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SPECIAL FUNCTIONS WITH APPLICATIONS IN NUMERICAL ANALYSIS

ABSTRACT

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Introduction

This thesis investigates the mathematical functions which arise in Analysis and Applied mathematical problems - the so called **Special functions** - as well as the mathematical theory of their approximations. It contains a theoretical theory of their approximations.

There are hundreds of special functions used in applied mathematics and computing sciences.

The algebraic aspect of the theory of special functions have not significantly changed since the nineteenth century.

Paul Turán remarked that special functions would be more appropriately label "useful functions".

Because of their remarkable properties, special functions have been used for several centuries, since they have numerous applications in astronomy, trigonometric functions which have been studied for over a thousand years. Even the series expansions for sine and cosine, as well as the arc tangent were known for long time ago from the fourteen century. Since then the subject of special functions has been continuously developed with contribution of several mathematicians including Euler, Legendre, Laplace, Gauss, Kummer, Riemann and Ramanujan. In the past several years the discoveries of new special functions and applications of this kind of functions to new areas of mathematics have initiated a great interest of this field. These discoveries include work in combinatorics, initiated by Schützeberg and Foata. Moreover, in recent years, particular cases of long familiar special functions have been clearly defined and applied as orthogonal polynomials.

The special functions have been studied in several volumes by the collective of mathematicians formed by G. E. Andrews, R. Askey and R. Roy (Cambridge University Press, Encyclopedia of Mathematics and its Applications).

There are important results from the past that must be included in this field because they are so useful. Then, there are recent developments that should be brought to the attention of those who could used them: we would wish to help educate the new generation of mathematicians and scientists so they can further develop and apply this subject. Specialized texts dealing with some of these developments have recently appeared: Petkovitek, Wilf and Zeilberger (1996), Macdonald (1995), Heckman and Schlicktrull (1994) and Vilenkin and Kliniyk (1992).

It is clear that the amount of knowledge about special functions is so great that only a small fraction of it can be included in one book. We decided to insist on hypergeometric functions and the associated hypergeometric series.

Several important facts about hypergeometric series were first found by Euler, Pfaff and Gauss. This last mathematician fully recognized their significance and gave a systematic account of these. A half century after Gauss, Riemann developed hypergeometric functions from a different point of view, which made available the basic formulas with a minimum of computations.

Another approach to hypergeometric functions using contour integrals was presented by the English mathematician E. W. Barnes in the first decade of the last century.

Hypergeometric functions have two very significant properties that add to their usefulness. They satisfy certain identities for special values of the function and they have transformation formulas.

In Combinatorial analysis hypergeometric identities classify single sums of products of binomial coefficients.

The arithmetic-geometric mean has recently been used to compute π to several million decimal places and earlier it played a great a role in Gauss theory of elliptic functions.

The gamma functions and beta integrals dealt with an essential understanding of hypergeometric functions.

The gamma function was introduced into mathematics by Euler in 1720 when he solved the problem of extending the factorial function to all real or complex numbers.

There are extensions of gamma and beta functions that are also very important.

The theory of special functions with its numerous beautiful formulas is very well suited to an algorithmic approach to mathematics. In the nineteenth century it was the ideal of Eisenstein and Kronecker to express and develop mathematical results by means of formulas. Before them, this attitude was common and best exemplified in the works of Euler, Jacobi and Gauss.

In the twentieth century, mathematics moved from this approach toward a more abstract and existential methods. In fact agreeing with Hardy that Ramanujan came 100 years too late, Littlewood wrote that "the great day of formulae seem to be over" (mentioned by Littlewood, 1986).

However, with the advent of computers and the consequent three of computational mathematics formulas are now once again playing a larger role in mathematics. We mention that beautiful, interesting and important formulas have been discovered since Ramanujan's time. These formulas are proving fertile and fruitful.

* * *

Our objective is to present a unified theory of special functions.

The thesis consists of five chapters, an introduction and a bibliography, containing 128 titles. First three of this chapters refer to the papers of the author, published in the journal "Studia Universitatis Babeş-Bolyai", Mathematica, containing theoretical elements useful in chapter four. The last chapter contains theoretical elements for a future research.

The chapter headings are as follows:

(0) Introduction, (1) The Gamma and Beta functions, (2) Classical orthogonal polynomials, (3) Numerical quadratures with multiple Gaussian nodes, (4) Applications of some special functions in Numerical Analysis, (5) The Zeta function (Riemann, Hurwitz). Integer values for even argument of $\zeta(z)$.

