"BABEŞ-BOLYAI" UNIVERSITY OF CLUJ-NAPOCA
FACULTY OF MATHEMATICS AND COMPUTER SCIENCE

# COHOMOLOGY OF GROUPS AND OF BLOCKS OF GROUP ALGEBRAS

**-Ph.D. thesis summary-**

by

**Constantin Cosmin Todea**

Scientific adviser:

**Prof. Ph.D. Andrei Mărcuş**



CLUJ NAPOCA

# Contents


**Introduction** **1**

**1 Preliminaries** **6**

1.1 Hochschild cohomology of symmetric algebras . . . 6
1.2 The cohomology of finite groups . . . 8
1.3 Blocks of group algebras, Brauer pairs and pointed groups . . . 9
1.4 The cohomology of blocks of finite groups . . . 10

**2 Stable elements in Hochschild cohomology of group algebra** **12**

2.1 Stable elements in Hochschild cohomology of symmetric algebras . . . 12
2.2 The transfer map between Hochschild cohomology algebras of group algebras . . . 14
2.3 The generalization of the diagonal induction map . . . 15
2.4 Stable elements in Hochschild cohomology of the group algebra . . . 17

**3 The restriction map in cohomology of blocks of finite groups** **19**

3.1 Stable elements in Hochschild cohomology of blocks . . . 20
3.2 Generalized Brauer pairs and pointed groups . . . 20
3.3 The generalized block cohomology . . . 23
3.4 Properties of the restriction map in block cohomology . . . 28
3.5 Varieties in the generalized block cohomology . . . 31

**4 An equivalent definition of cohomology of finite groups** **33**

4.1 Stable elements in the module of homomorphisms . . . 33
4.2 An equivalent definition of cohomology of finite groups . . . 34

**Bibliography** **36**

**KEYWORDS:** symmetric algebra, Hochschild cohomology algebra, transfer map, stable elements, cohomology algebra of a finite group, $G$-algebra, pointed group, block, defect group, fusion system, cohomology algebra of a block, restriction map, variety.

# Introduction

The cohomology of groups has a history which starts 100 years ago. Its origins are in theory of groups and in number theory and then it became an important component of algebraic topology. In the past 30 years group cohomology has developed a strong connection with the representations of finite groups. If at the beginning the connections were around the 1 and 2 degree cohomology, latter the study of ring cohomology became very important .

Representation theory has studied, initially the proprieties of abstract groups through some linear maps of some vector spaces. The representations involved were over the real field or over the complex field and they studied ordinary characters of finite groups, defined by Frobenius in 1896. In the same time L. E. Dickson has considered representations of finite groups with coefficients in a finite field. He proved that if the scalar field has characteristic $p$ and $p$ doesn' t divide the order of $G$ then the methods from ordinary representation theory can be successfully applied. If $p$ divides the order of $G$, Dickson proved that the theory is completely different and in this case we have modular representation theory.

Modular representation theory has been developed by R. Brauer between 1935 and 1977 which built the basics of what is known today as modular representation theory of finite groups. Brauer has defined and studied the basic concepts of block theory. J. A. Green has introduced in the '60 the notion of $G$-algebra where $G$ is a finite group, which can be used to study block theory and also module theory.

Another important step was realized in the '70, by J. L. Alperin, M. Broué şi L. Puig which start the study of $p$-local blocks and representations. Alperin and Brouè introduced the Brauer pairs, and these were used by Brouè and Puig to study nilpotent blocks. A good selection of main results and open problems in modular representation theory can be find in [24].

In [1] J. L. Alperin presents aspects of group cohomology which appears in modular representation theory. Following this line and the prediction from the title of that paper "Cohomology is representation theory", there were published numerous studies which applies homological algebra in representation theory and reverse.

In 1999 M. Linckelmann has published 2 papers [17] and [18], where he studies the properties of the cohomology ring associated with a block, defined similarly with the cohomology ring of a finite group by Cartan-Eilenberg stable elements method from [10]. Further, Linckelmann investigate the varieties associated to the cohomology ring of a block and he proves a Quillen stratification, similarly to the stratification obtained by G. S. Avrunin and L. L. Scott in [3], respectively by the creator of this theory, D. Quillen in [27] and [28].

In this thesis we will approach the cohomology algebra of a finite group and the cohomology algebra of a block. We will apply M. Linckelmann's method of embedding the cohomology algebra of a finite group into the Hochschild cohomology algebra of the group algebra, through a generalized induction map which we will investigate.

We will define the cohomology algebra of a block of a normal subgroup of the group $G$, which is $G$-stable, using generalized Brauer pairs and we will prove similar results obtained by Linckelmann in [17]. In a specific situation for the fixed block, we will obtain new results regarding the varieties associated to the cohomology algebra of the block defined by Linckelmann and the cohomology algebra defined by the author using generalized Brauer pairs.

The thesis is structured as follows. In *Chapter 1* we will give notations, notions and basic results which we will use in the following chapter. The main objectives covered by this chapter are: symmetric algebra, symmetric form, stable elements in group cohomology, Hochschild cohomology of symmetric algebra, blocks of group algebras, Brauer pairs, pointed groups and block cohomology. Main source used are: [4], [5], [11], [36] for homological algebra, [17], [22], [35] for modular representation theory, and [12] for finite groups.

*Chaper 2* is dedicated to characterize stable elements in Hochschild cohomology of group algebras. We will prove that under some hypothesis, the results proved by

M. Linckelmann in [17] remains true in a more general context of the generalized diagonal induction map with domain the cohomology algebra of the centralizer in the group $G$, of a representative of conjugacy class.

§2.1. In this section we define the normalized transfer $T_X$ associated with a bounded complex of $A-B$-bimodules $X$, between $\mathrm{HH}^*(B)$ and $\mathrm{HH}^*(A)$ where $A, B$ are two symmetric $R$-algebras. Also we will define the set of $X$-stable elements in $\mathrm{HH}^*(A)$, denoted by $\mathrm{HH}^*_X(A)$ and we will give proprieties satisfied by $T_X$ on $\mathrm{HH}^*_X(A)$, which moreover is a graded algebra. At the end of this section we will recall the embedding of the cohomology algebra of a finite group $G$ into the subalgebra of $M$-stable elements $\mathrm{HH}^*_M(RG)$, where $M = RG$ as $RG-RH$-bimodule and $H$ is a subgroup of $G$.

§2.2. Some of the above results we will explicite, by giving the definition of the transfer and of the diagonal induction map $\delta_G$. These objectives have been obtained by the author in [33].

§2.3. In the same paper [33], the author gives explicitly the generalized diagonal induction map, denoted $\gamma^G_{x_i}$ from $\mathrm{H}^*(C_G(x_i), R)$ to $\mathrm{HH}^*(RG)$, where $x_i$ is a representative of a conjugacy class of $G$. Next we will prove the compatibility of $\gamma^G_{x_i}$ with the restrictions and transfers in group cohomology.

§2.4. Using results from the same article [33], we will fix a working situation which we denote (†). In this situation [17, Proposition 4.8] is still true for $\gamma^G_{x_i}$. The sections 2.2, 2.3 and 2.4 contains new results obtained by the author in [33].

Through *chapter 3* we will work with $k$ an algebraically closed field of characteristic $p$, $G$ is a finite group with a normal subgroup $N$ and $c$ is a block of $kN$ which is $G$-stable. We will define and analyze, using generalized Brauer pairs, the generalized cohomology algebra of the block $c$ and a restriction map from this algebra to the usual cohomology algebra of the block $c$.

§3.1. In this section we will remind the main result from [17], that is Theorem 5.6, which proves the embedding of the cohomology algebra of a block into the subalgebra of stable elements in the Hochschild cohomology algebra of the block.

§3.2. We will define $(c, G)$-Brauer pairs (generalized Brauer pairs) and an order relation based on [14]. There is a link between $(c, G)$-defect groups and defect pointed groups of the $G$-algebra $kN$ which we analyze and then we will define the generalized

Brauer category, denoted $\mathcal{F}_{(P,e_P)}(G, N, c)$, where $(P, e_P)$ is a generalized $(c, G)$-Brauer pair. If $N$ is equal to $G$ we obtain the usual Brauer category.

§3.3. We will fix a working situation, denoted $(*)$, which always exists for the block $c$ and which allow us to find a defect pointed group $Q_\delta$ of $N_{\{c\}}$ with $Q_\delta \leq P_\gamma$, where $P_\gamma$ is a pointed defect group of $G_{\{c\}}$. Using generalized Brauer category we can define the "generalized" cohomology algebra of $c$ associated to $P_\gamma$ denoted $\mathrm{H}^*(G, N, c, P_\gamma)$, and in situation $(*)$ we have a restriction map, called *the restriction in block cohomology*, which we denote $res_{N,c}^{G,N,c}$. Next we will prove the main result of this section, Theorem 3.3.11, which proves that properties from [17, Teorem 5.6] remains true for generalized cohomology algebra. In the end of this section, we introduce a normal inclusion relation on $(c, G)$-Brauer pairs and we prove the third main Brauer's Theorem (Theorem 3.3.12) for $(c, G)$-Brauer pairs. Shortly, this says that if $c = c_0$ is the principal block of $N$ then the $(c_0, G)$-defect groups are the Sylow $p$-subgroups of $G$.

§3.4. Under situation $(*)$ from section 3.3 we will investigate the properties of the restriction map in block cohomology through a transfer map defined from Hochschild cohomology algebra of $kGc$ to Hochschild cohomology algebra of $kNc$. The fundamental result of this section is Theorem 3.4.10, which proves the compatibility of $res_{N,c}^{G,N,c}$ through $T_X$, where $X = kGc$ as $kNc - kGc$-bimodule.

§3.5. In this section we analyze the variety of the generalized cohomology algebra of $c$ associated to a finitely generated $kGc$-module $U$, denoted $V_{G,N,c}(U)$. Generally, considering the usual cohomology algebra of the block $c$, M. Linckelmann has studied the variety associated to $\mathrm{H}^*(N, c, Q_\delta)$ in [18]. Keeping the notations and the hypothesis of Theorem 3.4.10, we will prove that $V_{N,c}(U) = (r^*_{G,N,c})^{-1}(V_{G,N,c}(U))$. The sections 3.4 and 3.5 contains original results obtained by the author in [32].

