverse theorems for exponential spaces. In [9], Amanov obtained that the condition

$$\sup_{x\geq 0}\frac{\rho(x+\sqrt{x})}{\rho(x)}<\infty$$

upon the weight ρ, is necessary and sufficient for the uniform boundedness of the norms of the operators $S_n\colon C_\rho[0,\infty) \to C_\rho[0,\infty)$. He mentions that this condition implies the inequality

$$\rho(x) \le e^{\alpha\sqrt{1+x}}, \quad x \ge 0.$$

He, also, gives a characterization of the functions f which are uniformly approximated by $S_n f$ in the ρ-norm, using a weighted second order modulus of smoothness.

Corollary 3.17. For a real number $\alpha > 0$ and for $\rho(x) = (1+x)^\alpha$ the Baskakov operators $V_n\colon C_\rho[0,\infty) \to C_\rho[0,\infty)$ have the property that

$$\|V_n f - f\|_{\rho \to} 0, \text{ as } n \to \infty$$

if and only if

$$f(e^x - 1)e^{-\alpha x}, \text{ is uniformly continuous on } [0,\infty).$$

Moreover, for $f \in C_\rho[0,\infty)$ and for $n \ge 2$, we have

$$\|V_n f - f\|_\rho \le \|f\|_\rho \cdot \frac{\alpha C}{\sqrt{n-1}} + 2 \cdot \omega\left(f(e^t - 1)e^{-\alpha t}, \frac{1}{\sqrt{n-1}}\right).$$

where $C = \sup_{n\in\mathbb{N}} \frac{1}{2}\sqrt{\|V_n\rho^2\|_{\rho^2} + 2\|V_n\rho\|_\rho + 1} < \infty$ is a constant depending only on α.

Remark 3.18. The result of the Corollary 3.17 for the limit case, $\alpha = 0$, was obtained in [138], [140], [40] and [76] (see the Corollary 2.21).

Remark 3.19. Becker [16] studied the global approximation of functions from the polynomial weighted space and remarked that "polynomial growth is the frame best suited for global results for the Baskakov operators". The reason is that for the exponential weight $\rho(x) = e^{\beta x}$, the series $V_n(\rho, x)$ exists only for $x < (e^{\frac{\beta}{n}} - 1)^{-1}$. Nevertheless, Ditzian [46] gave some inverse results for functions with exponential growth.

Corollary 3.20. For $\alpha > 0$ and $\rho(x) = \left(\frac{1}{1-x}\right)^\alpha$ the Meyer-König and Zeller operators $M_n\colon C_\rho[0,1) \to C_\rho[0,1)$ have the property that

$$\|M_n f - f\|_\rho \to 0, \text{ when } n \to \infty$$

if and only if

$$f(1 - e^{-x})e^{-\alpha x} \text{ is uniformly continuous on } [0,\infty).$$

Moreover, for every $f \in C_\rho[0,1)$ and for every $n \ge 3$, one has

$$\|M_n f - f\|_\rho \le \|f\|_\rho \frac{\alpha C}{\sqrt{n}} + 2 \cdot \omega\left(f(1 - e^{-t})e^{-\alpha t}, \frac{1}{\sqrt{n}}\right),$$

where $C = \sup_{n\in\mathbb{N}} \frac{1}{2}\sqrt{\|M_n\rho^2\|_{\rho^2} + 2\|M_n\rho\|_\rho + 1} < \infty$ is a constant depending only on α.

Remark 3.21. The result of the Corollary 3.20 for the limit case, $\alpha = 0$, was obtained in [138], [140], [40] and [76] (see the Corollary 2.23).

