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FIXED POINT THEOREMS FOR NON-SELF OPERATORS AND APLICATIONS

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Contents

Keywords: singlevalued operator, multivalued operator, non-self operator, fixed point, strict fixed point, metric space, local fixed point theorem, homotopy principle, opem map principle, data dependence, well-posed problem, sets endowed with two metrics.

Introduction

The fixed point theory for singlevalued and multivalued operators is a domain of the non-linear analysis with a dynamic development in the last decades, proofs being a lot of monographs, proceedings and scientific articles appeared in these years. The subject of this Ph.D thesis also belongs to this important research domain. More precisely, the Ph.D. thesis approaches the study of the fixed point theory for non-self generalized contractions of singlevalued and multivalued type. Thus, the existence, the uniqueness, the data dependence as well as other characteristics of the fixed points are studied for generalized contractions defined on balls from complete metric spaces. A series of fixed points local theorems are stated and, as their consequences, some open operators theorems are established, theorems of invariance of the domain and continuation theorems. The case of spaces endowed with two metrics is being considered in the last chapter.

The study of fixed point theory for operators that goes from a subset to the hole space is extremely present in the specialised literature of the XX century. Beginning with Knaster-Kuratowski-Mazurkiewicz and Sperner papers (on the K^2M type operators) and the contributions of Borsuk (concerning the continuous and anti-podal operators), the study of fixed point theory for operators $f : Y \subset X \to X$ develops and gains consistency with the contributions of Leray-Schauder, Rothe, Granas, Krasnoselskii, K. Fan, Browder, R.F. Brown, etc. from the years 1940-1960. The main results refer to local fixed point theorems, to continuation and topological transversality principles, as well as, retraction theorems. Afterwards, important results, on metrical and topological theory of the fixed points for singlevalued and multivalued operators had M. Frigon, A. Granas (with the continuation theorems)

S. Reich (the case of weak interior contractions), I.A. Rus (with the theory of fixed point structures, based on retraction principles), R. Precup (with general theorems of continuation), R. Precup and D. O'Regan (Leray-Schauder type theorems), K. Deimling (the case of contractions with respect to measures of non-compactness and the case of condensator operators), etc. (see I.A. Rus, A. Petruşel, G. Petruşel [131])

This work is divided in four chapters, preceded by a list of symbols and followed by the bibliographical list of publications.

The first chapter, entitled **Preliminaries**, has the aim to remind some notions and basic results which are necessary in the presentation of the following chapters of this Ph.D. thesis. In writing this chapter, we used the following bibliographical sources: J.-P. Aubin, H. Frankowska [8], K. Deimling [36], J. Dugundji, A. Granas [49], S. Hu, N. S. Papageorgiou [54], W.A. Kirk, B. Sims [59], A. Petruşel [90], I.A. Rus [125], [121].

In the second chapter, entitled **Theory of Reich's type metric fixed point theorem** we will consider, in the first part, some basic results from the fixed point theory for Ćirić-Reich-Rus type singlevalued operators, results for which we will present a new proof method, using the technique introduced by R.S. Palais for the case of contractions. In the second part, we will present results which make up a theory of fixed point metrical theorem for Reich type multivalued operators. The author's own results are the following:

- in the section 2.1 : Lemma 2.1.1, Lemma 2.1.2, Lemma 2.1.3 and Theorem 2.1.2 (these results present a new method of proof for Reich's theorem, method which induces the approximation of fixed point with a stopping rule for the sequence of successive approximations), Lemma 2.1.4, Lemma 2.1.5 and Theorem 2.1.3 (where the above technique is used for φ-contractions). The results from this section extend to the case of Ćirić-Reich-Rus type operators and φ-contractions, the recent results given by R.S. Palais [82] for singlevalued contractions. These results were presented at the International Conference on Sciences (Oradea 12-14 November 2009) and published in the paper **T.A. Lazăr** [67];

• in the section 2.2 : Definition 2.2.2, Definition 2.2.3, Definition 2.2.4., Definition 2.2.5, Theorem 2.2.2- Theorem 2.2.5. These new notions present some abstract classes of multivalued operators and the results given here are both examples of the previous notions and the presentation of the theory of fixed point metric theorem for Reich type multivalued operators. The presented notions and results, extend similar notions and results from the papers A. Petruşel, I.A. Rus [94], [100], A. Petruşel, T.A. Lazăr [99], I.A. Rus, A. Petruşel, A. Sîntămărian [132] and J. Andres, J. Fišer [6]. The content of this section was presented at the international conference: The 11th Symposium on Symbolic and Numeric Algorithms for Scientific Computing (SYNASC09) (26-29 September 2009, Timişoara) and will appear in the volume of this conference [99], as well as, in a paper sent for publication **T.A. Lazăr**, G. Petruşel [68].

The third chapter of this paper is entitled **Fixed point theorems for non-self operators and applications** and has the aim to develop the fixed point theory for non-self generalized singlevalued and multivalued contractions, defined on balls from a metric space. Then, as applications of this theorems, data dependence, theorems of domain invariance and new principles of the open operators are presented. These were publieshed in the paper **T.A. Lazăr**, A. Petruşel and N. Shazhad, [65]. The author's own results in this chapter are:

• For the case of singlevalued operators: Remark 3.1.1, Remark 3.1.2, Theorem 3.1.4, Theorem 3.1.5, Theorem 3.1.6, Corollary 3.1.1, Theorem 3.1.7, Theorem 3.1.8, Theorem 3.1.9, Theorem 3.1.10;

• For the case of multivalued operators: Theorem 3.2.4, Theorem 3.2.5, Lemma 3.2.1, Theorem 3.2.6, Corollary 3.2.1, Corollary 3.2.2, Theorem 3.2.7, Remark 3.2.2, Theorem 3.2.8.

The results from this chapter extend and generalize some fixed point theorems for non-self operators and applications of these theorems in continuous data dependence results, theorems of domain invariance and homotopy theorems given by Agarwal-O'Regan-Precup [5], Andres-Górniewicz [7], L.Górniewicz [48], V. Berinde [14], [15], Bollenbacher-Hicks [19], Caristi [23], Browder [20], Chiş-Novac, Precup, I.A. Rus [29], R. Precup [103], [104], [104], Granas-Dugundji [37], [49], Frigon-Granas [44], Hegedüs [50], Hegedüs-Szilágyi [51], Jachymski-Jóźwik [58], Park-Kim [85], Reich [115], I.A. Rus [122], I.A. Rus-S. Mureşan [129], I.A. Rus-A. Petruşel, M.A. Şerban [133], A. Petruşel [87], M.Păcurar-I.A. Rus [86], Ts. Tsacev-V.G. Angelov [141], Walter [143], Węgrzyk [144], Ciric-Ume-Khan-Pathak [33], Ciric [32], Hicks-Saliga [53], Radovanović [114], Aamri-Chaira [1], Aubin-Siegel [9], Castañeda [24], Hicks-Rhoades [52], Hicks-Saliga [53], Jachymski [57], [56], Maciejewski [70], A. Petruşel, A. Sîntămărian [93], A. Petruşel [91].

The last chapter **Fixed point theorems on a set with two metrics**, presents a fixed point theory for singlevalued and multivalued operators defined on a set with two metrics. We prove local fixed point theorems for singlevalued operators (φ-contraction, Caristi-Browder operators, graphic contractions). There are given conditions for well-posedness of the fixed point problem, and, as aplication, is enounced a homotopy theorem for Ćirić type generalized multivalued contractions. More precisely, the author's own results from this chapter are:

• For the case of singlevalued operators: Theorem 4.1.1, Theorem 4.1.2, Theorem 4.1.3, Theorem 4.1.4, Theorem 4.1.5, and Theorem 4.1.6;

• For the case of multivalued operators: Theorem 4.2.1, Theorem 4.2.2, Theorem 4.2.3, Theorem 4.2.4, Theorem 4.2.5, Theorem 4.2.6, Theorem 4.2.7, Theorem 4.2.8, Corollary 4.2.1., Remarks 4.2.1 and 4.2.2.

The results presented in this chapter appeared in the papers: **T.A. Lazăr**, Fixed points for non-self nonlinear contractions and non-self Caristi type operators, Creative Math.& Inf. 17 (2008), No. 3, 446-451; **T.A. Lazăr**, D. O'Regan, A. Petruşel, Fixed points and homotopy results for Ciric-type multivalued operators on a set with two metrics, Bull. Korean Math. Soc. 45 (2008), No. 1, 6773; **T.A. Lazăr**, V.L. Lazăr, Fixed points for non-self multivalued operators on a set with two metrics, JP Journal of Fixed Point Theory and Applications 4 (2009), No. 3, 183-191. They extend and generalized some theorems of this type given by Agarwal-O'Regan [3], R.P. Agarwal, J.H. Dshalalow, D. O'Regan [4], Bucur-Guran [21], Bylka-Rzepecki [22], Chifu-Petrusel [27], [28], Ćirić [31], Feng-Liu [39], Frigon [43], Frigon-Granas [44], Matkowski [72], A.S. Mureşan [77], O'Regan-Precup [80], A. Petruşel-I.A. Rus [95], [96], Reich [117], I.A. Rus [126], [127], [120], B.Rzepecki [134], S.P.Singh [136], [137].

