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EQUATIONS IN SPACES OF MULTIVALUED FUNCTIONS

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Contents

Keywords: spaces of multivalued function, multivalued operators, derivability, integration, Picard operators, weakly Picard operators, semifixed sets, multivalued- φ - contractions, integral equations, functional-integral equations.

Introduction

The theme of this PhD Thesis is related to the study of certain classes of equations in spaces of multivalued functions. More specifically, after a study of some operatorial equations with multivalued operators in metric context and a presentation of several existence theorems, fixed and semifixed sets for set operators, in the second part of the thesis, we study qualitative properties (existence, uniqueness, data dependence, Hyers-Ulam-Rassias stability) for differential and integral equations in spaces of multivalued functions. The study is motivated by actuality of the topic (41 articles and 17 books in the last 10 years) and its importance: many problems of applied mathematics reduce to the study of such equations in spaces of multivalued functions.

The paper is divided into four chapters, followed by references.

The first chapter, entitled "Preliminaries", has the aim to remind some notions and basic results which are necessary for the following chapters of this Ph.D. thesis. In writing this chapter, we used the following bibliographical sources: J.-P. Aubin, A. Cellina [4], M. C. Anisiu [1], [2], K. Deimling [23], [24], J.-P. Aubin, H. Frankowska [5], M. Kisielewicz [44], G. Beer [9], S. Hu şi N.S. Papageorgiou [40], J. Dugundji, A. Granas [37], A. Petruşel [66], I. A. Rus [75], [77].

The second section of the first chapter presents concepts and results of the theory of multivalued operators. The notions and the results presented appear in classical works such as: J.-P. Aubin, H.Frankovska [5], G. Beer [9], C. Berge [10], C. Castaing, M. Valadier [13], F.S. De Blasi [19], L. Górniewicz [35], L. Górniewicz [36], C. J. Himmelberg [38], S. Hu, N. S. Papageorgiou [40], V. Lakshmikantham, T. Gnana Bhaskar, J. Vasundhara Devi [49], A. Petruşel, G. Petruşel [70], I.A. Rus, A. Petruşel, G. Petruşel [86].

The third section of this chapter entitled "Derivative and integrability of multivalued operators" presents concepts considered by many authors in their books: J.-P. Aubin, H.Frankovska [5], H.T. Banks, M.Q. Jacobs [8], F. S. De Blasi [19], G.N. Galanis, T.G. Bhaskar, V. Lakshmikantham and P.K.Palamides [31], V. Lakshmikantham, T. Gnana Bhaskar, J. Vasundhara Devi [49]. They are considered in different ways depending on the applications which involved them. The purpose of this paragraph is to present the concept of derivative and integration of the multivalued operator, in the sense of Hukuhara. The concept of derivative for multivalued operators was given in 1967 by Hukuhara [41], [42].

In the fourth section of the first chapter we present basic notions and results from the theory of Picard and weakly Picard operators.

Chapter two is entitled "Semifixed sets for multivalued contractions". In this chapter we present results concerning existence and uniqueness of semifixed sets for set-operators which satisfy some contraction conditions and some topological conditions. The results in this chapter extend some results presented in the following works: A.J. Brandao [12], A. Constantin [16], H. Covitz, S.B. Nadler jr.[17], F.S. De Blasi [19], [20], [21], [22], M. Frigon [27], [28], S. Kakutani [43], V. Lakshmikantham, A.N. Tolstonogov [47], V. Lakshmikantham, T. Gnana Bhaskar, J. Vasundhara Devi [49], A. Muntean [60].

Thus, in the first section, "Semifixed sets for multivalued operators", are presented some concepts relative to the semifixed sets for set-operators, notions introduced F.S. De Blasi in [21], [22].

In the second part of the chapter, in paragraph "Semifixed sets for multivalued φ-contractions", are proved some results on the existence of semifixed sets for set φ-contractions. The results which belong to the author are: Theorem 2.2.13, Theorem 2.2.14, Theorem 2.2.15 which are published in the paper I.C. Tişe [96]. In the last part is presented the generalized Hyers-Ulam stability in: Theorem 2.2.7 and Theorem 2.2.17, which are published in the paper I.C. Tişe [100].

The third chapter of this paper is called "Integral equations in spaces of multivalued functions". Here are presented some existence, uniqueness and data dependence results of solutions for integral equations and differential equations for multivalued function spaces and their applications.

In the first section, "Integrals equations in spaces of multivalued functions" are presented, with respect to integral equations in spaces of multivalued functions, existence theorems and uniqueness of solution of equations and continuous data dependence. Contributions of the author are: Theorem 3.1.3, Theorem 3.1.4, Theorem 3.1.7, Theorem 3.1.9, and they are published in the paper I.C. Tişe [98].

In the second part of the chapter is presented the notion of the Cauchy problem for differential equations in spaces of multivalued functions and are obtained results of existence and uniqueness for this problem through the fixed point method. Our contributions are Theorem 3.2.4, Theorem 3.2.5 appeared in the paper I.C.Tişe [99].

In the last part of the chapter, "Functional-integral equations in spaces of multivalued functions", we present the case of some functional-integral equations and we prove results of existence and uniqueness of the solution.

The contributions are Theorem 3.3.1, Theorem 3.3.2, Theorem 3.3.4 and are published in the paper I.C. Tişe [99].

Some Hyers-Ulam-Rassias stability results in the generalized sense for integral equations in spaces of multivalued functions are presented: Theorem 3.1.6, Theorem 3.1.11, Theorem 3.3.5, which are contained in the paper I.C. Tişe [100].

Chapter four is titled "Qualitative properties of solutions of differential equations in multivalued function spaces. The first paragraph of this chapter is dedicated to Gronwall type Lemmas and comparison theorems. In following section we discuss data dependence of the solution of differential equations in spaces of multivalued functions.

The contributions from this chapter are Theorem 4.1.3, Theorem 4.1.7, Theorem 4.1.9, Theorem 4.1.11, Theorem 4.2.1, results contained in the papers I.C Tişe [95], [97].