Corollary 3.22. For $\alpha > 0$ and $\rho(x) = 1 + x^\alpha$, the Post-Widder operators $P_n\colon C_\rho(0,\infty) \to C_\rho(0,\infty)$ have the property that

$$\|P_n f - f\|_\rho \to 0, \quad \text{when } n \to \infty$$

if and only if

$$f(e^x)e^{-\alpha x} \quad \text{is uniformly continuous on } (0,\infty).$$

Moreover, for every $f \in C_\rho(0,\infty)$ and every $n \geq 2$, one has

$$\|P_n f - f\|_\rho \leq \|f\|_\rho \frac{\alpha C}{\sqrt{n-1}} + 2 \cdot \omega \left(f(e^t)e^{-\alpha t}, \frac{1}{\sqrt{n-1}} \right),$$

where $C = \sup_{n\in\mathbb{N}} \frac{1}{2}\sqrt{\|P_n\rho^2\|_{\rho^2} + 2\|P_n\rho\|_\rho + 1} < \infty$ is a constant depending only on α.

Remark 3.23. The result of the Corollary 3.22 for the limit case, $\alpha = 0$, was obtained in [141] and in [40].

Corollary 3.24. For $\alpha > 0$ and $\rho(x) = 1 + x^\alpha$, the Gamma operators G_n defined on the space $C_\rho(0,\infty)$, $n \geq [2\alpha]$, have the property that

$$\|G_n f - f\|_\rho \to 0, \quad \text{when } n \to \infty$$

if and only if

$$f(e^x)e^{-\alpha x} \quad \text{is uniformly continuous on } (0,\infty).$$

Moreover, for every $f \in C_\rho(0,\infty)$ and every $n \geq [2\alpha]$, one has

$$\|G_n f - f\|_\rho \leq \|f\|_\rho \frac{\alpha C}{\sqrt{n}} + 2 \cdot \omega \left(f(e^t)e^{-\alpha t}, \frac{1}{\sqrt{n}} \right),$$

where $C = \sup_{n\in\mathbb{N}} \frac{1}{2}\sqrt{\|G_n\rho^2\|_{\rho^2} + 2\|G_n\rho\|_\rho + 1} < \infty$ is a constant depending only on α.

Remark 3.25. The result of the Corollary 3.24 for the limit case, $\alpha = 0$, was obtained in [140].

Corollary 3.26. For $\alpha > 0$ and $\rho(x) = e^{\alpha x}$ the Gauss-Weierstrass operators $W_n \colon C_\rho(\mathbb{R}) \to C_\rho(\mathbb{R})$ have the property that

$$\|W_n f - f\|_\rho \to 0, \quad \text{when } n \to \infty,$$

if and only if

$$f(x)e^{-\alpha x} \quad \text{is uniformly continuous on } \mathbb{R}.$$

Moreover, for every $f \in C_\rho(\mathbb{R})$ and for every $n \geq 1$, one has

$$\|W_n f - f\|_\rho \leq \|f\|_\rho \frac{\alpha C}{\sqrt{n}} + 2 \cdot \omega \left(f(t)e^{-\alpha t}, \frac{1}{\sqrt{n}} \right),$$

where $C = e^{\frac{\alpha^2}{2}} \sqrt{1 + \frac{\alpha^2}{4}\left(1 + e^{\frac{\alpha^2}{2}}\right)^2}$.

Remark 3.27. The result of the Corollary 3.26 for the limit case, $\alpha = 0$, was obtained in Holhoş [76] (see Corollary 2.25).

Corollary 3.28. For $\alpha > 0$ and $\rho(x) = e^{\alpha x}$ the Picard operators $\mathcal{P}_n \colon C_\rho(\mathbb{R}) \to C_\rho(\mathbb{R})$, $n \geq [2\alpha]+2$, have the property that

$$\|\mathcal{P}_n f - f\|_\rho \to 0, \quad \text{when } n \to \infty,$$

if and only if

$$f(x)e^{-\alpha x} \quad \text{is uniformly continuous on } \mathbb{R}.$$

Moreover, for every $f \in C_\rho(\mathbb{R})$ and for every $n \geq [2\alpha] + 2$, one has

$$\|\mathcal{P}_n f - f\|_\rho \leq \|f\|_\rho \frac{\alpha C}{n} + 2 \cdot \omega \left(f(t)e^{-\alpha t}, \frac{\sqrt{2}}{n} \right),$$

where $C > 0$ is a constant depending on α, but independent of n.

