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FACULTY OF MATHEMATICS AND COMPUTER SCIENCE

Fixed point methods for the study of semilinear evolution equations

Ph.D. Thesis

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Contents

1.1 Basic notations and results 11

1.1.1 Compactness and completely continuous operators 11

1.2 Sobolev Spaces 14

1.3 Eigenvalues and eigenvectors 14

1.4 The non-homogenous heat equation in $\mathbf{H}^{-1}(\boldsymbol{\Omega})$ 14

1.5 The non-homogenous wave equation in $\boldsymbol{H}^{-1}(\boldsymbol{\Omega})$. 15

2 Existence theory for systems of semilinear heat and wave equations 17

2.1 Systems of semilinear heat equations 17

2.1.1 Application of Perov's fixed point theorem 18

2.1.2 Application of Schauder's fixed point theorem 18

2.1.3 Application of Leray-Schauder's fixed point theorem 19

2.2 Systems of semilinear wave equations 20

2.2.1 Application of Perov's fixed point theorem 21

2.2.2 Application of Schauder's fixed point theorem 22

3 Existence theory for general semilinear evolution equations and systems 23

3.1 Semilinear evolution equations 24

3.1.1 The solution operator 24

Application of Banach's contraction principle 25

3.1.3 Application of Schauder's fixed point theorem 26

3.2 Semilinear evolution systems 27

3.2.1 Application of Perov's fixed point theorem 27

3.2.2 Application of Schauder's fixed point theorem 27

3.2.3 Examples 28

4 Nonlinear Schrödinger equations via fixed point principles 29

4.1 Linear Schrödinger equations 29

4.1.1 Introduction 29

4.1.2 The non-homogenous Schrodinger equation in $\boldsymbol{H}^{-1}(\boldsymbol{\Omega})$ 30

4.2 Schrödinger solution operator 31

4.2.1 Norm estimations 31

4.2.2 Compactness 32

4.3 Existence results for nonlinear Schrödinger equations 32

4.3.1 Application of Banach's fixed point theorem ... 32
4.3.2 Application of Schauder's fixed point theorem ... 33
4.3.3 Application of the Leray-Schauder fixed point theorem ... 34
5 Systems of nonlinear Schrödinger equations ... 36
5.1 Nonlinear Schrödinger equations ... 36
5.1.1 Application of Perov's fixed point theorem ... 37
5.1.2 Application of Schauder's fixed point theorem ... 38
5.1.3 Application of Leray-Schauder's fixed point theorem ... 39
Bibliography ... 41

Key words: evolution equations, fixed point, systems of nonlinear equations, compact operators, infinitesimal generators of a C_0-semigroups of contractions, convergent to zero matrices.

Introduction

Partial differential equations is a many-faceted subject. Created to describe the mechanical behavior of objects such as vibrating strings and blowing winds, it has developed into a body of material that interacts with many branches of mathematics, such as differential geometry, complex analysis, and harmonic analysis, as well as a ubiquitous factor in the description and elucidation of problems in mathematical physics.

The classical development of nonlinear functional analysis arose contemporaneously with the beginnings of linear functional analysis at about the beginning of the twentieth century in the work of such men as Picard, S.Bernstein, Ljapunov, E.Schimdt, and Lichtenstein and was motivated by the desire to study the existence and properties of boundary value problems for nonlinear partial differential equations. Its most classical tool was the Picard contraction principle (put in its sharpest form by Banach in his thesis of 1920- the Banach fixed-point theorem).

Beyond the early development of bifurcation theory by Ljapunov and E.Schimdt around 1905, the second, and even more fruitful, branch of the classical methods in nonlinear functional analysis was developed in the theory of the compact nonlinear mappings in Banach spaces in the late 1920's and early 1930's. These included Schauder's well-known fixed-point theorem and the extension of the Brower topological degree by Leray and Schauder in 1934 to mappings to Banach spaces of the form $I + C$ with C compact (as well as interesting related results of Caccioppoli on nonlinear Fredholm mappings).

The central role of compact mappings in this phase of the development of nonlinear functional analysis was due in part to the nature of the technical apparatus being developed, but also in part to a not fruitful tendency to see the theory of integral equations as the predestinated domain of application of the theory to be developed. Since, however, the more significant analytical problems lie in the somewhat different domain of boundary value problems for partial differential equations, and since the efforts to apply the theory of compact operators (and in particular the Leray-Schauder theory) to the latter problems have given rise to demands for ever more inaccessible (and sometimes invalid) *a priori* estimates in these problems; the hope of applying nonlinear functional analysis to the problems of this type centers on a general program of creating new theories for significant classes of nonlinear operators like infinitesimal generators of a C_0-semigroups of contractions or monotone-like operators.

In this thesis we are concerned with the study of semilinear boundary value problems using the operator approach based on abstract results from nonlinear functional analysis. The methods we use were initiated in early sixties by J.L.Lions and Temam for non-homogenous equations with the source term in $H^{-1}(\Omega)$, and Perov and Kibenko for semilinear operator systems and since then it was been extensively used for specific problems:

-boundary value problems for partial differential equation: J.-L.Lions [42], J.-L.Lions, E.Magenes [41], T.Cazenave [15], T.Cazenave, A.Haraux [16], T.Cazenave, F.B.Weissler [17], T.Kato [36], R.Temam [76], D.Bainov, E. Minchev and A.Myshkis [5], V.Barbu [6], H.Brezis and F.Browder [11], I.Bialynicki-Birula and J. Mycielski [14], L.C.Evans [21], R.Glassey [28], T.Kato [34], A.I.Perov, A.V.Kibenko [64], R.Precup [65], [66], [67].

-parabolic and elliptic boundary value problems: N.H.Pavel [62], D.Gilbarg and N.S.Trudinger [24], K.Tintarev [77]

-semilinear operator systems: A.I. Perov, A.V.Kibenko [64], R.Precup [65], [68], C.Avramescu [4], I.A. Rus [72], M.J. Ablowitz, B.Prinary and A.D.Trubatch [1], A.Domarkas [20].

-nonlinear semigroups and differential equations: V.Barbu [7], F.Kappel, H.Brezis and M.G.Crandall [39], A.C.McBride [52], N.H.Pavel [62], A.Pazy [63], I.Vrabie [81], [82],[83].

-nonlinear functional analysis and partial differential equations: H.Brezis [12], M.Clapp [18], P.Jebelean [33].

Related topics can be found in A.De Bouard [9], J.Bourgain [10], D.Bainov, E. Minchev [13], C.Cohen-Tannoudji, J.Dupont-Roc, G.Grynberg [19], R.P.Feynman [22], [23], J.Ginibre and G.Velo [25], [26], [27], A.Granas, J. Dugundji [29], H.Grosse and A.Martin [30].

The focus of this study is to find such operators for which we can prove the compactness property in order to apply contraction principles for different classes of partial differential equations and corresponding systems.

The classical boundary value problems (BVP) of mathematical physics include, besides the elliptic equations, the initial BVP for the heat equation and the Cauchy problem for the wave equation; in addition, following the development of quantum mechanics, the initial value problem for the Schrödinger equation. All these problems can be written in a common operatorial form:

$$Lu = Fu$$

where

(1) for the heat equation: $Lu = u_t - \tilde{a}\Delta u$
(2) for the wave equation: $Lu = u_{tt} - \tilde{a}\Delta u$
(3) for the Schrödinger equation: $Lu = iu_t + \Delta u$,

L is a differential operator, F a nonlinearity. We shell prove that the solution operator $L^{-1} = S$ of the nonlinear problem

$$\begin{cases} Lu = f(x) \\ u|_{t=0} = u_0 \end{cases}$$

exists and more it is completely continuous for all three types of equations.

The goal of this work is to make more precise the operator approach for some evolution partial differential equations and extend the theory to semilinear operator systems. More exactly, we shall precise basic properties, such as norm estimation and compactness, for the (linear) solution operator associated to the non-homogeneous linear evolution equations and we shall use them in order to apply the Banach, Schauder and Leray-Schauder theorems to the fixed point problems equivalent to Chauchy-Dirichlet problems for evolution equations. We extend these results to the corresponding semilinear operator systems. Banach's contraction principle on complete metric spaces will be replaced by Perov's fixed point theorem for completely continuous operators and the Leray-Schauder principle for completely continuous operators and set-contractions.

The thesis is divided into 5 chapters, each chapter containing several sections.

Chapter 1: Preliminaries.

In the first section of this chapter are reviewed some basic notations and results: the notions of compactness and completely continuous operators and some of their properties (R.Precup [68], H.Brezis [12]); the fixed point principles used throughout the thesis in order to prove existence results to semilinear operator equations and systems (A.Granas and J.Dugundji [29], A.I.Perov and A.V. Kibenko [64], R.Precup [67], I.A.Rus [72]. [73]) .

In the next two sections are presented embedding theorems for Sobolev Spaces of real-(or complex-) valued functions (H.Brezis [12], D. Gilbarg and N. S. Trudinger [24], J.-L. Lions [42], J.-L.Lions et E.Magenes [41]) and vector-valued Sobolev spaces (T. Cazenave, A.Haraux [16], J.-L. Lions [43], J.-L.Lions et E.Magenes [41]) ; continuous semigroups of operators (V.Barbu [7], A.Pazy [63], I.Vrabie [81], [82]).

Chapter 2: Existence theory for systems of semilinear heat and wave equations.

Motivation for the study of Chapter 2 consists of known results from A.I.Perov and A.V. Kibenko [64] for a vector version of the contraction principle applied for the heat and wave equations and it was extended recently in R.Precup [67] to other topics of nonlinear analysis. Our main goal in Chapter 2 is to extend the methods for heat and wave equation systems and to generalize the results from R.Precup [68].

In the section 2.1 we will present the first evolution equation for which we will apply our results. We shall precise basic properties, such as norm estimation and compactness, for the (linear) solution operator associated to the non-homogenous linear heat equation and we shell use them in order to apply the Perov's, Schauder and Leray-Schauder theorems to the fixed point problem equivalent to the system:

$$\begin{cases} Lu = F(u,v) & \\ Lv = G(u,v) & \text{on } Q \\ u(x,0) = 0, \;\; v(x,0) = 0 & \text{on } \Omega \\ u = v = 0 & \text{on } \Sigma \end{cases} \tag{0.0.1}$$

Here by Lu we mean $Lu = u_t - \Delta u$ such that $S = \mathrm{L}^{-1}$ and F, G are nonlinear operators. We seek weak solution to the problem (0.0.1) that is fixed point problem $\begin{bmatrix} u \\ v \end{bmatrix} = N\begin{bmatrix} u \\ v \end{bmatrix}$ in the space $C([0,T]; H^{-1}(\Omega)) \times \mathrm{C}([0,\mathrm{T}]; H^{-1}(\Omega))$, where $u, v \in C\big([0,T]; L^2(\Omega)\big)$, $N = (N_1, N_2)$ defined by

$$N_1 = S \circ F \text{ and } N_2 = S \circ G.$$

In the second section we apply the same program for the wave equation and the system:

$$\begin{cases} Lu = F(u,u',v,v') \\ Lv = G(u,u',v,v') \\ u(x,0) = 0, \;\; \dfrac{\partial u}{\partial t}(x,0) = 0 \\ v(x,0) = 0, \; \dfrac{\partial v}{\partial t}(x,0) = 0 \\ u = v = 0 \text{ on } \Sigma \end{cases} \tag{0.0.2}$$

in $E_0 = H_0^1(\Omega) \times L^2(\Omega)$ and $E = L^2(\Omega) \times H^{-1}(\Omega)$ endowed with the norms

$$|w|_{E_0} = \left(|w_1|_{H_0^1}^2 + |w_2|_{L^2}^2\right)^{1/2}$$

$$|w|_E = \left(|w_1|_{L^2}^2 + |w_2|_{H^{-1}}^2\right)^{1/2}$$

for any $w = [w_1, w_2]$. Here $w = \frac{\partial^2 w}{\partial t^2} - \Delta w$, $w_1 = (u, u'), w_2 = (v, v')$.