1. The Gamma and Beta functions

The gamma function was introduced into mathematics by Euler in 1720 when he solved the problem of expanding the factorial function to all real or complex numbers. This problem was apparently suggested by Daniel Bernoulli and Goldbach.

For all complex numbers $\neq 0, -1, -2, \ldots$, the gamma function $\Gamma(x)$ is defined by

$$\Gamma(z) = \lim_{k\to\infty} \frac{k! \cdot k^{z-1}}{(z)_k}. \tag{1.1.5}$$

where $(z)_k = z(z+1)\ldots(z+k-1)$, $k > 0$, $(z)_0 = 1$, $z \in \mathbb{R}$ or $\mathbb{C}$.

Immediate consequence of this definition are

$$\Gamma(z+1) = z\Gamma(z) \tag{1.1.6}$$

$$\Gamma(z+1) = z!, \quad z \in \mathbb{N} \tag{1.1.7}$$

$$\Gamma(1) = 1. \tag{1.1.8}$$

Over seventy years before Euler, Wallis (1656) attempt to compute the integral:

$$\int_0^1 \sqrt{1-z^2}dz = \frac{1}{2}\int_{-1}^1 (1-z)^{1/2}(1+z)^{1/2}dz.$$

Since this integral gives the area of a quarter circle, Wallis's aim was to obtain an expression for π. He found

$$\frac{\pi}{4} = \int_0^1 \sqrt{1-z^2}dz = \left[\Gamma\left(\frac{3}{2}\right)\right]^2.$$

The beta integral is defined by

$$B(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1}dt, \quad \operatorname{Re} z > 0,\ \operatorname{Re} w > 0.$$

This integral is symmetric in z and w as may be seen by the change of variable $u = 1 - t$. We also can write:

$$B(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)} \tag{1.6.8}$$

and

$$B(z,w) = \frac{\Gamma(w)}{\Gamma(z+w)} \int_0^\infty t^{z-1} e^{-t} dt. \tag{1.6.7}$$

The Euler integral of the second kind is:

$$\Gamma(z) = \int_0^\infty t^{z-1} \cdot e^{-t} dt; \quad \operatorname{Re} z > 0. \tag{1.1.9}$$

It is often taken as the definition of $\Gamma(z)$, $\operatorname{Re} z > 0$.

In the section "Numbers and polynomials of Bernoulli" we mention that frequently have to use polynomials which are related to the factorial and called Bernoulli's polynomials. These polynomials are of great importance in mathematical analysis and combinatorics. Bernoulli's polynomials are by various authors defined in slightly different ways.

We denote by $B_n(z)$ this polynomial of degree n. It satisfies at the same time the following two relations:

$$(\Delta B_n)(z) = nz^{n-1}, \quad B_n(0) = B_n \tag{1.5.7}$$

$$B_n'(z) = nB_{n-1}(z), \quad n \geq 1. \tag{1.5.11}$$

It is obvious that a polynomial with such simple properties must have important applications.

it can be seen that the polynomial $B_n(z)$ is perfectly determined by the above relations.

If the polynomial exists, then $B_n(z+h)$ will also be a polynomial of degree n and by the Taylor theorem we can write the unique expansion

$$B_n(z+h) = B_n(z) + \sum_{j=1}^{m} \frac{h^j}{j!} B_n^{(j)}(z) \tag{1.5.14}$$

It can be written also

$$B_n(z+h) = B_n(z) + \sum_{j=1}^{m} \binom{m}{j} h^j B_{n-j}(z) \tag{1.5.14'}$$

It follows:

$$B_n(z+h) = B_n(z) + \sum_{j=1}^{m} \binom{m}{j} h^{n-j} B_j(z).$$

Replacing $h = 1$, we obtain

$$\sum_{j=1}^{n} \binom{n}{j} B_{n-j}(z) = nz^{n-1}.$$

By this formula the polynomials $B_n(z)$ are perfect determined and we find

$$B_0(z) = 1, \quad B_1(z) = z - \frac{1}{2}, \quad B_2(z) = z^2 - z + \frac{1}{6}, \dots$$

The values of $B_n(z)$ for $z = 0$ are called Bernoulli numbers, that is

$$B_n = B_n(0).$$

By these formulas we find

$$B_0 = 1, \quad \sum_{j=1}^{1} \binom{n}{j} B_j = B_n \quad (n > 1)$$

This relation may be written in a convenient symbolical form:

$$(B+1)^n - B^n = 0, \quad n > 1.$$

The first Bernoulli's numbers are:

$$B_0 = 1, \quad B_1 = \frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}.$$

Bernoulli's polynomials may be expressed explicitly by Bernoulli's numbers. We have:

$$B_n(z) = \sum_{j=0}^{n} \binom{n}{j} B_j z^{n-j}$$

or in a symbolical form:

$$B_n(z) = (z+B)^n.$$

Bernoulli's numbers and polynomials satisfy a great many relations which are most obtained in symbolical form.