3.3.4 Moduli of continuity for weighted spaces

Many authors (see Examples 3.2 and 3.3) use the following modulus of continuity for the approximation on weighted spaces:

$$\omega^{\rho}(f,\delta) = \sup_{0<h\le\delta} \|\Delta_h f\|_{\rho}, \quad \text{where } \Delta_h(f,x) = f(x+h) - f(x).$$

But this modulus doesn't satisfy some properties of the classical moduli for functions defined on a compact set. This fact motivated Lopez-Moreno [97] to introduce the modulus

$$\Omega(f,\delta) = \sup_{0<h\le\delta} \sup_{x\ge 0} \frac{|f(x+h)-f(x)|}{\rho(x+h)},$$

for $\rho(x) = 1 + x^m$ and $f \in C_\rho[0,\infty)$. Amanov [9] gives a similar definition for the modulus of Ditzian and Totik:

$$\omega_1^{\varphi}(f,\delta)_{\rho} = \sup_{0<h\le\delta} \sup_{x\ge 0} \frac{|f(x+h\varphi(x))-f(x)|}{\rho(x+h\varphi(x))},$$

where $\rho\colon [0,\infty) \to [1,\infty)$ is an increasing function and $\varphi(x) = \sqrt{x}$.

In Theorema 3.9 the estimation of the remainder $f(x) - A_n(f,x)$ in ρ-norm, in approximating the function f by $A_n f$, where $(A_n)_{n\in\mathbb{N}}$ is a sequence of positive linear operators, is done using the quantities $\omega_{\varphi}\left(\frac{f}{\rho},\delta\right)$ and r_n, where $\omega_{\varphi}(\cdot,\cdot)$ is the modulus introduced in Definition 2.1, and $r_n = \sup_{x\ge\eta_n}\left|\frac{f(x)}{\rho(x)} - K_f\right|$ measures the speed of convergence in $\lim_{x\to\infty}\frac{f(x)}{\rho(x)} = K_f$ where $(\eta_n)_{n\ge1}$ is a sequence converging to infinity. In [13], Balász and Szabados use the following modululus

$$\Omega(f,A) = \sup_{x\ge y\ge A} |(f(x)-f(y)|,$$

calling it "modululus at infinity", which has the property that measures the speed of convergence in $\lim_{x\to\infty} f(x) = L < \infty$. This modulus doesn't need to know the limit of the function at infinity, as r_n does.

In Teorema 3.11, which is a recent result from 2010 (see [78]), the estimation of the remainder is done using only the weighted modulus $\omega_{\varphi}\left(\frac{f}{\rho},\delta\right)$, without the need to know the speed of convergence of the function f/ρ at infinity.

In [26, 27], Bustamante and Morales de la Cruz, construct some moduli on the space $C^*_\rho[0,\infty)$ using moduli on the space $C[0,1]$, by the relation

$$\Theta(f,\delta)_\rho = \Theta(\Phi f,\delta),$$

where Θ is a modulus on $C[0,1]$, and Φ is an isomorphism $\Phi\colon C^*_\rho[0,\infty) \to C[0,1]$.