In *Chapter 4* we will define a functor isomorphic with the functor $\mathrm{Hom}_{kG}(k, -)$ which allows us to have a new approach for the definition of cohomology of finite groups. The block cohomology doesn't have a global approach, that is a definition as a right (or left) derived functor of a specific functor, since the block algebra is not an algebra with augmentation. The results from chapter 4 may represent the first step for such an approach, which is realized by the author in [30]. In this chapter we consider $G$ a finite group, $P$ a Sylow $p$-subgroup of $G$ and $k$ a field of characteristic

$p$.

§4.1. Let $A, B$ be two $kG$-modules. The main result of this section is the isomorphism between $\mathrm{Hom}_{kG}(A,B)$ and the $k$-submodule of stable elements in $\mathrm{Hom}_{kP}(A,B)$, defined in Definition 4.1.2. This will be denoted by $\mathrm{Hom}_{kP}^{st}(A,B)$.

§4.2. We will define a functor $F_G$ from the category $\mathrm{Mod}(kG)$ to $\mathrm{Mod}k$ by $F_G(A) = \mathrm{Hom}_{kP}(k,A)$, for any $kG$-module $A$. This is isomorphic with $\mathrm{Hom}_{kG}(k,-)$. We obtain in the end the isomorphism $R^nF_G(k) \cong \mathrm{H}^n(G,k)$. The sections 4.1, 4.2 are based on the article [34].

◇

I'm grateful to Professor Andrei Mărcuş for the help and the answers give to some questions which appear during the writing of this thesis. I'm grateful for his patience and the for introducing me to this interaction area between representations of finite groups and homological algebra. Many thanks to the people from the Chair of Algebra from Faculty of Mathematics and Informatics in "Babeş-Bolyai" University and also to my colleagues from the Department of Mathematics of Technical University for the friendly atmosphere.

Finally i'm grateful to my parents, my brothers and my sister which help me in various way.

# Chapter 1

# Preliminaries

In this chapter we will present the basic notions and properties of Hochschild cohomology for symmetric algebras , and of finite group cohomology and also the main results regarding blocks of group algebras, which we will call shortly blocks. We will end the chapter presenting block cohomology, defined by Markus Linckelmann in 1999, in his paper [17]. In this chapter (and also in the entire thesis, if something else is not specified), all the algebras and rings are associative with unity and all the modules are finitely generated, left modules.

## 1.1 Hochschild cohomology of symmetric algebras

In this section we will present, shortly: the notion of *symmetric algebra*, two adjoint functors, the notion of *Hochschild cohomology algebra applied to the symmetric algebra* and we will end with the presentation of *the transfer map* between the two Hochschild cohomology algebras for symmetric algebras. First we will defined two pairs of adjoint functors associated with a bimodule over symmetric algebras and we will define the corresponding unities and counities. In the second part of the section we will give similarly results in a more general case of *bounded chain complexes*, of bimodules over symmetric algebras and we will define the transfer map between the Hoschild cohomology algebra of two symmetric algebras. We will exemplify with the case of the group algebra

In this section we consider $R$ a commutative ring, $A, B, C$ are symmetric $R$-algebras and $X$ is a bounded chain complex, of $A-B$-bimodules, proiective as left

$A$-modules and right $B$-modules .

**1.1.1. The pair of adjoint functors** $(X \otimes_B -, X^* \otimes_A -)$. The unity and counity of $(X \otimes_B -, X^* \otimes_A -)$ are the chain maps of $B-B$-bimodules, respectively $A-A$-bimodules

$$\varepsilon_X : B \to X^* \otimes_A X, \quad \eta_X : X \otimes_B X^* \to A.$$

**1.1.2. The pair of adjoint functors** $(X^* \otimes_A -, X \otimes_B -)$. The unity and counity of $(X^* \otimes_A -, X \otimes_B -)$ are the chain maps of $A-A$-bimodules, respectively $B-B$-bimodules

$$\varepsilon_{X^*} : A \to X \otimes_B X^*, \quad \eta_{X^*} : X^* \otimes_A X \to B.$$

The above and the following results include the case where $X = M$ is considered as a complex of $A-B$-bimodules concentrated in degree 0 (that is, $X_0 = M$ and the other components are 0). Next we will give the arguments for the group algebra to be a symmetric algebra (see [17, Example 2.6]). Let $H$ be a subgroup of $G$ and $M = RG$ as $RG - RH$-bimodule. The $R$-dual $M^*$ of $M$ is isomorphic with ${}_{RH}RG_{RG}$. Particularly, we have that $M^* \otimes_{RG} M \cong RG$ as $RH - RH$-bimodule. Using these results we obtain the adjunction maps of $M$ and $M^*$ (see [17, Example 2.6]).

**1.1.3. The Hochschild cohomology of an $R$-algebra $A$.** By [36, Chapter 9, Corollary 9.1.5] we will consider the following definition of the Hochschild cohomology of an algebra. *The Hochschild cohomology of an $R$-algebra $A$* is the algebra

$$\mathrm{HH}^*(A) = \mathrm{Ext}^*_{A \otimes A^0}(A).$$

By standard results from homological algebra, when $X$ is projective as left $A$-module and right $B$-module, we obtain that the complex $X^* \otimes_A \mathcal{P}_A \otimes X$ is a projective resolution of $X^* \otimes_A X$ in the abelian category of bounded complexes of $A-B$-bimodules. Further we get that the adjunction map $\varepsilon_X : B \longrightarrow X^* \otimes_A X$ lifts to a chain map, unique up to homotopy, which will be denoted

$$\varepsilon_X : \mathcal{P}_B \longrightarrow X^* \otimes_A \mathcal{P}_A \otimes_A X.$$

Similarly we have : $\eta_X, \varepsilon_{X^*}, \eta_{X^*}$.

**Definition 1.1.4** (Definition 2.9, [17])**.** Let $A, B$ be two symmetric $R$-algebras, $s \in A^*$, $t \in A^*$ the symmetric forms, ${}_AX_B$ a bounded complex of bimodules, projective as left and right bimodules. *The transfer map associated to* $X$ is the only graded linear map

$$t_X : \mathrm{HH}^*(B) \longrightarrow \mathrm{HH}^*(A),$$

which, for any $n \geq 0$, sends the homotopy class $[\xi]$ of $\xi : \mathcal{P}_B \longrightarrow \mathcal{P}_B[n]$ to the homotopy class $t_X[\xi] = [\eta_X[n] \circ (Id_X \otimes \xi \otimes Id_{X^*}) \circ \varepsilon_{X^*}]$, obtained by the composition of chain maps

$$\mathcal{P}_A \xrightarrow{\varepsilon_{X^*}} X \otimes_B \mathcal{P}_B \otimes_B X^* \xrightarrow{Id_X \otimes \xi \otimes Id_{X^*}} X \otimes_B \mathcal{P}_B[n] \otimes X^* \xrightarrow{\eta_X[n]} \mathcal{P}_A[n] \ .$$

## 1.2 The cohomology of finite groups

The algebraic definition of cohomology of groups, will follow the line from [4], but we remind also the approach of M. Linckelmann from [17]. In this section we consider $G$ to be a finite group, $R$ as a commutative ring and the group algebra $RG$ has a structure of $RG - RG$-bimodule given by the multiplication in $RG$. An $RG - RG$-bimodule $M$ has also a structure of $R(G \times G)$-module with $(x, y) \in G \times G$ acts on $m \in M$ by $xmy^{-1}$ (and reverse).

**Definition 1.2.1.** We call *the cohomology algebra of $G$ with coefficients in $R$* the algebra

$$\mathrm{H}^*(G, R) = \mathrm{Ext}^*_{RG}(R, R),$$

where $R$ is the trivial $RG$-module. Explicitly $\mathrm{Ext}^*_{RG}(R, R) = \bigoplus_{n \geq 0} \mathrm{Ext}^n_{RG}(R, R)$, with the multiplication in the ring given by the cup product (if we use cocycle) and

$$\mathrm{Ext}^n_{RG}(R, R) = \mathrm{H}^n(\mathrm{Hom}_{RG}(\mathcal{P}_R, R)),$$

where $\mathcal{P}_R$ is a projective resolution of $R$ as trivial $RG$-module.

We will use also, the alternative of chain maps in the definition of group cohomology, since it is easy to compute the product. From now

$$\mathrm{H}^n(G,R) \cong \mathrm{Hom}_{K(RG)}(\mathcal{P}_R, \mathcal{P}_R[n]).$$

**1.2.2.** The complex $\mathrm{Ind}_{\Delta G}^{G\times G}(\mathcal{P}_R)$ is a projective resolution of $RG$, thus we identify $\mathrm{Ind}_{\Delta G}^{G\times G}(\mathcal{P}_R) \cong \mathcal{P}_{RG}$, where $\mathcal{P}_{RG}$ is a projective resolution of $RG$ as $R(G\times G)$-module or as $RG-RG$-bimodule.

**Proposition 1.2.3** (Proposition 4.5,[17])**.** *Let $G$ be a finite group and $\mathcal{P}_R$ a projective resolution of $R$ as trivial $RG$-module. The map which sends $\tau \in \mathrm{Hom}_{C(RG)}(\mathcal{P}_R, \mathcal{P}_R[n])$ to $\mathrm{Ind}_{\Delta G}^{G\times G}(\tau)$ induces a $R$-algebra injective map*

$$\delta_G : \mathrm{H}^*(G,R) \longrightarrow \mathrm{HH}^*(RG), \quad \delta_G([\tau]) = [\mathrm{Ind}_{\Delta G}^{G\times G}(\tau)].$$

The map $\delta_G$ from the above proposition will be called "*the diagonal induction map*". The restriction and the transfer are compatible with the transfer defined between the Hochschild cohomology algebras of the group algebras through the diagonal induction map from Proposition 1.2.3, and these 2 results will be recalled in this section as [17, Proposition 4.6, Proposition 4.7].

## 1.3 Blocks of group algebras, Brauer pairs and pointed groups

In this section we will present the main results of blocks of group algebras. We are interested with the properties which bounds defect groups and Brauer pairs and also the approach with pointed groups of blocks. The initial sources of these results are the articles of J. Alperin and M. Broué, respectively M. Broué and L. Puig that is [2] and [8]. All these are exposed in [35], whose notations we will follow. First we will approach *idempotents* and *blocks* of group algebra and the notions of *G-algebra* and *pointed groups*. Secondly we will present the proprieties of blocks from [35], recalling from time to time the general context of pointed groups associated to blocks.