In conclusion, the contributions of this thesis appear in the following papers:

I.C. Tişe, *Data dependence of the solutions for set differential equations*, Carpathian J. Math., 23 (2007), No. 1-2, 192-195;

I.C. Tişe, *Semifixed sets for multivalued φ-contractions*, Creative Math.& Inf., 17 (2008), No. 3, 516-520;

I.C. Tişe, *Gronwall lemmas and comparison theorems for the Cauchy problem associated to a set differential equation*, Studia Universitatis Babes-Bolyai Mathematica, 54 (2009), No. 3, 161-169;

I.C. Tişe, *Set integral equations in metric spaces*, Mathematica Moravica, 13 (2009), 95-102;

I.C. Tişe, *A fixed point approach for functional-integral set equations*, accepted for publication in Demonstratio Mathematica, Vol. 44 (2011), No. 2, va apărea;

I.C. Tişe, *Ulam-Hyers-Rassias stability for set integral equations*, trimisă spre publicare.

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1

Preliminaries

1.1 Functional parts of a space metric space

1.2 Multivalued operators

1.3 Derivative and integrability of multivalued operators

The derivative for multivalued operators is considered by many authors in their works: J.-P. Aubin, H.Frankovska [5], H.T. Banks, M.Q. Jacobs [8], F. S. De Blasi [19], G.N. Galanis, T.G. Bhaskar, V. Lakshmikantham and P.K.Palamides [31], V. Lakshmikantham, T. Gnana Bhaskar, J. Vasundhara Devi [49]. They work in different ways, which depend of the applications.

The main purpose of this section is to present the concept of derivative in the sense of Hukuhara. We mention that the notion of derivative for a nonempty and compact set from a continuous function space was given from Bridgland in 1970. Banks and Jacobs also define in 1970 a notion of derivative for multivalued operators in normed spaces and present some results which are like the usual differential calculus, see [8].

In 1967 Hukuhara [41], [42] gave a definition for derivative for multivalued operators.

Definition 1.3.1 *(Hukuhara [41]) Let X an Banach spaces and $A, B \in P_{cp,cv}(X)$ is called difference sets A and B (note $A - B$) a set $C \in P_{b,cl,cv}(X)$ (if it exist) with properties $C + B = A$.*

This definition was called while the difference Hukuhara.

Definition 1.3.2 *(Hukuhara [41]) Let* $I \subset \mathbb{R}$ *and* $F : I \to P_{cp,cv}(\mathbb{R}^n)$ *multivalued operators. Then* F *is called H-differentiable (Hukuhara differentiable) in* $t_0 \in I$*, if exist* $D_H F(x_0) \in P_{cp,cv}(\mathbb{R}^n)$ *such that the limits:*

$$\lim_{\Delta t \to 0^+} \frac{F(t_0 + \Delta t) - F(t_o)}{\Delta t} \tag{1.3.1}$$

$$\lim_{\Delta t \to 0^+} \frac{F(t_0) - F(t_0 - \Delta t)}{\Delta t} \tag{1.3.2}$$

both exist and are equal to $D_H F(t_0)$*.*

Clearly, implicit in the definition of $D_H F(t_0)$ is the existence of the differences $F(t_o + \Delta t) - F(t_0)$, $F(t_0) - F(t_0 - \Delta t)$ for all $\Delta t > 0$ sufficiently small.

1.4 Picard operators and weakly Picard operators

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Semifixed sets for multivalued contractions

In this chapter we will prove some existence (and eventually uniqueness) results of the semifixed sets for set-operators which satisfy some contractive conditions and some topological conditions.

The results of this chapter extend and generalize some theorems from works by A.J. Brandao [12], A. Constantin [16], H. Covitz, S.B. Nadler jr.[17], F.S. De Blasi [19], [20], [21], [22], M. Frigon [27], [28], S. Kakutani [43], V. Lakshmikantham, A.N. Tolstonogov [47], V. Lakshmikantham, T. Gnana Bhaskar, J. Vasundhara Devi [49], A. Muntean [60].

The first part of the chapter presents some basic notions regarding semifixed sets for set-operators, notions introduced by de F.S. De Blasi in his works [21], [22]. In the second part of the chapter we prove some results regarding the existence of semifixed sets in the case of multivalued set-φ-contractions. The results belonging to the author from this chapter are: Theorem 2.2.13, Theorem 2.2.14, Theorem 2.2.15 and they are published in the paper I.C. Tişe [96]. In the last part the generalized Ulam-Hyers stability is presented in Theorem 2.2.7 and Theorem 2.2.17 and can be found in I.C. Tişe [100].

2.1 Semifixed sets for multivalued operators

In this section we present the notion of semifixed sets for multivalued operators, a concept introduced by F.S. De Blasi.

Let $(X, ||\cdot||)$ be a real Banach space, and $\mathcal{A}, \mathcal{B}$ be tow families of nonempty subsets of X and let $P(\mathcal{B})$ the family of all nonempty subset of $\mathcal{B}$.

We introduce in $P_{cp,cv}(X)$ the usual operations of addition and multiplication by nonnegative scalars, for $A, B \in P_{cp,cv}(X)$ and $\lambda \geq 0$ we have:

$A + B = \{a + b | a \in A, \quad b \in B\}$,

$\lambda A = \{\lambda a | a \in A\}$.

Clear, we have $A + B, \quad \lambda A \in P_{cp,cv}(X)$.

Moreover, if $A, B, C \in P_{cp,cv}(X)$ and $\lambda, \quad \mu \geq 0$ we have:

(i) $A + \{0\} = A$, where 0 the zero of Banach space X;

(ii) $A + B = B + A$;

(iii) $A + (B + C) = (A + B) + C$;

(iv) $1 \cdot A = A$;

(v) $\lambda(\mu A) = (\lambda\mu)A$;

(vi) $\lambda(A + B) = \lambda A + \lambda B$;

(vii) $(\lambda + \mu)A = \lambda A + \mu A$.