In [62], Gadjiev and Aral use the following modulus

$$\Omega_\rho(f,\delta) = \sup_{\substack{x,y\ge0\\ |\rho(x)-\rho(y)|\le\delta}} \frac{|f(x)-f(y)|}{[|\rho(x)-\rho(y)|+1]\rho(x)},$$

where $\rho \in C^1[0,\infty)$, $\rho(0) = 1$ and $\inf_{x\ge0}\rho'(x) > 0$ and $f \in C_\rho[0,\infty)$, to prove the following theorem

Theorem 3.29. *Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of positive linear operators. If $\|A_n e_0 - 1\|_\rho = \alpha_n$, $\|A_n\rho - \rho\|_\rho = \beta_n$ and $\left\|A_n\rho^2 - \rho^2\right\|_{\rho^2} = \gamma_n$ and if $\alpha_n + 2\beta_n + \gamma_n \to 0$, then*

$$\|A_n f - f\|_{\rho^4} \le 16\cdot\Omega_\rho(f, \sqrt{\alpha_n + 2\beta_n + \gamma_n}) + \|f\|_\rho\,\alpha_n,$$

for every $f \in C^*_\rho[0,\infty)$ and n large enough.

Motivated by this last result, we introduce a similar modulus to obtain better results.

Definition 3.30. Let $\varphi\colon [0,\infty) \to [0,\infty)$ be a strictly increasing, continuous function with the property that $\varphi(0) = 0$ and $\lim_{x\to\infty} \varphi(x) = \infty$ and set $\rho(x) = 1+\varphi^2(x)$. For a function $f \in C_\rho[0,\infty)$ and for $\delta \geq 0$, we introduce the following modulus

$$\omega_\varphi^\rho(f,\delta) = \sup_{\substack{t,x\geq 0\\ |\varphi(t)-\varphi(x)|\leq\delta}} \frac{|f(t)-f(x)|}{\rho(t)+\rho(x)}. \tag{3.7}$$

Remark 3.31. If $\varphi(x) = x$, then $\omega_\varphi^\rho(f,\cdot)$ is equivalent with the modulus $\Omega(f,\cdot)$ defined by

$$\Omega(f,\delta) = \sup_{x\geq 0,\ |h|\leq\delta} \frac{|f(x+h)-f(x)|}{(1+h^2)(1+x^2)}.$$

This is true because

$$\omega_\varphi^\rho(f,\delta) \leq \Omega(f,\delta) \leq 3\cdot\omega_\varphi^\rho(f,\delta),$$

the first inequality being true for $\delta \leq \frac{1}{\sqrt{2}}$ and the second one for every $\delta \geq 0$.

Proposition 3.32. $\lim\limits_{\delta\searrow 0} \omega_\varphi^\rho(f,\delta) = 0$, for every function $f \in C_\rho^*[0,\infty)$.

Proposition 3.33. For every $\delta \geq 0$ and $\lambda \geq 0$, we have

$$\omega_\varphi^\rho(f,\lambda\delta) \leq (2+\lambda)\cdot\omega_\varphi^\rho(f,\delta).$$

Proposition 3.34. For every $f \in C_\rho[0,\infty)$, for $\delta > 0$ and for every $x,t \geq 0$

$$|f(t)-f(x)| \leq (\rho(t)+\rho(x))\left(2+\frac{|\varphi(t)-\varphi(x)|}{\delta}\right)\omega_\varphi^\rho(f,\delta).$$

Theorem 3.35 (Holhoş [73])**.** *Let $A_n : C_\rho[0,\infty) \to B_\rho[0,\infty)$ be a sequence of positive linear operators with the property that*

$$\begin{aligned}
\left\|A_n\varphi^0-\varphi^0\right\|_{\rho^0} &= a_n,\\
\left\|A_n\varphi-\varphi\right\|_{\rho^{\frac12}} &= b_n,\\
\left\|A_n\varphi^2-\varphi^2\right\|_{\rho} &= c_n,\\
\left\|A_n\varphi^3-\varphi^3\right\|_{\rho^{\frac32}} &= d_n,
\end{aligned}$$

where $(a_n)_{n\geq1}, (b_n)_{n\geq1}, (c_n)_{n\geq1}$ and $(d_n)_{n\geq1}$ are convergent to 0. Then