Let $G$ be a finite group, $k$ be a field of characteristic $p$, which divides the order of $G$ and $A$ a $G$-algebra. We will remind in this section: *the relative trace map* $\mathrm{Tr}_H^G$ with $H$ a subgroup of $G$; *pointed groups*; *the inclusion* of pointed groups; *the Brauer map* $\mathrm{Br}_P^A$ where $P$ is a $p$-subgroup of $G$; *$b$-Brauer pairs* where $b$ is a block of $kG$; *the inclusion* of $b$-Brauer pairs; *the principal block*, which we denote $b_0$.

The properties of the defect group of the principal block are exposed in the third main Brauer's Theorem which we recall in this thesis by [35, Theorem 40.17].

## 1.4 The cohomology of blocks of finite groups

The first part of this section describes the fusion systems of $p$-groups which have as basic examples: *the fusion system associated with a Sylow $p$-subgroup* of a group and that *associated with a block* of a finite group algebra. The originally definition was given by L. Puig (which called them full Frobenius systems), and these were developed by Broto, Levi şi Oliver and [7] (who called them saturated fusion systems). We will follow the notion of *fusion system* and the results from [21]. In the last years, the theory of fusion systems (saturated fusion systems) represents a growing domain which creates an interaction between group theory and algebraic topology.

Most of the properties of fusion systems were systemized by their inventator, L. Puig in [26]. Next we give the most important concept of this thesis , that is *the cohomology algebra of a block*, defined by M. Linckelmann in [17]. We will also give some properties of this algebra.

**1.4.1. The fusion system associated with a block.** Let $b$ be a block of $kG$ and $(P,e)$ maximal $b$-Brauer pair. For any $Q$, a subgroup of $P$ there is a block $e_Q$ of $kC_G(Q)$ such that $(Q,e_Q)\leq(P,e)$. We will denote by $\mathcal{F}_{(P,e)}(G,b)$ the category on $P$ with morphisms the group homomorphisms $\varphi:Q\to R$ for which there is $x\in G$ with ${}^x(Q,e_Q)\leq(R,e_R)$ (equivallently ${}^xe_Q=e_{{}^xQ}$) such that $\varphi(u)=xux^{-1}$, for any $u\in Q$. By [21, Theorem 2.4] we have that $\mathcal{F}_{(P,e)}(G,b)$ is a fusion system on $P$. If $b$ is the principal block, by third main Brauer's theorem we get that $\mathcal{F}_{(P,e)}(G,b)=\mathcal{F}_P(G)$, where $\mathcal{F}_P(G)$ is the fusion system on $P$ with maps from $Q$ to $R$ induced by conjugation with elements in $x \in G$ such that ${}^xQ \leq R$.

Next we give the definition of the cohomology algebra of a block.

**Definition 1.4.2** (Definition 5.1, [17])**.** Let $b$ be a bloc of $kG$ and $P_\gamma$ un defect pointed group of $G_{\{b\}}$. We call *the cohomology algebra of the block $b$ associated with $P_\gamma$* the subalgebra

$$\mathrm{H}^*(G, b, P_\gamma)$$

of $\mathrm{H}^*(P, k)$, which consists in all elements $[\zeta] \in \mathrm{H}^*(P, k)$ which satisfy $res^P_Q([\zeta]) = res_\varphi([\zeta])$, for any subgroup $Q$ in $P$ and any $\varphi \in \mathrm{Hom}_{\mathcal{F}_{(P,e_P)}(G,b)}(Q, P)$.

# Chapter 2

# Stable elements in Hochschild cohomology of group algebra

The second chapter is dedicated to apply some of the above results to group algebra. We will define the notion of stable element in Hochschild cohomology algebra and explicitly will give the adjunction and transfer map for group algebra, defined in section 1.1. In section 2.3, using the language of chain map we will explicate the generalization of the diagonal induction map, defined in [29] with cocycles. A new result will be proved, which show that the image of such a map (a generalization of diagonal induction map) is in the subalgebra of stable elements. We obtain a similar result to Linckelmann's from [17]. The sections 2.3, 2.4 contains original results published by the author in [33].

## 2.1 Stable elements in Hochschild cohomology of symmetric algebras

We will present basic results which characterize stable elements in Hochschild cohomology of symmetric algebras and we study their link with transfer maps.

**Definition 2.1.1.** Let $A, B$ be two symmetric $R$-algebras with symmetric forms $s \in A^*, t \in B^*$ and $X$ a bounded complex of $A-B$-bimodules with projective components as left and right modules.

i) The element $\pi_X = (\eta_X \circ \varepsilon_{X^*})(1_A) \in Z(A)$, the image of $1_A$ under the composition:

$$A \xrightarrow{\varepsilon_{X^*}} X \otimes_B X^* \xrightarrow{\eta_X} A$$

is called *the projectiv element relative to $X$.*

ii) If $\pi_X$ is invertible in $Z(A)$ we denote by $T_X : \mathrm{HH}^*(B) \longrightarrow \mathrm{HH}^*(A)$ the graded linear map defined by $T_X([\tau]) = \pi_X^{-1} t_X([\tau]), \quad \tau \in \mathrm{HH}^*(B)$, which we call *the normalized transfer associated to $X$.*

iii) An element $[\zeta] \in \mathrm{HH}^*(A)$ is called $X$-*stable* if there is $[\tau] \in \mathrm{HH}^*(B)$ such that for all positive $n \in \mathbb{Z}$, the next diagram is commutative up to homotopy

$$\begin{tikzcd} \mathcal{P}_A \otimes_A X \arrow[r, "\cong"] \arrow[d, "\zeta_n \otimes Id_X"] & X \otimes_B \mathcal{P}_B \arrow[d, "Id_X \otimes \tau_n"] \\ \mathcal{P}_A[n] \otimes_A X \arrow[r, "\cong"] & X \otimes_B \mathcal{P}_B[n] \end{tikzcd} \quad , \tag{2.1.1}$$

where $\zeta_n, \tau_n$ are the degree $n$ components of $\zeta, \tau$ and the horizontal maps are the natural homotopy equivalences $\mathcal{P}_A \otimes_A X \cong X \otimes_B \mathcal{P}_B$ which lifts the natural isomorphisms $A \otimes_A X \cong X \otimes_B B$. We denote by $\mathrm{HH}^*_X(A)$ the set of $X$-stable elements in $\mathrm{HH}^*(A)$.

In [17] there are some results which give us the proprieties of stable elements. That is: a lemma which give us equivalent conditions to the condition 2.1.1 of defining a stable element and a proposition which proves that the transfer maps stable elements to stable elements.

We end this section with three, very useful results: [17, Corollary 3.8], [17, Example 3.9], [17, Proposition 4.8] .

## 2.2 The transfer map between Hochschild cohomology algebras of group algebras

The main scop of this section is to give explicily, using element definitions, the characterization of unity and counity map associated to $M = RG$ and $M^* = RG$ as $RG - RH$-bimodule, respectively $RH - RG$-bimodule, where $H$ is a subgroup of $G$. In the same manner we will explicite the transfer maps $t_M$ and $t_{M^*}$.

We take $\mathcal{P}_R$ a projective resolution of $R$ as trivial $RG$-module. Since $RG$ is free as left $RH$-module we have that $\mathrm{Res}^G_H \mathcal{P}_R$ remains a projective resolution of $R$ as trivial $RH$-module. Thus any element $[\tau] \in \mathrm{H}^n(H, R)$ can be represented by a chain map $\tau : \mathrm{Res}^G_H \mathcal{P}_R \longrightarrow \mathrm{Res}^G_H \mathcal{P}_R[n]$. By 1.2.2, any element $[\tau] \in \mathrm{HH}^n(RG)$ is considered as represented by a chain map $\tau : \mathrm{Ind}^{G \times G}_{\Delta G} \mathcal{P}_R \longrightarrow \mathrm{Ind}^{G \times G}_{\Delta G} \mathcal{P}_R[n]$. We take $\mathrm{Ind}^{H \times H}_{\Delta H} R$ as $RH - RH$-bimodule by

$$h_1 \cdot [(x, y) \otimes_{R\Delta H} 1_R] \cdot h_2 = (h_1 x, h_2^{-1} y) \otimes_{R\Delta H} 1_R,$$

where $h_1, h_2, x, y \in H$.

We explicite the unity $\varepsilon_{M^*}$ and counity $\eta_M$ such that we have the detailed definition of $t_M$:

**Definition 2.2.1.** *The transfer associated to* $M$ is the unique graded linear map

$$t_M : \mathrm{HH}^*(RH) \longrightarrow \mathrm{HH}^*(RG),$$

which sends the homotopy class $[\tau]$, of the chain map

$$\tau : \mathrm{Ind}^{H \times H}_{\Delta H} \mathcal{P}_R \longrightarrow \mathrm{Ind}^{H \times H}_{\Delta H} \mathcal{P}_R[n]$$

to the homotopy class $[\eta_M[n] \circ (Id_M \otimes_{RH} \tau \otimes_{RH} Id_{M^*}) \circ \varepsilon_{M^*}]$, for any $n \geq 0$.

Similarly we will characterize the lifting to resolutions of $\varepsilon_M$ and $\eta_{M^*}$.

**Definition 2.2.2.** *The transfer associated to* $M^*$ is the unique graded linear map

$$t_{M^*} : \mathrm{HH}^*(RG) \longrightarrow \mathrm{HH}^*(RH),$$

which sends the homotopy class $[\tau]$, of the chain map

$$\tau : \mathrm{Ind}_{\Delta G}^{G\times G}\mathcal{P}_R \longrightarrow \mathrm{Ind}_{\Delta G}^{G\times G}\mathcal{P}_R[n]$$

to the homotopy class $[\eta_{M^*}[n] \circ \tau \circ \varepsilon_M]$, for any $n \geq 0$.