The analysis of (0.0.1) and (0.0.2) will be a vector-valued one and will use matrices instead of constants. The result is an existence theory derived from contractions principles.

Our contributions in this chapter are: Lemma 2.1.1, Theorem 2.1.2, Theorem 2.1.3, Theorem 2.1.4, Theorem 2.2.1, Theorem 2.2.2.

Chapter 3: Existence theory for general semilinear evolution equations and systems.

Our arguments are related to same ideas found in Chapter 2 but extended to the case of general evolution equation systems.

This chapter has two main sections. The contributions from first section are based on results of I.Vrabie [81], [82]. We present some properties of the solution operator like the complete continuity, crucial result for next section. In section 2 we present two fixed point results for the system of semilinear equations:

$$\begin{cases} u_t - Au = F(u,v) \\ v_t - Au = G(u,v) \\ u(0) = v(0) = 0 \end{cases} \tag{0.0.3}$$

in a Banach space X. Here $Lu = u_t - Au$ and $A: D(A) \subset X \to X$ is the infinitesimal generator of a C_0 semigroup of contractions.L is a linear operator such that $S = \mathrm{L}^{-1}$ and F, G are nonlinear operators. We seek weak solution of the fixed point problem $\begin{bmatrix} u \\ v \end{bmatrix} = N\begin{bmatrix} u \\ v \end{bmatrix}$ in the space X^2, where$u, v \in C([0,T]; X) \times C([0,T]; X)$, $N = (N_1, N_2)$ defined by

$$N_1 = S \circ F \text{ and } N_2 = S \circ G.$$

In particular we will present results derived from Perov's fixed point theorem and Schauder's contraction principle. The results from this section contain as a particular case the results of R. Precup (see [68]).

Our contributions in this chapter are: Theorem 3.1.2, Theorem 3.1.3 and Theorem 3.2.1.

Chapter 4: Nonlinear Schrödinger equations via fixed point principles.

This chapter deals with weak solvability of the Cauchy-Dirichlet problem for the perturbated Schrodinger equation:

$$\begin{cases} u_t - i\Delta u = F(u) & in\ \Omega \times (0,\mathrm{T}) \\ u(x,0) = 0 & in\ \ \Omega \\ u = 0 & on\ \partial\Omega \times (0,\mathrm{T}) \end{cases} \tag{0.0.4}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain and F is a general nonlinear operator which, in particular, can be a superposition operator, a delay operator, or an integral operator. Specific Schrödinger equations arise as models from several areas of physics. The problem is a classical one (see [15], [36], [42], [41] and [76]) and our goal here is to make more precise the operator approach based on abstract results from nonlinear functional analysis. More exactly, we shall precise basic properties, such as norm estimation and compactness, for the (linear) solution operator associated to the non-homogeneous linear Schrödinger equation and we shall use them in order to apply the Banach, Schauder and Leray-Schauder theorems to the fixed point problem. The same programme has been applied to discuss nonlinear perturbations of the heat and wave equations in [65] and [66].

Theorems 4.2.1, 4.3.1, 4.3.2, and 4.3.3, are the original results contained in Chapter 4 of this work. These theorems are included in M.Manole and R.Precup [49].

Chaper 5: Systems of nonlinear Schrödinger equations.

In the first section we will present the Schrödinger equation for which we will apply our results:

$$\begin{cases} u_t - i\Delta u = f & on\ Q = \Omega \times (0,\mathrm{T}) \\ u(x,0) = 0 & on\ \ \Omega \\ u = 0 & on\ \Sigma = \partial\Omega \times (0,\mathrm{T}) \end{cases} \tag{0.0.5}$$

We shall precise basic properties, such as norm estimation and compactness, for the (linear) solution operator associated to the non-homogenous linear Schrödinger equation obtained in Chapter 4 and we shell use them in order to apply the Perov, Schauder and Leray-Schauder theorems to the fixed point problem equivalent to the system :

$$\begin{cases} Lu = F(u,v) \\ Lv = G(u,v) \\ u(x,0) = 0,\ \ v(x,0) = 0 \end{cases} \tag{0.0.6}$$

in $H^{-1}(\Omega)$. Here by Lu we mean $Lu = u_t - i\Delta u$ such that $S = \mathrm{L}^{-1}$ and F,G are nonlinear operators. We seek weak solution to problem (0.0.6) that is fixed point problem $\begin{bmatrix} u \\ v \end{bmatrix} = N\begin{bmatrix} u \\ v \end{bmatrix}$ in the space $C([0,T];H^{-1}(\Omega)) \times \mathrm{C}([0,\mathrm{T}];H^{-1}(\Omega))$, where $u,v \in C\big([0,T];L^2(\Omega)\big)$, $N = (N_1,N_2)$ defined by

$$N_1 = S \circ F \text{ and } N_2 = S \circ G.$$

The original results are stated in theorems 5.1.1, 5.1.2, and 5.1.3, These theorems are included in paper Manole [50].

Preliminaries

Our arguments and proofs rely essentially on one of the following basic results in nonlinear analysis.

1.1 Basic notations and results

1.1.1 Compactness and completely continuous operators

1.1.2 Fixed point principles

In order to study nonlinear differential equations we are going to apply the following theorems (see Cazenave [14], Vrabie [82] and Precup [66]) in Chapters 2, 3 and 4. After we show the existence and compactness of the solution operator for the heat equation, wave equation, a general evolution equation and Schrödinger equation we can apply the next result to our evolution equations. In Chapter 2 of this work we use results due to Precup [66] and [68]. The first one is Banach's contraction principle

Theorem 1.1.11 (Banach) *Let (X,d) be a complete metric space and* $F: X \to X$ *. If there exists a constant L < 1 such that* $d(F(x), F(y)) \leq Ld(x, y)$ *for all* $x, y \in X$*, then* F *has a unique fixed point* $x_0 \in X$*; i.e., there exists a unique* $x_0 \in X$ *such that* $F(x_0) = x_0$.

The next two theorems are the known as the fixed point theorem of Schauder. In applications the second variant of Schauder's theorem is most useful.

Theorem 1.1.12 (Schauder) *Let K be a nonempty, convex and bounded set in a Banach space X and let* $T: K \to K$ *be a continuous operator. Then T has at least one fixed point in K, i.e. there exists at least one* $u \in K$ *such that* $T(u) = u$.

Theorem 1.1.13 (Schauder) *Let D be nonempty, convex and bounded, closed set in a Banach space X and let* $T: D \to D$ *be a completely continuous operator. Then T has at least one fixed point in D.*

The next fixed point theorem is the Leray-Schauder principle. In the applications one of the drawbacks of Schauder's fixed point theorem is the invariance condition $T(D) \subset D$ which has to be guaranteed for a bounded, closed and convex subset D of a Banach space. The Leray-Schauder principle makes it possible to avoid such a condition and requires instead that a 'boundary condition' is satisfied.

Theorem 1.1.14 ***(Leray–Schauder).*** *Let X be a Banach space, K a bounded open subset of X with* $0 \in K$, *and* $N : \overline{K} \to X$ *a completely continuous operator. If* $u \neq \lambda N(u)$ *for all* $u \in \overline{K} \setminus K$ *and* $\lambda \in (0,1)$, *then N has at least one fixed point.*

In applications the Leray-Schauder principle is usually used together with the so called '*a priori*' bounds technique:

Suppose we wish to solve the operator equation

$$N(u) = u, \; u \in K, \tag{1.1.1}$$

Where K is closed, convex subset of a Banach space $(X, \|\cdot\|_X)$, and $N: K \to K$ is completely continuous. Then we look at the set of all solutions to the one-parameter family of equations

$$u = (1-\lambda)u_0 + \lambda N(u), \quad u \in K, \tag{1.1.2}$$

when $\lambda \in (0,1)$. Here $u_0 \in K$ is fixed (in most cases $u_0 = 0$). If this set is bounded, i.e., there exists $R > 0$ such that

$$\|u - u_0\|_X < R$$

whenever u solves (1.1.2) for some $\lambda \in (0,1)$, then we let U be the intersection of K with the open ball $B(u_0, R)$ from X. Thus, Theorem 1.1.14 applies and guarantees the existence of s solution to (1.1.1).

The last fixed point theorem we present is Perov's theorem. The Banach contraction principle was generalized in A.I.Perov and A.V.Kibenko [64] for contractive maps on spaces endowed with vector-valued metrics. We present first some basic notions and results and afterwards we will state Perov's theorem.

Let X be a nonempty set. By a *vector-valued metric* on X we mean a map $d: X \times X \to \mathbb{R}^n$ with the following properties:

i) $d(u,v) \geq 0$ for all $u, v \in X$; if $d(u,v) = 0$ then $u = v$.
ii) $d(u,v) = d(v,u)$ for all $u, v \in X$;
iii) $d(u,v) \leq d(u,w) + d(w,v)$ for all $u, v, w \in X$.

A set X endowed with a vector-valued metric is said to be a generalized metric space. For the generalized metric spaces the notions of a convergent sequence, completeness, open subset and closed subset are similar to those for usual metric spaces.

Definition 1.1.15 Let (X, d) be a generalized metric space. A map $N: X \to X$ is said to be *contractive* if there exists a matrix $M \in M_{n\times n}(\mathbb{R}_+)$ such that

$$M^k \to 0 \text{ as } k \to \infty \tag{1.1.3}$$

and

$$d(N(u), N(v)) \leq Md(u, v)$$

for all $u, v \in X$. A matrix M which satisfies (1.1.3) is said to be *convergent to zero*.

Lemma 1.1.16 *(see Precup* [68]*) Let M be a square matrix of nonnegative numbers. The following statements are equivalent:*

(i) *M is a matrix convergent to zero.*

(ii) *I-M is non-singular and*

$(I - M)^{-1} = I + M + M^2 + \cdots$.

(iii) $|\lambda| < 1$ *for every* $\lambda \in \mathbb{C}$ *with det*$(M - \lambda I) = 0$.

(iv) $I - M$ *is non-singular and* $(I - M)^{-1}$ *has nonnegative elements.*

Theorem 1.1.15 ***(Perov)*** *Let (E,d) be a complete generalized metric space with* $d: E \times E \to \mathbb{R}^n$, *and let* $N: E \to E$ *be such that*

$$d\big(N(u), N(v)\big) \leq Md(u, v) \tag{1.1.4}$$

for all $u, v \in E$ *and some square matrix M of nonnegative numbers. If the matrix M is convergent to zero, that is* $M^k \to 0$ *as* $k \to \infty$ *then N has a unique fixed point* u *and*

$$d(N^k(v), u) \leq M^k(I - M)^{-1}d(N(v), v) \tag{1.1.5}$$

for every $v \in E$ *and* $k \geq 1$.