In the end, I want to thank my scientific advisor, prof. univ. dr. Adrian Petruşel, for his careful guidance and permanent encouragement I received during this period, to the members of the Differential Equations Research Group, as well as, to all the collaborators of the Nonlinear Operators and Differential Equations research seminar, for the help and collaboration they offered me. From all together and from each other I have learnt a lot. My formation as a teacher and a researcher was made at the Faculty of Mathematics and Computer Science from "Babeş-Bolyai" University Cluj-Napoca. I want to thank all my teachers once again.

Cluj-Napoca, february 2010 Drd. Tania Angelica Lazăr

0.1 List of symbols

A.

Let X a nonempty set and an operator $f : X \to X$.

$\mathcal{P}(X) := \{Y|Y \subseteq X\}$ the set of X subsets

$P(X) := \{Y \subseteq X|Y \neq \emptyset\}$ the set of nonempty subsets of X

$1_X : X \to X, 1_X(x) = x$ the identity operator

$I(f) := \{A \in P(X)|f(A) \subset A\}$ the set of invariant subsets

$Fix(f) := \{x \in X|x = f(x)\}$ the fixed point set of f

$f^0 = 1_X, \cdots f^{n+1} = f \circ f^n$, $n \in \mathbb{N}$ the iterates of f

$O(x; f) := \{x, f(x), \cdots , f^n(x), \cdots\}$ the orbit of f relativly to x

$O(x, y; f) := O(x; f) \cup O(y; f)$

If Y is another nonempty set, then

$$\mathbf{M}(X, Y) := \{f : X \to Y| \text{ f is an operator } \} \text{ and}$$

$$\mathbf{M}(Y) := \mathbf{M}(Y, Y).$$

B.

Let (X, d) a metric space.

$\widetilde{B}(x, R) := \{y \in X|d(x, y) \leq R\}$the closed ball centerd in $x \in X$ and radius $R > 0$

$B(x, R) := \{y \in X|d(x, y) < R\}$ the open ball centerd in $x \in X$ and radius $R > 0$

$diam(A) := \sup\{d(a, b)|a, b \in A\} \in \mathbb{R}_+ \cup \{+\infty\}$,where $A \subset X$

$P_b(X) := \{Y \in P(X)|\delta(Y) < +\infty\}$

$P_{op}(X) := \{Y \in P(X) | Y \text{ is open}\}$

$P_{cl}(X) := \{Y \in P(X) | Y = \overline{Y}\}$

$P_{cp}(X) := \{Y \in P(X) | Y \text{ is a compact set }\}$

$D_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{+\infty\}$

$$D_d(A, B) = \begin{cases} \inf\{d(a, b) | \ a \in A, \ b \in B\}, & \text{if } A \neq \emptyset \neq B \\ 0, & \text{if } A = \emptyset = B \\ +\infty, & \text{if } A = \emptyset \neq B \text{ or } A \neq \emptyset = B. \end{cases}$$

In particular, $D_d(x_0, B) = D_d(\{x_0\}, B)$ (where $x_0 \in X$).

$\delta_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{+\infty\}$,

$$\delta_d(A, B) = \begin{cases} \sup\{d(a, b) | \ a \in A, \ b \in B\}, & \text{if} A \neq \emptyset \neq B \\ 0, & \text{otherway} \end{cases}$$

$\rho_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{+\infty\}$,

$$\rho_d(A, B) = \begin{cases} \sup\{D_d(a, B) | \ a \in A\}, & \text{if } A \neq \emptyset \neq B \\ 0, & \text{if } A = \emptyset \\ +\infty, & \text{if } B = \emptyset \neq A \end{cases}$$

$H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{+\infty\}$,

$$H_d(A, B) = \begin{cases} \max\{\rho_d(A, B), \rho_d(B, A)\}, & \text{if } A \neq \emptyset \neq B \\ 0, & \text{if } A = \emptyset = B \\ +\infty, & \text{if } A = \emptyset \neq B \text{ or } A \neq \emptyset = B. \end{cases}$$

is called generalized Pompeiu-Hausdorff metric.

C.

Let X a Banach space.

$P_{cv}(X) := \{Y \in P(X) | Y \text{ is a convex set }\}$

$P_{cp,cv}(X) := \{Y \in P(X) | Y \text{ is a compact and convex set }\}$

$\|A\| := H(A, \{0\}),\ A \in P_{b,cl}(X).$

D.

Let (X, d) metric space, $Y \in P(X)$ and $\varepsilon > 0$. Then:

$V^0(Y, \varepsilon) := \{x \in X | \inf\limits_{y \in Y} d(x, y) < \varepsilon\}$

$V(Y, \varepsilon) := \{x \in X | \inf\limits_{y \in Y} d(x, y) \leq \varepsilon\}$

If X, Z are nonempty sets, then

$$T : X \multimap Z \text{ or } T : X \to \mathcal{P}(Z)$$

denote a multivalued operator from X to Z.

$DomT := \{x \in X | T(x) \neq \emptyset\}$

$T(Y) := \bigcup\limits_{x \in Y} T(x)$, for $Y \in P(X)$

$I(T) := \{A \in P(X) | T(A) \subset A\}$

$I_b(T) := \{Y \in I(T) | \delta(Y) < +\infty\}$

$I_{b,cl}(T) := \{Y \in I(T) | \delta(Y) < +\infty, Y = \overline{Y}\}$

$I_{cp}(T) := \{Y \in I(T) | Y \text{ is compact }\}$

$T^{-1}(z) := \{x \in X | z \in T(x)\}$

$GrafT := \{(x, z) \in X \times Z | z \in T(x)\}$

$T^-(W) := \{x \in X | T(x) \cap W \neq \emptyset\}$, for $W \in P(Z)$

$T^+(W) := \{x \in DomT | T(x) \subset W\}$, for $W \in P(Z)$

$T^+(\emptyset) = \emptyset,\ T^-(\emptyset) = \emptyset.$

The sequence of succesive aproximation coordinating to T that starts from $x \in X$ is $(x_n)_{n\in\mathbb{N}} \subset X$, defined by

$$x_0 = x, \text{ and } x_{n+1} \in T(x_n), \text{ for } n \in \mathbb{N}.$$

If $T : X \to P(X)$ is a multivalued operator, then for $x \in X$ we have the iterate of T:

$$T^0(x) = x, T^1(x) = T(x), \cdots, T^{n+1}(x) = T(T^n(x)).$$

A point $x \in X$ is caled fixed point (respectively strict fixed point) for T if

$$x \in T(x) \text{ (respectively } \{x\} = T(x)).$$

Note $Fix(T)$ (or $SFix(T)$) the fixed point set (respectively the strict fixed point set) for the multivalued operator T, i. e.

$Fix(T) := \{x \in X | x \in T(x)\}$ the fixed point set of T

$SFix(T) := \{x \in X | \{x\} = T(x)\}$ the strict fixed point set of T.

If X, Y are two nonempty sets, then note

$$\mathbf{M}^0(X,Y) := \{T \mid T : X \to \mathcal{P}(Y)\} \text{ and}$$

$$\mathbf{M}^0(Y) := \mathbf{M}^0(Y,Y).$$

Chapter 1

Preliminaries

In this chapter we present some notions and results which we will use in the next chapters of this Ph.D. thesis. In writting it, we studied the next works: J.-P. Aubin, H. Frankowska [8], K. Deimling [36], J. Dugundji, A. Granas [49], S. Hu, N. S. Papageorgiou [54], W.A. Kirk, B. Sims [59], A. Petruşel [90], I. A. Rus [125], [121].

1.1 Classes of singlevalued operators

1.2 Classes of multivalued operators

Chapter 2

Theory of Reich's type metric fixed point theorem

We will consider in the first part some base results from the fixed point theory for Reich type operators, results for which we will present a new demonstration method, using the technique introduced by R.S. Palais for the case of contractions. In the second part we will present results which make up a theory of fixed point metrical theorem for Reich type multivalued operators.

2.1 The fixed point approximation for classes of generalized contraction

In a recent paper of R.S. Palais it is given another proof of the Banach contraction principle. The result is accompanied by a stopping rule for the sequence of the successive approximations. This new proof is based on a a-contraction notion form for an operator $f : X \to X$, i.e.

$$d(x_1, x_2) \leq \frac{1}{1-a}[d(x_1, f(x_1)) + d(x_2, f(x_2))], \text{ for all } x_1, x_2 \in X.$$

Thorem 2.1.1 *(Banach, Cacciopoli, R.S. Palais [82]) If (X, d) is a complete metric space and $f : X \to X$ is an a-contraction, then f has a unique fixed point $x^* \in X$*

and for any x in X, the sequence $(f^n(x))_{n\in\mathbb{N}}$ converges to x^* as $n \to +\infty$. Moreover, we have:

i) $d(f^n(x), x^*) \leq \frac{a^n}{1-a} d(x, f(x))$, *for all* $x \in X$;

ii) (R.S. Palais [82])(stopping rule) for every $x \in X$ and for every $\epsilon > 0$ we have that $d(f^n(x), x^*) < \epsilon$, for each $n > \frac{log(\epsilon)+log(1-a)-log(d(x,f(x)))}{log(a)}$.