**2.2.3. The explicitness of "diagonal induction map $\delta_G$".** By Proposition 1.2.3 there is an injective homomorphism of $R$-algebras

$$\delta_G : \mathrm{H}^*(G, R) \longrightarrow \mathrm{HH}^*(RG), \delta_G([\tau]) = [\mathrm{Ind}_{\Delta G}^{G\times G}(\tau)],$$

where $[\tau] \in \mathrm{H}^n(G, R)$ corresponds to $\tau : \mathcal{P}_R \longrightarrow \mathcal{P}_R[n]$.

## 2.3 The generalization of the diagonal induction map

Using cocycle language S.F. Siegel and S.W. Witherspoon give an additive decomposition of the cohomology algebra of a group (which acts as automorphisms on a second group) with coefficients in group algebra. They describe the isomorphism which determine this decomposition and when the groups are equals and we have the conjugation action we obtain the decomposition from [5, Theorem 2.11.2] extended to graded algebras. In this section we will characterize, using the language of chain maps, the injective homomorphisms of algebras which appear in [29, Lemma 4.2], denoted $\gamma_i$. Through this chapter for $G$ a finite group we choose $\{x_i \mid i \in \{1, \ldots, r\}\}$ a system of representatives of those $r$ conjugacy classes of $G$, with a fixed representative $x_i$, for $i \in \{1, \ldots, r\}$.

If $[\tau] \in \mathrm{H}^n(C_G(x_i), R)$ is represented by a chain map

$$\tau : \mathrm{Res}_{C_G(x_i)}^{G}\mathcal{P}_R \longrightarrow \mathrm{Res}_{C_G(x_i)}^{G}\mathcal{P}_R[n]$$

we define the next chain map between projective resolutions of $RG$ as $R(G \times G)$-modules

$$\gamma_{x_i}^{G}(\tau) : \mathrm{Ind}_{\Delta G}^{G\times G}\mathcal{P}_R \longrightarrow \mathrm{Ind}_{\Delta G}^{G\times G}\mathcal{P}_R[n],$$

$$\gamma_{x_i}^G(\tau)((x,y)\otimes_{R\Delta G} z) = (x,y)\sum_{g\in[G/C_G(x_i)]}(gx_i,g)\otimes_{R\Delta G}\tau(g^{-1}z) \tag{2.3.1}$$

pentru $x, y \in G, z \in \mathcal{P}_R$.

**Proposition 2.3.1.** *i) For any $\tau$ a chain map as above, the map $\gamma_{x_i}^G(\tau)$ is well defined and is a chain map.*

*ii) For any class $[\tau] \in \mathrm{H}^n(C_G(x_i), R)$ we have that $\gamma_{x_i}^G([\tau]) = [\gamma_{x_i}^G(\tau)]$ well defined.*

The proposition 2.3.1 allow us to give the following definition of the graded de $R$-algebras homomorphisms from (2.3.1).

**Definition 2.3.2.** Let $G$ be a finite group and $x_i$ is a representative of a conjugacy class of $G$. The homomorphism of $R$-algebras

$$\gamma_{x_i}^G : \mathrm{H}^*(C_G(x_i), R) \longrightarrow \mathrm{HH}^*(RG)$$

is the unique graded linear map $\gamma_{x_i}^G([\tau]) = [\gamma_{x_i}^G(\tau)]$, where $[\tau] \in \mathrm{H}^n(C_G(x_i), R)$ is represented by a chain map $\tau : \mathrm{Res}_{C_G(x_i)}^G \mathcal{P}_R \longrightarrow \mathrm{Res}_{C_G(x_i)}^G \mathcal{P}_R[n]$. This homomorphism is called *the generalized diagonal induction map relative to $x_i$.*

It is clear that if $x_i = 1$ then $C_G(1) = G$ and by 2.2.3 we obtain that $\gamma_1^G = \delta_G$. The next proposition is a translation of [17, Proposition 4.7] to the generalized diagonal induction map.

**Proposition 2.3.3.** *Let $G$ be a finite group and $x_i$ a representative of a conjugacy class of $G$, $H$ a subgroup of $G$ such that $x_i \in H$. Then $x_i$ is a representative of a conjugacy class of $H$, $C_H(x_i) \leq C_G(x_i)$ and the next diagram is commutative*

$$\begin{tikzcd} \mathrm{H}^*(C_H(x_i), R) \arrow[r, "\mathrm{tr}_{C_H(x_i)}^{C_G(x_i)}"] \arrow[d, "\gamma_{x_i}^H"] & \mathrm{H}^*(C_G(x_i), R) . \arrow[d, "\gamma_{x_i}^G"] \\ \mathrm{HH}^*(RH) \arrow[r, "t_M"] & \mathrm{HH}^*(RG) \end{tikzcd}$$

## 2.4 Stable elements in Hochschild cohomology of the group algebra

In this section we add to the above hypothesis the following situation:

Situation (†). *Let $G$ be a finite group, $H$ a subgroup of $G$ and $x_i$ an element of $H$, a representative of a $G$-conjugacy class. We suppose that there is a system of representatives of left cosets of $C_H(x_i)$ in $H$ which remains a system of representatives of left cosets of $C_G(x_i)$ in $G$.*

We ask now if there are groups in situation (†). Next we give an example of a group and a subgroup which are in this situation.

**Example 2.4.1.** Let $G$ be the dihedral group of order $4n$ denoted $D_{2n}$, where $n$ is an odd positive integer. We have the explicit description

$$D_{2n} = \{1, x, x^2, \dots, x^{2n-1}, y, xy, x^2y, \dots, x^{2n-1}y\}.$$

We choose $x_i = y$ and $H = \{1, y, x^2, x^4, \dots, x^{2n-2}, x^2y, x^4y, \dots, x^{2n-2}y\}$ a subgroup of $G$.

Moreover as we can notice from Example 2.4.1 we have the next lemma.

**Lemma 2.4.2.** *If we are in situation (†) then any system of representatives of left cosets of $C_H(x_i)$ in $H$ is a system of representatives of left cosets of $C_G(x_i)$ in $G$.*

In situation (†) we have that $\gamma^G_{x_i}$ is compatible with a specific restriction.

**Proposition 2.4.3.** *If we are in situation (†) then we have the following commutative diagram*

$$\begin{array}{ccc}
\mathrm{H}^*(C_G(x_i), R) & \xrightarrow{\mathrm{res}^{C_G(x_i)}_{C_H(x_i)}} & \mathrm{H}^*(C_H(x_i), R) \\
\downarrow{\scriptstyle \gamma^G_{x_i}} & & \downarrow{\scriptstyle \gamma^H_{x_i}} \\
\mathrm{HH}^*(RG) & \xrightarrow{t_{M^*}} & \mathrm{HH}^*(RH)
\end{array}.$$

In the next theorem, which is the main result of this section, we prove that in situation (†) we have a similar embedding to [17, Proposition 4.8] $\mathrm{Im}\gamma_{x_i}^G \subset \mathrm{HH}_M^*(RG)$, where $M$ is the regular $RG-RH$-bimodule.

**Theorem 2.4.4.** *In situation* (†) *the following statements are true:*

*i) For any positive integer $n$, and any chain map $\tau \in \mathrm{Hom}_{C(RG)}(\mathcal{P}_R, \mathcal{P}_R[n])$ we have that the following diagram is a commutative homotopy:*

$$\begin{tikzcd} RG \otimes_{RH} \mathrm{Ind}_{\Delta_H}^{H\times H}(\mathcal{P}_R) \otimes_{RH} RG \arrow[r, "\eta_M"] \arrow[d, "Id_M \otimes_{RH} \gamma_{x_i}^H(\tau) \otimes_{RH} Id_{M^*}"] & \mathrm{Ind}_{\Delta G}^{G\times G}(\mathcal{P}_R) \arrow[d, "\gamma_{x_i}^G(\tau)"] \\ RG \otimes_{RH} \mathrm{Ind}_{\Delta_H}^{H\times H}(\mathcal{P}_R[n]) \otimes_{RH} RG \arrow[r, "\eta_M[n]"] & \mathrm{Ind}_{\Delta G}^{G\times G}(\mathcal{P}_R[n]) \end{tikzcd} .$$

*ii)* $\mathrm{Im}\,\gamma_{x_i}^G \subset \mathrm{HH}_M^*(RG)$.

# Chapter 3

# The restriction map in cohomology of blocks of finite groups

We investigate in this chapter the cohomology algebra of a block, defined by M. Linckelmann in [17], in a similar way to group cohomology, using the "stable elements Cartan-Eilenberg method". We will use for this the language of fusion systems reminded in section 1.4. Through this chapter we consider $k$ an algebraically closed field of characteristic $p$ (a prime number) and $G$ a finite group. Let $N$ be a normal subgroup of $G$ and $c$ a $G$-stable block of $kN$, under conjugation. In this situation, using results noticed by R. Kessar and R. Stancu in [14], we define *the "generalized" cohomology algebra of the block* $c$ and a *restriction map* to the usual cohomology algebra of the block $c$. We will analyze this restriction map through transfer maps between Hochschild cohomology of the algebra $kGc$ and the usual Hochschild cohomology algebra of the block $c$.

The first section of this chapter presents basic results obtained by M. Linckelmann in [17] regarding block cohomology and the embedding of this into the subalgebra of some stable elements of Hochschild cohomology algebra of the block. The second section contains original results with proprieties of generalized Brauer pairs, obtained by the author in [31] and [32]. In the third section we will study the generalized block cohomology. The sections four and five decries the compatibility of the restriction map in block cohomology with the transfer map between Hochschild cohomology algebras in some situations, and also the proprieties of the varieties associated to the generalized cohomology of blocks. The last three sections contains original results obtained by the author in [32].

## 3.1 Stable elements in Hochschild cohomology of blocks

In this section we will give the main result from [17], which proves the embedding of the block cohomology algebra into the subalgebra of stable elements in Hochschild cohomology algebra of that block. For some families of blocks this embedding is studied in [25]. We will give the proofs of these results in section 3.3, in a more general case, by imitating the proofs of Linckelmann from [17].

## 3.2 Generalized Brauer pairs and pointed groups

In this section we describe the proprieties of the generalized Brauer pairs, which are associate to a block of a normal subgroup in $G$. The generalized Brauer pairs forms a fusion system which has as a normal subsystem the usual Brauer category. We will give some proprieties which connects generalized Brauer pairs with pointed groups and we will end with the third main Theorem of Brauer for generalized Brauer pairs. Many of these results are obtained by the author in [31] and [32].