1.2 Sobolev Spaces

1.3 Eigenvalues and eigenvectors

1.4 The non-homogenous heat equation in $\boldsymbol{H^{-1}(\Omega)}$

The next lemmas and theorems are used in Chapter 2 in order to justify the complete continuity of the solution operator S for the non-homogenous heat equation in $H^{-1}(\Omega)$.

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u = f & on\ Q = \Omega \times (0, \mathrm{T}) \\ u(x,0) = g_0 & on\ \ \Omega \\ u = 0 & on\ \Sigma = \partial\Omega \times (0, \mathrm{T}) \end{cases} \tag{1.4.1}$$

In this way we have the theoretical support to extend the theory for the systems of semilinear operators. We refer to Precup [65] and [66] .

Theorem 1.4.1 (Lions) *If* $f \in L^2\big(0,T; H^{-1}(\Omega)\big)$ *and* $g_0 \in L^2(\Omega)$, *then there is an unique function*

$$u \in L^2\big(0,T; H_0^1(\Omega)\big) \cap C\big([0,T]; L^2(\Omega)\big)$$

such that for all $v \in H_0^1(\Omega)$ *the function* $(u(t), v)_{L^2}$ *is absolutely continuous on* $[0,T]$ *and*

$$\begin{cases} \dfrac{d}{dt}(u(t), v)_{L^2} + (u(t), v)_{H_0^1} = (f(t), v)\ a.p.t.\ t \in [0,T] \\ \qquad\qquad u(0) = g_0. \end{cases}$$

Moreover, for any $\in [0,T]$ *, we have*

$$\frac{1}{2}|u(t)|_{L^2}^2 - \frac{1}{2}|g_0|_{L^2}^2 + \int_0^t |u(\tau)|_{H_0^1}^2 d\tau = \int_0^t \big(f(\tau), u(\tau)\big) d\tau.$$

The estimation theorem that follows implies, on the one hand the continuously dependence of f and g_0 of the solution u of the problem (1.4.1), and, on the other hand guarantees the nonexpansivity of the solution operator from $L^2\big(0,T; H^{-1}(\Omega)\big)$ to $L^2\big(0,T; H_0^1(\Omega)\big)$ and from $L^2\big(0,T; H^{-1}(\Omega)\big)$to $C\big([0,T]; L^2(\Omega)\big)$.

Theorem 1.4.2 *Let* $f \in L^2\big(0,T; H^{-1}(\Omega)\big)$ *and* $g_0 \in L^2(\Omega)$. *If* u *is the solution of the problem* (1.4.1) *then for any* $t \in [0,T]$ *we have*

$$|u|_{C\left([0,t]; L^2(\Omega)\right)} \le \sqrt{2|g_0|^2 + |f|^2}$$

$$|u|_{L^2\left(0,t; H_0^1(\Omega)\right)} \le \frac{1}{2}\left(|f| + \sqrt{|f|^2 + 2|g_0|^2}\right)$$

where $|f| = |f|_{L^2\left(0,t;H^{-1}(\Omega)\right)}$ *and* $|g_0| = |g_0|_{L^2(\Omega)}$.

In particular, for $g_0 = 0$ *and* $t \in [0,T]$, *the following inequalities hold:*

$$|u|_{C\left([0,t];L^2(\Omega)\right)} \leq |f|_{L^2\left(0,t;H^{-1}(\Omega)\right)}$$

$$|u|_{L^2\left(0,t;H_0^1(\Omega)\right)} \leq |f|_{L^2\left(0,t;H^{-1}(\Omega)\right)}.$$

Theorem 1.4.3 *Let* $g_0 \in L^2(\Omega)$ *and*

$$\Phi: C\big([0,T]; L^2(\Omega)\big) \to L^2\big(0,T; H^{-1}(\Omega)\big)$$

be a map for which exists a constant $a \in \mathbb{R}_+$ *such that the following inequality holds for all* $u, v \in L^2\big(0,T; H^{-1}(\Omega)\big)$

$$|\Phi(u)(t) - \Phi(v)(t)|_{H^{-1}(\Omega)} \leq a|u(t) - v(t)|_{L^2(\Omega)}\ a.e. on\ [0,T].$$

Then there exists a unique solution u to problem (2.0.1), i.e.,

$$u \in L^2\big(0,T; H_0^1(\Omega)\big) \cap C\big([0,T]; L^2(\Omega)\big)$$

With the property that for all $v \in H_0^1(\Omega)$ *the map* $(u(t), v)_{L^2}$ *is absolutely continuous on* $[0,T]$ *and*

$$\begin{cases} \dfrac{d}{dt}(u(t), v) + (u(t), v)_{H_0^1} = (\Phi(u)(t), v)\ a.p.t.\ t \in [0,T] \\ u(0) = g_{0.} \end{cases}$$

Theorem 1.4.4 *The solution operator S is completely continuous from* $L^2\big(0,T; H^{-1}(\Omega)\big)$ *to* $L^2\big(0,T; L^p(\Omega)\big)$ *for* $(2^*)' \leq p < 2^*$ *if* $n \geq 3$ *and for any* $p \geq 1$ *if* $n = 1$ *or* $n = 2$.

1.5 The non-homogenous wave equation in $H^{-1}(\Omega)$.

Another group of theorems and lemmas will be useful to insure the complete continuity of the solution operator S for the non-homogenous wave equation. We refer to Precup [66].

Theorem 1.5.1 (Lions-Mangenes) *If* $f \in L^2\big(0,T; H^{-1}(\Omega)\big)$, $g_0 \in L^2(\Omega)$ *and* $g_1 \in H^{-1}(\Omega)$, *then there exists an unique function u such that*

$$u \in C\big([0,T]; L^2(\Omega)\big) \cap C^1\big([0,T]; H^{-1}(\Omega)\big)$$

$$u(0) = g_0,\ \ u'(0) = g_1$$

$$\int_0^T (u,h)_{L^2} dt = \int_0^T (f,v)dt + (g_1, v(0)) - \big(g_0, v'(0)\big)_{L^2}$$

for any map $h \in L^2\big(0,T; L^2(\Omega)\big)$, *where* $v \in C\big([0,T]; H_0^1(\Omega)\big) \cap C^1\big([0,T]; L^2(\Omega)\big)$ *is the solution of the problem*

$$\begin{cases} \dfrac{\partial^2 v}{\partial t^2} - \Delta v = h & on\ Q \\ v(x,T) = \dfrac{\partial v}{\partial t}(x,T) = 0 & on\ \Omega \\ v = 0 & on\ \Sigma \end{cases} . \tag{1.5.1}$$

Definition 1.5.1 By a (weak or generalized) solution of the Cauchy-Dirichlet problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u = f & in\ Q \coloneqq \Omega \times (0,\infty) \\ u(x,0) = g_0(x),\ \dfrac{\partial u}{\partial t}(x,0) = g_1(x) & in\ \Omega \\ u = 0 & on\ \Sigma \coloneqq \partial\Omega \times (0,\infty) \end{cases} \tag{1.5.2}$$

where $f \in L^2\big(0,T; H^{-1}(\Omega)\big)$, $g_0 \in L^2(\Omega)$ *and* $g_1 \in H^{-1}(\Omega)$, *we mean the function u defined in the theorem* 1.5.1

Remarque 1.5.2 *If* $f \in L^2\big(0,T; H^{-1}(\Omega)\big)$, $g_0 \in L^2(\Omega)$ *and* $g_1 \in H^{-1}(\Omega)$, *then the weak solution of the problem* (1.5.2) satisfies:

$$u \in C\big([0,T]; H_0^1(\Omega)\big) \cap C^1\big([0,T]; L^2(\Omega)\big)$$

$$|u(t)|_{H_0^1}^2 \le 3\left(|g_0|_{H_0^1}^2 + |g_1|_{L^2}^2 + t\int_0^t |f(s)|_{H^{-1}}^2 ds\right)$$

$$|u'(t)|_{L^2}^2 \le 3\left(|g_0|_{H_0^1}^2 + |g_1|_{L^2}^2 + t\int_0^t |f(s)|_{H^{-1}}^2 ds\right).$$

2 Existence theory for systems of semilinear heat and wave equations

Our main goal in this chapter is to extend the method used by Precup in [67] and to generalize his results to the semilinear operator systems for heat and wave equations.

In the first part of this chapter we consider the nonlinear Cauchy-Dirichlet problem for systems of heat equations. Our framework is based on existence and uniqueness results of R.Precup (see [65]) and we establish the existence of a weak solution for the system of semilinear heat equations. Next we apply the fixed-point theory for this type of systems and we present existence results via Perov, Schauder and Leray-Schauder principles. Our approach relies on compact operators theory combined with matrices that converge to zero method. We use the same programme for a nonlinear wave equations and also, we establish existence results via fixed-point principles.

2.1 *Systems of semilinear heat equations*

Let Ω be an open and bounded subset of $\mathbb{R}^N$, $0 < T < \infty$ and consider the Cauchy-Dirichlet problem for the heat equation:

$$\begin{cases} u_t - \Delta u = f \quad on\ Q = \Omega \times (0, \mathrm{T}) \\ \quad u(x,0) = 0 \ \ on\ \ \Omega \\ u = 0 \qquad\qquad on\ \Sigma = \partial\Omega \times (0, \mathrm{T}) \end{cases} \tag{2.0.1}$$

By Theorems 1.3.1, 1.3.2 and 1.3.3 in R.Precup [67] we can associate to (2.1.1) the solution operator

$$S: L^2\big(0,T; H^{-1}(\Omega)\big) \to L^2\big(0,T; H_0^1(\Omega)\big) \cap C\big([0,T]; L^2(\Omega)\big),$$

defined by $Sf = u$ where $u \in L^2\big(0,T; H_0^1(\Omega)\big) \cap C\big([0,T]; L^2(\Omega)\big)$ is the weak solution of problem (2.1.1) .

In the first part of this chapter we are concerned with the existence of solutions for the system of semilinear heat equations:

$$\begin{cases} \qquad Lu = F(u,v) on\ Q \\ \qquad Lv = G(u,v) on\ Q \\ u(x,0) = 0, \ \ v(x,0) = 0\ on\ \Omega \\ \qquad\quad u = v = 0 \end{cases} \tag{2.0.2}$$

Here by Lu we mean $Lu = u_t - \Delta u$ such that $S = \mathrm{L}^{-1}$ and F, G are nonlinear operators. We seek weak solution to problem (2.0.2) that is fixed point problem $\begin{bmatrix} u \\ v \end{bmatrix} = N\begin{bmatrix} u \\ v \end{bmatrix}$ in the space $C([0,T]; L^2(\Omega)) \times \mathrm{C}([0,\mathrm{T}]; L^2(\Omega))$, where $N = (N_1, N_2)$ is defined by

$$N_1 = S \circ F \text{ and } N_2 = S \circ G.$$

2.1.1 **Application of Perov's fixed point theorem**

Our first result is an existence, uniqueness and approximation theorem. First we present an useful result.

Lemma 2.1.1 *Let* $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ *be a square matrix of nonnegative numbers. Then for* $\theta > 0$ *large enough the matrix*

$$M := \begin{bmatrix} a_1/\sqrt{\theta} & b_1/\sqrt{\theta} \\ a_2/\sqrt{\theta} & b_2/\sqrt{\theta} \end{bmatrix}$$

is convergent to zero.