It can be seen that $B_{2j}(z)$ is symmetrical about the point $z = \frac{1}{2}$ and that $B_{2j+1}\left(\frac{1}{2}\right) = 0$. On the other way it can be seen that $B_{2j+1} = 0$ $(j > 0)$.

The Bernoulli polynomial satisfy the recurrence relation:

$$\sum_{j=1}^{n} \binom{n}{j} B_{n-j}(z) = nz^{n-1}, \quad n \in \mathbb{N}.$$

If we consider the function of complex variable

$$G_z(t) = e^{tz}\frac{t}{e^t - 1}. \tag{1.5.15}$$

we can write

$$G_z(t) = \sum_{n=0}^{\infty} \frac{A_n(z)}{n!} t^n$$

where:

$$A_n(z) = G_z^{(n)}(t)$$

and more:

$$G_z(t) = e^{tz}\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(z)}{n!} t^n, \quad |t| < 2\pi \tag{1.5.16}$$

with

$$B_n(z) = G_z^{(n)}(t) \tag{1.5.17}$$

Replacing $z = 0$ we get

$$G_0(t) = g(t) = \frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n, \quad |t| < 2\pi. \tag{1.5.18}$$

G_x is called the generating function of the Bernoulli polynomials.

The Bernoulli polynomials satisfy the relation:

$$B_n(1 - z) = (-1)^n B_n(z), \quad n \in \mathbb{N},\ z \in \mathbb{C}. \tag{1.5.19}$$

For $n = 2k$ we obtain:

$$B_{2k}(1 - z) = B_{2k}(z).$$

Consequently, the graphic of the function $w = B_{2k}(z)$ is symmetric with respect to $x = \frac{1}{2}$.

The Bernoulli numbers of odd degree are all equal with zero.

2. Classical orthogonal polynomials

For simplicity, we select as fundamental interval $[-1,1]$. The relevant weight function is:

$$w(x) = (1-x)^\alpha(1+x)^\beta, \quad \alpha > -1, \quad \beta > -1$$

for which we obtain the Jacobi polynomials $J_m^{(\alpha,\beta)}(x)$.

If the exponents are between -1 and 0 then we can write

$$\int_{-1}^{1}(1-x)^\alpha(1+x)^\beta dx = 2^{\alpha+\beta+1}\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}.$$

The following special selection of α and β carry special names for the Jacobi polynomials $J_m^{(\alpha,\beta)}(x)$.

If $\alpha = \beta = 0$ then it corresponds to Legendre polynomials

$$L_m(x) = \frac{1}{2^m \cdot m!}[(x^2-1)^m]^{(m)} \tag{2.3.1}$$

It is known as Rodrigues formula.

The corresponding recurrence relation has the following form:

$$\widetilde{L}_{m+1}(x) = x\widetilde{L}_m(x) - \frac{m^2}{4m^2-1}\widetilde{L}_{m-1}(x) \tag{2.3.2}$$

If $\alpha = -\dfrac{1}{2}$ we obtain the Chebyshev polynomials of the first kind

$$J_m^{(-\frac{1}{2},-\frac{1}{2})}(x) = \frac{(-1)^m}{2^m \cdot m!}\sqrt{1-x^2}[(1-x^2)^{-\frac{1}{2}}]^{(m)} \tag{2.4.1}$$

or equivalently except a multiplicative constant:

$$T_m(x) = \cos(m \arccos x) \tag{2.4.2}$$

For $\alpha = \dfrac{1}{2}$ we get the Cebyshev polynomials of the second kind

$$U_m(x) = \frac{1}{m+1} T'_{m+1}(x) = \frac{\sin[(m+1)\arccos x]}{\sin(\arccos x)} \tag{2.5.4}$$

If $\alpha = \beta$ we obtain the ultraspherical polynomials:

$$J_m^{(\alpha,\alpha)}(x) = \frac{1}{2^m \cdot m!} \frac{(-1)^m}{(1-x^2)^\alpha} [(1-x^2)^{m+\alpha}]^{(m)} \tag{2.2.1}$$