Let $N$ be a normal subgroup of $G$, $c$ be a block of $kN$ , which is $G$-stable and $k$ an algebraically closed field. We will denote by: $A = kG$ as interior $G$-algebra, $A_1 = kN$ as $G$-algebra and $kN$ is the usual interior $N$-algebra. We know that $N_{\{c\}}$ and $G_{\{c\}}$ are pointed groups on $A_1$.

**3.2.1. Brauer map on $A_1$.** For any $p$-subgroup $Q$ of $G$, the canonical projection from $kN$ to $kC_N(Q)$ induces a surjective homomorphism of algebras from $(kN)^Q$ to $kC_N(Q)$, *the Brauer map for* $A_1$, denoted $\mathrm{Br}_Q^N$. Explicitly $\mathrm{Br}_Q^N(x) = x$ if $x \in C_N(Q)$ and $\mathrm{Br}_Q^N(x) = 0$ if $x \notin C_N(Q)$.

**Definition 3.2.2.** A $(c, G)$*-Brauer pair (generalized Brauer pair)* is a pair $(Q, e_Q)$ where $Q$ is a $p$-subgroup of $G$ such that $\mathrm{Br}_Q^N(c) \neq 0$ and $e_Q$ is a block of $kC_N(Q)$ such that $\mathrm{Br}_Q^N(c)e_Q \neq 0$. If $G = N$ then we obtain that a $(c, G)$-Brauer pair becomes a $c$-Brauer pair.

**Definition 3.2.3.** If $(R, e_R)$ and $(Q, e_Q)$ are two $(c, G)$-Brauer pairs, we say that $(Q, e_Q)$ is *included* in $(R, e_R)$ and we denote $(Q, e_Q) \leq (R, e_R)$, if $Q \leq R$ and for any primitive idempotent $i \in (kN)^R$ such that $\mathrm{Br}_R^N(i)e_R \neq 0$ we have that $\mathrm{Br}_Q^N(i)e_Q \neq 0$.

**3.2.4.** $(c, G)$**-defect groups.** By [8, Theorem 1.14] we know that $G$ acts transitively on the set of maximal $(c, G)$-Brauer pairs. Equivalently, all maximal $(c, G)$-Brauer pairs are $G$-conjugate. If $(P, e_P)$ is a maximal $(c, G)$-Brauer pair then $P$ is called $(c, G)$*-defect group*, and if $N = G$ we obtained that $P$ is the defect group of $c$.

We notice that $N_{\{c\}}$ and $G_{\{c\}}$ are pointed groups on $A_1$, with the property $N_{\{c\}} \leq G_{\{c\}}$. We are now in [15, Proposition 5.3], which we apply to obtaine the following situation.

**3.2.5. Defect pointed groups on** $A_1$**.** $P_\gamma$ is the defect pointed group of $G_{\{c\}}$ on $A_1$ if and only if $\overline{P} = PN/N$ is a Sylow $p$-subgroup of $\overline{G} = G/N$ and there is $Q_\delta$ a pointed defect group of $N_{\{c\}}$ on $kN$ as $N$-algebra such that $Q_\delta \leq P_\gamma$. In this case $Q = P \cap N$.

**Proposition 3.2.6.** *Let* $P_\gamma$ *be a pointed defect group of* $G_{\{c\}}$ *on* $A_1$*. Then there is a unique* $(c, G)$*-Brauer pair* $(P, e_P)$ *such that* $\mathrm{Br}_P^N(i)e_P \neq 0$*, for any* $i \in \gamma$*. Moreover* $(P, e_P)$ *is a maximal* $(c, G)$*-Brauer pair, thus* $P$ *is a* $(c, G)$*-defect group.*

**Definition 3.2.7** (Definition 3.3, [14])**.** Let $N$ be a normal subgroup $G$, $c$ a $G$-stable block of $kN$ and $(P, e_P)$ is a maximal $(c, G)$-Brauer pair. For a subgroup $Q$ of $P$ let $e_Q$

be the unique block $kC_N(Q)$ such that $(Q, e_Q) \leq (P, e_P)$. We denote $\mathcal{F}_{(P,e_P)}(G, N, c)$ the category on $P$ with morphisms $\mathrm{Hom}_{\mathcal{F}_{(P,e_P)}(G,N,c)}(Q, R)$ given by the set

$$\{\varphi : Q \longrightarrow R \mid \varphi(u) = gug^{-1}, \forall u \in Q, g \in G,^g (Q, e_Q) \leq (R, e_R)\}.$$

$\mathcal{F}_{(P,e_P)}(G, N, c)$ is called the *generalized Brauer category* and is a fusion system by [14, Theorem 3.4]. If $N = G$ we obtain the fusion system associated to the block $c$ of $kG$ denoted $\mathcal{F}_{(P,e_P)}(G, c)$, by 1.4.1.

In the next proposition we keep the notations and the working situation given by 3.2.5.

**Proposition 3.2.8.** *Let $P_\gamma$ be the pointed defect group of $G_{\{c\}}$ on $A_1$ and $Q_\delta = (P \cap N)_\delta \leq P_\gamma$ defect pointed group corresponding to $N_{\{c\}}$, by Remark 3.2.5. Then there is a maximal $(c, G)$-Brauer pair $(P, e_P)$ such that $\mathrm{Br}_P^N(i)e_P \neq 0$, for any $i \in \gamma$, and a unique maximal $c$-Brauer pair $(Q, e_Q)$ such that $\mathrm{Br}_Q^N(j)e_Q \neq 0$, for any $j \in \delta$. Moreover $(Q, e_Q) \leq (P, e_P)$ and $\mathcal{F}_{(Q,e_Q)}(N, c)$ is a normal subsystem in $\mathcal{F}_{(P,e_P)}(G, N, c)$.*

**Lemma 3.2.9.** *Let $c$ be a $G$-stable block of $kN$ and $P_\gamma$ is a defect pointed group of $G_{\{c\}}$ with $i \in \gamma$. Then the homomorphism of $kGc - kGc$-bimodules*

$$kGi \otimes_{kP} ikG \longrightarrow kGc$$

*given by the multiplication in $kGc$, splits.*

Under the hypothsesis 3.2.5 let $A_\delta = jAj$ where $j \in (ckN)^Q$ is a primitive idempotent such that $\mathrm{Br}_Q^N(j) \neq 0$. Then $A_\delta$ is a $k$-subalgebra of $A$ and the interior $Q$-algebra $jA_1j$ is called *the source algebra* of $cA_1$. $A_\gamma$ is the interior $P$-algebra $ikGi$, where $i \in \gamma$. With these notations we have the following proposition, which can be obtained as a consequence of [23, Proposition 3.2]. We will give a different proof here.

**Proposition 3.2.10.** *If $P_\gamma$ is a defect pointed group of $G_{\{c\}}$ on $A_1$ then $A_\gamma$ as interior $P$-algebra is Morita equivalent with $A_{\{c\}}$.*

In [23] A. Mărcuş noticed that M. Linckelmann's results from [19, 7.1, 7.7] generalizes for twisted group algebras case. Similar results are proved in [16]. In our case we have the following two propositions, whose proofs follow Linckelmann's approach.

**Proposition 3.2.11.** *Let $P_\gamma$ be a defect pointed group $G_{\{c\}}$ on $A_1$ and $i$ is a source idempotent. For any two subgroups $R, S$ of $P$ and any indecomposable direct summand $W$ of $ikGi$ as $kR{-}kS$-bimodule there is an element $x \in G$ and $\varphi : T \to S$ with $\varphi(u) = x^{-1}ux$, for any $u \in T$, where $T = R \cap {}^xS$ such that $W \cong k[RxS] \cong kR \otimes_{kT^\varphi} (kS)$.*

**Corollary 3.2.12.** *Any indecomposable direct summand of $ikGi$ as $kP{-}kP$-bimodule is $kP \otimes_{kT^\varphi} (kP)$ unde $\varphi : T \to P$ cu $\varphi(u) = x^{-1}ux$, for all $u \in T$ and $T = P \cap {}^xP$.*

**Proposition 3.2.13.** *Let $P_\gamma$ be a defect pointed group of $G_{\{c\}}$ on $A_1$ and $i$ a source idemptent. Let $R, S$ be two subgroups of $P$. If $\varphi : T \to S$ is a group homomorphism such that the $kR - kS$-bimodule $kR \otimes_{kT^\varphi} (kS)$ is an indecomposable direct summand of $ikGi$ then $\varphi$ is injective homomorphism $\varphi \in \mathrm{Hom}_{\mathcal{F}_{(P,e_P)}(G,N,c)}(T, S)$.*

## 3.3 The generalized block cohomology

In this section, using results obtained in the above section, we will fix a working situation in which the definition of block cohomology of $c$ using generalized Brauer pairs is possible. Moreover in this situation is possible to define a restriction map between these two cohomology. For principal blocks, as we expect, this restriction becomes the usual restriction map from the cohomology of $G$ to that of $N$. In the end of this section we will prove, that in our situation, results from section 3.1, which links cohomology of $c$ with Hochschild cohomology of the block algebra $kNc$ are still true.

By 3.2.5 and Proposition 3.2.8 in the next sections we will work under the hypothesis of the following situation:

**Situation(∗).** *Let $G$ be a finite group, $N$ be a normal subgroup of $G$ and $c$ a $G$-stable block of $kN$. Let $P_\gamma$ be a pointed defect group of $G_{\{c\}}$ on $A_1$ and $Q_\delta = (P \cap N)_\delta \leq P_\gamma$ the corresponding defect pointed group of $N_{\{c\}}$. Then there is a unique maximal $(c, G)$-Brauer pair $(P, e_P)$ such that $\mathrm{Br}_P^N(i)e_P \neq 0$, for any $i \in \gamma$, and there is a unique maximal $c$-Brauer pair $(Q, e_Q)$ such that $\mathrm{Br}_Q^N(j)e_Q \neq 0$, for any $j \in \delta$. Moreover we have that $(Q, e_Q) \leq (P, e_P)$.* Similarly, in situation (∗), we can define the generalized block cohomology of $c$, which if $N = G$ becomes the usual cohomology $\mathrm{H}^*(N, c, Q_\delta)$.