$$\|A_nf-f\|_{\rho^{\frac32}} \leq (7+4a_n+2c_n)\cdot\omega_\varphi^\rho(f,\delta_n)+\|f\|_\rho\, a_n$$

for every $f \in C_\rho[0,\infty)$, where the sequence $(\delta_n)_{n\geq1}$ is given by

$$\delta_n = 2\sqrt{(a_n+2b_n+c_n)(1+a_n)}+a_n+3b_n+3c_n+d_n.$$

Chapter 4

Inequalities related to linear operators and their applications

4.1 Approximation by rational functions with prescribed numerator using positive linear operators

In this section, we present two results for the approximation of functions using rational expressions with prescribed numerator: first, for functions which keep a constant sign over the entire interval of the approximation, which are approximated by reciprocals of polynomials, i.e. numerator equals 1, and second, for functions which change sign, and the numerator of the approximants is a polynomial that change sign in the same points as the function to be approximated. These results, namely Theorem 4.2 and Theorem 4.5) are not new (see [96], [95], [94], [155]), but the technique used to derive them is new and unitary. In the papers mentioned above, the authors use the Jackson operators and an inequality which can be obtained using Chebyshev polynomials. We obtain these results using a sequence of positive linear operators with "good" properties. The proofs of Lemma 4.1, Lemma 4.3 and Lemma 4.4 are new.

Let $A_n\colon C[0,1] \to \Pi_n$ be a sequence of positive linear operators. Suppose that A_n satisfies the properties:

1. $A_n(e_i, x) = e_i(x), \quad i = 0, 1, \text{ where } e_i(x) = x^i,$
2. $A_n((t-x)^2, x) \leq C \cdot \dfrac{\phi^2(x)}{n^2}, \quad \text{where } \phi(x) = \sqrt{x(1-x)},$
3. $\|A_n f - f\| \leq C \cdot \omega_1^{\phi}\left(f, \dfrac{1}{n}\right),$
4. $A_n(f, x) \geq f(x), \quad 0 < x < 1, \text{ for every convex function } f \text{ on } (0,1),$
5. $|A_n(f,x) - f(x)| \leq C \cdot \omega_2\left(f, \dfrac{\phi(x)}{n}\right),$
6. $\|A_n f - f\| \leq C \cdot \omega_2^{\phi}\left(f, \dfrac{1}{n}\right).$

An example of such operators are $A_n = H_{2\left[\frac{n-1}{2}\right]+1}\colon C[0,1] \to \Pi_n$, $n \geq 3$, where H_n are the Gavrea operators presented in Example 1.10.

Lemma 4.1. For the operators A_n with the above properties, we have

$$A_n(|f(t) - f(x)|^2, x) \leq C \cdot \left[\omega_1^{\phi}\left(f, \frac{1}{n}\right)\right]^2.$$

Theorem 4.2. *Let $f \in C[0,1]$ be a nonconstant and nonnegative function. Then, there is a sequence of polynomials $p_n \in \Pi_n$ such that*

$$\left\| f - \frac{1}{p_n} \right\| \leq C \cdot \omega_1^{\phi}\left(f, \frac{1}{n}\right).$$

Lemma 4.3. For any $0 < b_1 < b_2 < \cdots < b_\ell < 1$, $\ell \geq 1$, set

$$\rho(x) = (x - b_1)(x - b_2) \cdots (x - b_\ell).$$

Then, there is a polynomial $S_n \in \Pi_n$, such that for every $x \in [0,1]$ and every $n \geq \ell$, we have

$$0 \leq 1 - \frac{|\rho(x)|}{S_n(x)} \leq \min\left(1, \frac{C\ell}{n} \sum_{j=1}^{\ell} \frac{\phi(x)}{|x - b_j|}\right).$$

Lemma 4.4. There exists an absolute constant $C > 0$ such that for every $t, x \in [0,1]$ and every $f \in C[0,1]$, we have

$$|f(t) - f(x)| \cdot \min\left(1, \frac{\max(\phi(t), \phi(x))}{n|t - x|}\right) \leq C \cdot \omega_1^{\phi}\left(f, \frac{1}{n}\right).$$