We give also the Christoffel-Darboux formula:

$$\mathcal{K}_m(x,t) = \sum_{k=1}^{m} \widehat{P}_k(x)\widehat{P}_k(t) = \sqrt{\gamma_{m+1}} \frac{\widehat{P}_{m+1}(x)\widehat{P}_m(t) - \widehat{P}_{m+1}(t)\widehat{P}_m(x)}{x-t}. \tag{2.0.30}$$

In the section 2.6 one present the Laguerre orthogonal polynomials

$$L_m^{[\alpha]}(x) = x^{-\alpha} e^x (x^{m+\alpha} e^{-x})^{(m)}, \quad x \in [0, +\infty) \tag{2.6.1}$$

which corresponds to the weight function

$$w(x) = x^\alpha e^{-x}, \quad \alpha > -1 \tag{2.6.2}$$

The corresponding recurrence relation is

$$\widetilde{L}_{m+1}^{[\alpha]}(x) = [x - (2m+\alpha+1)]\widetilde{L}_m^{[\alpha]}(x) - m(m+\alpha)\widetilde{L}_{m-1}^{[\alpha]}(x). \tag{2.6.7}$$

Section 2.7 from the second chapter refers to the Hermite orthogonal polynomials

$$H_m(x) = (-1)^m e^{x^2} (e^{-x^2})^{(m)} \tag{2.7.1}$$

with the weight function

$$w(x) = e^{-x^2} \tag{2.7.2}$$

3. Numerical quadrature formulas using multiple Gaussian nodes

Section 3.1 contain a presentation of the s-orthogonal polynomials.

We denote by $\{P_{n,s}(x)\}$ the sequence of s-orthogonal polynomials characterized by the condition:

$$\int_a^b w(x) P_{m,s}^{2s+1}(x) x^k dx = 0, \quad k = \overline{0, m-1} \tag{3.1.1}$$

In the case of interval $(-1, 1)$ we have

$$\int_{-1}^1 [P_{m,s}(x)]^{2s+2} = \frac{2}{1 + (2s+2)m} \tag{3.1.11}$$

This formula was given by Ghizetti and Ossicini [40].

The s-orthogonal polynomials $P_{n,s}(x)$ minimizes the integral

$$F(a_0, a_1, \ldots, a_{m-1}) = \int_{-\infty}^{\infty} w(x)][P_{m,s}(x)]^{2s+2} dx \tag{3.1.13}$$

where

$$P_{m,s}(x) = x^m + a_{m-1}x^{m-1} + \ldots + a_1 x + a_0 \tag{3.1.13'}$$

G. V. Milovanović [66] has presented a method for the construction of the s-orthogonal polynomials.

S. Bernstein has proved [10] that for any nonnegative integer $s \in \mathbb{N}$ which minimizes $F(a_0, a_1, \ldots, a_{m-1})$ we obtain the Chebyshev orthogonal polynomial of the first kind

$$\widetilde{T}_m(x) = \frac{1}{2^{m-1}} T_m(x) \tag{3.1.15}$$

Section 3.2 is devoted to the study of the Gauss-Turán quadrature formulas.

In general such a formula is of the form:

$$\int_a^b w(x)f(x)dx = \sum_{k=1}^{m}\sum_{j=0}^{2s} A_{k,j} f^{(j)}(x_k) + R(f) \tag{3.2.1}$$

and has the degree of exactness $N = 2(s+1)n - 1$.

Turán [120] has constructed such a quadrature formula for the interval $[-1, 1]$ and the weight function $w(x) = 1$.

G. Vincenti [121] has presented a procedure for the evaluation of the coefficients of the s-orthogonal polynomials.

G. Milovanović [66], [67] has presented a stable procedure for the construction of the s-orthogonal polynomials $P_{n,s}(x)$.

The Gauss-Bernstein-Turán quadrature formula is

$$\int_{-1}^{1} \frac{f(x)dx}{\sqrt{1-x^2}} = \sum_{k=1}^{m}\sum_{j=0}^{2s} A_{k,j} f^{(j)}\left(\cos\frac{2k-1}{2m}\pi\right) + R(f) \tag{3.2.4}$$

Section 3.3 referes to σ-orthogonal polynomials.