It is worth noticing that the above properties, except (vii), remain valid in the space $P_{cp}(X)$.

DEFINITION 2.1.1 *A set $\mathcal{A} \subset P_{cp,cv}(X)$ is called convex if $A, B \in \mathcal{A}$ and $t \in [0, 1]$, imply $(1-t)A+tB \in \mathcal{A}$.*

Let X be a Banach space. The space $P_{cp}(P_{cp}(X))$ endowed with the Pompeiu-Hausdorff metric $\mathcal{H}$ induced by the metric H of $P_{cp}(X)$,

$$\mathcal{H}(\mathcal{A}, \mathcal{B}) := max\{e(\mathcal{A}, \mathcal{B}), e(\mathcal{B}, \mathcal{A})\},$$

where $e(\mathcal{A}, \mathcal{B}) := \sup\limits_{A \in \mathcal{A}} \inf\limits_{B \in \mathcal{B}} H(A, B)$, and $e(\mathcal{B}, \mathcal{A}) := \sup\limits_{B \in \mathcal{B}} \inf\limits_{A \in \mathcal{A}} H(B, A)$.

REMARK 2.1.2 $P_{cp,cv}(P_{cp,cv}(X)) \subset P_{cp}(P_{cp}(X))$.

DEFINITION 2.1.3 *Let $(X, \mathcal{H})$ be a metric space. The multivalued operator $\phi : X \to P_{cp}(P_{cp}(X))$, is called upper semicontinuous (resp. lower semicontinuousif, for every $x_0 \in X$ and $\varepsilon > 0$ there exists an open neighborhood V of x_0 such that $\mathcal{H}(\phi(x), \phi(x_0)) < \varepsilon$ (resp. $\mathcal{H}(\phi(x_0), \phi(x)) < \varepsilon$), for every $x \in V$.*

ϕ is called continuous if it is both, upper and lower semicontinuous.

Let $(X, ||\cdot||)$ be a norm space. For $A, B \in P_{cp}(X)$ note:

$$D(A,B) = inf\{||a-b|| \; |a \in A, b \in B\}.$$

We have the next properties for $\lambda \in \mathbb{R}$ and $A, A', B, B' \in P_{cp}(X)$:

1. $D(A,B) = D(B,A)$;
2. $D(A,B) = 0$ if and only if $A \cap B \neq \emptyset$;
3. $D(\lambda A, \lambda B) = |\lambda| D(A,B)$;
4. $D(A,B) \leq D(A', B') + H(A,A') + H(B,B')$;
5. $H(A,B) \leq diam(A) + diam(B) + D(A,B)$.
6. the function D is continuous on $P_{cp}(X) \times P_{cp}(X)$:

$$|\sup_{B\in\mathcal{B}} D(A,B) - \sup_{B\in\mathcal{B}} D(A',B)| \leq H(A,A').$$

Define set

$$\Delta(\mathcal{A},\mathcal{B}) = max\{f(\mathcal{B},\mathcal{A}), f(\mathcal{A},\mathcal{B})\},$$

for $\mathcal{A}, \mathcal{B} \in P_{cp}(P_{cp}(X))$,

$$\text{where } f(\mathcal{A},\mathcal{B}) = \inf_{A\in\mathcal{A}} \sup_{B\in\mathcal{B}} D(A,B) \text{ and } f(\mathcal{B},\mathcal{A}) = \inf_{B\in\mathcal{B}} \sup_{A\in\mathcal{A}} D(B,A).$$

Remark, if $\Delta(\mathcal{A},\mathcal{B}) = 0$ then exist $A' \in \mathcal{A}$ and $B' \in \mathcal{B}$ such that $A' \cap B \neq \emptyset$, for every $B \in \mathcal{B}$ and $B' \cap A \neq \emptyset$, for every $A \in \mathcal{A}$.

The purpose of this section is to present some semifixed set teorems for multivalued operators with compact and convex vales, for $\mathfrak{X}$ a Banach space, defined by:

$$\phi : P_{cp,cv}(\mathfrak{X}) \longrightarrow P_{cp,cv}(P_{cp,cv}(\mathfrak{X})).$$

Definition 2.1.4 *Let $\phi : \mathcal{A} \to P(\mathcal{B})$ such that there exists on $F \in \phi(A)$ satisfying a relation of the type*

$$A \subset F, A \supset F, A \cap F \neq \emptyset,$$

for any set $A \in \mathcal{A}$ is called a semifixed set of multivalued ϕ.
Moreover, a fixed set for ϕ is any set $A \in \mathcal{A}$ satisfying $A \in \phi(A)$.

PROPOSITION 2.1.5 *(F.S. De Blasi[22]) Let $\mathcal{A}$ be a nonempty compact convex subset of $P_{cp,cv}(\mathfrak{X})$, and let $\phi : \mathcal{A} \to P_{cp,cv}(P_{cp,cv}(\mathfrak{X}))$ be an upper semicontinuous multifunction with values $\phi(X) \subset \mathcal{A}$, for every $X \in \mathcal{A}$. Then there exists at least one set $A \in \mathcal{A}$ such that $A \in \phi(A)$.*

THEOREM 2.1.6 *(F.S. De Blasi[22]) Let $\mathcal{A}$ be a nonempty compact convex subset of $P_{cp,cv}(\mathfrak{X})$ and let $\phi : \mathcal{A} \to P_{cp,cv}(P_{cp,cv}(\mathfrak{X}))$ be an upper semicontinuous multivalued operators satisfying the following condition:*

(i) for every $X \in \mathcal{A}$, there exists a set $F \in \phi(X)$ such that $F \cap (\bigcup\limits_{Z \in \mathcal{A}} Z) \neq \emptyset$.