Theorem 4.5. *There exists a positive constant C with the following property: if $f \in C[0,1]$ changes sign in exactly $\ell \geq 1$ points from $[0,1]$, say $b_1, b_2, \ldots, b_\ell$, then, for every $n \geq 2\ell$, there is a polynomial $p_n \in \Pi_n$, having the same sign as f in $(b_\ell, 1)$ and such that that for every $x \in [0,1]$,*

$$\left| f(x) - \frac{(x - b_1)(x - b_2) \ldots (x - b_\ell)}{p_n(x)} \right| \leq C\ell^2 \omega_1^{\phi}\left(f, \frac{1}{n}\right).$$

4.2 An inequality for a linear discrete operator

In [86], Kuang Jichang proved the following inequality

$$\frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right) > \frac{1}{n+1} \sum_{k=1}^{n+1} f\left(\frac{k}{n+1}\right) > \int_0^1 f(x)dx,$$

where f is a strictly increasing and convex (or concave) function on $(0,1]$.

In [67], using positive linear operators of Bernstein-Stancu type, I. Gavrea obtained the inequality

$$\frac{1}{n} \sum_{k=1}^{n} f\left(x_{k-1,n-1}\right) - \frac{1}{n+1} \sum_{k=1}^{n+1} f\left(x_{k-1,n}\right) \geq 0, \tag{4.1}$$

for an increasing convex function f and for the points $x_{i,n},\ i = 0, 1, \ldots, n$ from $[0,1]$, which satisfy the properties

$$0 \leq x_{0,n} \leq x_{1,n} < \cdots < x_{n,n} \leq 1$$

$$x_{k-1,n} \leq x_{k-1,n-1} \leq x_{k,n}$$

$$x_{0,n-1} \geq x_{0,n} \text{ 'si } x_{n-1,n-1} \geq x_{n,n}$$

$$(n-k)(x_{k,n-1} - x_{k,n}) \geq k(x_{k,n} - x_{k-1,n-1}),$$

for $n \geq 1$ and for every $k = 1, 2, \ldots, n$.

For related inequalities see [3],[67],[109],[112]. In [3], the authors presented, in a chronological order, these inequalities and some recent results related to them.

In this section, we want to prove an inequality for a discrete linear operator and obtain necessary and sufficient conditions over the points $x_{i,n}$ in order to obtain inequality (4.1). We deduce, also a weighted majorization inequality.

Let $m \geq 3$ be an integer and let $I_m = \{1, 2, \dots, m\}$. Consider $(z_k)_{k \in I_m}$ a strictly decreasing sequence of real numbers from $[0, 1]$. Let A be the linear functional defined by

$$A[f] = \sum_{k=1}^{m} a_k f(z_k), \tag{4.2}$$

where a_k are real numbers and f is a real function defined on $[0, 1]$. We want to find the necessary and sufficient conditions over z_k, such that

$$A[f] \geq 0, \tag{4.3}$$

holds for some classes of convex functions. We have the following result

Theorem 4.6 (Holhoş [75])**.** *Consider the following conditions*

$$A[e_0] = 0 \tag{4.4}$$

$$A[e_1] \geq 0 \tag{4.5}$$

$$A[e_1] \leq 0 \tag{4.6}$$

$$\sum_{i=1}^{k} a_i(z_i - z_{k+1}) \geq 0, \text{ for every } k \in I_{m-2}. \tag{4.7}$$

$$\sum_{i=k}^{m} a_i(z_i - z_{k-1}) \geq 0, \text{ for every } k \in I_m \setminus \{1, 2\}. \tag{4.8}$$

Then
a) (4.3) holds for every increasing convex function f iff (4.4), (4.5) and (4.7) hold;
b) (4.3) holds for every decreasing convex function f iff (4.4), (4.6) and (4.7) hold;
c) (4.3) holds for every convex function f iff (4.4), (4.5), (4.6) and (4.7) hold;
d) (4.3) holds for every increasing concave function f iff (4.4), (4.5) and (4.8) hold;
e) (4.3) holds for every decreasing concave function f iff (4.4), (4.6) and (4.8) hold;
f) (4.3) holds for every concave function f iff (4.4), (4.5), (4.6) and (4.8) hold.