Let $\sigma = (s_1, s_2, \ldots, s_m)$, $m \in \mathbb{N}$, be a sequence of integers numbers. We consider the nodes (x_k), $k = \overline{1,m}$ such that $a \le x_1 < x_2 < \ldots < x_m \le b$, having the multiplicities $2s_1 + 1$, $2s_2 + 1, \ldots$, $2s_m + 1$. Here we study the quadrature formula

$$\int_a^b w(x)f(x)dx = \sum_{k=1}^{m}\sum_{j=0}^{2s_k} A_{k,j} f^{(j)}(x_k) + R(f), \tag{3.3.1}$$

having the degree of exactness $N = 2S + 2m - 1$, where

$$S = s_1 + s_2 + \ldots + s_m.$$

This degree N can be obtained if

$$\int_a^b w(x)\prod_{\nu=1}^{m}(x - x_\nu)^{2s_\nu - 1} x^k dx = 0, \quad k = \overline{0, m-1}. \tag{3.3.3}$$

The polynomials satisfying these conditions are called σ-orthogonal polynomials.

A general quadrature formula of the form (3.3.1) in thesis was introduced and investigated by L. Chakalov [15] and T. Popoviciu [83].

For the construction of a quadrature formula of the form (3.3.1) in thesis we can start from the Lagrange-Hermite interpolation formula

$$f(x) - (L_H f)(x) = (Rf)(x) \tag{3.3.8}$$

where:

$$(L_H f)(x) = L\begin{pmatrix} x_k & , & t_j & , & x & ; f \\ 2s_k+1 & & 1 & & 1 & \end{pmatrix} \tag{3.3.4}$$

and

$$(Rf)(x) = u(x)v(x)\begin{bmatrix} x_1 & ,\dots, & x_m & , & t_1 & ,\dots, & t_m & , & x & ; f \\ 2s_1+1 & & 2s_m+1 & & 1 & & 1 & & 1 & \end{bmatrix} \tag{3.3.9}$$

while

$$\begin{aligned} v(x) &= (x-t_1)(x-t_2)\dots(x-t_m) \\ u(x) &= (x-x_1)^{2s_1+1}(x-x_2)^{2s_2+1}\dots(x-x_m)^{2s_m+1} \end{aligned} \tag{3.3.6}$$

Because we can write

$$(L_H f)(x) = v(x)L_H\begin{pmatrix} x_k & , & x & ; f_1 \\ 2s_k+1 & & 1 & \end{pmatrix} + u(s)L\begin{pmatrix} t_j & , & x & ; f_2 \\ 1 & & 1 & \end{pmatrix} \tag{3.3.5}$$

where:

$$\begin{aligned} f_1 \equiv f_1(x) &= f(x)/v(x) \\ f_2 \equiv f_2(x) &= f(x)/u(x) \end{aligned} \tag{3.3.6}$$

Multiplying by the weight function $w(x)$ and integrating on the interval (a, b) we get a quadrature formula of the form

$$I(w; f) = F(f) + \phi(f) + E(f) \tag{3.3.10}$$

where $E(f) = I(w, (Rf)(x))$ while

$$\phi(f) = \sum_{j=1}^{m} B_j f(t_j) \tag{3.3.11}$$

Now, we want to choose the nodes x_k such that we have $B_j = 0$ $(j = \overline{1, n})$.

For this it is necessary that the polynomial $u(x)$ to be orthogonal on (a, b) with respect to the weight function $w(x)$ with any polynomial of degree $n - 1$.

By integration we obtain for the coefficients of this quadrature formula the expression:

$$A_{k,j} = \int_a^b w(x) l_{k,j}(x) dx, \quad k = \overline{1,m},\ j = \overline{0, 2s_k} \tag{3.3.17}$$

where:

$$l_{k,j}(x) = \frac{(x-x_k)^j}{j!} \left[\sum_{\nu=0}^{2s_k - j} \frac{(x-x_k)^\nu}{\nu!} \left(\frac{1}{u_k(x)} \right)_{x_k}^{(\nu)} \right] u_k(x) \tag{3.3.18}$$

and

$$u_k(x) = u(x)/(x-x_k)^{2s_k+1} \tag{3.3.19}$$

The σ-orthogonal polynomials:

$$P_{m,\sigma}(x) = \prod_{\nu=1}^{m} (x - x_\nu^{m,\sigma}) \tag{3.3.20}$$

can be obtained minimizing the following integral:

$$\int_{-\infty}^{\infty} w(x) \prod_{\nu=1}^{m} (x - x_\nu)^{2s_\nu+2} dx. \tag{3.3.23}$$

The section 3.4 has the title: "The generalization given by D. D. Stancu for the quadrature formula of Gauss-Turán-Chakalov-Popoviciu".