Theorem 2.1.2 *Let* $F, G: C([0,T]; L^2(\Omega)) \times C([0,T]; L^2(\Omega)) \to L^2(0,T; H^{-1}(\Omega))$ *be continuous operators. Assume that*

$$|F(u_1,v_1)(t) - F(u_2,v_2)(t)|_{H^{-1}} \leq a_1|u_1(t) - u_2(t)|_{L^2} + b_1|v_1(t) - v_2(t)|_{L^2}$$

and (2.1.1)

$$|G(u_1,v_1)(t) - G(u_2,v_2)(t)|_{H^{-1}} \leq a_2|u_1(t) - u_2(t)|_{L^2} + b_2|v_1(t) - v_2(t)|_{L^2}$$

for every $(u_1,v_1),(u_2,v_2) \in C([0,T]; L^2(\Omega)) \times C([0,T]; L^2(\Omega))$ *and some nonnegative constants* a_1, a_2, b_1, b_2 .

Then (2.0.2) *has an unique solution* $(u,v) \in C([0,T]; L^2(\Omega)) \times C([0,T]; L^2(\Omega))$.

2.1.2 **Application of Schauder's fixed point theorem**

The next theorem is an existence result derived from Schauder's fixed point principle, assuming that nonlinearities F and G have a growth at most linear.

Theorem 2.1.3

Let $, G: L^2([0,T]; L^2(\Omega)) \times L^2([0,T]; L^2(\Omega)) \to L^2(0,T; H^{-1}(\Omega))$. *Assume that F and G are continuous and satisfy the growth conditions.*

$$|F(u_1,u_2)(t)|_{H^{-1}} \leq a_1|u_1(t)|_{L^2} + b_1|u_2(t)|_{L^2} + h_1 \tag{2.1.5}$$

and

$$|G(u_1,u_2)(t)|_{H^{-1}} \leq a_2|u_1(t)|_{L^2} + b_2|u_2(t)|_{L^2} + h_2$$

For all $u = (u_1,u_2) \in C(0,T;L^2(\Omega)) \times C(0,T;L^2(\Omega))$ *, where* $a_i, b_i, h_i \in \mathbb{R}_+$*. Then* (2.0.2) *has at least one solution* $u = (u_1,u_2)$*,*

$u \in L^2(0,T;H_0^1(\Omega)) \cap C([0,T];L^2(\Omega)) \times L^2(0,T;H_0^1(\Omega)) \cap C([0,T];L^2(\Omega))$.

2.1.3 Application of Leray-Schauder's fixed point theorem

The next result is based on the Leray-Schauder principle. We shall look for a weak solution to system (2.0.2).

Theorem 2.1.4

Let $F, G: L^2([0,T];L^2(\Omega)) \times L^2([0,T];L^2(\Omega)) \to L^2(0,T;H^{-1}(\Omega))$*. Assume that* F *and* G *are continuous and admit the decompositions* $F = F_0(u,v) + f$ *and* $G = G_0(u,v) + g$ *such that the following conditions are satisfied for all* $u, v \in L^2(0,T;H_0^1(\Omega))$*, any* $t \in [0,T]$ *, some constants* $a_1, a_2, b_1, b_2 \in \mathbb{R}_+$ *such that* $\lambda_1 > \left(a_1 + b_2 + \sqrt{(a_1-b_2)^2 + 4a_2b_1}\right)/2$ *, and* $f, g \in H^{-1}(\Omega)$*:*

$$\begin{aligned}(F_0(u,v)(t), u(t)) &\leq a_1|u(t)|_{L^2}^2 + b_1|v(t)|_{L^2}^2 \\ (G_0(u,v)(t), v(t)) &\leq a_2|u(t)|_{L^2}^2 + b_2|v(t)|_{L^2}^2\end{aligned} \tag{2.1.7}$$

Then (2.0.2) *has at least one solution* $(u,v) \in L^2(0,T;H_0^1(\Omega)) \times L^2(0,T;H_0^1(\Omega))$.

Example 2.1.1 Let $F_0, G_0: L^p(\Omega) \times L^p(\Omega) \to L^{(2^*)'}(\Omega)$ be continuous maps satisfying the following conditions:

$$|F_0(u,v)|_{L^{(2^*)'}} \leq c_1|v|_{L^p}^{\frac{p}{(2^*)'}} \text{ and } |G_0(u,v)|_{L^{(2^*)'}} \leq c_2|v|_{L^p}^{\frac{p}{(2^*)'}}, \tag{2.1.11}$$

for all $u, v \in L^2(0,T;H_0^1(\Omega))$;

$$(F_0(u,v), u) \leq a_1|u|_{L^2}^2 + b_1|v|_{L^2}^2$$

an (2.1.12)

$$(G_0(u,v), v) \leq a_2|u|_{L^2}^2 + b_2|v|_{L^2}^2$$

for all $u, v \in L^2(0,T;H_0^1(\Omega))$

for some $\lambda_1 > \frac{\left(a_1+b_2+\sqrt{(a_1-b_2)^2+4a_2b_1}\right)}{2}$, $a_1, a_2, b_1, b_2 > 0, (2^*)' \leq p \leq 2^*$ if $n \geq 3$ and $p \geq 1$ for $n = 2$ and $n = 1$.Then the maps $F, G: L^2\big(0,T; L^p(\Omega)\big) \times L^2\big(0,T; L^p(\Omega)\big) \to L^2\big(0,T; H^{-1}(\Omega)\big)$ given by $F(u)(t) = F_0\big(u(t)\big)$ and by $G(u)(t) = G_0\big(u(t)\big)$ satisfy all assumptions of Theorem 2.1.4.

Example 2.1.2 Let $f, g: \Omega \times \mathbb{R} \to \mathbb{R}$ be two functions such that $f(.,\tau), g(.,\tau)$ is measurable for every $\tau \in \mathbb{R}$, $f(x,.), g(x,.)$ are continuous for a.e. $x \in \Omega$ and there are $c_1, c_2, \alpha \in \mathbb{R}_+$, $1 \leq \alpha < 2^* - 1$ such that

$$|f(x,\tau)| \leq c_1|\tau|^\alpha \text{ and } |g(x,\tau)| \leq c_2|\tau|^\alpha; \tag{2.1.13}$$

$$\tau f(x,\tau) \leq 0 \text{ and } \tau g(x,\tau) \leq 0 \tag{2.1.14}$$

for a.e. $x \in \Omega$ and all $\tau \in \mathbb{R}$. Then the superposition operators $F, G: L^p(\Omega) \times L^p(\Omega) \to H^{-1}(\Omega)$ given by $F(u,v) = f\big(.,v(.)\big)$, and $G(u,v) = g\big(.,v(.)\big)$, with $p = \alpha(2^*)'$, satisfies the conditions of the previous example. Indeed, condition (2.1.13) guarantee that the superposition operators are well-defined, continuous from $L^p(\Omega) \times L^p(\Omega)$ to $L^{(2^*)'}(\Omega)$, and satisfy (2.1.11). Also (2.1,14) guarantees (2.1.12) with $a_1, a_2, b_1, b_2 = 0$.

Example 2.1.3 The functions $f(x,\tau) = -|\tau|^{\alpha-1}\tau \; and \; g(x,\tau) = -|\tau|^{\beta-1}\tau$ $(\tau \in \mathbb{R},)$, where $1 \leq \alpha, \beta < 2^* - 1$, satisfy all the assumptions from Example 2.1.2.

2.2 ***Systems of semilinear wave equations***

Now we shall look for weak solutions for another evolution system.

Let Ω be an open and bounded subset of $\mathbb{R}^N$, $0 < T < \infty$ and consider Cauchy-Dirichlet boundary problem:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \Delta u = f \quad on\ Q = \Omega \times (0,\mathrm{T}) \\ u(x,0) = 0,\ \frac{\partial u}{\partial t}(x,0) = 0 \quad on\ \Omega \\ u = 0 \qquad\qquad on\ \Sigma = \partial\Omega \times (0,\mathrm{T}) \end{cases} \tag{2.2.1}$$

By Theorem 1.4.1 and Remark 1.4.2 we can associate to (2.2.1) the solution operators

$S_0: L^2\big(0,T; H^{-1}(\Omega)\big) \to C\big([0,T]; H_0^1(\Omega)\big) \cap C^1\big([0,T]; L^2(\Omega)\big)$,

$S: L^2\big(0,T; H^{-1}(\Omega)\big) \to C\big([0,T]; H_0^1(\Omega) \times L^2(\Omega)\big)$,

defined by $S_0 f = u, Sf = [u, u']$, where u is the solution of (2.2.1).

In this section we are concerned with the existence of solution for the system of semilinear wave equations:

$$\begin{cases} Lu = F(u,v) on\ Q \\ Lv = G(u,v) on\ Q \\ u(x,0) = 0,\ \frac{\partial u}{\partial t}(x,0) = 0\ on\ \Omega \\ v(x,0) = 0,\ \frac{\partial v}{\partial t}(x,0) = 0\ on\ \Omega \\ \mathrm{w}_1 = \mathrm{w}_2 = 0 \text{ on } \Sigma \end{cases} \tag{2.2.2}$$

Here $Lw = \frac{\partial^2 w}{\partial t^2} - \Delta w$. We will make some notations for the following spaces: $E_0 = H_0^1(\Omega) \times L^2(\Omega)$ and $E = L^2(\Omega) \times H^{-1}(\Omega)$ endowed with the norms

$$|w|_{E_0} = \left(|u|_{H_0^1}^2 + |v|_{L^2}^2\right)^{1/2}$$

$$|w|_E = \left(|u|_{L^2}^2 + |v|_{H^{-1}}^2\right)^{1/2}$$

for any $w = [u,v]$.

2.2.1 **Application of Perov's fixed point theorem**

Theorem 2.2.1 *Let* $F, G: C([0,T]; E)^2 \to L^2\left(0,T; H^{-1}(\Omega)\right)$ *be continuous operators. Assume that* there exists $a_1, a_2, b_1, b_2 \in \mathbb{R}_+$ *such that if* $w_1 = [u_1, u_2], w_2 = [v_1, v_2]$ *thus*

$$|F(w_1)(t) - F(w_2)(t)|_{H^{-1}} \leq a_1 |u_1(t) - u_2(t)|_{E_0} + b_1 |v_1(t) - v_2(t)|_{E_0}$$

and (2.2.3)

$$|G(w_1)(t) - G(w_2)(t)|_{H^{-1}} \leq a_2 |u_1(t) - u_2(t)|_{E_0} + b_2 |v_1(t) - v_2(t)|_{E_0}$$

for every $w_1 = [u_1, u_2]$, $w_2 = [v_1, v_2] \in C([0,T]; E)$.*Then* (2.2.2) *has a unique solution* $w = (w_1, w_2)$, *with* $w_1, w_2 \in C\left([0,T]; H_0^1(\Omega)\right) \cap C^1\left([0,T]; L^2(\Omega)\right)$.

The next theorem is an existence result derived from Schauder's fixed point principle, assuming that nonlinearities F and G have a growth at most linear.

2.2.2 **Application of Schauder's fixed point theorem**

The next theorem is an existence result for the problem (2.2.2) derived from Schauder's fixed point principle, assuming that nonlinearities F and G have a growth at most linear.