Remark 4.7. In [109], T. Popoviciu proves the case c) from the Theorem 4.6. The author, also, generalizes the result to the class of convex functions of order n.

Corollary 4.8. Let $n \geq 1$ be an integer and let x_i, $i \in I_n$ and y_j, $j \in I_{n+1}$ be two increasing sequences of points from $[0, 1]$. Let A be the linear functional defined by

$$A[f] = \frac{1}{n} \sum_{k=1}^{n} f(x_k) - \frac{1}{n+1} \sum_{k=1}^{n+1} f(y_k). \tag{4.9}$$

If

$$\begin{gathered} x_1 \geq y_1, \\ x_n \geq y_{n+1}, \\ (n-i)(x_{i+1} - y_{i+1}) \geq i(y_{i+1} - x_i), \text{ for } i \in I_{n-1}, \\ (n+1-i)(x_i - y_i) \geq i(y_{i+1} - x_i), \text{ for } i \in I_n, \end{gathered}$$

then $A[f] \geq 0$, for every increasing convex or concave function $f : [0, 1] \to \mathbb{R}$.

Corollary 4.9. Let $n \geq 1$ be an integer and let x_i, $i \in I_n$ and y_j, $j \in I_{n+1}$ be two increasing sequences of points from $[0,1]$ such that $x_k \geq y_k$ for $k \in I_n$. Let A be the linear functional defined by

$$A[f] = \frac{1}{n}\sum_{k=1}^{n} f(x_k) - \frac{1}{n+1}\sum_{k=1}^{n+1} f(y_k), \tag{4.10}$$

a) If $x_n \geq y_{n+1}$ and

$$(n-i)(x_{i+1} - y_{i+1}) \geq i(y_{i+1} - x_i), \text{ for } i \in I_{n-1},$$

then $A[f] \geq 0$, for every increasing convex function $f : [0,1] \to \mathbb{R}$.
b) If

$$(n+1-i)(x_i - y_i) \geq i(y_{i+1} - x_i), \text{ for } i \in I_n,$$

then $A[f] \geq 0$, for every increasing concave function $f : [0,1] \to \mathbb{R}$.

Corollary 4.10. Let $(a_n)_{n\in\mathbb{N}}$ be a positive increasing sequence of real numbers such that the sequence $\left(n\left(1 - \frac{a_n}{a_{n+1}}\right)\right)_{n\in\mathbb{N}}$ is increasing ($\left(n\left(\frac{a_{n+1}}{a_n}\right) - 1\right)_{n\in\mathbb{N}}$ is increasing). Then

$$\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{a_k}{a_n}\right) \geq \frac{1}{n+1}\sum_{k=1}^{n+1} f\left(\frac{a_k}{a_{n+1}}\right)$$

for every $f : [0,1] \to \mathbb{R}$ increasing convex (concave) function.

Remark 4.11. The result of the Corollary 4.10 was obtained for the first time in [113].