Then there exists at least one set $A \in \mathcal{A}$ such that:

$$A \cap F \neq \emptyset, \quad for \quad some \quad F \in \phi(A). \tag{2.1.1}$$

THEOREM 2.1.7 *(F.S. De Blasi[22]) Let $\mathcal{A}$ be a nonempty compact convex subset of $P_{cp,cv}(\mathfrak{X})$, $\phi : \mathcal{A} \to P_{cp,cv}(P_{cp,cv}(\mathfrak{X}))$ be an upper semicontinuous multivalued operators satisfying the following condition for every $X \in \mathcal{A}$, there exist $F \in \phi(X)$ and $Z \in \mathcal{A}$ such that: $F \cap (\bigcup\limits_{Z \in \mathcal{A}} Z) \neq \emptyset$ and $Z \subset F$ (resp. $Z \supset F$).*
Then there exists at least one set $A \in \mathcal{A}$ such that

$$A \subset F \ (resp.\ A \supset F) \ for\ some\ F \in \phi(A). \tag{2.1.2}$$

2.2 Semifixed sets for multivalued φ-contractions

In the first part of this paragraph we present the fixed point theorem Matkowski-Rus ([54], [81]) for φ-contractions. Our results extend some previous theorems given by F. S. De Blasi in [21], [22], M. Frigon [29], M. Frigon, A. Granas [30].

DEFINITION 2.2.1 *(I.A. Rus [77]) A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a comparison function if it satisfies:*

(i) φ is monotone increasing;

(ii) $(\varphi^n(t))_{n \in \mathbb{N}}$ converges to 0, for all $t > 0$.

DEFINITION 2.2.2 *(I.A. Rus [77]) A comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be:*

(i) a strict comparison function if it satisfies $t - \varphi(t) \to \infty$, for $t \to \infty$;

(ii) a strong function if it satisfies $\sum\limits_{n=1}^{\infty} \varphi^n(t) < \infty$, for all $t > 0$.

Remark 2.2.3 *If $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a comparison function then $\varphi(0) = 0$ and $\varphi(t) < t$, for all $t > 0$.*

Example 2.2.4 *(I.A. Rus [77]) The function $\varphi_1 : \mathbb{R}_+ \to \mathbb{R}_+$, $\varphi_1(t) = at$ (where $a \in]0, 1[$) and $\varphi_2 : \mathbb{R}_+ \to \mathbb{R}_+$, $\varphi_2(t) = \frac{t}{1+t}$ is a comparison function.*

Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a comparison function. We note

$$\varphi_\eta := sup\{t \in \mathbb{R}_+ | \ \ t - \varphi(t) \le \eta\}.$$

Definition 2.2.5 *(I.A. Rus [82]) Let (X, d) be a metric space. A mapping $A : X \to X$ is a φ-contraction if φ is a comparison function and*

$$d(A(x), A(y)) \le \varphi(d(x, y)), \text{ for all } x, y \in X.$$

We present the concept of generalized Ulam-Hyers stability.

Let (X, d) be a metric space, $A : X \to X$ be a operator, and we consider the following differential equation:

$$x = A(x), \ x \in X \tag{2.2.3}$$

and for $\varepsilon > 0$ inequality

$$d(y, A(y)) \le \varepsilon. \tag{2.2.4}$$

Definition 2.2.6 *(I.A. Rus [82]) The equation (2.2.3) is generalized Ulam-Hyers stability if there exists $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ increasing and continuous in 0 with $\psi(0) = 0$, such that: for each $\varepsilon > 0$ and for each solution $y^* \in X$ of (2.2.4) there exists a solution $x^* \in X$ of (2.2.3) such that:*

$$d(y^*, x^*) \le \psi(\varepsilon).$$

In the case that $\psi(t) := ct$, $c > 0$, for all $t \in \mathbb{R}_+$, the equation (2.2.3) is said to be Ulam-Hyers stability.

Theorem 2.2.7 *(J. Matkowski [54], I.A. Rus [81], I.C. Tişe [100])*
Let (X, d) be a complete metric space and $A : X \to X$ an φ-contraction. Then:

(i) $F_A = \{x_A^*\}$ and $A^n(x) \to x_A^*$ as $n \to \infty$, for all $x \in X$, i.e., A is a Picard operator.

(ii) $F_A = F_{A^n} = \{x_A^*\}$, for all $n \in \mathbb{N}^*$, i.e., A is Bessaga operator.

(iii) If φ is a strong comparison function, note $\alpha_n := \sum\limits_{k=0}^{n-1} \varphi^k(t)$ and $s(d(x, A(x))) := \sum\limits_{k \ge 0} \varphi^k(t)$ then $d(A^n(x), x_A^*) \le s(t) - \alpha_n$, for all $x \in X$ and $n \in \mathbb{N}^*$.

(iv) *If φ is a strict comparison function then*

$d(x, x_A^*) \leq \varphi_{d(x,A(x))}$, *for all* $x \in X$.

(v) $\sum\limits_{n\in\mathbb{N}} d(A^n(x), A^{n+1}(x)) \leq s(d(x, A(x))$, *for all* $x \in X$.

(vi) *We have* $\sum\limits_{n\in\mathbb{N}} d(A^n(x), x_A^*) \leq s(d(x, x_A^*))$, *for all* $x \in X$.

(vii) *If the function* $\psi(t) = t - \varphi(t)$ *satisfies the condition* $\psi(u_n) \to 0$, *for* $n \to \infty$ *then* $u_n \to 0$, *when* $n \to \infty$ *and if* $(x_n)_{n\in\mathbb{N}} \subset X$ *such that* $d(x_n, A(x_n)) \to 0$, $n \to \infty$ *then* $x_n \to x_A^* \in F_A$, *for* $n \to \infty$, *i.e., the fixed point problem is well posed.*

(viii) *Let* $(x_n) \subseteq X$ *such that* $(d(x_{n+1}, A(x_n)))_{n\in\mathbb{N}}$ *is converges to 0. Then exists* $x \in X$ *such that* $d(x_n, A^n(x)) \to 0$, *for* $n \to \infty$ *(i.e. the operator* A *has the limit shadowing property).*

(ix) *If* $(x_n)_{n\in\mathbb{N}} \subset X$ *is bounded sequence then* $A^n(x_n) \to x_A^*$, *for* $n \to \infty$.