In the paper [92] D. D. Stancu has introduced and investigated a general quadrature using multiple of fixed and Gaussian nodes having the form:

$$I(f) = \phi(f) + R(f) \tag{3.4.1}$$

where:

$$I(f) = I(f; w) = \int_a^b w(x) f(x) dx \tag{3.4.2}$$

and

$$\phi(f) = \sum_{k=1}^{m} \sum_{j=0}^{2s_j} A_{k,j} f^{(j)}(x_k) + \sum_{i=1}^{r} \sum_{\nu=0}^{r_i} B_{i,\nu} f^{(\nu)}(a_i) \tag{3.4.3}$$

while the polynomial of fixed nodes is

$$\omega(x) = \prod_{i=1}^{\pi} (x - a_i)^{r_i+1} \tag{3.4.4}$$

and the polynomial of Gaussian nodes is of the form:

$$u(x) = \prod_{k=1}^{m} (x - x_k)^{2s_k+1} \tag{3.4.5}$$

The above quadrature formula has the maximum degree of exactness $D = M + N + m - 1$, where:

$$M = \sum_{i=1}^{r} (r_i + 1), \quad N = \sum_{k=1}^{m} (2s_k + 1) \tag{3.4.6}$$

if and only if the polynomial $u(x)$ is orthogonal with respect to the weight function $w(x) \cdot \omega(x)$ with any polynomial of degree $m - 1$.

In order to find the nodes x_k we can consider the function of n variables

$$F(t_1, t_2, \ldots, t_m) = I(w; U) = \int_a^b w(x)\omega(x)(x - t_1)^{2s_1+2} \ldots (x - t_m)^{2s_m+2} dx \tag{3.4.24}$$

This function is continuous and positive. Consequently it has a relative minimum. We can find it solving the system of equations

$$\frac{1}{2s_k + 2} \cdot \frac{\partial F}{\partial x_k} = I(P_k) = 0 \tag{3.4.25}$$

where

$$P_k = \omega(x) \prod_{k=1}^{m} (x - x_k)^{2s_k+2} \frac{1}{x - x_k} \tag{3.4.26}$$

We have:

$$\frac{\partial F}{\partial x_k} = 0; \quad \frac{\partial^2 F}{\partial x_k^2} > 0; \quad i, k = \overline{1, m}; \ i \neq k \tag{3.4.28}$$

For the remainder $R(f)$ was found the following expression:

$$R(f) = \frac{f^{(M+N+m)}(\xi)}{(M + N + m)!} \int_a^b w(x)u^2(x)\omega(x)dx. \tag{3.4.29}$$

after making $t_j \mapsto x_j$, $j = \overline{1, m}$, and assuming that the function f has a continuous derivative of order $M + N + n$ on (a, b).

4. Applications of some special functions in Numerical Analysis

In section 4.1 is presented a linear positive operator of D. D. Stancu

$$(S_n^\alpha f)(x) = S_n^\alpha(f,x) := \sum_{k=0}^{n} w_{n,k}^\alpha(x) f\left(\frac{k}{n}\right) \tag{4.1.1}$$

where

$$w_{n,k}^\alpha(x) = \binom{n}{k} \frac{x^{(k,-\alpha)}(1-x)^{(n-k,-\alpha)}}{1^{(n,-\alpha)}}, \quad \alpha \in \mathbb{R}. \tag{4.1.2}$$

For $\alpha = 0$ it reduces to the Bernstein operators, while $\alpha = \alpha_n = \frac{1}{n}$ one gets the Lagrange interpolation operator.

Section 4.2 is devoted to a probabilistic methods using the Markov-Polya distribution.

This distribution can be obtained by the following modification of the Bernoulli scheme. An urn contains (a) white and (b) black balls. One draws one ball at random. Then it is replaced and one adds (c) balls of the same color. This procedure is repeated n times. Assuming that X is the random variable which takes on the value k $(0 \le k \le n)$ if during n trials one obtains exactly k times (a) white ball then

$$P(k; n, a, b, c) = \binom{n}{k} \frac{a(a+c)\dots[a+(k-1)c]b(b+c)\dots[b+(n-k-1)c]}{(a+b)(a+b+c)\dots[a+b+(n-1)c]} \tag{4.2.2}$$

Now by introducing the notations

$$\begin{aligned} x &:= \frac{a}{a+b}, \ x \text{ - variabil} \\ \alpha &= \frac{c}{a+b}, \ \alpha = \text{ const} \end{aligned} \tag{4.2.3}$$

we can see that

$$w^{\alpha}_{n,k}(x) = \binom{n}{k}\frac{x^{(k,\alpha)}(1-x)^{(n-k,-\alpha)}}{1^{(n,-\alpha)}} \tag{4.2.5}$$