In the next, we extend the above approach to the case of Reich type operators and to the case of φ-contractions. This results were published in the paper T.A. Lazăr [67].

Definition 2.1.1 *The operator $f : X \to X$ is called* **Ćirić-Reich-Rus type operator** *or (a, b, c)-Ćirić-Reich-Rus type contraction, if $\exists a, b, c \in \mathbb{R}_+$ with $a + b + c < 1$ such that*

$$d(f(x_1), f(x_2)) \leq ad(x_1, x_2) + bd(x_1, f(x_1)) + cd(x_2, f(x_2)),\ \forall\ x_1, x_2 \in X.$$

Using the Reich type condition, we obtain the central inequality of this section:

$$d(x_1, x_2) \leq \frac{1}{1-a}[(1 + b)d(x_1, f(x_1)) + (1 + c)d(x_2, f(x_2))]. \qquad (2.1.3)$$

Lemma 2.1.1 *(S. Reich [116], I.A. Rus [121], L. Ćirić [30], T.A. Lazăr [67]) Let (X, d) be a metric space and $f : X \to X$ be a Reich type operator. Then, the sequence $(f^n(x))_{n\in\mathbb{N}}$ is a Cauchy, for each $x \in X$.*

Lemma 2.1.2 *(S. Reich [116], I.A. Rus [121], L. Ćirić [30], T.A. Lazăr [67]) Let (X, d) be a metric space and $f : X \to X$ be a Reich type operator. Then $card(Fix(f)) \leq 1$.*

Let (X, d) a metric space and $f : X \to X$. Then f is caled Picard operator if $Fix(f) = \{x^*\}$ and the sequence of succesive approximation $(f^n(x))_{n\in\mathbb{N}}$ for f, starting from any $x \in X$,converges to x^*. Moreover f is called c-Picard operator, if $c > 0$, f is a Picard operator and holds the relation:

$$d(x, x^*) \leq c\ d(x, f(x)),\ \text{for any } x \in X.$$

Lemma 2.1.3 *(T.A. Lazăr [67]) Let* (X,d) *a metric space and* $f : X \to X$ *a c-Picard operator. Then* $d(x_1,x_2) \leq c[d(x_1,f(x_1)) + d(x_2,f(x_2))]$ *,* $\forall x_1,x_2 \in X$*.*

Using the above results, we have the following theorem:

Thorem 2.1.2 *(S. Reich [116], I.A. Rus [121], L. Ćirić [30], T.A. Lazăr [67]) If* (X,d) *is a complete metric space and* $f : X \to X$ *is a Reich type operator, then* f *has a unique fixed point* $x^* \in X$ *and for any* x *in* X *the sequence* $(f^n(x))_{n\in\mathbb{N}}$ *converges to* x^**. Moreover, we have:*

i) $d(f^n(x),x^*) \leq \frac{1+b}{1-a}\alpha^n d(x,f(x))$*, for each* $x \in X$*.*

ii) (stopping rule) for each $x \in X$ *and every* $\epsilon > 0$*, we have that*

$$d(f^n(x),x^*) < \epsilon, \text{ for each } n > \left[\frac{log\epsilon + log(1-a) - log(1+b) - logd(x,f(x))}{log(a+b) - log(1-c)}\right] + 1.$$

We will consider now the case of φ-contractions for which has been obtained some related results.

For this we recall some notions:

Definition 2.1.2 *The function* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *is called a comparison function if* φ *is increasing and* $\varphi^n(t) \to 0$ *as* $n \to \infty$ *for* $t > 0$*. As a consequence, we have that* $\varphi(0) = 0$ *and* $\varphi(t) < t$*, for each* $t > 0$*.*

Example 2.1.1 *Well-known examples of comparison functions are* $\varphi(t) = at$ *(with* $a \in [0,1[$*),* $\varphi(t) = \frac{t}{1+t}$ *and* $\varphi(t) = log(1+t)$*, for* $t \in \mathbb{R}_+$

Definition 2.1.3 *Let* (X,d) *be a metric space. Then,* $f : X \to X$ *is said to be a* φ*-contraction if* φ *is a comparison function and*

$$d(f(x_1),f(x_2)) \leq \varphi(d(x_1,x_2)), \text{ for all } x_1,x_2 \in X.$$

for witch we obtain the relation

$$d(x_1,x_2) \leq \psi^{-1}(d(x_1,f(x_1)) + d(x_2,f(x_2))) \qquad (2.1.5)$$

where $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, $\psi(t) := t - \varphi(t)$ is strictly increasing and onto.

Based on this relations we can prove the following lemmas.

Lemma 2.1.4 *(J. Matkowski [72], I.A. Rus [124], T.A. Lazăr [67]) Let (X, d) be a metric space and let $f : X \to X$ be a φ-contraction. Then, $card(Fix(f)) \leq 1$.*

Lemma 2.1.5 *(J. Matkowski [72], I.A. Rus [124], T.A. Lazăr [67]) Let (X, d) be a metric space and let $f : X \to X$ be a φ-contraction. Suppose that $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, $\psi(t) = t - \varphi(t)$ is continuous, strictly increasing and onto. Then, for any x in X, the sequence $(f^n(x))_{n\in\mathbb{N}}$ is Cauchy.*

The main result for φ-contractions, is the following theorem (see also J. Matkowski [72], I.A. Rus [124] and J. Jachymski, I. Jóźwik [58]).

Thorem 2.1.3 *(J. Matkowski [72], I.A. Rus [124], T.A. Lazăr [67]) If X is a complete metric space, $f : X \to X$ is a φ-contraction and if the function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, $\psi(t) = t - \varphi(t)$ is continuous, strictly increasing and onto, then f has a unique fixed point $x^* \in X$ and for any $x \in X$ the sequence $(f^n(x))_{n\in\mathbb{N}}$ converges to x^*. Moreover, we have that*

$$d(f^n(x), x^*) \leq \psi^{-1}(\varphi^n(d(x, f(x)))), \text{ for each } x \in X.$$

with

Stopping Rule: For arbitary $x \in X$, consider $n \in \mathbb{N}^*$ satisfying the relation $\varphi^n(d(x, f(x))) < \psi(\epsilon)$. Then, for each $\epsilon > 0$ we get that $d(f^n(x), x^*) < \epsilon$.

2.2 Theory of Reich's type fixed point theorem

First we recal the Picard multivalued operator concept, given by A. Petruşel and I.A. Rus in [94].

Definition 2.2.1 *(A. Petruşel, I.A. Rus in [94]) Let* (X, d) *a metric space and* $T : X \to P(X)$*. The opetaor* T *is called* **Picard Multivalued operator(PM operator)** *if:*

(i) $SFix(T) = Fix(T) = \{x^*\}$

(ii) $T^n(x) \overset{H_d}{\to} \{x^*\}$*, as* $n \to \infty$, $\forall\ x \in X$.

Sufficient condition for $T : X \to P_{cl}(X)$ or $T : X \to P_{b,cl}(X)$ beeing PM operator are given in the paper A. Petruşel, I.A. Rus [94].

Let us introduce some new abstract classes of multivalued operators, for wich we will present some examples. The notions and the results presented, are related to the papers A. Petruşel, T.A. Lazăr [99] and T.A. Lazăr, G. Petruşel [68]

Definition 2.2.2 *(A. Petruşel, T.A. Lazăr [99]) Let* (X, d) *metric space and* $T : X \to P(X)$*. Then* T *satisfy the* **(PM2) property** *if:*

(i) $SFix(T) = Fix(T) = \{x^*\}$

(ii) for $\forall\ x \in X$ *and* $\forall\ y \in T(x)$, $\exists (x_n)_{n\in\mathbb{N}}$ *the sequence of the succesive approximation of* T *that starts by* $x_0 := x$ *and* $x_1 := y$ *such that* $x_n \overset{d}{\to} x^*$*, as* $n \to \infty$.

Definition 2.2.3 *(A. Petruşel, T.A. Lazăr [99]) Let* (X, d) *metric space and* $T : X \to P(X)$*. Then* T *satisfy the* **(PM3) property** *if there exists* $x^* \in Fix(T)$ *such that* $T^n(x) \overset{H_d}{\to} \{x^*\}$*, as* $n \to \infty$, $\forall\ x \in X$.

Definition 2.2.4 *(A. Petruşel, T.A. Lazăr în [99]) Let* (X, d) *metric space and* $T : X \to P(X)$*. Then* T *satisfy the* **(PM4) property** *if there exists* $x^* \in Fix(T)$ *and exists* $x_0 \in X$ *such that* $T^n(x_0) \overset{H_d}{\to} \{x^*\}$*, when* $n \to \infty$.

Definition 2.2.5 *(A. Petruşel, T.A. Lazăr [99]) Let* (X, d) *metric space and* $T : X \to P(X)$*. Then* T *satisfy the* **(PM5) property** *if exists* $x^* \in Fix(T)$ *such that for every* $x \in X$ *and every* $y \in T(x)$ *exists* $(x_n)_{n\in\mathbb{N}}$*, the succesiv approximations sequence for* T*, that begins from* $x_0 := x$ *and* $x_1 := y$ *such that* $x_n \overset{d}{\to} x^*$*, when* $n \to \infty$*.*

Another notion, close related to the above given notions, is the weakly Picard operator given by I.A. Rus, A. Petruşel and A. Sîntămărian in [132].