(x) *Let* φ *be a strict comparison function. If* $B : X \to X$ *is such that there exists* $\eta > 0$ *with* $d(A(x), B(x)) \leq \eta$ *for all* $x \in X$. *Then* $x_B^* \in F_B$ *imply* $d(x_A^*, x_B^*) \leq \varphi_\eta$.

(xi) *If* φ *is a strict comparison function,* $A_n : X \to X$, $A_n \overset{unif}{\to} A$, $n \to \infty$. *Let* $x_n \in F_{A_n}$, $n \in \mathbb{N}$ *and* $\{x_A^*\} = F_A$. *Then* $x_n \to x_A^*$ *as* $n \to \infty$.

(xii) *If* $(X, ||\cdot||)$ *is a Banach space, then* 1_X, $d(x, y) = ||x - y||$ *then* $1_X - A : X \to X$ *is surjective.*

(xiii) *If* $\psi(t) = t - \varphi(t)$ *is strict increasing and surjective, then the fixed point equation*

$$x = A(x), \quad x \in X$$

is generalized Ulam-Hyers stability.

REMARK 2.2.8 *If choose* $\varphi(t) := at$ *(where* $a \in [0, 1)$*) then (iii) by last Theorem imply Theorem 1.1. by I.A. Rus [81]. More specifically, because*

$$\begin{aligned} d(A^n(x), x_A^*) &\leq \sum_{k\geq 0} \varphi^k(d(x, A(x))) - \sum_{k=0}^{n-1} \varphi^k(d(x, A(x))) \\ &= \sum_{k\geq 0} a^k d(x, A(x)) - \sum_{k=0}^{n-1} a^k d(x, A(x)) \\ &= d(x, A(x))\frac{1}{1-a} - d(x, A(x))\frac{a^n - 1}{a - 1} = \frac{a^n}{1-a} d(x, A(x)). \end{aligned}$$

imply $d(A^n(x), x_A^*) \leq \frac{a^n}{1-a} d(x, A(x))$, *for all* $x \in X$ *and* $n \in \mathbb{N}^*$.

REMARK 2.2.9 *The results of this section extend and generalize some theorems from works by: A. Petrusel, I.A. Rus [72], A. Petruşel, A. Sîntămărian [73], I.A. Rus, S. Mureşan [84], I.A. Rus, A. Petruşel and A. Sîntămărian [85], J. Saint-Raymond [88].*

DEFINITION 2.2.10 *The metric space (X, d) is precompact (totally bounded) if and only if for all $\varepsilon > 0$ there exists a finite cover $(F_i)_{i\in\{1,\ldots,n\}}$ in X such that $A \subseteq F_i$ we have $diam(A) < \varepsilon$. Note F depend on ε.*

The purpose is to presents existence for the solution of a semifixed set theorem for set φ-contraction.

DEFINITION 2.2.11 *(F.S. De Blasi[22]) A multivalued operator $\phi : \mathcal{A} \to P_{cp}(P_{cp}(\mathfrak{X}))$ is said strong compact if its range $\phi(\mathcal{A})$ is precompact in $P_{cp}(P_{cp}(\mathfrak{X}))$.*

DEFINITION 2.2.12 *(I.C. Tişe [96]) Let $\mathcal{A}$ be a subset of $P_{cp}(P_{cp}(\mathfrak{X}))$. Then $\phi : \mathcal{A} \to P_{cp}(P_{cp}(\mathfrak{X}))$ is said to be a set φ-contraction if $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a comparison function and*

$$\Delta(\phi(X), \phi(Y)) \leq \varphi(D(X, Y)), \text{ for all } X, Y \in \mathcal{A}.$$

The first main result is:

THEOREM 2.2.13 *(I.C. Tişe [96]) Let $\mathfrak{X}$ be a Banach space, $\mathcal{A}$ be a close subset of $P_{cp}(P_{cp}(\mathfrak{X}))$ and let $\phi : \mathcal{A} \to P_{cp}(P_{cp}(\mathfrak{X}))$ be a strong compact and upper semicontinuous multivalued, with values $\phi(X) \subset \mathcal{A}$ for every $X \in \mathcal{A}$.*
Suppose there exists a comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that satisfying the following condition:

$$\Delta(\phi(X), \phi(Y)) \leq \varphi(D(X, Y)) \text{ for every } X, Y \in \mathcal{A}. \tag{2.2.5}$$

Then there exists $A \in \mathcal{A}$ and $F \in \phi(A)$ such that:

$$A \cap F \neq \emptyset. \tag{2.2.6}$$

Another main result is:

THEOREM 2.2.14 *(I.C. Tişe [96]) Let $\mathfrak{X}$ be a Banach space, $\mathcal{A} \subset P_{cp}(\mathfrak{X})$ and $\phi : \mathcal{A} \to \mathcal{A}$ be a continuous map satisfying the following conditions:*

(i) $\phi(\mathcal{B})$ is precompact in $\mathcal{A}$ for every bounded set $\mathcal{B} \subset \mathcal{A}$;

(ii) there exists $M > 0$ such that $diam(\phi(X)) \leq M$, for every $X \in \mathcal{A}$.

(iii) there exists a comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that the function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, $\psi(t) = t - \varphi(t)$ is strictly increasing, onto and for which the following assertion is satisfied:

$$D(\phi(X), \phi(Y)) \leq \varphi(D(X, Y)) \text{ for all } X, Y \in \mathcal{A}. \tag{2.2.7}$$

Then there exists $A \in \mathcal{A}$ such that $A \cap \phi(A) \neq \emptyset$.

We will use the fixed point Theorem of J. Matkowski and I.A. Rus as it is present in Theorem 2.2.7 (i).

We will present now an application of the integral equation in the spaces of multivaled functions.

Let $\mathbf{B}_r := \{X \in P_{cp,cv}(\mathbb{R}^n) | diam(X) \leq r\}$, where $r > 0$.