**Definition 3.3.1.** *The generalized cohomology algebra of the block $c$ of $N$ associated to $P_\gamma$* is the subalgebra

$$\mathrm{H}^*(G, N, c, P_\gamma)$$

of $\mathrm{H}^*(P, k)$ which consists of the elements $[\zeta] \in \mathrm{H}^*(P, k)$ satisfying the stability condition $res_\varphi[\zeta] = res_R^P[\zeta]$, for any subgroup $R$ of $P$ and any group homomorphism

$$\varphi : R \to P \ \texttt{în}\ \mathcal{F}_{(P,e_P)}(G, N, c).$$

Since all defect groups of $G_{\{c\}}$ are $G$-conjugate, it follows that the above algebra, is up to an isomorphism independent of choosing $P_\gamma$.

**Proposition 3.3.2.** *In situation (∗), for any $[\zeta] \in \mathrm{H}^*(G, N, c, P_\gamma)$ we have that*

$$res_Q^P([\zeta]) \in \mathrm{H}^*(N, c, Q_\delta).$$

Using Proposition 3.3.2, we define a restriction map from generalized block cohomology of $c$ to block cohomology of $c$.

**Definition 3.3.3.** In situation (∗) we define *the restriction in block cohomology*

$$res_{N,c}^{G,N,c} : \mathrm{H}^*(G, N, c, P_\gamma) \longrightarrow \mathrm{H}^*(N, c, Q_\delta),$$

by $res_{N,c}^{G,N,c}([\zeta]) = res_Q^P([\zeta])$, for any $[\zeta] \in \mathrm{H}^*(G, N, c, P_\gamma)$.

**3.3.4. The multiplicity algebra; the multiplicity module.** We consider $B = kNc$, which is a primitive $G$-algebra (the unity of $B$, that is $c$ is a primitive idempotent of $B^G$, thus $B^G$ is a local ring). Moreover $B$ is the localization of $G_{\{c\}}$ in $A_1$ and $P_\gamma$ is a defect of $B$. We remind that $S(\gamma) = B^P/m_\gamma$ is a simple $k$-algebra which we call *the multiplicity algebra*, where $m_\gamma = J(B^P)$ is the only maximal ideal $B^P$ such that $\gamma \nsubseteq m_\gamma$. Then $S(\gamma) \simeq \mathrm{End}_k(V(\gamma))$, where $V(\gamma)$ is a simple $B^P$-module which we call *the multiplicity module.*

In the following section we denote by $\overline{N} = N_G(P_\gamma)/P$ and we can consider $\overline{C} = C_N(P)/Z(P) \cap N$. We notice that $\overline{C} = C_N(P)/P \cap C_N(P) \simeq PC_N(P)/P$, which is a subgroup of $\overline{N}$.

**Lemma 3.3.5.** *Under the conditions 3.3.4 it is true that the multiplicity module $V(\gamma)$ is simple and projective as $k\overline{C}$-modul. Moreover we have that $p$ doesn't divide $\mid \overline{N}/\overline{C} \mid$.*

**Proposition 3.3.6.** *Let $P_\gamma$ be a defect pointed group of $G_{\{c\}}$ and $(P, e_P)$ the only maximal $(c, G)$-Brauer pair with the property that $\mathrm{Br}_P^N(i)e_P \neq 0$. Then it follows that $Z(P) \cap N$ is a pointed defect group of $e_P$. Particularly $e_P$ is a nilpotent block of $kC_N(P)$.*

**Proposition 3.3.7.** *Let $P_\gamma$ be a defect pointed group of $G_{\{c\}}$ and $(P, e_P)$ the only maximal $(c, G)$-Brauer pair with the property that $\mathrm{Br}_P^N(i)e_P \neq 0$, for any $i \in \gamma$. Then $P_\gamma$ is the only pointed group on $A_1$ with the above property and moreover we have that $N_G(P, e_P) = N_G(P_\gamma)$. In this situation $N_G(P, e_P)/PC_N(P) \cong \overline{N}/\overline{C}$.*

**Proposition 3.3.8.** *Let $N$ be a normal subgroup of $G$, let $c$ be a $G$-stable block of $kN$, $P_\gamma$ a pointed defect group of $G_{\{c\}}$ and $i \in \gamma$. We consider $ikGi$ as $kP - kP$-bimodule avd $[\zeta] \in \mathrm{H}^*(G, N, c, P_\gamma)$.*

*i) We have that $t_{ikGi}(\delta_P([\zeta])) = \frac{dim_k(ikGi)}{|P|}\delta_P([\zeta])$; particularly $\pi_{ikGi} = \frac{dim_k(ikGi)}{|P|}1_{kP}$.*

*ii) For any positive integer n the following diagram is commutative up to homotopy:*

$$\begin{tikzcd}
\mathcal{P}_{kP} \arrow[r, "\varepsilon_{ikGi}"] \arrow[d, "\delta_P(\zeta_n)"] & (ikGi) \otimes_{kP} \mathcal{P}_{kP} \otimes_{kP} (ikGi) \arrow[d, "Id\otimes\delta_P(\zeta_n)\otimes Id"] \\
\mathcal{P}_{kP}[n] \arrow[r, "\varepsilon_{ikGi}[n]"] & (ikGi) \otimes_{kP} \mathcal{P}_{kP}[n] \otimes_{kP} (ikGi)
\end{tikzcd},$$

*where $\zeta_n$ is the degree $n$ component of $\zeta$. Particularly $\delta_P([\zeta])$ is $ikGi$-stable.*

**Remark 3.3.9.** Let $N$ be a normal subgroup of $G$, let $c$ be a $G$-stable block of $kN$ and $P_\gamma$ pointed defect group of $G_{\{c\}}$ with $i \in \gamma$. Then there is an isomorphism of $kP - kGc$-bimodule $(kGi)^* \cong ikG$.

We apply [17, 6.6] in the particular case of $A = kGc, B = kP$ as symmetric $k$-algebras with the usual symmetric forms $s$, respectively $t$ and $M = kGi$, and obtain descriptions of the unity an counity.

**Lemma 3.3.10.** *Let $N$ be a normal subgroup of $G$, let $c$ be a $G$-stable block of $kN$ and $P_\gamma$ pointed defect group of $G_{\{c\}}$ with $i \in \gamma$. Using the identification from Remark 3.3.9 and since by Proposition 3.2.10 the multiplication in $kGc$ induces an isomorphism $ikG \otimes_{kGc} kGi \cong ikGi$ it follows that the adjunction maps $kGi$ and its dual $ikG$ are given as follows:*

$\epsilon_{kGi} : kP \longrightarrow ikGi$ *maps* $u \in P$ *to* $ui$;

$\eta_{kGi} : kGi \otimes_{kP} ikG \longrightarrow kGc$ *given by multiplication in* $kGc$;

$\epsilon_{ikG} : kGc \longrightarrow kGi \otimes_{kP} ikG$ *maps* $a \in kGc$ *to* $\displaystyle\sum_{x\in[G/P]} axi \otimes ix^{-1}$;

$\eta_{ikG} : ikGi \longrightarrow kP$ *maps* $b \in ikGi$ *to* $\displaystyle\sum_{u\in P} s(bu^{-1})u$.

*Moreover we have that* $\pi_{kGi} = \mathrm{Tr}_P^G(i)$ *and* $\pi_{ikG} = s(i)1_{kP} = \frac{dim_k(ikG)}{|G|}1_{kP}$.

Next we will give the main result of this chapter, which says the the generalized block cohomology algebra of a $G$-stable block embeds through the diagonal map into the subalgebra of stable elements of Hochschild cohomology algebra of $kGc$. The result is similar to [17, Teorema 5.6].

**Theorem 3.3.11.** *Let $N$ be a normal subgroup of $G$, let $c$ be a $G$-stable block of $kN$ and $P_\gamma$ pointed defect group of $G_{\{c\}}$ with $i \in \gamma$. Let $(P, e_P)$ be the maximal $(c, G)$-pereche Brauer and let $kGi$ and $ikG$ as $kGc - kP$ respectively $kP - kGc$-bimodule.*

*i) We have that $\pi_{kGi} = \mathrm{Tr}_P^G(i) \in Z(kGc)^\times$ and $\pi_{ikG} = \frac{dim_k(ikG)}{|G|} 1_{kP} \in k^* 1_{kP}$.*

*ii) If $[\zeta] \in \mathrm{H}^*(G, N, c, P_\gamma)$ then $\delta_P([\zeta])$ is $ikG$-stable in $\mathrm{HH}^*(kP)$.*

*iii) The homomorphism $T_{kGi} \circ \delta_P$ induces an injective homomorphism of graded $k$-algebras*

$$\mathrm{H}^*(G, N, c, P_\gamma) \xrightarrow{\delta_P} \mathrm{HH}^*_{ikG}(kP) \xrightarrow{T_{kGi}} \mathrm{HH}^*_{kGi}(kGc)$$

We end this section with third main Brauer's Theorem for $(c, G)$-Brauer pairs. This result is the main result proved by the author in[31], and the proof just imitates the proof from [35, Teorema 40.17]; we will use an normal inclusion relation between $(c, G)$-Brauer pairs.

**Theorem 3.3.12.** *Let $c = c_0$ be the principal block of $kN$, where $N$ is a normal subgroup of $G$ and $Q$ is a $p$-subgroup of $G$. Then we have that:*

*a) The principal block $c_0$ is $G$-stable.*

*b) $\mathrm{Br}_Q^N(c_0)$ is an primitive idempotent in $Z(kC_N(Q))$ and is the principal block of $kC_N(Q)$.*

*c) $(Q, e_Q)$ is a $(c_0, G)$-Brauer pair if and only if $e_Q$ is the principal block of $kC_N(Q)$.*

*d)* $(c_0, G)$*-defect groups are the Sylow* $p$*-subgroups of* $G$*.*

## 3.4 Properties of the restriction map in block cohomology

In this section we will analyze the proprieties of the generalized restriction map, defined in section 3.3. If we are in situation $(*)$ we will study this restriction map through the transfer map between Hochschild cohomology algebras of $kGc$ and of block ideal $kNc$.