Definition 2.2.6 *(I.A. Rus-A. Petruşel-A. Sîntămărian [132]) Let* (X, d) *a metric space. Then,* $T : X \to P(X)$ *is called a* **multivalued weakly Picard operator (briefly MWP operator)** *if for each* $x \in X$ *and each* $y \in T(x)$ *there exists a sequence* $(x_n)_{n\in\mathbb{N}}$ *in* X *such that:*

i) $x_0 = x,\ x_1 = y$*;*

ii) $x_{n+1} \in T(x_n)$*, for all* $n \in \mathbb{N}$*;*

iii) the sequence $(x_n)_{n\in\mathbb{N}}$ *is convergent and its limit is a fixed point of* T*.*

Definition 2.2.7 *Let* (X, d) *be a metric space and* $T : X \to P(X)$ *be an MWP operator. Then we define the multivalued operator* $T^\infty : GraphT \to P(Fix(T))$ *by the formula* $T^\infty(x, y) = \{\ z \in Fix(T) \mid$ *there exists a sequence of successive approximations of* T *starting from* (x, y) *that converges to* $z\ \}$*.*

Definition 2.2.8 *Let* (X, d) *be a metric space and* $T : X \to P(X)$ *an MWP operator. Then* T *is said to be a c-multivalued weakly Picard operator (briefly c-MWP operator) if and only if there exists a selection* t^∞ *of* T^∞ *such that* $d(x, t^\infty(x, y)) \leq c\ d(x, y)$*, for all* $(x, y) \in GraphT$*.*

Other results for multivalued weakly Picard operators are presented in the paper [91].

Starting by these results we have the following result for the case of multivalued Reich's operators.

We recall a fixed point theorem for single-valued Ćirić-Reich-Rus type operators, that will be used to prove the main result of this section.

Thorem 2.2.1 *(S. Reich [116], I.A. Rus [121], L. Ćirić [30]) Let (X,d) be a complete metric space and $f : X \to X$ be a Ćirić-Reich-Rus (a,b,c)-contraction, i.e., there exist $a,b,c \in \mathbb{R}_+$ with $a+b+c<1$ such that*

$$d(f(x),f(y)) \leq ad(x,y)+bd(x,f(x))+cd(y,f(y)), \text{ for each } x,y \in X.$$

Then f is a Picard operator, i.e., we have:

(i) $Fix(f)=\{x^*\}$;

(ii) for each $x \in X$ the sequence $(f^n(x))_{n\in\mathbb{N}}$ converges in (X,d) to x^*.

Our main result concerning the theory of the Reich's fixed point theorem is the following.

Thorem 2.2.2 *(T.A. Lazăr, G. Petruşel [68]) Let (X,d) be a complete metric space and $T : X \to P_{cl}(X)$ be a Reich-type multivalued (a,b,c)-contraction, i.e., there exist $a,b,c \in \mathbb{R}_+$ with $a+b+c<1$ such that*

$$H_d(T(x),T(y)) \leq ad(x,y)+bD_d(x,T(x))+cD_d(y,T(y)), \text{ for each } x,y \in X.$$

Denote $\alpha := \frac{a+b}{1-c}$. Then we have:

(i) $Fix(T) \neq \emptyset$;

(ii) T is a $\frac{1}{1-\alpha}$-multivalued weakly Picard operator;

(iii) Let $S : X \to P_{cl}(X)$ be a Reich-type multivalued (a,b,c)-contraction and $\eta > 0$ such that $H_d(S(x),T(x)) \leq \eta$, for each $x \in X$.
Then $H_d(Fix(S),Fix(T)) \leq \frac{\eta}{1-\alpha}$;

(iv) Let $T_n : X \to P_{cl}(X)$, $n \in \mathbb{N}$ be a sequence of Reich-type multivalued (a,b,c)-contraction, such that $T_n(x) \overset{H_d}{\to} T(x)$ as $n \to +\infty$, uniformly with respect to $x \in X$. Then, $Fix(T_n) \overset{H_d}{\to} Fix(T)$ as $n \to +\infty$.

If, moreover $T(x) \in P_{cp}(X)$, for each $x \in X$, then we additionally have:

*(v) (***Ulam-Hyers stability of the inclusion** $x \in T(x)$*) Let* $\epsilon > 0$ *and* $x \in X$ *be such that* $D_d(x, T(x)) \leq \epsilon$*. Then there exists* $x^* \in Fix(T)$ *such that* $d(x, x^*) \leq \frac{\epsilon}{1-\alpha}$*;*

(vi) $\hat{T} : (P_{cp}(X), H_d) \to (P_{cp}(X), H_d)$, $\hat{T}(Y) := \bigcup_{x \in Y} T(x)$ *is a set-to-set* (a, b, c)*-contraction and (thus)* $Fix(\hat{T}) = \{A_T^*\}$*;*

(vii) $T^n(x) \overset{H_d}{\to} A_T^*$ *as* $n \to +\infty$*, for each* $x \in X$*;*

(viii) $F_T \subset A_T^*$ *and* $Fix(T)$ *is compact;*

(ix) $A_T^* = \bigcup_{n \in \mathbb{N}^*} T^n(x)$*, for each* $x \in Fix(T)$*.*

A second result for Reich-type multivalued (a, b, c)-contractions is given for the case $SFix(T) \neq \emptyset$, as follows.

Thorem 2.2.3 *(T.A. Lazăr, G. Petruşel [68] Let* (X, d) *be a complete metric space and* $T : X \to P_{cl}(X)$ *be a Reich-type multivalued* (a, b, c)*-contraction with* $SFix(T) \neq \emptyset$*. Then, the following assertions hold:*

(x) $Fix(T) = SFix(T) = \{x^*\}$*;*

*(xi) (***Well-posedness of the fixed point problem with respect to** D_d*) If* $(x_n)_{n \in \mathbb{N}}$ *is a sequence in* X *such that* $D(x_n, T(x_n)) \to 0$ *as* $n \to \infty$*, then* $x_n \overset{d}{\to} x^*$ *as* $n \to \infty$*;*

*(xii) (***Well-posedness of the fixed point problem with respect to** H_d*) If* $(x_n)_{n \in \mathbb{N}}$ *is a sequence in* X *such that* $H_d(x_n, T(x_n)) \to 0$ *as* $n \to \infty$*, then* $x_n \overset{d}{\to} x^*$ *as* $n \to \infty$*.*

Another result for Reich type multivalued operators is the following:

Thorem 2.2.4 *(T.A. Lazăr, G. Petruşel [68]) Let* (X, d) *be a complete metric space and* $T : X \to P_{cp}(X)$ *be a Reich-type multivalued* (a, b, c)*-contraction such that* $T(Fix(T)) = Fix(T)$*. Then we have:*

(xiii) $T^n(x) \overset{H_d}{\to} Fix(T)$ *as* $n \to +\infty$*, for each* $x \in X$*;*

(xiv) $T(x) = Fix(T)$, *for each* $x \in Fix(T)$*;*

(xv) If $(x_n)_{n\in\mathbb{N}} \subset X$ *is a sequence such that* $x_n \xrightarrow{d} x^* \in Fix(T)$ *as* $n \to \infty$ *and* T *is* H_d*-continuous, then* $T(x_n) \xrightarrow{H_d} Fix(T)$ *as* $n \to +\infty$.

For compact metric spaces, we have:

Thorem 2.2.5 *(T.A. Lazăr, G. Petruşel [68]) Let* (X, d) *be a compact metric space and* $T : X \to P_{cl}(X)$ *be a* H_d*-continuous Reich-type multivalued* (a, b, c)*-contraction. Then*

(xvi) If $(x_n)_{n\in\mathbb{N}}$ *is a sequence in* X *such that* $D_d(x_n, T(x_n)) \to 0$ *as* $n \to \infty$*, then there exists a subsequence* $(x_{n_i})_{i\in\mathbb{N}}$ *of* $(x_n)_{n\in\mathbb{N}}$ $x_{n_i} \xrightarrow{d} x^* \in Fix(T)$ *as* $i \to \infty$ *(***The fixed point problem is well-posed respect to** D_d **in generalized way***).*

Remark 2.2.1 *For the case* $b = c = 0$*, the above results goes to the theorems proved in: A. Petruşel, I.A. Rus [100] and A. Petruşel, T.A. Lazăr [99]*

Remark 2.2.2 *The next mentioned papers were studied in obtaining the above results: Ayerbe-Benavides-Acedo [10], M.F. Barnsley [11], [12], K. Border [18], Chang-Yen [25], Y.-Q. Chen [26], Espinola-Petruşel [38], Fraser-Nadler jr. [40], M. Fréchet [41], Glăvan-Guţu [46], Gobel-Kirk [47], Lasota-Myjak [60], Latif-Beg [61], J.T. Markin [71], Moţ-A.Petruşel-G.Petruşel [75], Petruşel-Rus [92], I.A. Rus [123], J.Saint-Raymond [135], N.S. Papageorgiou [83], L. Pasicki [84], A. Muntean [76], A. Sîntămărian [138], Taradaf-Yuan [140], H-K. Xu [145], Yamaguti-Hata-Kigani [146].*

Chapter 3

Fixed point theorems for non-self operators and aplications

The aim of this chapter is to develop the fixed point theory for non-self generalized singlevalued and multivalued contractions, defined on balls from metric spaces. Thus, in this chapter theorems of data dependence, theorems of domain invariance and new principles of the open operators are presented. These were publieshd in the paper [65], i.e., **T.A. Lazăr**, A. Petruşel, N. Shahzad: Fixed points for non-self operators and domain invariance theorems, Nonlinear Analysis 70 (2009) 117-125.