The set $\mathbf{B}_r$ endowed with the Pompeiu-Hausdorff metric, is convex and complete.

Let $I = [a, b]$, fie $F : I \times I \times \mathbf{B}_r \to \mathbf{B}_{r/2}$, and a set $A \in \mathbf{B}_{r/2}$.

Consider the integral equation

$$X(t) = A + \int_a^b F(t, s, X(s))ds. \tag{2.2.8}$$

By a solution of equation (2.2.8) we understand a continuous function $X : I \to \mathbf{B}_r$, which satisfies (2.2.8) for every $t \in I$.

THEOREM 2.2.15 *(I.C. Tişe [96]) Let $F : I \times I \times \mathbf{B}_r \to \mathbf{B}_{r/2}$ be continuous and suppose there exist a comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ and a function $p : I \times I \to \mathbb{R}_+$ such that:*

$$H(F(t, s, X), F(t, s, Y)) \leq p(t, s)\varphi(H(X, Y))$$

for every $t, s \in I$, $X, Y \in \mathbf{B}_r$, where $\max\limits_{t \in I} \int_a^b p(t, s) \leq 1$.
Then, for each $A \in \mathbf{B}_{r/2}$ the integral equation (2.2.8) has a unique solution $X(\cdot, A) : I \to \mathbf{B}_r$ with depends continuously on A.

DEFINITION 2.2.16 *(I.C. Tişe [100]) Let $F : [a, b] \times [a, b] \times P_{cp,cv}(\mathbb{R}^n) \to P_{cp,cv}(\mathbb{R}^n)$ and $A \in P_{cp,cv}(\mathbb{R}^n)$. Integral equation:*

(2.2.8) $\quad X(t) = A + \int_a^b F(t, s, X(s))ds, \quad t \in [a, b]$

is generalized Ulam-Hyers stability if there exists a function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ increasing and continuous in 0 with $\psi(0) = 0$ such that for each $\varepsilon > 0$ and for each solution $Y^* \in C([a, b], P_{cp,cv}(\mathbb{R}^n))$ of

$$H(Y(t), A + \int_a^b F(t, s, Y(s))ds) \leq \varepsilon, \quad t \in [a, b]$$

there exists a solution X^* of the equation (2.2.8) such that we have

$$||X^* - Y^*||_{C([a,b],P_{cp,cv}(\mathbb{R}^n))} \leq \psi(\varepsilon).$$

In the case that $\psi(t) = ct$, $c > 0$ the equation (2.2.8) is said to be Ulam-Hyers stability.

THEOREM 2.2.17 *(I.C. Tişe [100]) The assumptions Theorem 2.2.15, in addition assume the function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, $\psi(t) = t - \varphi(t)$ is strict increasing and surjective. Then integral equation (2.2.8) is generalized Ulam-Hyers stability.*

3

Integrals equations in spaces of multivalued functions

The aim of this chapter is to present some existence, uniqueness and data dependence results of solutions of integral and differential equations in multivalued function spaces.

There exist only a few works in the literature which study integral equations in multivalued function spaces, but there are many works regarding the problems associated to differential equations in multivalued function spaces. The study of some Cauchy problems associated to differential equations in multivalued function spaces can be regarded through the study of of some equivalent integral equation in such spaces. This is the way we will consider the Cauchy problem in this work. From this perspective the study of integral equations in the multivalued function space is essential.

In the second part of the chapter is presented the notion of the Cauchy problem for differential equations in spaces of multivalued functions and are obtained results of existence and uniqueness for this problem through the fixed point method. Our contributions are Theorem 3.2.4, Theorem 3.2.5 appeared in the paper I.C.Tişe [99].

In the first section, "Integrals equations in spaces of multivalued functions" are presented, with respect to integral equations in spaces of multivalued functions, existence theorems and uniqueness of solution of equations and continuous data dependence. Contributions of the author are: Theorem 3.1.3, Theorem 3.1.4, Theorem 3.1.7, Theorem 3.1.9, which are published in the paper I.C. Tişe [98].

In the second part of the chapter is presented the notion of the Cauchy problem for differential equations in spaces of multivalued functions and are obtained results of existence and uniqueness for this problem through the fixed point method. Our contributions are Theorem 3.2.4, Theorem 3.2.5

appeared in the paper I.C.Tişe [99]

In the last part of the chapter, "Functional-integral equations in spaces of multivalued functions", we present the case of some functional-integral equations and we prove results of existence and uniqueness of the solution. Our contributions are: Theorem 3.3.1, Theorem 3.3.2, Theorem 3.3.4 and are published in the paper I.C. Tişe [99].

Some Hyers-Ulam-Rassias stability results in the generalized sense for integral equations in spaces of multivalued functions are presented in Theorem 3.1.6, Theorem 3.1.11, Theorem 3.3.5, appeared in the paper I.C. Tişe [100].

The results of this section extend and generalize some theorems from works by: A. J. Brandao Lopes Pinto, F. S. De Blasi, F. Iervillino [12], A. Cernea [14], F. S. De Blasi [22], T. Gnana Bhaskar, J. Vasundhara Devi [33], C.J. Himmelberg, F.S. Van Vieck [39], V. Lakshmikantham, T. Gnana Bhaskar, J. Vasundhara Devi [48], [49], N Lungu [50], [51], N. Lungu, I.A. Rus [52], V. Lupulescu [53], D. O'Regan, A. Petruşel [64], A. Petruşel [67], [68], A. Petruşel, G. Petruşel, G. Moţ [71], R. Precup [74], I.A. Rus, A. Petrusel, G. Petrusel [87].

3.1 Integrals equations in spaces of multivalued functions

We consider the following integral equations in spaces of multivalued functions:

$$X(t) = \int_a^b K(t,s,X(s))ds + X_0(t), \quad t \in [a,b] \tag{3.1.1}$$

$$X(t) = \int_a^t K(t,s,X(s))ds + X_0(t), \quad t \in [a,b], \tag{3.1.2}$$

where $K : [a,b]\times[a,b]\times P_{cp,cv}(\mathbb{R}^n) \to P_{cp,cv}(\mathbb{R}^n)$ is a continuous operator, and $X_0 \in C([a,b], P_{cp,cv}(\mathbb{R}^n))$.