Theorem 2.2.2 $F, G: C([0,T]; L^2(\Omega) \times E)^2 \to L^2\big(0,T; H^{-1}(\Omega)\big)$. *Assume that F and G are continuous and satisfy the growth conditions.*

$$|F(u_1,u_2)(t)|_{H^{-1}} \leq a_1|u_1(t)|_{E_0} + b_1|u_2(t)|_{E_0} + h_1 \tag{2.2.6}$$

and

$$|G(u_1,u_2)(t)|_{H^{-1}} \leq a_2|u_1(t)|_{E_0} + b_2|u_2(t)|_{E_0} + h_2$$

for all $\in C([0,T]; E)$, $u = (u_1,u_2)$, *where* $a_i, b_i, h_i \in \mathbb{R}_+$. *Then* (2.2.2) *has at least one solution* $u = (u_1,u_2), u_1,u_2 \in C\big([0,T]; H_0^1(\Omega)\big) \cap C^1\big([0,T]; L^2(\Omega)\big)$.

3 Existence theory for general semilinear evolution equations and systems

Starting with the existence and uniqueness result for the a general non-homogenous evolution equation with the source term in a Banach space, X, we present existence results for the a nonlinear evolution system via Perov and Schauder fixed point theorems. In the first part of this chapter we consider a non-homogenous problem $\begin{cases} u' = Au + f \\ u(0) = 0 \end{cases}$ to which we can associate a solution operator $S: L^1(0,T;X) \to C([0,T];X) \subset L^p(0,T;X)$, according to Vrabie ([82], pp.142). Based on the same results we prove the compactness of the solution operator necessary to apply Schauder fixed point theorem for both, semilinear operator equation and the operator system.

In this chapter we are concerned with the existence of solutions for the system of semilinear equations

$$\begin{cases} u' - Au = F(u,v) \\ v' - Av = G(u,v) \\ u(0) = 0, \;\; v(0) = 0 \end{cases} \tag{3.0.1}$$

in a Banach space X.

Our goal in this chapter is to apply the same methods as in Chapter 2 now for general evolution equation systems. That means that to work with the operator $L = u' - Au$, where the operator A is the infinitesimal generator to a C_0-semigroup.

Here L is a linear operator such that $S = \mathrm{L}^{-1}$ and F, G are nonlinear operators. We seek weak solution of the fixed point problem $\begin{bmatrix} u \\ v \end{bmatrix} = N\begin{bmatrix} u \\ v \end{bmatrix}$ in the space $C([0,T],X)^2$, where $u, v \in C([0,T];X) \times C([0,T];X)$, $N = (N_1, N_2)$ defined by

$$N_1 = S \circ F \text{ and } N_2 = S \circ G.$$

The interest of (3.0.1) lies moreover in the fact that it can be regarded as an abstract model for particular systems describing specific processes as mechanics and dynamical systems. In the first section we present existence results for semilinear operator equations of the type

$$\begin{cases} u' = Au + F(u) \\ u(0) = 0 \end{cases} \tag{3.0.2}$$

Where $A: D(A) \subseteq X \to X$ is the infinitesimal generator to a C_0-semigroup $\{\mathcal{T}(t); t \geq 0\}$.

X is a Banach space and we define next Banach spaces:

$\mathcal{L}(X) = \{\mathcal{T}(t);\ \mathcal{T}: X \to X$ linear and continuous$\}$ endowed with the operatorial norm:

$$\|\mathcal{T}\|_{\mathcal{O}} = \sup_{\|x\| \leq 1} \|\mathcal{T}x\|$$

for every $\mathcal{T} \in \mathcal{L}(X)$ and $C([0,T];X)$ the space of continuous functions endowed with the norm:

$$\|f\|_{\infty} = sup\{\|f(t)\|; t \in [0,T]\}.$$

F is a continuous operator defined

$$F: C([0,T];X) \subset L^p(0,T;X) \to L^1(0,T;X)$$

The analysis of system (3.0.1) will be a vector-valued one and we will use matrices instead of constants, as it was initiated in A.I. Perov and A.V. Kibenko [64] for a vector version of the contraction principle and it was extended recently in R.Precup [68] to other topics of nonlinear analysis. Moreover, the theory can be easily extended to systems of n operator equations with $n \geq 2$, and contains, as a particular case, the theory from the Section 3.1.2 and 3.1.3 for a single equation.

3.1 Semilinear evolution equations

It is well-known (see Vrabie [82], pp.142) that one can associate to the non-homogenous problem

$$\begin{cases} u' = Au + f \\ u(0) = 0 \end{cases} \tag{3.1.1}$$

the solution operator

$$S: L^1(0,T;X) \to C([0,T];X) \subset L^p(0,T;X)$$

given by $Sf = u$ where $u: [0,T] \to X$ defined by the so-called *variation of constants formula*

$$u(t) = \int_0^t \mathcal{T}(t-s) f(s) ds \quad \text{for each } t \in [0,T] \tag{3.1.2}$$

is the C_0-solution to the problem (3.1.1).

3.1.1 The solution operator

The next lemmas are used in order to apply Schauder fixed point theorem, more exactly for proving the complete continuity of the solution operator S.

Lemma 3.1.1 *Let $f \in L^1(0,T;X)$ and $S: L^1(0,T;X) \to C([0,T];X) \subset L^p(0,T;X)$ the solution operator to the problem* (3.1.1)*. Then the solution operator S is nonexpansive from* $L^1(0,T;X)$

to $C([0,T];X)$. In particular S maps bounded subsets of $L^1(0,T;X)$ into bounded subsets of $C([0,T];X)$.

Lemma 3.1.2 (Vrabie [82] pp.143) *Let $A: D(A) \subseteq X \to X$ be the infinitesimal generator to a C_0-semigroup of contractions $\{\mathcal{T}(t); t \geq 0\}$, and let $\mathcal{F}$ be an uniforme integrable subset of $L^1(0,T;X)$. Then $S\mathcal{F}$ is relatively compact in $C([0,T];X)$ if and only if there exists a dense subset D of $[0,T]$ such that, for any $t \in D$, the family section $S\mathcal{F}$ in t, $(S\mathcal{F})(t) = \{(Sf)(t); f \in \mathcal{F}\}$ is relatively compact in X.*

See Vrabie [82, p.143] for a proof.

Lemma 3.1.3 (Vrabie [82] p.147) *Let $A: D(A) \subseteq X \to X$ be the infinitesimal generator to a C_0-semigroup of contractions $\{\mathcal{T}(t); t \geq 0\}$, and let $\mathcal{F}$ be a bounded subset of $L^1(0,T;X)$. Then $S\mathcal{F}$ is relatively compact in $L^p(0,T;X)$ for any $p \in [1,+\infty)$ if and only if for any $\varepsilon > 0$ there exists a relatively compact subset C_ε of X such as for any $f \in \mathcal{F}$ there exists a subset $E_{\varepsilon,f}$ of $[0,T]$ those Lebesgue measure is less then ε and such as $(S\mathcal{F})(t) \in C_\varepsilon$ for any $f \in \mathcal{F}$ and $t \in [0,T] \backslash E_{\varepsilon,f}$.*

Lemma 3.1.4 (Gutman) *An uniformely integrable family $\mathcal{F}$ from $L^p(0,T;X)$ is relatively compact if and only if :*

(i) *$\mathcal{F}$ is p-equiintegrable;*

(ii) *$\varepsilon > 0$ there exists a relatively compact subset C_ε of X such as for any $f \in \mathcal{F}$ there exists a subset $E_{\varepsilon,f}$ of $[0,T]$ those Lebesgue measure is less then ε and such as $(S\mathcal{F})(t) \in C_\varepsilon$ for any $f \in \mathcal{F}$ and $t \in [0,T] \backslash E_{\varepsilon,f}$*

Lemma 3.1.5 *(Baras-Hassan-Veron) Let $A: D(A) \subseteq X \to X$, (X is a Banach space)be the infinitesimal generator of a compact contractions C_0-semigroup. Then for any bounded subset $\mathcal{F}$ from $L^1(0,T;X)$ and for any $p \in [1,+\infty)$ the following set $S\mathcal{F} := \{Sf; f \in \mathcal{F}\}$ is relatively compact in $L^p(0,T;X)$.*

Theorem 3.1.1 *The solution operator S is completely continuous from $L^1(0,T;X)$ to $L^p(0,T;X)$ for any $p \in [1,\infty)$.*

Application of Banach's contraction principle

We will apply here Banach's fixed point theorem in order to obtain the existence of solution to problem (3.1.1)

Theorem 3.1.2 *Let $F: C([0,T];X) \subset L^p(0,T;X) \to L^1(0,T;X)$ be a continuous map for which there is a constant $a \in \mathbb{R}_+$ such that the following inequality holds*

$$|F(u)(t) - F(v)(t)|_X \leq a|u(t) - v(t)|_X \tag{3.1.5}$$

for all $u, v \in C([0,T];X)$ and any $t \in [0,T]$.

Then there exists at least one solution to the problem (3.1.1).

3.1.3 Application of Schauder's fixed point theorem

The next existence result comes from Schauder fixed point theorem. The Lipschitz condition on the nonlinear term F in Theorem 3.1.2 is weakened to a growth condition at most linear.

Theorem 3.1.3 *Let $F: L^p(0,T;X) \to L^1(0,T;X)$ be a continuous map for which there are constants $a, b \in \mathbb{R}_+$ such that the following inequality holds*

$$|F(u)(t)|_X \leq a|u(t)|_X + b$$

for all $u \in C([0,T];X)$ and any $t \in [0,T]$.

Then there exists at least one solution to the problem (3.1.1).

Example 3.1.1 Let $\Omega \subset \mathbb{R}^n$ a bounded domain, $n \geq 1$, and let $f: \mathbb{R}_+ \times \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ and $p \in [1, +\infty]$ such that

(a) $f(.,.,\tau)$ is measurable for each $\tau \in \mathbb{R}$,

(b) continuous for a.e. $x \in \Omega$, and

(c) for each $T > 0$ there exist constants $a_T > 0$ and $b_T \in \mathbb{R}$ such that

$$|f(t,x,\tau)| \leq a_T|\tau|^p + b_T$$

for a.e $x \in \Omega$ and all $\tau \in \mathbb{R}$.

Then the operator $F: L^p(0,\mathrm{T};\mathrm{X}) \to L^1(0,\mathrm{T};\mathrm{X})$ defined by

$$F(u)(x) := f(t,x,u(x))$$

satisfies all the assumptions of Theorem 3.1.3.

3.2 **Semilinear evolution systems**

In this section we are concerned with the existence of solution for the system of semilinear evolution equations:

$$\begin{cases} Lu = F(u,v) \\ Lv = G(u,v) \\ u(0) = 0, \ \ v(0) = 0 \end{cases} \tag{3.2.1}$$

in a Banach space X. Here $Lu = u' - Au$ is a linear operator such that $S = \mathrm{L}^{-1}$, F, G are nonlinear operators and A is the infinitesimal generator to a C_0-semigroup of contractions $\{\mathcal{T}(t); t \geq 0\}$. We seek weak solution of the fixed point problem $\begin{bmatrix} u \\ v \end{bmatrix} = N\begin{bmatrix} u \\ v \end{bmatrix}$ in the space X^2, where $u, v \in C([0,T];X) \times C([0,T];X)$, $N = (N_1, N_2)$ defined by

$$N_1 = S \circ F \text{ and } N_2 = S \circ G.$$

Our first result is an existence, uniqueness and approximation theorem.

3.2.1 **Application of Perov's fixed point theorem**

Theorem 3.2.1 Let $F, G: C([0,T];X)^2 \to L^1(0,T;X)$. *Assume that*

$$\|F(u_1, v_1) - F(u_2, v_2)\| \leq a_1\|u_1 - u_2\| + b_1\|v_1 - v_2\|$$

and (3.2.2)

$$\|G(u_1, v_1) - G(u_2, v_2)\| \leq a_2\|u_1 - u_2\| + b_2\|v_1 - v_2\|$$

for every $u, v \in C([0,T];X) \times C([0,T];X)$, $u = (u_1, u_2)$, $v = (v_1, v_2)$ *and* a_1, a_2, b_1, b_2 *are nonnegative constants* .