The linear operator $(S^{\alpha}_n f)(x)$ can be expressed by means of finite differences

$$(S^{\alpha}_n f)(x) = f(0) + \sum_{j=1}^{n}\binom{n}{j}\frac{x(x+\alpha)\dots[x+(j-1)\alpha]}{(1+\alpha)(1+2\alpha)\dots[1+(j-1)\alpha]}\Delta^{j}_{\frac{1}{n}}f(0) \tag{4.2.18}$$

where:

$$\Delta^{j}_{\frac{1}{n}}f(0) = \sum_{\nu=0}^{j}(-1)^{\nu}\binom{j}{\nu}f\left(\frac{j-\nu}{m}\right). \tag{4.2.18$'$}$$

In the case $\alpha = 0$ it was given by G. Lorentz.

For $\alpha > 0$ we can give a representation using the Beta function

$$(S^{\alpha}_n f)(x) = \frac{1}{B\left(\frac{x}{\alpha};\frac{1-x}{\alpha}\right)}\int_0^1 t^{\frac{x}{\alpha}}(1-t)^{\frac{1-x}{\alpha}}(B_n f)(t)dt \tag{4.3.1}$$

Using a Lupaş's result we obtain:

$$(S^{\alpha}_n f)(x) = (\mathcal{B}_n f)(x) + \frac{\alpha x(1-x)}{1+\alpha}[x_0, x_1, x_2; \mathcal{B}_n, f] \tag{4.3.2}$$

and with divided differences we have:

$$(S^{\alpha}_n f)(x) = f(0) + \sum_{j=0}^{n} A_{n,j}(f)x(x+\alpha)\dots[x+(j-1)\alpha] \tag{4.3.2$'$}$$

In section 4.4 are given the Stancu operators of two variables.

There are presented the following operators Stancu-Baskakov, Stancu-Meyer-König and Zeller.

$$(V^{\alpha}_m f)(x) = \sum_{k=0}^{\infty} v^{\alpha}_{m,k}(x) f\left(\frac{k}{m}\right), \quad x \geq 0 \tag{4.5.4}$$

where

$$(W^{\alpha}_m f)(x) = \sum_{k=0}^{\infty} p^{\alpha}_{m,k}(x) f\left(\frac{k}{m+k}\right) \tag{4.5.9}$$

Then is presented the operator of D. D. Stancu using the Beta distribution of second kind

$$(L_m f)(x) = (T_{mx,m+1}f)(x) = \frac{1}{B(mx, m+1)}\int_0^{\infty} f(t)\frac{t^{mx-1}dt}{(1+t)^{mx+m+1}} \tag{4.5.21}$$

and for $f \in C[0,\infty)$ is evaluated the order of approximation using the moduli of continuity of first and second order.

Section 4.6 has the following title: "Construction of the operators of approximations using the approximation formulas of Abel-Jensen".

One starts with the celebraten generalization of the Newton binomial formula, given in 1826 by the outstanding mathematical genius Niels Henrik Abel [Journal für Reine und Mathematik **1**(1826), 159-160], namely

$$(u+n)^n = \sum_{k=0}^{n} \binom{n}{k} u(u-k\beta)^{k-1}(v+k\beta)^{n-k} \tag{4.6.4}$$

where β is a nonnegative parameter.

There are also mentioned the Abel type formulas

$$(u+v+n\beta)^n = \sum_{k=0}^{n} \binom{n}{k} u(u+k\beta)^{k-1}(v+(n-k)\beta)^{n-k} \tag{4.6.10}$$

$$(u+v+n\beta)^n = \sum_{k=0}^{n} \binom{n}{k} (u+k\beta)^{k} v[v+(n-k)\beta]^{n-k-1} \tag{4.6.11}$$

Jensen [49] has obtained a new symmetrical identity of Abel

$$[u+v(u+v+n\beta)]^{n-1} = \sum_{k=0}^{n} \binom{n}{k} u(u+k\beta)^{k-1} v[v+(n-k)\beta]^{n-k-1} \tag{4.6.8}$$

The American mathematician H. W. Gould [43] gave the following generalization of the Vandermonde formula

$$\binom{u+v+n\beta}{n} = \sum_{k=0}^{n} \binom{n+k\beta}{k} \binom{v+(n-k)\beta}{n-k} \frac{v}{v+(n-k)\beta}$$

which can be written, by using the factorial powers, under the form:

$$(u+n+n\beta)^{[n]} = \sum_{k=0}^{n} \binom{n}{k} (u+k\beta)^{[k]} u(v+(n-k)\beta)^{[n-k-1]}$$