First we remind some notations and results which are implicitly in 3.1 and 1.1. Let $A, B, C$ be three symmetric $R$-algebras, $X$ be a bounded complex of finitely generated $A-B$-bimodules, projective as left $A$-modules and right $B$-modules, $Y$ be a bounded complex of $B-C$-bimodules, projective as left $B$-module and right $C$-module. We will denote that $[\zeta] \in \mathrm{HH}^*(A)$ is $X$-stable with $[\zeta] \otimes_A 1_X = 1_X \otimes_B [\tau]$, where $[\tau] \in \mathrm{HH}^*(B)$ which satisfy Definition 2.1.1.

**Proposition 3.4.1.** *With the above hypothesis:*

*i)* $\pi_{X \otimes_B Y} = t_X^0(\pi_Y)$*.*

*ii) If* $X'$ *is a direct summand* $X$ *then* $\mathrm{HH}^*_{X^*}(B) \subset \mathrm{HH}^*_{X'^*}(B)$*. Moreover if* $\pi_X, \pi_{X'}$ *are invertible then the normalized transfer map* $T_{X'}$ *coincides* $T_X$ *on* $\mathrm{HH}^*_{X^*}(B)$*.*

The next proposition follows from [13].

**Proposition 3.4.2.** *If* $\pi_X, \pi_Y, \pi_{X \otimes_B Y}$ *are invertible then* $T_X \circ T_Y$ *coincides with* $T_{X \otimes_B Y}$ *on* $\mathrm{HH}^*_{Y^* \otimes_B X^*}(C)$*.*

**3.4.3. Particular case of complex in situation** $(*)$**.** Let $A = kNc$, $B = kGc$, and $X = ckGc = kGc$ as $A-B$-bimodule with $X^* = ckGc = ckG$ as $B-A$-bimodule. Let $M = kP$ as $kP-kQ$-bimodule with $M^* = kP$ respectively $kQ-kP$-bimodule. We know that $\pi_M = [P : Q]1_{kP} \in Z(kP)$ is not invertible and $\pi_{M^*} = 1_{kQ} \in Z(kQ)^\times$.

**Proposition 3.4.4.** *In the hypothesis 3.4.3 the following statements are true:*

*i)* $\pi_X = c, \quad \pi_{X^*} = [G:N]c.$

*ii)* $\pi_X \in Z(kNc)$ *is invertible.* $\pi_{X^*}$ *is invetible in* $Z(kGc)$ *if and only if* $p$ *doesn't divide* $[G:N]$.

In situation $(*)$ since $Q_\delta \leq P_\gamma$ we choose $i \in \gamma$ and $j \in \delta$ such that $j = ij = j\ i$. Next we choose $Y = kGi$ as $B - kP$-bimodule and $Z = kNj$ as $A - kQ$-bimodule. Then we have the following descriptions:

$$ikGj = ikG \otimes_B X^* \otimes_A kNj = Y^* \otimes_B X^* \otimes_A Z;$$

$$jkGi = Z^* \otimes_A X \otimes_B Y.$$

**Lemma 3.4.5.** *The following statements are true:*

*a)* $\mathrm{HH}^*_{ikGj}(kP) \subset \mathrm{HH}^*_{Y^* \otimes_B X^*}(kP)$.

*b)* $T_Y(\mathrm{HH}^*_{Y^* \otimes_B X^*}(kP)) \subset \mathrm{HH}^*_{X^*}(kGc)$.

**Lemma 3.4.6.** *With the above notations the following statements are true:*

*a)* $kQ - kQ$*-bimodulul* $jkNj$ *is a direct summand of* $ikGj$.

*b)* $A - kP$*-bimodulul* $X \otimes_B Y$ *is a direct summand* $Z \otimes_{kQ} jkGi$.

*c) We have that* $M$ *is isomorphic with a direct summand of* $ikGj$ *as* $kP - kQ$*-bimodule.*

**3.4.7. A commutative diagram given by the restriction in block cohomology.** By c) from Lema 3.4.6 we will identify $M$ with a direct summand $ikGj$. Since

$\pi_{M^*} = 1_{kQ}$ the normalized transfer $T_{M^*}$ is $t_{M^*}$. We suppose that we are in situation $(*)$, then by [17, Propoziţia 4.7] the following diagram of graded $k$-algebras is commutative:

$$\begin{tikzcd} \mathrm{H}^*(G,N,c,P_\gamma) \arrow[r, "\delta_P"] \arrow[d, "res^{G,N,c}_{N,c}"] & \mathrm{HH}^*(kP) \arrow[d, "T_{M^*}"] \\ \mathrm{H}^*(N,c,Q_\delta) \arrow[r, "\delta_Q"] & \mathrm{HH}^*(kQ) \end{tikzcd} .$$

We obtain a different diagram in the next remark, where abusively we denote by $T_{M^*}$ the surjective map $R_{M^*}$.

**Remark 3.4.8.** With the hypothesis from 3.4.7 the following diagram of homomorphisms of graded $k$-algebras is commutative:

$$\begin{tikzcd} \mathrm{H}^*(G,N,c,P_\gamma) \arrow[r, "\delta_P"] \arrow[d, "res^{G,N,c}_{N,c}"] & \mathrm{HH}^*_M(kP) \arrow[d, "T_{M^*}"] \\ \mathrm{H}^*(N,c,Q_\delta) \arrow[r, "\delta_Q"] & \mathrm{HH}^*_{M^*}(kQ) \end{tikzcd} .$$

**Proposition 3.4.9.** *We suppose that $\delta_P(\mathrm{H}^*(G,N,c,P_\gamma)) \subseteq \mathrm{HH}^*_{ikGj}(kP)$, where $ikGj$ is $kP - kQ$-bimodule. Then the following diagram of homomorphisms of graded $k$-algebras is commutative:*

$$\begin{tikzcd} \mathrm{H}^*(G,N,c,P_\gamma) \arrow[r, "\delta_P"] \arrow[d, "res^{G,N,c}_{N,c}"] & \mathrm{HH}^*_{ikGj}(kP) \cap \mathrm{HH}^*_M(kP) \arrow[d, "T_{M^*}"] \\ \mathrm{H}^*(N,c,Q_\delta) \arrow[r, "\delta_Q"] & \mathrm{HH}^*_{jkNj}(kQ) \cap \mathrm{HH}^*_{M^*}(kQ) \end{tikzcd} .$$

**Theorem 3.4.10.** *In situation $(*)$ we choose $i \in \gamma$ and $j \in \delta$ such that $j = ij = ji$. We suppose that $\delta_P(\mathrm{H}^*(G,N,c,P_\gamma)) \subset \mathrm{HH}^*_{ikGj}(kP)$. Then the following diagram of homomorphisms of graded $k$-algebras is commutative::*

$$\begin{tikzcd}
\mathrm{H}^*(G,N,c,P_\gamma) \arrow[r, "T_{kGi}\circ\delta_P"] \arrow[d, "res^{G,N,c}_{N,c}"] & \mathrm{HH}^*_{X^*}(kGc) \arrow[d, "T_X"] \\
\mathrm{H}^*(N,c,Q_\delta) \arrow[r, "T_{kNj}\circ\delta_Q"] & \mathrm{HH}^*(kNc)
\end{tikzcd} .$$

If $c$ is the principal block of $kN$ then $\delta_P(\mathrm{H}^*(G,N,c,P_\gamma)) \subset \mathrm{HH}^*_{ikGj}(kP)$, the property from Theorem 3.4.10 is satisfied; in this case the above diagram is commutative.

## 3.5 Varieties in the generalized block cohomology

In this section we follow the notations and hypothesis from Theorem 3.4.10. The main articles where is studied the variety associated to the usual block cohomology algebra are: [18], [20], [6]. We will end with a theorem which links the varieties associated to a module through the restriction in blocks cohomology.

### 3.5.1. Varieties associated to modules for generalized block cohomology.

The generalized block cohomology algebra $\mathrm{H}^*(G,N,c,P_\gamma)$ is a finitely generated algebra, graded commutative. We denote the maximal ideal spectrum by $V_{G,N,c}$, and this is called *the variety* of this algebra. Let $U$ be a finitely generated $kGc$-module and let $I^*_{G,N,c,P_\gamma}(U)$ be the kernel of the composition of graded $k$-algebras

$$\mathrm{H}^*(G,N,c,P_\gamma) \xrightarrow{T_{kGi}\circ\delta_P} \mathrm{HH}^*(kGc) \xrightarrow{\alpha_U} \mathrm{Ext}^*_{kGc}(U,U) ,$$

where $\alpha_U$ is the functor induced by $-\otimes_{kGc} U$. The variety $V_{G,N,c}(U)$ is defined as the subvariety of $V_{G,N,c}$, which consists of the maximal ideals containing $I^*_{G,N,c,P_\gamma}(U)$. We will denote still by $U$ the structure of $U$ as $kNc$-module and by $I^*_{N,c,Q_\delta}$ the kernel of the composition

$$\mathrm{H}^*(N,c,Q_\delta) \xrightarrow{T_{kNj}\circ\delta_Q} \mathrm{HH}^*(kNc) \xrightarrow{\alpha_U} \mathrm{Ext}^*_{kNc}(U,U) .$$

The variety $V_{N,c}(U)$ is the subvariety of $V_{N,c}$ which consists in all maximal ideals containing $I^*_{N,c,Q_\delta}$. The cohomology variety $V_G(U)$ associated to $U$, introduced by Carlson in [9], is defined as the subvariety of the maximal ideals spectrum of $\mathrm{H}^*(G,k)$ (denoted with $V_G$) determined by $I^*_G(U)$. Here $I^*_G(U)$ is the kernel of the homomorphism of graded $k$-algebras $\mathrm{H}^*(G,k) \longrightarrow \mathrm{Ext}^*_{kGc}(U,U)$ induced by the functor $-\otimes_k U$.

Next we prove the following proposition which is an analogous result to [20, Theorem 2.1] and a lemma which provides us a stratification of the variety $V_{G,N,c}(U)$. These two results allow us to prove the main theorem of this section.