A solution of integral equations in spaces of multivalued functions (3.1.1) and (3.1.2) means a continuous function $X : [a,b] \to P_{cp,cv}(\mathbb{R}^n)$ which satisfies (3.1.1) respectively (3.1.2), for each $t \in [a,b]$.

The aim of this section is to present some notices use in the chapter.

LEMMA 3.1.1 *(A. Petruşel [66]) Let X be a Banach space. Then $H(A+C, B+D) \leq H(A,B) + H(C,D)$, for $A,B,C,D \in P(X)$.*

THEOREM 3.1.2 *(V. Lakshmikantham [49]) Let $F, G : [a,b] \to P_{cp,cv}(\mathbb{R}^n)$ Aumann integral operators. Then*

$$H(\int_a^b F(t)dt, \int_a^b G(t)dt) \leq \int_a^b H(F(t),G(t))dt.$$

We consider on $C([a,b], P_{cp,cv}(\mathbb{R}^n))$, the metrics:

$$H_*^C(X,Y) := \max_{t\in[a,b]} H(X(t),Y(t)).$$

The pairs $(C([a,b], P_{cp,cv}(\mathbb{R}^n)), H_*^C)$ has the complet metric space.

On first result is an existence and uniqueness theorem for the solution of the integral equation (3.1.1).

THEOREM 3.1.3 *(I.C. Tişe [98])*

Let $K : [a,b] \times [a,b] \times P_{cp,cv}(\mathbb{R}^n) \to P_{cp,cv}(\mathbb{R}^n)$ be a multivalued operator. Suppose that:

(i) K is continuous on $[a,b] \times [a,b] \times P_{cp,cv}(\mathbb{R}^n)$ and $X_0 \in C([a,b], P_{cp,cv}(\mathbb{R}^n))$;

(ii) $K(t,s,\cdot)$ is Lipschitz, i.e. there exists $L_K \geq 0$ such that:

$$H(K(t,s,A), K(t,s,B)) \leq L_K H(A,B),$$

for all $A, B \in P_{cp,cv}(\mathbb{R}^n)$ and for all $t, s \in [a,b]$;

(iii) $L_K(b-a) < 1$.

Then the integral equation

$$X(t) = \int_a^b K(t,s,X(s))ds + X_0(t)$$

has a unique solution.

A data dependence result for the solution of integral equation (3.1.1) is:

THEOREM 3.1.4 *(I.C. Tişe [98]) Let $K_1, K_2 : [a,b] \times [a,b] \times P_{cp,cv}(\mathbb{R}^n) \to P_{cp,cv}(\mathbb{R}^n)$, be a continuous and $X_0, Y_0 \in C([a,b], P_{cp,cv}(\mathbb{R}^n))$. Consider the following equations:*

$$X(t) = \int_a^b K_1(t,s,X(s))ds + X_0(t), \tag{3.1.3}$$

$$Y(t) = \int_a^b K_2(t,s,Y(s))ds + Y_0(t). \tag{3.1.4}$$

Suppose:

(i) there exists $L_{K_1} \geq 0$ such that

$$H(K_1(t,s,A), K_1(t,s,B)) \leq L_{K_1} H(A,B),$$

for all $A, B \in P_{cp,cv}(\mathbb{R}^n)$, $t, s \in [a,b]$ with $L_{K_1}(b-a) < 1$ (denote by X^* the unique solution of the equation (3.1.3));

(ii) there exists $\eta_1, \eta_2 > 0$ such that:

(a) $H(K_1(t,s,U), K_2(t,s,U)) \leq \eta_1$, for all $(t,s,U) \in [a,b] \times [a,b] \times P_{cp,cv}(\mathbb{R}^n)$,

(b) $H(X_0(t), Y_0(t)) \leq \eta_2$, for all $t \in [a,b]$;

(iii) there exists $Y^* \in C([a,b], P_{cp,cv}(\mathbb{R}^n))$ a solution of the a equation (3.1.4).

Then

$$H_*^C(X^*, Y^*) \leq \frac{\eta_2 + \eta_1(b-a)}{1 - L_{K_1}(b-a)}.$$

An auxiliary result.

LEMMA 3.1.5 *(I.A.Rus [83])Let $h \in C([a,b], \mathbb{R}_+)$ and $\beta > 0$ with $\beta(b-a) < 1$. If $u \in C([a,b], \mathbb{R}_+)$ satisfies*

$$u(t) \leq h(t) + \beta \int_a^b u(s)ds, \text{ for all } t \in [a,b],$$

then

$$u(t) \leq h(t) + \beta(1 - \beta(b-a))^{-1} \int_a^b h(s)ds, \text{ for all } t \in [a,b].$$

A result for generalized Ulam-Hyers-Rassias stability for integral equation (3.1.1)is:

THEOREM 3.1.6 *(I.C. Tişe [100]) Consider equations (3.1.1).*
We suppose that:

(i) $K : [a,b] \times [a,b] \times P_{cp,cv}(\mathbb{R}^n) \to P_{cp,cv}(\mathbb{R}^n)$ is continuous multivalued operator and $X_0 \in C([a,b], P_{cp,cv}(\mathbb{R}^n))$;

(ii) $K(t,s,\cdot)$ is Lipschitz, i.e. there exists $L_K \geq 0$ such that:

$$H(K(t,s,A), K(t,s,B)) \leq L_K H(A,B),$$

for all $A, B \in P_{cp,cv}(\mathbb{R}^n)$ and for all $t,s \in [a,b]$;

(iii) $L_K(b-a) < 1$;

(iv) $\varphi \in C([a,b], (0,+\infty))$.