3.1 Fixed point theorems for singlevalued non-self operators and aplications

The following local fixed point result is an easy consequence of the Banach-Caccioppoli fixed point principle.

Thorem 3.1.1 *(Granas-Dugundji [49], pp. 11) Let* (X, d) *a complete metric space,* $x_0 \in X$ *and* $r > 0$*. If* $f : B(x_0; r) \to X$ *is an* a*-contraction and* $d(x_0, f(x_0)) <$

$(1-a)r$, then f has a unique fixed point.

Let us remark that if $f : \widetilde{B}(x_0; r) \to X$ is an a-contraction such that $d(x_0, f(x_0)) \leq (1-a)r$, then $\widetilde{B}(x_0; r) \in I(f)$ and again f has a unique fixed point in $\widetilde{B}(x_0; r)$.

Definition 3.1.1 *Let E be a Banach space and $Y \subset E$. Given an operator $f : Y \to E$, the operator $g : Y \to E$ defined by $g(x) := x - f(x)$ is called the field associated with f.*

Definition 3.1.2 *An operator $f : Y \to E$ is said to be open if for any open subset U of Y the set $f(U)$ is open in E too.*

As a consequence of the above result one obtains the following domain invariance theorem for contraction type fields.

Thorem 3.1.2 *(see Dugundji-Granas [49], pp. 11) Let E be a Banach space and Y be an open subset of E. Consider $f : U \to E$ be an a-contraction. Let $g : U \to E$ $g(x) := x - f(x)$, the associated field. Then:*

(a) $g : U \to E$ is an open operator;

(b) $g : U \to g(U)$ is a homeomorphism. In particular, if $f : E \to E$, then the associated field g is a homeomorphism of E into itself.

In the next part of this section, it is given some generalization of the mentioned known results for classes of generalized contractions.

The next result is known in literature as J. Matkowski and I.A. Rus theorem.

Thorem 3.1.3 *(J. Matkowski [72], I. A. Rus [124]) Let (X, d) be a complete metric space and $f : X \to X$ a φ-contraction.*
Then $Fix(f) = \{x^*\}$ and $f^n(x_0) \to x^*$ when $n \to \infty$, for all $x_0 \in X$.

Using the φ-contraction defined on an open ball, we obtain first a fixed point theorem for this kind of operator. As consequence, a domain invariance theorem for φ-contractive fields follows. For a similar results see also Agarwal-O'Regan-Shahzad [2].

Thorem 3.1.4 *(T.A. Lazăr, A. Petruşel and N. Shazhad [65]) Let (X,d) be a complete metric space, $x_0 \in X$ and $r > 0$. Let $f : \widetilde{B}(x_0;r) \to X$ be a φ-contraction such that $d(x_0, f(x_0)) < r - \varphi(r)$. Then*

$$Fix(f) \cap B(x_0;r) = \{x^*\}.$$

Thorem 3.1.5 *(T.A. Lazăr, A. Petruşel and N. Shazhad [65]) Let E a Banach space and $U \in P_{op}(E)$. Suppose $f : U \to E$ is a φ-contraction. Let $g : U \to E$ be the field associated with f.*

Then:

(a) $g : U \to E$ is an open operator;

(b) $g : U \to g(U)$ is a homeomorphism. In particular, if $f : E \to E$, then the associated field g is a homeomorphism of E into itself.

Remark 3.1.1 *(T.A. Lazăr, A. Petruşel and N. Shazhad [65])*

a) If we consider $\varphi(t) = at$, for each $t \in \mathbb{R}_+$ (where $a \in [0,1[$) then the above theorem is Theorem 3.1.2.
Other choices for φ can be, for example, $\varphi(t) = \frac{t}{1+t}$ and $\varphi(t) = \ln(1+t)$, $t \in \mathbb{R}_+$.

b) Theorem 3.1.5 extends Theorem 2.1 of Dugundji-Granas in [37], where the continuity of φ is additionally imposed.

c) Theorem 3.1.5 could be compared with Theorem 10 of Jachymski-Jóźwik in [58], where the space E need not be complete and U is not assumed to be open, but the function $\psi(t) := t - \varphi(t)$, $t \in \mathbb{R}_+$ is assumed to be increasing.

In the next we present some continuous data dependence result for non-self φ-contractions. These results are in connection to some more general results (that were appeard latter) in A. Chis-Novac, R. Precup, I.A. Rus [29].

Thorem 3.1.6 *(T.A. Lazăr, A. Petruşel and N. Shazhad [65]) Let* (X, d) *be a complete metric space,* $x_0 \in X$ *and* $r > 0$*. Let* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *be such that the function* $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ *defined by* $\psi(t) = t - \varphi(t)$*, is strictly increasing and onto. Let* $f, g : \widetilde{B}(x_0; r) \to X$*. Suppose that:*

i) $d(x_0, f(x_0)) < r - \varphi(r)$*;*

ii) f *is a* φ*-contraction (denote by* x_f^* *its unique fixed point, see Theorem 3.1.4);*

iii) there exists $x_g^* \in Fix(g)$*;*

iv) $d(f(x), g(x)) \leq \eta$*, for each* $x \in \widetilde{B}(x_0; r)$*.*

Then we have $d(x_f^*, x_g^*) \leq \psi^{-1}(\eta)$*. Moreover,* $\psi^{-1}(\eta) \to 0$ *as* $\eta \to 0$*.*

Remark 3.1.2 *(T.A. Lazăr, A. Petruşel and N. Shazhad [65]) In particular, if* φ *is a continuous comparison function and* $\lim_{t \to +\infty} \psi(t) = +\infty$*, then* ψ *is a self-bijection on* $\mathbb{R}_+$*.*

Thorem 3.1.7 *(T.A. Lazăr, A. Petruşel and N. Shazhad [65]) Let* (X, d) *be a complete metric space,* $x_0 \in X$ *and* $r > 0$*. Let* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *be such that the function* $\psi : \mathbb{R}_+ \to \mathbb{R}_+$*, defined by* $\psi(t) = t - \varphi(t)$*, is increasing. Let* $f, f_n : \widetilde{B}(x_0; r) \to X$*,* $n \in \mathbb{N}$ *be such that* $(f_n)_{n \in \mathbb{N}}$ *converges uniformly to* f*, as* $n \to +\infty$*.*

Suppose that:

i) $d(x_0, f(x_0)) < r - \varphi(r)$*;*

ii) f *is a* φ*-contraction;*

iii) $Fix(f_n) \neq \emptyset$ *for each* $n \in \mathbb{N}$*.*

Then, for $x_n \in Fix(f_n)$*,* $n \in \mathbb{N}$ *we have* $x_n \to x^*$*, as* $n \to +\infty$ *(where* $\{x^*\} = B(x_0; r) \cap Fix(f)$*).*

Consider now the case of the Caristi-type operators. The result is in connection with a version of Caristi's theorem given by Bollenbacher and Hicks in [19].

Thorem 3.1.8 *(T.A. Lazăr, A. Petruşel and N. Shazhad [65]) Let* (X, d) *be a complete metric space,* $x_0 \in X$ *and* $r > 0$*. Let* $\varphi : X \to \mathbb{R}_+$ *be a function such that* $\varphi(x_0) < r$*. Consider* $f : B(x_0; r) \to X$ *such that* $d(x, f(x)) \leq \varphi(x) - \varphi(f(x))$*, for each* $x \in B(x_0; r) \cap O_f(x_0)$*. If* f *has a closed graph or the function* $x \mapsto d(x, f(x))$*,* $x \in B(x_0; r)$ *is lower semicontinuous, then* $Fix(f) \neq \emptyset$*.*

Corollyar 3.1.1 *(T.A. Lazăr, A. Petruşel and N. Shazhad [65]) Let* (X, d) *be a complete metric space,* $f : X \to X$*,* $x_0 \in X$ *and* $r > 0$*. Suppose that there exists* $a \in]0, 1[$ *such that* $d(f(x), f^2(x)) \leq a \cdot d(x, f(x))$*, for each* $x \in B(x_0; r) \cap O_f(x_0)$ *and* $d(x_0, f(x_0)) < (1 - a)r$*. If* f *has a closed graph or the function* $x \mapsto d(x, f(x))$*,* $x \in B(x_0; r)$ *is lower semicontinuous then* $Fix(f) \neq \emptyset$*.*

Another consequence of Caristi's theorem for operators defined on a ball is the next result.