Then (3.2.1) *has a unique solution* $u = (u_1, u_2) \in C([0,T];X) \times C([0,T];X)$.

3.2.2 **Application of Schauder's fixed point theorem**

The next theorem is an existence result derived from Schauder's fixed point principle, assuming that nonlinearities F and G have a growth at most linear.

Theorem 3.2.2 Let $F, G: F: L^p(0,T;X)^2 \to L^1(0,T;X)$. *Assume that F and G are continuous and satisfy the growth conditions.*

$$|F(u_1,u_2)(t)|_X \le a_1|u_1(t)|_X + b_1|u_2(t)|_X + h_1(t)$$

and (3.2.4)

$$|G(u_1,u_2)(t)|_X \le a_2|u_1(t)|_X + b_2|u_2(t)|_X + h_2(t)$$

for all $= (u_1,u_2) \in C([0,T];X) \times C([0,T];X)$ *, where* $a_i, b_i, h_i \in \mathbb{R}_+$. *Then* (3.2.1) *has at least one solution* $u = (u_1,u_2) \in C([0,T];X) \times C([0,T];X)$.

3.2.3 **Examples**

Example 3.2.1 Let $F, G: L^p(0,T;X)^2 \to L^1(0,T;X)$ be two continuous maps for which there exists the constants $a_1, a_2, b_1, b_2 \in \mathbb{R}_+$ with

$$|F(u_1,u_2)(t)|_X \le a_1|u_1(t)|_X + b_1|u_2(t)|_X + h_1(t)$$

and

$$|G(u_1,u_2)(t)|_X \le a_2|u_1(t)|_X + b_2|u_2(t)|_X + h_2(t)$$

Then the maps $F_1, G_1: F: L^p(0,T;X)^2 \to L^1(0,T;X)$ given by

$F_1(u)(t) = F(u(t))$ and $G_1(u)(t) = G(u(t))$. $\big((u_1,u_2), \in C([0,T];X) \times C([0,T];X)\big)$

satisfy all the assumptions of Theorem 3.2.2.

Example 3.2.2 Let $\Omega \subset \mathbb{R}^n$ a bounded domain, $n \ge 1$, and set $f, g: \mathbb{R}_+ \times \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ and $p \in [1, +\infty]$ be a function such that $f(t,\cdot,\tau)$ and $g(t,x,\cdot)$ are measurable for every $\tau \in \mathbb{R}$,

continuous for a.e. $x \in \Omega$, $f(t,.,\tau)g(t,\cdot,0) \in L^1(0,\mathrm{T};\mathrm{X})$ and there are the constants $a_0, \mathrm{b}_0 \in \mathbb{R}_+$ with

$$|f(t,x,\tau) - f(t,x,0)| \le a_0|\tau|$$

$$|g(t,x,\tau) - g(t,x,0)| \le b_0|\tau|$$

for a.e $x \in \Omega$ and all $\tau \in \mathbb{R}$. Then the operators $F, G: L^p(0,\mathrm{T};\mathrm{X}) \to L^1(0,\mathrm{T};\mathrm{X})$ defined by

$F(u,v) \coloneqq f(t,\cdot,u(\cdot))$ and $G(u,v) \coloneqq g(t,\cdot,u(\cdot))$

satisfy all the assumptions from previous example.

4 Nonlinear Schrödinger equations via fixed point principles

Starting with the existence and uniqueness result for the non-homogenous Schrödinger equation with the source term in $\mathrm{H}^{-1}(\Omega)$, we present existence results for the nonlinear perturbated Schrödinger equation via Banach, Schauder and Leray-Schauder fixed point theorems. In the first part of this chapter we consider a Cauchy-Dirichlet problem for Schrödinger equation. Our framework is based on the theory of Sobolev space $H^{-1}(\Omega)$ and we establish the existence of a weak solution based on an existence and uniqueness result of J.L.Lions [42]. We include a proof adapted from Temam [76] and Precup [66] for completeness. Next we will associate to the Cauchy-Dirichlet problem the solution operator $S: L^2(0,T;H^{-1}(\Omega)) \to L^2(0,T;H_0^1(\Omega)) \cap \mathrm{C}([0,\mathrm{T}];\mathrm{H}^{-1}(\Omega))$. We focus on the complete continuity of this operator on $L^2(0,T;L^2(\Omega))$ in order to prove a result concerning a superlinear problem. It is established by means of Leray-Schauder fixed point theorem. (see Precup [66]).

4.1 Linear Schrödinger equations

4.1.1 Introduction

This chapter deals with weak solvability of the Cauchy-Dirichlet problem for the perturbated Schrödinger equation:

$$\begin{cases} u_t - i\Delta u = F(u) & in\ \Omega \times (0,\mathrm{T}) \\ u(x,0) = 0 & in\ \ \Omega \\ u = 0 & on\ \partial\Omega \times (0,\mathrm{T}) \end{cases} \tag{4.1.1}$$

Here $\Omega \subset \mathbb{R}^n$ is a bounded domain and F is a general nonlinear operator which, in particular, can be a superposition operator, a delay operator, or an integral operator. Specific Schrödinger equations arise as models from several areas of physics. The problem is a classical one (see [15], [36], [42], [41] and [76]) and our goal here is to make more precise the operator approach based on abstract results from nonlinear functional analysis. More exactly, we shall precise basic properties, such as norm estimation and compactness, for the (linear) solution operator associated to the non-homogeneous linear Schrödinger equation and we shall use them in order to apply the Banach, Schauder and Leray-Schauder theorems to the fixed point problem equivalent to problem (4.1.1). The same program me has been applied to discuss nonlinear perturbations of the heat and wave equations in [65] and [66].

Compared to [65] and [66], here all spaces consist of complex-valued functions. Thus $L^2(\Omega)$ is the space of all complex-valued measurable functions u with $\int_\Omega |u(x)|^2 dx < \infty$ endowed with inner product and norm

$$(u, v)_{L^2} = \int_\Omega u(x)\overline{v(x)}dx, \quad |u|_{L^2} = \left(\int_\Omega |u(x)|^2 dx\right)^{\frac{1}{2}}.$$

Also the Sobolev space of complex-valued functions $H_0^1(\Omega)$ is endowed with inner product and norm

$$(u, v)_{H_0^1} = \int_\Omega \left(\sum_{k=1}^{n} \frac{\partial u}{\partial x_k}\frac{\partial \bar{v}}{\partial x_k}\right)dx, \quad |u|_{H_0^1} = (u, u)_{H_0^1}^{\frac{1}{2}}.$$

As usual by $H^{-1}(\Omega)$ we denote the dual of $H_0^1(\Omega)$, that is the space of all linear continuous complex-valued functional on $H_0^1(\Omega)$. The duality between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$ is defined as follows: for $f \in H^{-1}(\Omega)$ and $u \in H_0^1(\Omega)$, (f, u) stands for the valued of f at $\bar{u}$; in particular, if $f \in L_{loc}^1(\Omega)$, then $(f, u) = \int_\Omega f\bar{u}dx$, and if $f \in L^2(\Omega)$, then $(f, u) = (f, v)_{L^2}$. Recall that $-\Delta$ is an isometry between spaces $H_0^1(\Omega)$ and $H^{-1}(\Omega)$.

Throughout this chapter by λ_k and ϕ_k $(k = 1,2, \dots)$ we mean the eigenvalues and eigenfunctions of $-\Delta$. Thus

$$\begin{cases} -\Delta\phi_k = \lambda_k\phi_k \text{ in } \Omega \\ \phi_k = 0 \ \ on\ \partial\Omega. \end{cases}$$

We also assume that $|\phi_k|_{L^2} = 1$. Then the systems $(\phi_k)_{k\geq 1}, \left(\frac{1}{\sqrt{\lambda_k}}\phi_k\right)_{k\geq 1}$ are orthonormal and complete in $L^2(\Omega)$ and $H_0^1(\Omega)$, respectively. Recall in addition Poincaré's inequality

$$|u|_{L^2} \leq \frac{1}{\sqrt{\lambda_k}}|u|_{H_0^1}, \ u \in H_0^1(\Omega). \tag{4.1.2}$$

4.1.2 The non-homogenous Schrodinger equation in $\boldsymbol{H^{-1}(\Omega)}$

First we need the following lemma, the version for the complex-valued functions of a result from [65], which is a realization to $H^{-1}(\Omega)$ of Parseval's relation and of the completeness property of eigenfunctions ϕ_k. We include its proof for the sake of completeness.

Lemma 4.1.1 (i) *For any $u \in H^{-1}(\Omega)$, one have*

$$u = \sum_{k=1}^{\infty}(u, \phi_k)\,\phi_k \quad (in\ \ H^{-1}(\Omega)) \tag{4.1.3}$$

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{\lambda_k}} |(u, \phi_k)|^2 = |u|_{H^{-1}}^2 \tag{4.1.4}$$

(ii) *If* $u \in L^2(0, T; H^{-1}(\Omega))$, *then*

$$u = \sum_{k=1}^{\infty} (u, \phi_k)\, \phi_k \quad (in\ \ L^2(0, T; H^{-1}(\Omega))).$$

Consider now the Cauchy-Dirichlet problem for the non-homogeneous Schrödinger equation

$$\begin{cases} u_t - i\Delta u = f & in\ \Omega \times (0, \mathrm{T}) \\ u(x, 0) = g_0(x) & in\ \ \Omega \\ u = 0 & on\ \partial\Omega \times (0, \mathrm{T}) \end{cases} \tag{4.1.5}$$

We have the following existence and uniqueness result. Its proof uses arguments patterned from [41], [76] and [66].

Theorem 4.1.1 *If* $f \in L^2(0, T; H^{-1}(\Omega))$ *and* $g_0 \in L^2(\Omega)$ *, then there exists an unique function u such that*

$$u \in L^2(0, T; H_0^1(\Omega)) \cap C([0, T]; L^2(\Omega)),\ u' \in L^2\big(0, T; H^{-1}(\Omega)\big) \tag{4.1.6}$$

$$and\ \begin{cases} (u'(t), v) \quad + i(u(t), v)_{H_0^1} = (f(t), v)\ a.e.\ on\ [0, T], for\ all\ v \in H_0^1(\Omega), \\ u(0) = g_0 \end{cases} \tag{4.1.7}$$

By the (weak or generalized) *solution* of the Cauchy-Dirichlet problem (4.1.5), when $f \in L^2(0, T; H^{-1}(\Omega)$ and $g_0 \in L^2(\Omega)$, we mean the function u which satisfies ((4.1.6) and (4.1.7). The mapping

$$S: L^2(0, T; H^{-1}(\Omega)) \to L^2(0, T; H_0^1(\Omega)) \cap \mathrm{C}([0, \mathrm{T}]; L^2(\Omega))$$

given by $Sf = u$, where u is the unique solution of problem (4.1.5) for $g_0 = 0$, is called the *solution operator* of Cauchy-Dirichlet problem for the Schrödinger equation.

4.2 Schrödinger solution operator

4.2.1 Norm estimations

The following estimation theorem guarantees the nonexpansivity and Lipschitz property of the solution operator S from $L^2(0, T; H^{-1}(\Omega))$ to $L^2(0, T; H_0^1(\Omega))$ and $\mathrm{C}([0, \mathrm{T}]; L^2(\Omega))$, respectively.