**Theorem 3.5.2.** *We keep the notations and the assumptions from Theorem 3.4.10.*

*(a) The restriction map in block cohomology $res^{G,N,c}_{N,c} : \mathrm{H}^*(G,N,c,P_\gamma) \longrightarrow \mathrm{H}^*(N,c,Q_\delta)$ induces a finite map $(res^{G,N,c}_{N,c})^*$, which we denote by $r^*_{G,N,c} : V_{N,c} \longrightarrow V_{G,N,c}$.*

*b) For any finitely genrated $kGc$-module $U$, we have that*

$$V_{N,c}(U) = (r^*_{G,N,c})^{-1}(V_{G,N,c}(U)).$$

# Chapter 4

# An equivalent definition of cohomology of finite groups

Let $G$ be a finite group, $k$ a field of characteristic $p$ and $P$ a Sylow $p$-subgroup of $G$. The fusion system of $P$ in $G$ is defined by 1.4.1. In the first chapter of this section we will obtained a similar result to the embedding of the cohomology of a finite group into the submodule of stable elements in the cohomology of a Sylow subgroup, but for $kG$-modules. In the second section we will prove an isomorphism of the functor Hom, a left exact functor which appear in the definition of group cohomology, with a new functor defined by stable elements in the $k$-submodule of homomorphisms of $kP$-modules.

The chapter is entirely developed on the results obtained by the author în [34].

## 4.1 Stable elements in the module of homomorphisms

Let $A, B$ be two $kG$-modules. If $\varphi : H \longrightarrow G$ is a homomorphism of finite groups, where $H, G$ are two groups, then there is *the restriction through* $\varphi$

$$res_{\varphi} : \mathrm{Hom}_{kG}(A, B) \longrightarrow \mathrm{Hom}_{kH}(A, B) \quad f \mapsto res_{\varphi}(f),$$

where $res_\varphi(f)$ is $f$ considered as homomorphisms of $RH$-modules. The structure of $RH$-module is given by $\varphi$ (i.e. $ha = \varphi(h)a$, for $a \in A$ and $h \in H$).

First we prove a similar proposition to [11, Corollary 4.2.7]

**Proposition 4.1.1.** *Let $P$ be a Sylow $p$-subgroup of $G$ and $A, B$ be two $kG$-modules. Then $f$ is in* $\mathrm{Im} res_P^G$ *if and only if*

$$res^P_{P\cap {}^gP}(f) = res^{{}^gP}_{P\cap {}^gP}(g^*(f)), \quad \forall g \in G. \tag{4.1.1}$$

**Definition 4.1.2.** A homomorphism $f \in \mathrm{Hom}_{kP}(A, B)$ which satisfy condition (4.1.1) is called *stable*. We denote by $\mathrm{Hom}^{st}_{kP}(A, B)$ the $k$-submodule of stable elements.

Since $\mathrm{Tr}_P^G \circ res_P^G = [G : P]id$ and $[G : P]$ is invertible in $k$, by Proposition 4.1.1 we obtain the following corollary.

**Corollary 4.1.3.** *With the above hypothesis the following statements are true:*

*1)* $\mathrm{Hom}_{kG}(A, B)$ *is isomorphic to the $k$-submodule of stable elements in* $\mathrm{Hom}^{st}_{kP}(A, B)$.

*2)* $\mathrm{Hom}^{st}_{kP}(A, B) = \{f \in \mathrm{Hom}_{kP}(A, B) \mid res^P_R(f) = res_\varphi(f), \forall \varphi \in \mathrm{Hom}_{\mathcal{F}_P(G)}(R, P)\}$.

## 4.2 An equivalent definition of cohomology of finite groups

The usual definition of group cohomology is $\mathrm{H}^n(G, k) = \mathrm{R}^n\mathrm{Hom}_{kG}(k, -)(k)$ as the right $n$-th derived functor of the covariant functor Hom, which is in this case, the left exact, covariant and additive functor

$$\mathrm{Hom}_{kG}(k, -) : \mathrm{Mod}(kG) \longrightarrow \mathrm{Mod}k.$$

Using Corollary 4.1.3 we will define a new functor $F_G : \mathrm{Mod}(kG) \longrightarrow \mathrm{Mod}k$.

#### 4.2.1. An isomorphic functor to $\mathrm{Hom}_{kG}(k,-)$.

If $A$ is a $kG$-module we denote by $F_G(A)$ the $k$-submodule of stable elements in $\mathrm{Hom}_{kP}(k,A)$, that is

$$F_G(A)=\{f\in \mathrm{Hom}_{kP}(k,A) \mid res_R^P(f)=res_{\varphi}(f), \forall \varphi\in \mathrm{Hom}_{\mathcal{F}_P(G)}(R,P)\}.$$

If $\partial : A\longrightarrow B$ is a homomorphism of $kG$-modules we define by $F_G(\partial)$ the homomorphism of $k$-modules

$$F_G(\partial):F_G(A)\longrightarrow F_G(B),\quad F_G(\partial)(f)=\partial\circ f.$$

It follows that $F_G(A)$ is an additive covariant functor. The Corollary 4.1.3 allow us to obtain the natural isomorphism of functors $F_G\cong \mathrm{Hom}_{kG}(k,-)$, which implies that $F_G$ is a left exact functor.

The relevance of the following proposition is given by the isomorphism from the proof. We hope that this approach allow us to apply the same method to block cohomology. This is analyzed in [30].

**Proposition 4.2.2.** *There is the well defined homomorphism of $k$-modules*

$$\psi:\mathrm{R}^nF_G(k)\longrightarrow \mathrm{H}^n_{st}(\mathcal{F}_P(G)),$$

*cu $\psi(f+\mathrm{Im}F_G(\delta^{n-1}))=[f]$, for any $f\in \mathrm{Ker}F_G(\delta^n)$.*

Now it is easy to check the following corollary.

**Corollary 4.2.3.** *There is the isomorphism of $k$-modules $R^nF_G(k)\cong \mathrm{H}^n(G,k)$.*

# Bibliography


[1] J. L. Alperin, *Cohomology is representation theory*, Proceedings of the Symposium in Pure Mathematics **47** (1987), no. 1, 3–11.

[2] J. L. Alperin and M. Broué, *Local methods in block theory*, The Annals of Mathematics **110** (1979), no. 1, 143–157.

[3] G. S. Avrunin and L. L. Scott, *Quillen stratification for modules*, Inv. Math **66** (1982), 277–286.

[4] D. J. Benson, *Representations and Cohomology I*, Cambridge University Press, Cambridge, 1991.

[5] ———, *Representations and Cohomology II*, Cambridge University Press, Cambridge, 1991.

[6] D. J. Benson and M. Linckelmann, *Vertex and source determine the block variety of an indecomposable module*, J. Pure Appl. Algebra **197** (2005), 11–17.

[7] C. Broto, R. Levi, and L. Oliver, *The homotopy theory of fusion systems*, J. Amer. Math. Soc. **16** (2003), 779–856.

[8] M. Broué and L. Puig, *Characters and local structures in G-algebras*, J. of Alg. **63** (1980), 306–317.

[9] J. F. Carlson, *The varieties and the cohomology ring of a module*, J. Algebra **85** (1983), 104–143.

[10] H. Cartan and S. Eilenberg, *Homological Algebra*, Princeton University Press, New Jersey, 1956.

[11] L. Evens, *The Cohomology of Groups*, Clarendon Press, Oxford, New York, Tokyo, 1991.

[12] D. Gorenstein, *Finite Groups*, Chelsea Publishing Company, New York, 1980.

[13] H. Kawai and H. Sasaki, *Cohomology algebras of blocks of finite groups and Brauer correspondence*, Alg. and Repres. Theory **9** (2006), 497.

[14] R. Kessar and R. Stancu, *A reduction theorem for fusion systems of blocks*, J. Algebra **319** (2008), 806–823.

[15] B. Külshammer and L. Puig, *Extensions of nilpotent blocks*, Inventiones Mathematicae **102** (1990), 17–71.

[16] M. Linckelmann, *On derived equivalences and local structure of blocks of finite groups*, Tr. J. of Mathematics **22** (1998), 93–107.

[17] ———, *Transfer in Hochschild cohomology of blocks of finite groups*, Alg. and Repres. Theory **2** (1999), 107–135.

[18] ———, *Varieties in block theory*, J. Algebra **215** (1999), 460–480.

[19] ———, *On splendid derived and stable equivalences between blocks of finite groups*, J. Algebra **242** (2001), 819–843.

[20] ———, *Quillen stratification for block varieties*, J. Pure Appl. Algebra **172** (2002), 257–270.

[21] ———, *Simple fusion systems and the Solomon* 2*-local groups*, J. Algebra **296** (2006), no. 2, 385–401.

[22] ———, *Introduction to fusion systems*, Group Representation Theory, EPFL Press, Lausanne (2007), 79–113.

[23] A. Mărcuş, *Twisted group algebras, normal subgroups and derived equivalences*, Alg. Repres. Theory **4** (2001), 25–54.

[24] ———, *Modular Representation Theory of Finite Groups*, Editura EFES, Cluj Napoca, 2002.

[25] J. Pakianathan and S. W. Witherspoon, *Hochschild cohomology and Linckelmann cohomology for blocks of finite groups*, J. Pure Appl. Algebra **178** (2003), 87–100.

[26] L. Puig, *Frobenius Categories versus Brauer Blocks*, Birkhäuser, Basel, Boston, Berlin, 2009.

[27] D. Quillen, *The spectrum of an equivariant cohomology ring: I*, The Annals of Math. **94** (1971), no. 3, 549–572.

[28] ———, *The spectrum of an equivariant cohomology ring: II*, The Annals of Math. **94** (1971), no. 4, 573–602.

[29] S. F. Siegel and S. W. Witherspoon, *The Hochschild cohomology ring of a group agebra*, Proc. London Math. Soc. **79** (1999), 131–157.

[30] **C. C. Todea**, *A global approach for cohomology of blocks of finite groups*, J. Algebra (trimis spre publicare).

[31] ———, *Remarks on generalized Brauer pairs*, Mathematica, Tome (va apărea).

[32] ———, *Restriction between cohomology algebras of blocks of finite groups*, Alg. and Repres. Theory.

[33] ———, *New stable elements in Hochschild cohomology algebra*, Mathematica, Tome **75** (2010), no. 1, 75–84.

[34] ———, *Remarks on definition of group cohomology of finite groups*, Automat. Comput. Appl. Math. **19** (2010), no. 1, 7–11.

[35] J. Thévenaz, *G-algebras and Modular Representation Theory*, Clarendon Press, Oxford, 1995.

[36] C. A. Weibel, *An Introduction to Homological Algebra*, Cambridge University Press, 1994.