Thorem 3.1.9 *(T.A. Lazăr, A. Petruşel and N. Shazhad [65]) Let* (X, d) *be a complete metric space and* $f : X \to X$ *be an operator with bounded orbits. Suppose that there exists* $a \in [0, 1[$ *such that* $diam O_f(f(x)) \leq a \cdot diam O_f(x)$*, for each* $x \in X$*. If* f *has a closed graph or the function* $x \mapsto diam O_f(x)$ *is lower semicontinuous, then* $Fix(f) \neq \emptyset$*.*

Next result is again a local one:

Thorem 3.1.10 *(T.A. Lazăr, A. Petruşel and N. Shazhad [65]) Let* (X, d) *be a complete metric space,* $x_0 \in X$ *and* $r > 0$*. Let* $f : X \to X$ *be an operator with bounded orbits. Suppose that there exists* $a \in [0, 1[$ *such that* $diam O_f(f(x)) \leq a \cdot diam O_f(x)$*, for each* $x \in B(x_0; r) \cap O_f(x_0)$ *and* $diam O_f(x_0) < (1 - a)r$*. If* f *has a closed graph or the function* $x \mapsto diam O_f(x)$*,* $x \in B(x_0; r)$ *is lower semicontinuous, then* $Fix(f) \neq \emptyset$*.*

3.2 Fixed point theorems for multivalued non-self operators and aplications

The following result was given, basically, by Frigon and Granas in [44] (see also [7]).

Thorem 3.2.1 *(Frigon-Granas [44]) Let (X, d) be a complete metric space, $x_0 \in X$ and $r > 0$. Let $F : B(x_0; r) \to P_{cl}(X)$ be a multivalued a-contraction such that $D_d(x_0, F(x_0)) < (1-a)r$. Then $Fix(F) \neq \emptyset$.*

Remark 3.2.1 *If in the above theorem, instead of the assumption $D_d(x_0, F(x_0)) < (1-a)r$ we put the stronger one $\delta(x_0, F(x_0)) < (1-a)r$, then we get that the closed ball $\tilde{B}(x_0; s)$ is invariant with respect to F and the existence of a fixed point, results from theorem Covitz-Nadler [34].*

Let X be a Banach space, $U \in P_{op}(X)$ and $F : U \to P(X)$ a multivalued operator. Then we denote by G the multivalued field associated with F, i. e. $G : U \to P(X)$, given by $G(x) = x - F(x)$.

An open operator principle for multivalued a-contractions is the following theorem (see also [7]):

Thorem 3.2.2 *(Andres-Gorniewicz [7]) Let X be a Banach space, $x_0 \in X$ and $r > 0$. Let $F : B(x_0, r) \to P_{b,cl}(X)$ be a multivalued a-contraction and let $G : B(x_0, r) \to P_{b,cl}(X)$ be the multivalued field associated with F.*
Then $G(B(x_0, r))$ is an open subset in X.

The aim of this section is to generalize both Theorem 3.2.1 and Theorem 3.2.2 for the case of some multivalued generalized contraction.

The following result is known in the literature as Węgrzyk's theorem (see [144]).

Thorem 3.2.3 *Let (X, d) be a complete metric space and $F : X \to P_{cl}(X)$ be a multivalued φ-contraction, where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a strict comparison function.*

Then $Fix(F)$ is nonempty and for any $x_0 \in X$ there exists a sequence of successive approximations of F starting from x_0 which converges to a fixed point of F.

A local result for φ-contractions was obtain through the next theorem:

Thorem 3.2.4 *(T.A. Lazăr, A. Petruşel and N. Shazhad în [65]) Let (X, d) be a complete metric space, $x_0 \in X$ and $r > 0$. Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a strict comparison function such that the function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, $\psi(t) := t - \varphi(t)$ is strictly increasing, continuous in r and $\sum_{n=1}^{\infty} \varphi^n(\psi(s)) \leq \varphi(s)$, for each $s \in]0, r[$. Let $F : B(x_0; r) \to P_{cl}(X)$ be a multivalued φ-contraction such that $D_d(x_0, F(x_0)) < r - \varphi(r)$. Then $Fix(F) \neq \emptyset$.*

A similar result is the following theorem.

Thorem 3.2.5 *(T.A. Lazăr, A. Petruşel and N. Shazhad [65]) Let (X, d) be a complete metric space, $x_0 \in X$ and $r > 0$. Let $F : \widetilde{B}(x_0; r) \to P_{cl}(X)$ be a multivalued φ-contraction such that $\delta(x_0, F(x_0)) < r - \varphi(r)$. Then $Fix(F) \cap B(x_0; r) \neq \emptyset$.*

An auxiliary result is:

Lemma 3.2.1 *(T.A. Lazăr, A. Petruşel and N. Shazhad [65]) Let X a normed space. Then for $\forall\ x, y \in X$ and $\forall\ A \in P_{cl}(X)$ we have the relation: $D_d(x, A + y) = D_d(y, x - A)$.*

Using Theorem 3.2.4 we can prove the following result for the open multivalued operators.

Thorem 3.2.6 *(T.A. Lazăr, A. Petruşel and N. Shazhad [65])*
Let X be a Banach space and $U \in P_{op}(X)$. Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a strict comparison function, such that the function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, $\psi(t) := t - \varphi(t)$ is strictly increasing and continuous on $\mathbb{R}_+$.

Suppose that there exists $r_0 > 0$ such that $\sum_{n=1}^{\infty} \varphi^n(\psi(s)) \leq \varphi(s)$, for each $s \in]0, r_0[$. Let $F : U \to P_{cl}(X)$ be a multivalued φ-contraction.

Then, the multivalued field G associated with F is open.

Taking into account that a multivalued φ-contraction is lower semicontinuous and hence the image of a connected set is connected, we have:

Corollyar 3.2.1 *(T.A. Lazăr, A. Petruşel and N. Shazhad [65]) Let X be a Banach space and $U \in P(X)$ a domain (i. e. open and connected). Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a strict comparison function such that the function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, $\psi(t) := t - \varphi(t)$ is strictly increasing and continuous on $\mathbb{R}_+$. Suppose that there exists $r_0 > 0$ such that $\sum_{n=1}^{\infty} \varphi^n(\psi(s)) \leq \varphi(s)$, for each $s \in]0, r_0[$. Let $F : U \to P_{cl}(X)$ be a multivalued φ-contraction. Let G be the multivalued field associated with F. Then $G(U)$ is a domain too.*

Using the same prove principle and Węgrzyk's theorem, we have the next result (see [122]):

Corollyar 3.2.2 *(T.A. Lazăr, A. Petruşel and N. Shazhad [65]) Let X be a Banach space and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a strict comparison function. Let $F : X \to P_{cl}(X)$ be a multivalued φ-contraction. Then, the multivalued field G associated with F is surjective.*

We will consider now the case of multivalued Meir-Keeler operators and we obtain:

Thorem 3.2.7 *(T.A. Lazăr, A. Petruşel and N. Shazhad [65]) Let (X, d) be a metric space, $x_0 \in X$ and $r > 0$. Let $F : X \to P_{cp}(X)$ be a multivalued Meir-Keeler operator. Suppose that $\delta(x_0, F(x_0)) \leq \eta(r)$, where $\eta(r)$ denotes the positive number corresponding to $r > 0$ by Def.1.2.8. point 9). Then:*

(i) $B(x_0, r + \eta(r)) \in I(F)$;

(ii) If (X, d) is complete, then there exists $x^* \in Fix(F) \cap \widetilde{B}(x_0, r + \eta(r))$.

Remark 3.2.2 *(T.A. Lazăr, A. Petruşel and N. Shazhad [65]) If the operator F, in the above theorem, is singlevalued, then one obtains a result given by Park and Kim in [85].*

Let us establish now a fixed point result for multivalued Caristi type operators defined on a ball.

Thorem 3.2.8 *(T.A. Lazăr, A. Petruşel and N. Shazhad [65]) Let (X, d) be a complete metric space, $x_0 \in X$ and $r > 0$. Let $\varphi : X \to \mathbb{R}_+$ be a function such that $\varphi(x_0) < r$. Consider $F : B(x_0; r) \to P_{cl}(X)$ such that for each $x \in B(x_0; r)$ there exists $y \in F(x)$ such that $d(x, y) \leq \varphi(x) - \varphi(y)$. If F has a closed graph, then $Fix(F) \neq \emptyset$.*

Remark 3.2.3 *The results of this section extend and generalize some theorems from works by J. Andres-L. Górniewicz [7], J.P. Aubin-J. Siegel [9], M. Frigon-A. Granas [44], M. Maciejewski [70], A. Petruşel [89], [88], [93], I.A. Rus [119].*

Chapter 4

Fixed point theorems on a set with two metrics

The aim of this chapter is to present a fixed point theory for single and multivalued operators defined on a set with two metrics.

4.1 Fixed point theorems for singlevalued operators

We start this section presenting the well posed fixed point problem notion for singlevalued operators.

Definition 4.1.1 *(Reich-Zaslawski, I.A. Rus [127]) Let (X, d) a metric space, $Y \subseteq X$ and $f : Y \to X$. We said that the fixed point problem is well-posed for f relatively to the metric d, if $Fix(f) = x^*$ and for any sequence $(x_n)_{n \in (N)}$ from Y, such that $d(x_n, f(x_n)) \to 0$ as $n \to \infty$, we have $x_n \xrightarrow{d} x^*$ as $n \to \infty$.*

The first of this section is an extension of the Matkowski-Rus (3.1.3) theorem to the case of a set X endowed with two metrics.