Corollary 4.12. Let $n \geq 1$ be an integer and let x_i, $i \in I_n$ and y_j, $j \in I_{n+1}$ be two strictly increasing sequences of points from $[0,1]$, with the property

$$0 \leq y_1 < x_1 < y_2 < \cdots < y_n < x_n \leq y_{n+1} \leq 1. \tag{4.11}$$

Let A be the linear functional defined by

$$A[f] = \alpha\sum_{k=1}^{n} f(x_k) - \beta\sum_{k=1}^{n+1} f(y_k), \tag{4.12}$$

where α and β are positive real numbers. Then $A[f] \geq 0$ for every increasing convex or concave function $f : [0,1] \to \mathbb{R}$, if and only if

$$\alpha = \frac{c}{n} \text{ and } \beta = \frac{c}{n+1}, \text{ where } c > 0,$$
$$(n+1)(x_1 + x_2 + \cdots + x_k) - n(y_1 + \cdots + y_k) \geq k y_{k+1}, \text{ for every } k \in I_n,$$
$$(n+1)(x_k + \cdots + x_n) - n(y_{k+1} + \cdots + y_{n+1}) \geq (n+1-k)x_k, \text{ for } k \in I_n.$$

Remark 4.13. Let $n \geq 1$ be an integer and let x_i, $i \in I_n$ and y_j, $j \in I_{n+1}$ be two strictly increasing sequences of points from $[0,1]$, with the property

$$0 \leq y_1 \leq x_1 < y_2 < \cdots < y_n < x_n \leq y_{n+1} \leq 1.$$

The condition

$$(n+1)(x_k + \cdots + x_n) - n(y_{k+1} + \cdots + y_{n+1}) \geq (n+1-k)x_k, \text{ for } k \in I_n,$$

is equivalent with

$$\sum_{i=k}^{n-1}[(n-i)(x_{i+1} - y_{i+1}) - i(y_{i+1} - x_i)] \geq 0, \text{ for every } k \in I_{n-1} \quad \text{and} \quad x_n = y_{n+1},$$

and the condition

$$(n+1)(x_1+x_2+\cdots+x_k)-n(y_1+\cdots+y_k)\geq ky_{k+1},\ \text{for every } k\in I_n$$

is equivalent with

$$\sum_{i=1}^{k}[(n+1-i)(x_i-y_i)-i(y_{i+1}-x_i)]\geq 0,\ \text{for every } k\in I_n.$$

Remark 4.14. Using Corollary 4.12 and Remark 4.13 we deduce the result obtained in [67], the one presented in the beginning of this section.

Remark 4.15. If $y_1\leq y_2\leq\cdots\leq y_{n+1}$ are the roots of a polynomial P of degree $n+1$ and $x_1\leq x_2\leq\cdots\leq x_n$ the roots of the derivative of the polynomial P, then we have the inequalities

$$(n+1)(x_1+x_2+\cdots+x_k)-n(y_1+\cdots+y_k)\geq ky_{k+1},\ \text{for every } k\in I_{n-1}.$$

See [109], for details.

Corollary 4.16. Let $n\geq 1$ be an integer and let $x_i, y_i,\ i\in I_n$ be two decreasing sequences of points from $[0,1]$ and $p_i, q_i,\ i\in I_n$ be real numbers such that (p_i) majorizes (q_i) (i.e. $p_1+\cdots+p_k\geq q_1+\cdots+q_k$ for every $k\in I_{n-1}$ and $p_1+\cdots+p_n=q_1+\cdots+q_n$). If the following conditions are satisfied:

$$\sum_{i=1}^{k}q_ix_i\geq\sum_{i=1}^{k}q_iy_i, \text{for every } k\in I_{n-1},$$

$$\sum_{i=1}^{k}p_ix_i\geq\sum_{i=1}^{k}p_iy_i,\ \text{for every } k\in I_n,$$

and

$$\sum_{i=1}^{n}p_ix_i=\sum_{i=1}^{n}q_iy_i,$$

then, for every convex function $f:[0,1]\to\mathbb{R}$, we have

$$\sum_{k=1}^{n}p_kf(x_k)\geq\sum_{k=1}^{n}q_kf(y_k).$$

Corollary 4.17. If f is a convex function on $[0,1]$, then

$$B_n(f,x)\geq B_{n+1}(f,x),\quad \text{for every } x\in[0,1] \text{ and } n\geq 1.$$

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