Theorem 4.2.1 *Let* $f \in L^2(0,T;H^{-1}(\Omega))$. *Then for every* $t \in [0,T]$ *one has*

$$|Sf|_{L^2(0,t;H_0^1(\Omega))} \leq |f|_{L^2(0,T;H^{-1}(\Omega))} \tag{4.2.1}$$

and

$$|Sf|_{C([0,t];L^2(\Omega))} \leq \sqrt{2}|f|_{L^2(0,t;H^{-1}(\Omega))} \tag{4.2.2}$$

4.2.2 Compactness

This section deals with the complete continuity of the solution operator S. We shall also use the following result (see [65, p 255] and [82, p 307]):

Lemma 4.2.1 *Let X, B and Y be Banach spaces with the inclusion* $X \subset B$ *compact and* $B \subset Y$ *continuous. If a set F is bounded in* $L^p(0,T;X)$ *and relatively compact in* $L^p(0,T;Y)$, *where* $1 \leq p \leq \infty$, *then F is relatively compact in* $L^p(0,T;B)$

Theorem 4.2.2 *The solution operator S is completely continuous from* $L^2(0,T;H^{-1}(\Omega))$ *to* $L^2(0,T;L^p(\Omega))$ *for* $(2^*)' = \frac{2n}{n+2} \leq p \leq 2^* = \frac{2n}{n-2}$ *if* $n \geq 3$ *and for any* $p \geq 1$ *if n=1 or n=2.*

4.3 Existence results for nonlinear Schrödinger equations

4.3.1 Application of Banach's fixed point theorem

Our first result of existence and uniqueness for the semi-linear problem (4.1.1) is established by means of Banach fixed point theorem.

Theorem 4.3.1 *Let* $g_0 \in L^2(\Omega)$ *and* $F: C([0,T];L^2(\Omega)) \to L^2(0,T;H^{-1}(\Omega))$ *be a map for which there exists a constant* $a \in \mathbb{R}_+$ *such that the following inequality holds for all* $u, v \in C\big([0,T];\ L^2(\Omega)\big)$:

$$|F(u)(t) - F(v)(t)|_{H^{-1}} \leq a|u(t) - v(t)|_{L^2} \ a.e. on\ [0,T]. \tag{4.3.1}$$

Then there exists a unique solution u to problem (4.1.1), *i.e.,*

$$u \in L^2(0,T;H_0^1(\Omega)) \cap C\big([0,T];L^2(\Omega)\big)\,, \quad u' \in L^2\big(0,T;H^{-1}(\Omega)\big)$$

and

$$\begin{cases} (u'(t), v) + i(u(t), v)_{H_0^1} = (F(u)(t), v) \quad a.e.\ on\ [0,T],\ \ for\ all\ v \in H_0^1(\Omega) \\ \qquad\qquad u(0) = g_0 \end{cases}$$

Example 4.3.1 Let $G: L^2(\Omega) \to H^{-1}(\Omega)$ be a map for which there exists a constant $a \in \mathbb{R}_+$ wit

$$|G(u) - G(v)|_{H^{-1}(\Omega)} \leq a|u - v|_{L^2(\Omega)}\,,\ u, v \in L^2(\Omega)\,. \tag{4.3.2}$$

Then the map $F: C([0,T]; L^2(\Omega)) \to L^2(0,T; H^{-1}(\Omega))$ given by

$$F(u)(t) = G\big(u(t)\big)\ \ (u \in C\big([0,T]; L^2(\Omega)\big), t \in [0,T])$$

satisfies all the assumptions of Theorem 4.3.1.

Example 4.3.2 Let $g: \Omega \times \mathbb{R} \to \mathbb{R}$ be a function such that $g(.\,,\tau)$ is measurable for each $\tau \in \mathbb{R}$, $g(.\,,0) \in H^{-1}(\Omega)$ and there is a constant $a_0 \in \mathbb{R}_+$ with

$$|g(x,\tau_1) - g(x,\tau_2)| \leq a_0|\tau_1 - \tau_2|$$

For a.e. $x \in \Omega$ and all $\tau_1, \tau_2 \in \mathbb{R}$. Then the operator $G: L^2(\Omega) \to H^{-1}(\Omega)$ defined by

$$G(u) = g\big(.\,, u(.)\big)$$

satisfies all the assumptions from the previous example.

4.3.2 Application of Schauder's fixed point theorem

The next existence result comes from Schauder's fixed point theorem. The Lipschitz condition on the nonlinear term F in Theorem 4.3.1 is weakened to a growth condition at most linear.

Theorem 4.3.2 *Let $g_0 \in L^2(\Omega)$ and $F: L^2(0,T; L^2(\Omega)) \to L^2(0,T; H^{-1}(\Omega))$ be a continuous smap for which there exists a constant $a \in \mathbb{R}_+$ such that the following inequality holds for all $u \in C\big([0,T];\ L^2(\Omega)\big)$*

$$|F(u)(t) - F(0)(t)|_{H^{-1}} \leq a|u(t)|_{L^2}\ a.e. on\ [0,T].$$

Then there exists at least one solution to problem (4.1.1).

Example 4.3.3 Let $G: L^2(\Omega) \to H^{-1}(\Omega)$ be a continuous map for which there exists a constant $a \in \mathbb{R}_+$ with

$$|G(u) - G(0)|_{H^{-1}} \leq a|u|_{L^2}\,,\ u \in L^2(\Omega)\,. \tag{4.3.3}$$

Then the map $F: C([0,T]; L^2(\Omega)) \to L^2(0,T; H^{-1}(\Omega))$ given by

$$F(u)(t) = G\big(u(t)\big)\ \ (u \in L^2\big(0,T; L^2(\Omega)\big), t \in [0,T])$$

satisfies all the assumptions of Theorem 4.3.2.

Example 4.3.4 Let $g: \Omega \times \mathbb{R} \to \mathbb{R}$ be a function such that $g(.,\tau)$ is measurable for every $\tau \in \mathbb{R}$, $g(x,.)$ is continuous for a.e. $x \in \Omega$, $g(.,0) \in H^{-1}(\Omega)$ and there is a constant $a_0 \in \mathbb{R}_+$ with

$$|g(x,\tau) - g(x,0)| \leq a_0|\tau|$$

for a.e. $x \in \Omega$ and all $\tau \in \mathbb{R}$. Then the operator $G: L^2(\Omega) \to H^{-1}(\Omega)$ defined by

$$G(u) = g\big(.,u(.)\big)$$

satisfies all the assumptions from the previous example.

4.3.3 Application of the Leray-Schauder fixed point theorem

Our next existence result established by means of the Leray-Schauder fixed point theorem (see [66]).

Theorem 4.3.3 *Let $g_0 \in L^2(\Omega)$ and $F: L^2(0,T;L^p(\Omega)) \to L^2(0,T;H^{-1}(\Omega))$, where $(2^*)' \leq p \leq 2^*$ if $n \geq 3$ and $p \geq 1$ for $n = 2$ and $n = 1$. Assume that F is continuous and bounded (sends bounded sets into bounded sets) and there are constants $a, b \in \mathbb{R}_+, a < \lambda_1$ and $f \in H^{-1}(\Omega)$ such that if $F_0 := F - f$ the following conditions holds*

$$(F_0(u)(t), u(t)) \leq a|u(t)|_{L^2}^2 + b \tag{4.3.4}$$

for every $u \in L^2(0,T;L^p(\Omega))$ and a.e. $t \in [0,T]$. Then there exists at least one solution to problem (4.1.1)

Example 4.3.5 Let $G: L^p(\Omega) \to L^{(2^*)'}(\Omega)$ be a continuous map satisfying the following conditions:

$$|G(u) - G(0)|_{L^{(2^*)'}} \leq c_0|v|_{L^p}^{\frac{p}{(2^*)'}},\ for\ all\ v \in L^p(\Omega); \tag{4.3.6}$$

$$(G(v), v) \leq a|v|_{L^2}^2 + b\ for\ all\ v \in L^p(\Omega), \tag{4.3.7}$$

for some $a < \lambda_1$, $b, c \geq 0, (2^*)' \leq p \leq 2^*$ if $n \geq 3$ and $p \geq 1$ for $n = 2$ and $n = 1$.Then the map $F: L^2(0,T;L^p(\Omega)) \to L^2(0,T;H^{-1}(\Omega))$ given by $F(u)(t) = G\big(u(t)\big)$ satisfies all assumptions of Theorem 4.3.3.

Example 4.3.6 Let $g: \Omega \times \mathbb{R} \to \mathbb{R}$ be a function such that $g(.,\tau)$ is measurable for every $\tau \in \mathbb{R}$, $g(x,.)$ is continuous for a.e. $x \in \Omega$ and there are $c_0, \alpha \in \mathbb{R}_+$, $1 \leq \alpha < 2^* - 1$ such that

$$|g(x,\tau)| \leq c_0|\tau|^{\alpha}; \tag{4.3.8}$$

$$\tau g(x,\tau) \leq 0 \tag{4.3.9}$$

for a.e. $x \in \Omega$ and all $\tau \in \mathbb{R}$. Then the superposition operator $G: L^p(\Omega) \to H^{-1}(\Omega)$ given by $G(v) = g\big(.,v(.)\big)$, with $p = \alpha(2^*)'$, satisfies the conditions of the previous example.

Example 4.3.7 The function $g(x,\tau) = -|\tau|^{\alpha-1}\tau$ ($\tau \in \mathbb{R}$,), where $1 \leq \alpha < 2^* - 1$, satisfies all the assumptions from Example 4.3.6.

5 Systems of nonlinear Schrödinger equations

Using the existence linear theory from chapter 4 we establish existence results for the nonlinear perturbed Schrödinger operator systems via Perov, Schauder and Leray-Schauder fixed point theorems. The abstract frame-work are related to Lebesgue-Sobolev spaces. The proofs are based on the method of convergent to zero matrices of Precup [68]. Our results particularize the general theory presented in this work.

This chapter deals with the weak solvability of semilinear operator systems using the method of matrices that converge to zero. Our starting point is related to the Schrödinger solution operator, for which we established properties of compactness in Chapter 4.

5.1 *Nonlinear Schrödinger equations*

Let Ω be an open and bounded subset of $\mathbb{R}^N$, $0 < T < \infty$ and consider the Chaucy-Dirichlet problem for the linear Schrödinger equation:

$$\begin{cases} u_t - i\Delta u = f & on\ Q = \Omega \times (0, \mathrm{T}); \\ u(x,0) = 0 & on\ \ \Omega; \\ u = 0 & on\ \Sigma := \partial\Omega \times (0, \mathrm{T}). \end{cases} \tag{5.1.1}$$

By Theorems 4.1.1, 4.2.1 and 4.2.2 we can associate to this problem the solution operator

$$S: L^2\big(0,T; H^{-1}(\Omega)\big) \to L^2\big(0,T; L^2(\Omega)\big) \cap C\big([0,T]; L^2(\Omega)\big),$$

defined by $Sf = u$ where $u \in L^2\big(0,T; L^2(\Omega)\big) \cap C\big([0,T]; L^2(\Omega)\big)$ is the weak solution of the problem (5.1.1) .