Thorem 4.1.1 *(T.A. Lazăr [63]) Let X be a nonempty set, and d, d' two metrics on X, Suppose that:*

i) (X, d) *is a complete metric space;*

ii) there exists $c > 0$ *such that* $d(x, y) \leq cd'(x, y)$ *for each* $x, y \in X$.

Let $f : X \to X$ *be a* φ*-contraction with respect to* d' *and suppose that* $f : (X, d) \to (X, d)$ *is continuous. Then*

A) $Fix(f) = \{x^*\}$.

B) If additionally, the mapping $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, $\psi(t) := t - \varphi(t)$ *is continuous, strictly increasing and onto, then the fixed point problem for* f *is well posed with respect to* d'.

A local result of this type is:

Thorem 4.1.2 *(T.A. Lazăr [63]) Let* X *be a nonempty set, and* d, d' *two metrics on* X*. Suppose that*

i) (X, d) *is a complete metric space,*

ii) there exists $c > 0$ *such that* $d(x, y) \leq cd'(x, y)$ *for each* $x, y \in X$.

Let $x_0 \in X$, $r > 0$ *and* $f : \bar{B}^d_{d'}(x_0; r) \to X$ *be a* φ*-contraction respect to* d'*. Suppose that* $d'(x_0, f(x_0)) < r - \varphi(r)$ *and* $f : (X, d) \to (X, d)$ *continuous. Then:*

A) $Fix(f) \cap \bar{B}^d_{d'}(x_0, r) = \{x^*\}$.

B) If additionally, we suppose that $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, $\psi(t) := t - \varphi(t)$ *is continuous, strictly increasing and onto, then the fixed point problem for* f *is well posed with respect to* d' *metrics.*

Consider now the case of the Caristi-type operators.

Thorem 4.1.3 *(T.A. Lazăr [63]) Let* X *be a nonempty set, and* d, d' *two metrics on* X*. Suppose that:*

i) (X, d) *is a complete metric space;*

ii) there exists $c > 0$ *such that* $d(x, y) \leq cd'(x, y)$ *for each* $x, y \in X$.

Let $x_0 \in X, r > 0$ and $\varphi : X \to \mathbb{R}_+$ be a function such that $\varphi(x_0) < r$. Consider $f : \bar{B}^d_{d'}(x_0; r) \to X$ such that $d'(x, f(x)) \leq \varphi(x) - \varphi(f(x))$, for each $x \in \bar{B}^d_{d'}(x_0; r)$. If f has closed graph with respect to the metric d or the function $x \mapsto d(x, f(x))$, $x \in \bar{B}^d_{d'}(x_0; r)$ is lower semicontinuous, then $Fix(f) \neq \emptyset$.

The following three results are applications of the above local theorem of Caristi type.

Thorem 4.1.4 *(T.A. Lazăr [63]) Let X be a nonempty set, and d, d' two metrics on X. Suppose that:*

i) (X, d) is a complete metric space;

ii) there exists $c > 0$ such that $d(x, y) \leq cd'(x, y)$ for each $x, y \in X$;

Let $f : X \to X$, be an operator and let $x_0 \in X$. Suppose that there exists $a \in]0, 1[$ such that:

a) $d'(f(x), f^2(x)) \leq a \cdot d'(x, f(x))$, for each $x \in \bar{B}^d_{d'}(x_0; r)$

b) $d'(x_0, f(x_0)) < (1 - a)r$.

If f has a closed graph with respect to the metric d or the function $x \mapsto d(x, f(x))$, $x \in \bar{B}^d_{d'}(x_0; r)$ is lower semicontinuous, then $Fix(f) \neq \emptyset$.

Another consequence of Caristi's theorem for operators defined on a ball will be considered now.

Thorem 4.1.5 *(T.A. Lazăr [63]) Let X be a nonempty set, and d, d' two metrics on X, Suppose that:*

i) (X, d) is a complete metric space;

ii) there exists $c > 0$ such that $d(x, y) \leq cd'(x, y)$ for each $x, y \in X$.

Let $f : X \to X$ be an operator with bounded orbits. Suppose that there exists $a \in [0, 1[$ such that $diamO^{d'}_f(f(x)) \leq a \cdot diamO^{d'}_f(x)$, for each $x \in X$. If f has *closed graph with respect to the metric d or the function $x \mapsto diamO_f^d(x)$ is lower semicontinuous, then $Fix(f) \neq \emptyset$.*

Next result is again a local one:

Thorem 4.1.6 *(T.A. Lazăr [63]) Let X be a nonempty set, and d, d' two metrics on X, Suppose that:*

i) (X, d) is a complete metric space;

ii) there exists $c > 0$ such that $d(x, y) \leq cd'(x, y)$ for each $x, y \in X$.

Let $x_0 \in X$ and $r > 0$, $f : X \to X$ be an operator with bounded orbits. Suppose that there exists $a \in [0, 1[$ such that $diamO_f^{d'}(f(x)) \leq a \cdot diamO_f^{d'}(x)$, for each $x \in \bar{B}_{d'}^d(x_0; r) \cap O_f(x_0)$ and $diamO_f^{d'}(x_0) < (1 - a)r$. If f has closed graph with respect to the metric d or the function $x \mapsto diamO_f^d(x)$, $x \in \bar{B}_{d'}^d(x_0; r)$ is lower semicontinuous, then $Fix(f) \neq \emptyset$.

4.2 Fixed point theorems for multivalued operators

4.2.1 The φ-multivalued contraction case

Using the known Węgrzyk's ([144]), Theorem 3.2.3, we obtain the following local result on a set with two metrics.

Thorem 4.2.1 *(T.A. Lazăr şi V.L. Lazăr [66]) Let X be a nonempty set and d, d' two metrics on X, $x_0 \in X$, $r > 0$. Suppose that:*

i) (X, d) is a complete metic space;

ii) there exists $c > 0$ such that $d(x, y) \leq cd'(x, y)$ for each $x, y \in X$;

iii) $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a strict comparison function such that the function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, $\psi(t) := t - \varphi(t)$ strict increasing and continuous with $\sum\limits_{n=0}^{\infty} \varphi^n(\psi(r)) \leq \varphi(r)$.

Let $F : \bar{B}^d_{d'}(x_0; r) \to P_{cl}(X)$ be a multivalued φ-contraction such that $D_{d'}(x_0, F(x_0)) < r - \varphi(r)$. Suppose that $F : \bar{B}^d_{d'}(x_0; r) \to P((X, d))$ has closed graph. Then $Fix(F) \neq \emptyset$.

As a consequence we have the next result on open balls.

Thorem 4.2.2 *(T.A. Lazăr şi V.L. Lazăr [66]) Let X be a nonempty set and d, d' two metrics on X, $x_0 \in X$, $r > 0$. Suppose that :*

i) (X, d) is a metric space;

ii) there exists $c > 0$ such that $d(x, y) \leq cd'(x, y)$ for each $x, y \in X$;

iii) $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a strict comparison function such that the function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, $\psi(t) := t - \varphi(t)$ is strictly increasing and continuous in r, with $\sum\limits_{n=0}^{\infty} \varphi^n(\psi(s)) \leq \varphi(s)$ for all $s \in]0, r[$.

Let $F : B_{d'}(x_0; r) \to P_{cl}(X)$ be a multivalued φ-contraction respect to metric d' such that $D_{d'}(x_0, F(x_0)) < r - \varphi(r)$. Then $Fix(F) \neq \emptyset$.

Using the above theorem we can obtain an open operator principle in a Banach space.

Thorem 4.2.3 *(T.A. Lazăr şi V.L. Lazăr [66]) Let X be a Banach space, $\|\cdot\|, \|\cdot\|'$ two norms on X. Let d, d' the metrics induced by normes $\|\cdot\|, \|\cdot\|'$. Let U an open set with respect to the norm $\|\cdot\|'$ and let $F : U \to P_{cl}(X)$ a multivalued operator. Suppose that:*

i) there exists $c > 0$ such that $d(x, y) \leq cd'(x, y)$ for all $x, y \in X$;

ii) $F : U \to P_{cl}(X)$ is a φ-contraction with respect to the norm $\|\cdot\|$ (i.e. $H_{d'}(F(x_1), F(x_2)) \leq \varphi(\|x_1 - x_2\|')$ for all $x_1, x_2 \in U$;

iii) $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a strict comparison function, such that the function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, $\psi(t) := t - \varphi(t)$ is strictly increasing and continuous on $\mathbb{R}_+$ and there exists $r_0 > 0$ such that $\sum_{n=1}^{\infty} \varphi^n(\psi(s)) \leq \varphi(s)$, for each $s \in]0, r_0[$.

Then, the multivalued field $G : U \to P(X), G(x) = x - F(x)$ is an open operator.

Let us establish now a fixed point result for multivalued Caristi type operators defined on a ball with two metrics.