The factorial power of a non-negative order n and increment h of u is defined by the formula

$$u^{(n,h)} = u(u-h)\dots[u-(n-1)h] \tag{4.6.1}$$

By using the preceding combinatorial identities one can introduces the following basic polynomials:

$$s_{m,k}^{\alpha,\beta}(x) = \frac{1}{1^{(m,-\alpha)}} \sum_{k=0}^{m} \binom{m}{k} x(x-k\beta)^{(k-1,-\alpha)}(1-x+k\beta)^{(m-k,-\alpha)} \tag{4.6.14}$$

$$q_{m,k}^{\alpha,\beta}(x) = \frac{1}{(1+m\beta)^{[m-1,-\alpha]}} \sum_{k=0}^{m} \binom{m}{k} x(x+k\beta)^{(k-1,-\alpha)}(1-x)[1-x+(n-k)\beta]^{(m-k,-\alpha)} \tag{4.6.15}$$

$$p_{m,k}^{\alpha,\beta}(x) = \frac{1}{(1+m\beta)^{(m,-\alpha)}} \sum_{k=0}^{m} \binom{m}{k} x(x+k\beta)^{(k-1,-\alpha)}[1-x+(m-k)\beta]^{(m-k,-\alpha)} \tag{4.6.16}$$

$$r_{m,k}^{(\alpha,\beta)}(x) = \frac{1}{(1+m\beta)^{(m,-\alpha)}} \sum_{k=0}^{m} \binom{m}{k} (x+k\beta)^{(k,-\alpha)}(1-x)[1-x+(m-k)\beta]^{(m-k-1,-\alpha)} \tag{4.6.17}$$

By using these basic polynomials I constructed the following linear positive operators, corresponding to a function $f \in C[0,1]$

$$\begin{aligned} S_m^{\alpha,\beta,\gamma,\delta} f &= \sum_{k=0}^{m} s_{m,k}^{\alpha,\beta}(x) f\left(\frac{k+\gamma}{m+\delta}\right) \\ Q_m^{\alpha,\beta,\gamma,\delta} f &= \sum_{k=0}^{m} q_{m,k}^{\alpha,\beta}(x) f\left(\frac{k+\gamma}{m+\delta}\right) \\ P_m^{\alpha,\beta,\gamma,\delta} f &= \sum_{k=0}^{m} p_{m,k}^{\alpha,\beta}(x) f\left(\frac{k+\gamma}{m+\delta}\right) \\ R_m^{\alpha,\beta,\gamma,\delta} f &= \sum_{k=0}^{m} r_{m,k}^{\alpha,\beta}(x) f\left(\frac{k+\gamma}{m+\delta}\right) \end{aligned} \tag{4.6.18}$$

where we have $0 \leq \gamma \leq \delta$.

In the case $\beta = \gamma = \delta = 0$ these operators reduce to the Stancu operator S_m^{α} introduced and investigated in the paper [95]. This operator was further investigated by Della Vecchia [23], Mastroianni G. and Occorsio M. R. [61] and others.

5. The Zeta function (Riemann, Hurwitz). Integer values for even argument of $\zeta(z)$

Here are given the expression for the values of the function $\zeta(z)$ for even integers by using the Bernoulli numbers

$$\zeta(-n) = (-1)^n \frac{B_{n+1}}{n+1} \tag{5.5.3}$$

which are irrational. The calculation of the values of the function $\zeta(z)$ of even positive number has been studied. This was a more difficult because it was necessary to know the nature of these values.

The first notable result was found by the French mathematician Apery in 1979. In [6] he has proved that $\zeta(3)$ is an irrational number. It was difficult to extend the method of this author to other even integers. Some results were found by T. Rivoal. He shows that there exist an infinity of irrational numbers in the sequence $\zeta(2k+1)$, $k \in \mathbb{N}$.

In 2001 the Russian mathematician W. Zudilin has proved that any set of the form $\zeta(s+2), \zeta(s+4), \ldots, \zeta(8s-3), \zeta(8s-1)$ with even $s > 1$ contains at least an irrational number.

He also proves that at least one of four values $\zeta(5)$, $\zeta(7)$, $\zeta(9)$ and $\zeta(11)$ is irrational.

In section 5.1 we present a new proof given by the mathematician M. Prevost [85] for the irrationality of $\zeta(2)$ and $\zeta(3)$ using Padé approximants.

In the final part of the thesis I express my appreciation and gratitude to the scientific advisor for helping me in elaboration and writing in final form the text of this thesis.

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