In the first part we are concerned with the existence of solutions for the following system of semilinear Schrödinger equations:

$$\begin{cases} Lu = F(u,v) & on\ \ \Omega \times (0, \mathrm{T}); \\ Lv = G(u,v) & on\ \Omega \times (0, \mathrm{T}); \\ u(x,0) = 0, v(x,0) = 0 & on\ \Omega; \\ u = v = 0 & on\ \Sigma. \end{cases} \tag{5.1.2}$$

Here by Lu we mean $Lu = u_t - i\Delta u$ such that $S = \mathrm{L}^{-1}$ and F, G are nonlinear operators. We seek a weak solution to (5.1.2), that is a fixed point of the problem $\begin{bmatrix} u \\ v \end{bmatrix} = N\begin{bmatrix} u \\ v \end{bmatrix}$, where $u, v \in C\big([0,T]; L^2(\Omega)\big)$, $N = (N_1, N_2)$ defined by

$$N_1 = S \circ F, \quad N_2 = S \circ G.$$

5.1.1 **Application of Perov's fixed point theorem**

Our first result is an existence, uniqueness and approximation theorem.

Theorem 5.1.1 *Let* $F, G: C\big([0,T]; L^2(\Omega)\big) \times C\big([0,T]; L^2(\Omega)\big) \to L^2\big(0,T; H^{-1}(\Omega)\big)$ *be continuous operators*. *Assume that*

$$|F(u_1,v_1)(t) - F(u_2,v_2)(t)|_{H^{-1}} \le a_1|u_1(t) - u_2(t)|_{L^2} + b_1|v_1(t) - v_2(t)|_{L^2}$$

and (5.1.3)

$$|G(u_1,v_1)(t) - G(u_2,v_2)(t)|_{H^{-1}} \le a_2|u_1(t) - u_2(t)|_{L^2} + b_2|v_1(t) - v_2(t)|_{L^2}$$

for every $(u_1,v_1),(u_2,v_2) \in C\big([0,T]; L^2(\Omega)\big) \times C\big([0,T]; L^2(\Omega)\big), t \in [0,T]$ *and some nonnegative constants* a_1, a_2, b_1, b_2 .

Then (5.1.2) *has a unique solution* $(u,v) \in C\big([0,T]; L^2(\Omega)\big) \times C\big([0,T]; L^2(\Omega)\big)$.

Example 5.1.1 *Let* $F_1, G_1: L^2(\Omega) \times L^2(\Omega) \to H^{-1}(\Omega)$ *be two maps for which there exists the constants* $a_1, a_2, b_1, b_2 \in \mathbb{R}_+$ *with*

$$|F_1(u_1,v_1) - F_1(u_2,v_2)|_{H^{-1}} \le a_1|u_1 - u_2|_{L^2} + b_1|v_1 - v_2|_{L^2}$$

and

$$|G_1(u_1,v_1) - G_1(u_2,v_2)|_{H^{-1}} \le a_2|u_1 - u_2|_{L^2} + b_2|v_1 - v_2|_{L^2}$$

for all $u_1, v_1, u_2, v_2 \in L^2(\Omega)$

Then the maps $F, G: C([0,T]; L^2(\Omega)) \times C\big([0,T]; L^2(\Omega)\big) \to L^2(0,T; H^{-1}(\Omega))$ *given by*

$$F(u,v)(t) = F_1(u(t),v(t)) \text{ and } G(u,v)(t) = G_1(u(t),v(t))$$

for $(u,v) \in C\big([0,T]; L^2(\Omega)\big) \times C\big([0,T]; L^2(\Omega)\big)$ *satisfy all the assumptions of Theorem* 5.1.1.

5.1.2 **Application of Schauder's fixed point theorem**

The next theorem is an existence result derived from Schauder's fixed point principle, assuming that nonlinearities F and G have a growth at most linear.

Theorem 5.1.2*Let* $F, G: C\big([0,T]; L^2(\Omega)\big) \times C\big([0,T]; L^2(\Omega)\big) \to L^2\big(0,T; H^{-1}(\Omega)\big)$. *Assume that F and G are continuous and satisfy the growth conditions.*

$$|F(u_1,u_2)(t)|_{H^{-1}} \leq a_1|u_1(t)|_{L^2} + b_1|u_2(t)|_{L^2} + h_1$$

and (5.1.7)

$$|G(u_1,u_2)(t)|_{H^{-1}} \leq a_2|u_1(t)|_{L^2} + b_2|u_2(t)|_{L^2} + h_2$$

for every $u = (u_1,u_2) \in C\big(0,T; L^2(\Omega)\big) \times C\big(0,T; L^2(\Omega)\big)$ *,*$t \in [0,T]$, *where* $a_i, b_i, h_i \in \mathbb{R}_+$. *Then* (5.1.2) *has at least one solution* $u = (u_1,u_2)$*,*

$u \in L^2\big(0,T; H_0^1(\Omega)\big) \cap C\big([0,T]; L^2(\Omega)\big) \times L^2\big(0,T; H_0^1(\Omega)\big) \cap C\big([0,T]; L^2(\Omega)\big)$.

Example 5.1.2 *Let* $F_1, G_1: L^2(\Omega) \times L^2(\Omega) \to H^{-1}(\Omega)$ *be two continuous maps for which there exists the constants* $a_1, a_2, b_1, b_2 \in \mathbb{R}_+$ *with*

$$|F_1(u_1,u_2)|_{H^{-1}} \leq a_1|u_1|_{L^2} + b_1|u_2|_{L^2} + h_1$$

and

$$|G_1(u_1,u_2)|_{H^{-1}} \leq a_2|u_1|_{L^2} + b_2|u_2|_{L^2} + h_2$$

Then the maps $F, G: C([0,T]; L^2(\Omega)) \times C\big([0,T]; L^2(\Omega)\big) \to L^2(0,T; H^{-1}(\Omega))$ *given by*

$$F(u)(t) = F_1\big(u(t)\big) \text{ and } G(u)(t) = G_1\big(u(t)\big)$$

for $u = (u_1,u_2) \in C\big([0,T]; L^2(\Omega)\big) \times C\big([0,T]; L^2(\Omega)\big)$ *satisfy all the assumptions of Theorem* 5.1.2.

Example 5.1.3 Let $f, g: \Omega \times \mathbb{R} \to \mathbb{R}$ be two functions such that $f(.,\tau)$ and $g(.,\tau)$ are measurable for every $\tau \in \mathbb{R}$, $f(.,\tau), g(x,.)$ are continuous for a.e. $x \in \Omega$, $f(.,\tau), g(.,0) \in H^{-1}(\Omega)$ and there are the constants $a_0, b_0 \in \mathbb{R}_+$ with

$$|f(x,\tau) - f(x,0)| \leq a_0|\tau|$$

and

$$|g(x,\tau) - g(x,0)| \leq b_0|\tau|$$

for a.e. $x \in \Omega$ and all $\tau \in \mathbb{R}$. Then the operators $F, G: C([0,T]; L^2(\Omega)) \times C([0,T]; L^2(\Omega)) \to L^2(0,T; H_0^1(\Omega))$ defined by

$$F(u,v) = f(.,u(.)) \text{ and } G(u,v) = g(.,u(.))$$

satisfy all the assumptions from the previous example.

5.1.3 **Application of Leray-Schauder's fixed point theorem**

The next result is based on the Leray-Schauder principle. We seek a weak solution to the system (5.1.2).

Theorem 5.1.3 *Assume that* F *and* G *are continuous and admit the decompositions* $F = F_0(u,v) + f$ *and* $G = G_0(u,v) + g$ *such that the following conditions are satisfied for all* $u, v \in L^2(0,T; H_0^1(\Omega))$, *any* $t \in [0,T]$, *some constants* $a_1, a_2, b_1, b_2 \in \mathbb{R}_+$ *such that* $\lambda_1 > \left(a_1 + b_2 + \sqrt{(a_1 - b_2)^2 + 4a_2 b_1}\right)/2$, *and* $f, g \in H^{-1}(\Omega)$*:*

$$(F_0(u,v)(t), u(t)) \leq a_1 |u(t)|_{L^2}^2 + b_1 |v(t)|_{L^2}^2$$

$$(G_0(u,v)(t), v(t)) \leq a_2 |u(t)|_{L^2}^2 + b_2 |v(t)|_{L^2}^2 \tag{5.2.9}$$

Then (5.1.2) *has at least one solution*

$$(u,v) \in L^2(0,T; H_0^1(\Omega)) \times L^2(0,T; H_0^1(\Omega)).$$

Example 5.1.4 Let $F_0, G_0: L^p(\Omega) \times L^p(\Omega) \to L^{(2^*)'}(\Omega)$ be continuous maps satisfying the following conditions:

$$|F_0(u,v)|_{L^{(2^*)'}} \leq c_1 |v|_{L^p}^{\frac{p}{(2^*)'}} \text{ and } |G_0(u,v)|_{L^{(2^*)'}} \leq c_2 |v|_{L^p}^{\frac{p}{(2^*)'}}, \tag{5.2.13}$$

for all $u, v \in L^2(0,T; H_0^1(\Omega))$;

$$(F_0(u,v)(t), u(t)) \leq a_1 |u(t)|_{L^2}^2 + b_1 |v(t)|_{L^2}^2$$

and (5.1.14)

$$(G_0(u,v)(t), v(t)) \leq a_2 |u(t)|_{L^2}^2 + b_2 |v(t)|_{L^2}^2$$

for all $u, v \in L^2(0,T; H_0^1(\Omega))$

for some $\lambda_1 > \frac{\left(a_1+b_2+\sqrt{(a_1-b_2)^2+4a_2b_1}\right)}{2}$, $a_1, a_2, b_1, b_2 > 0, (2^*)' \leq p \leq 2^*$ if $n \geq 3$ and $p \geq 1$ for $n = 2$ and $n = 1$.Then the maps $F, G: L^2(0,T; L^p(\Omega)) \times L^2(0,T; L^p(\Omega)) \to L^2(0,T; H^{-1}$given by $F(u)(t) = F_0\big(u(t)\big)$ *and* by $G(u)(t) = G_0\big(u(t)\big)$ satisfy all assumptions of Theorem 5.1.3.

Example 5.1.5 Let $f, g: \Omega \times \mathbb{R} \to \mathbb{R}$ be two functions such that $f(.,\tau), g(.,\tau)$ is measurable for every $\tau \in \mathbb{R}$, $f(x,.), g(x,.)$ are continuous for a.e. $x \in \Omega$ and there are $c_1, c_2, \alpha \in \mathbb{R}_+$, $1 \leq \alpha < 2^* - 1$ such that

$$|f(x,\tau)| \leq c_1|\tau|^\alpha \text{ and } |g(x,\tau)| \leq c_2|\tau|^\alpha; \tag{5.1.15}$$

$$\tau f(x,\tau) \leq 0 \text{ and } \tau g(x,\tau) \leq 0 \tag{5.1.16}$$

for a.e. $x \in \Omega$ and all $\tau \in \mathbb{R}$. Then the superposition operators $F, G: L^p(\Omega) \times L^p(\Omega) \to H^{-1}(\Omega)$ given by $F(u,v) = f\big(., v(.)\big)$, and $G(u,v) = g\big(., v(.)\big)$, with $p = \alpha(2^*)'$, satisfies the conditions of the previous example.

Example 5.1.6 The functions $f(x,\tau) = -|\tau|^{\alpha-1}\tau$ *and* $g(x,\tau) = -|\tau|^{\beta-1}\tau$ ($\tau \in \mathbb{R}$,), where $1 \leq \alpha, \beta < 2^* - 1$, satisfy all the assumptions from Example 5.1.5.

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