Thorem 4.2.4 *(T.A. Lazăr şi V.L. Lazăr [66]) Let X be a nonempty set, and d, d' two metrics on X, $x_0 \in X$ and $r > 0$. Suppose that:*

i) (X, d) is a complete metric space;

ii) there exists $c > 0$ such that $d(x, y) \leq cd'(x, y)$ for each $x, y \in X$;

Let $\varphi : X \to \mathbb{R}_+$ be a function with $\varphi(x_0) < r$. Consider $F : \bar{B}^d_{d'}(x_0; r) \to P_{cl}(X)$ such that $\forall x \in \bar{B}^d_{d'}(x_0; r)$ there exists $y \in F(x)$ such that $d'(x, y) \leq \varphi(x) - \varphi(y)$. If F is a closed operator with respect to the metric d, then $Fix(F) \neq \emptyset$.

4.2.2 The Ćirić type multivalued operators

Let (X, d) a metric space $T : X \to P_{cl}(X)$ multivalued operator. For $x, y \in X$, we denote

$$M_d^T(x, y) := \max\{d(x, y), D_d(x, T(x)), D_d(y, T(y)), \frac{1}{2}[D_d(x, T(y)) + D_d(y, T(x))]\}.$$

A slight modified variant of Ćirić's theorem (see [31]) is the following.

Thorem 4.2.5 *Let (X, d) a complet metric space and $T : X \to P_{cl}(X)$ a multivalued operator that satisfies the following condition:*

there exists $\alpha \in [0, 1[$ such that $H_d(T(x), T(y)) \leq \alpha \cdot M_d^T(x, y)$, for each $x, y \in X$.

Then $Fix(T) \neq \emptyset$ and for each $x \in X$ and each $y \in T(x)$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that:

(1) $x_0 = x,\ x_1 = y$;

(2) $x_{n+1} \in T(x_n),\ n \in \mathbb{N}$;

(3) $x_n \xrightarrow{d} x^* \in T(x^*)$, as $n \to \infty$;

(4) $d(x_n, x^*) \leq \frac{(\alpha p)^n}{1-\alpha p} \cdot d(x_0, x_1)$, for each $n \in \mathbb{N}$ (where $p \in]1, \frac{1}{\alpha}[$ is arbitrary).

In connection to the above theorem we present a data dependence theorem for Ćirić-type multivalued operators.

Thorem 4.2.6 *(T.A. Lazăr, D. O'Regan and A. Petruşel [64]) Let (X, d) be a complete metric space and $T_1, T_2 : X \to P_{cl}(X)$ two multivalued operators. Suppose that:*

i) there exists $\alpha_i \in [0, 1[$ such that

$$H_d(T_i(x), T_i(y)) \leq \alpha_i \cdot M_d^{T_i}(x, y), \text{ for each } x, y \in X, \text{ for } i \in \{1, 2\};$$

ii) there exists $\eta > 0$ such that $H_d(T_1(x), T_2(x)) \leq \eta$, for each $x \in X$.

Then $Fix(T_1) \neq \emptyset \neq Fix(T_2)$ and $H_d(Fix(T_1), Fix(T_2)) \leq \frac{\eta}{1-\max\{\alpha_1,\alpha_2\}}$.

We continue the section with a local version of Ćirić's theorem on a set with two metrics. We recall first the concept of well-posed problem for multivalued operators.

Definition 4.2.1 *(I.A. Rus-A. Petruşel [96], [97]) Let (X,d) a metric space, $Y \subseteq X$ and $T : Y \to P_{cl}(X)$ a multivalued operator. We said that the fixed point problem is well-posed in the generalized sense*

a) relativ to D_d: if $Fix(T) \neq \emptyset$ and for every sequence $(x_n)_{n\in\mathbb{N}} \subset Y$ such that $D_d(x_n, T(x_n)) \to 0$ as $n \to +\infty$ we have that $x_n \xrightarrow{d} x \in Fix(T)$ as $n \to +\infty$

b) relativ to H_d: if $SFix(T) \neq \emptyset$ and for every sequence $(x_n)_{n\in\mathbb{N}} \subset Y$ such that $H_d(x_n, T(x_n)) \to 0$ as $n \to +\infty$ we have that $x_n \xrightarrow{d} x \in SFix(T)$ as $n \to +\infty$

Thorem 4.2.7 *(T.A. Lazăr, D. O'Regan and A. Petruşel [64]) Let X be a nonempty set, $x_0 \in X$ and $r > 0$. Suppose that d, d' two metrics on X and $T : \overline{B}^d_{d'}(x_0, r) \to P(X)$ is a multivalued operator. We suppose that:*

i) (X,d) is a complete metric space;

ii) there exists $c > 0$ such that $d(x,y) \leq cd'(x,y)$, for each $x, y \in X$;

iii) if $d \neq d'$ then $T : \overline{B}^d_{d'}(x_0, r) \to P(X^d)$ has closed graph with respect to d, while
if $d = d'$ then $T : \overline{B}^d_{d}(x_0, r) \to P_{cl}(X^d)$;

iv) there exists $\alpha \in [0,1[$ such that $H_{d'}(T(x), T(y)) \leq \alpha M^T_{d'}(x,y)$, for each $x, y \in \overline{B}^d_{d'}(x_0, r)$;

v) $D_{d'}(x_0, T(x_0)) < (1-\alpha)r$.

Then:

(A) there exists $x^* \in \overline{B}^d_{d'}(x_0, r)$ such that $x^* \in T(x^*)$;

(B) if $SFix(T) \neq \emptyset$ and $(x_n)_{n\in\mathbb{N}} \subset \overline{B}^d_{d'}(x_0, r)$ is such that $H_{d'}(x_n, T(x_n)) \to 0$ as $n \to +\infty$, then $x_n \xrightarrow{d'} x \in SFix(T)$ as $n \to +\infty$ (i.e. the fixed point problem is well-posed in the generalized sense for T with respect to $H_{d'}$).

Remark 4.2.1 *(T.A. Lazăr, D. O'Regan and A. Petruşel [64]) Theorem 4.2.7 holds if the condition (ii) is replaced by:*

(ii') if $d' \not\geq d$ *then for each* $\epsilon > 0$ *there exists* $\delta > 0$ *such that for each* $x, y \in \overline{B}^d_{d'}(x_0, r)$ *with* $d'(x,y) < \delta$ *we have* $d(u,v) < \epsilon$*, for each* $u \in T(x)$ *and* $v \in T(y)$*.*

A homotopy result for Ćirić-type multivalued operators on a set with two metrics is the following theorem.

Thorem 4.2.8 *(T.A. Lazăr, D. O'Regan şi A. Petruşel [64]) Let* (X, d) *be a complete metric space and* d' *another metric on* X *such that there exists* $c > 0$ *with* $d(x,y) \leq cd'(x,y)$*, for each* $x, y \in X$*. Let* U *be an open subset of* (X, d') *and* V *be a closed subset of* (X, d)*, with* $U \subset V$*.*

Let $G : V \times [0,1] \to P(X)$ *be a multivalued operator such that the following conditions are satisfied:*

i) $x \notin G(x,t)$*, for each* $x \in V \setminus U$ *and each* $t \in [0,1]$*;*

ii) there exists $\alpha \in [0,1[$*, such that for each* $t \in [0,1]$ *and each* $x, y \in V$ *we have:*

$$H_{d'}(G(x,t), G(y,t)) \leq \alpha M^{G(\cdot,t)}_{d'}(x,y);$$

iii) there exists a continuous increasing function $\phi : [0,1] \to \mathbb{R}$ *such that*

$$H_{d'}(G(x,t), G(x,s)) \leq |\phi(t) - \phi(s)| \text{ for all } t, s \in [0,1] \text{ and each } x \in V;$$

iv) $G : V^d \times [0,1] \to P(X^d)$ *is closed.*

Then $G(\cdot, 0)$ *has a fixed point if and only if* $G(\cdot, 1)$ *has a fixed point.*

We can obtain a special case of T.4.2.8, useful in applications, if we take $d = d'$.

Corollyar 4.2.1 *(T.A. Lazăr, D. O'Regan şi A. Petruşel [64]) Let* (X, d) *be a complete metric space,* U *be an open subset of* X *and* V *be a closed subset of* X*, with*

$U \subset V$. Let $G : V \times [0,1] \to P(X)$ be a closed multivalued operator such that the following conditions are satisfied:

i) $x \notin G(x,t)$, for each $x \in V \setminus U$ and each $t \in [0,1]$;

ii) there exists $\alpha \in [0,1[$, such that for each $t \in [0,1]$ and each $x, y \in V$ we have:

$$H_d(G(x,t), G(y,t)) \leq \alpha M_d^{G(\cdot,t)}(x,y);$$

iii) there exists a continuous increasing function $\phi : [0,1] \to \mathbb{R}$ such that

$$H_d(G(x,t), G(x,s)) \leq |\phi(t) - \phi(s)| \text{ for all } t, s \in [0,1] \text{ and each } x \in V;$$

Then $G(\cdot,0)$ has a fixed point if and only if $G(\cdot,1)$ has a fixed point.

Remark 4.2.2 *(T.A. Lazăr, D. O'Regan and A. Petruşel [64]) Usually in Corollary 4.2.1 we take $Q = \overline{U}$. Notice that in this case condition (i') becomes:*

(i') $x \notin G(x,t)$, for each $x \in \partial U$ and each $t \in [0,1]$.

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