Then integral equation (3.1.1) has the generalize Ulam-Hyers-Rassias stability, i.e., if $X \in C([a,b], P_{cp,cv}(\mathbb{R}^n))$ have the property

$$H(X(t), \int_a^b K(t,s,X(s))ds) \leq \varphi(t), \text{ for all } t \in [a,b]$$

there exists $c_\varphi > 0$ *such that*

$$H(X(t), X^*(t)) \leq c_\varphi \cdot \varphi(t), \text{ for all } t \in [a, b]$$

(where X^* *denote a unique solution of a equation (3.1.1) obtained according to Theorem 3.1.3).*

We will prove now an existence result for the solution of the integral equation (3.1.2).

We consider on $C([a, b], P_{cp,cv}(\mathbb{R}^n))$ the metric of Bielecki type:

$$H_*^B(X, Y) := \max_{t\in[a,b]}[H(X(t), Y(t))e^{-\tau(t-a)}], \quad \text{with} \ \ \tau > 0 \text{ is arbitrary.}$$

The pair $(C([a, b], P_{cp,cv}(\mathbb{R}^n)), H_*^B)$ forms a complete metric space.

THEOREM 3.1.7 *(I.C. Tişe [98]) Consider an integral equation (3.1.2).Let* $K : [a, b]\times[a, b]\times P_{cp,cv}(\mathbb{R}^n) \to P_{cp,cv}(\mathbb{R}^n)$ *be a multivalued operator and* $X_0 \in C([a, b], P_{cp,cv}(\mathbb{R}^n))$*. Suppose that:*

(i) K *is continuous on* $[a, b] \times [a, b] \times P_{cp,cv}(\mathbb{R}^n)$*;*

(ii) $K(t, s, \cdot)$ *is Lipschitz, i.e. there exists* $L_K \geq 0$ *such that*

$$H(K(t, s, A), K(t, s, B)) \leq L_K H(A, B),$$

for all $A, B \in P_{cp,cv}(\mathbb{R}^n)$ *and* $t, s \in [a, b]$*.*

Then the integral equation (3.1.2),

$$X(t) = \int_a^t K(t, s, X(s))ds + X_0(t)$$

has a unique solution.

REMARK 3.1.8 *Such results is obtained and other techniques for Hammersteins type equation appears in the paper [94].*

A data dependence result is:

THEOREM 3.1.9 *(I.C. Tişe [98])*

Let $K_1, K_2 : [a, b] \times [a, b] \times P_{cp,cv}(\mathbb{R}^n) \to P_{cp,cv}(\mathbb{R}^n)$ *be continuous,* $X_0, Y_0 \in C([a, b], P_{cp,cv}(\mathbb{R}^n))$*.*
Consider the following integral equations:

$$X(t) = \int_a^t K_1(t, s, X(s))ds + X_0(t) \tag{3.1.5}$$

$$Y(t) = \int_a^t K_2(t, s, Y(s))ds + Y_0(t). \tag{3.1.6}$$

Suppose:

(i) $H(K_1(t,s,A), K_1(t,s,B)) \leq L_{K_1} H(A,B)$, *for all* $A, B \in P_{cp,cv}(\mathbb{R}^n)$ *and* $t, s \in [a,b]$, *where* $L_{K_1} \geq 0$ *(denote by* X^* *the unique solution of the equation (3.1.5));*

(ii) there exists $\eta_1, \eta_2 > 0$, *such that:*

(a) $H(K_1(t,s,U), K_2(t,s,U)) \leq \eta_1$, *for all* $(t,s,U) \in [a,b] \times [a,b] \times P_{cp,cv}(\mathbb{R}^n)$,

(b) $H(X_0(t), Y_0(t)) \leq \eta_2$, *for all* $t \in [a,b]$;

(iii) there exists Y^* *a solution of the equation (3.1.6).*

Then

$$H_*^B(X^*, Y^*) \leq \frac{\eta_2 + \eta_1(b-a)}{1 - \frac{L_{K_1}}{\tau}} \ \textit{(where } \tau > L_{K_1}\textit{)}.$$

An auxiliary result.

LEMMA 3.1.10 *(I.A.Rus [83]) Let* J *be an interval in* $\mathbb{R}$, $t_0 \in J$ *and* $h, k, u \in C(J, \mathbb{R}_+)$. *If*

$$u(t) \leq h(t) + \left| \int_{t_0}^t k(s)u(s)ds \right|, \ \textit{for all } t \in J,$$

then

$$u(t) \leq h(t) + \left| \int_{t_0}^t h(s)k(s)e^{\left|\int_s^t k(\sigma)d\sigma\right|}ds \right|, \ \textit{for all } t \in J.$$

A result of generalized Ulam-Hyers-Rassias stability for integral equation (3.1.2).

THEOREM 3.1.11 *(I.C. Tişe [100]) Consider the equation (3.1.2).*
Suppose:

(i) $K : [a,b] \times [a,b] \times P_{cp,cv}(\mathbb{R}^n) \to P_{cp,cv}(\mathbb{R}^n)$ *be continuous multivalued operator and* $X_0 \in C([a,b], P_{cp,cv}(\mathbb{R}^n))$;

(ii) $K(t,s,\cdot)$ *is Lipschitz, i.e. there exists* $L_K \geq 0$ *such that:*

$$H(K(t,s,A), K(t,s,B)) \leq L_K H(A,B),$$

for all $A, B \in P_{cp,cv}(\mathbb{R}^n)$ *and for all* $t, s \in [a,b]$;

(iii) there exists $\varphi \in C([a,b], (0,+\infty))$ *and* $\eta_\varphi > 0$ *such that* $\int_a^t \varphi(s)ds \leq \eta_\varphi \cdot \varphi(t)$ *for all* $t \in [a,b]$.

Then the integral equation (3.1.2) has the generalized Ulam-Hyers-Rassias stability, i.e., if $X \in C([a,b], P_{cp,cv}(\mathbb{R}^n))$ *has property*

$$H(X(t), \int_a^t K(t,s,X(s))ds) \leq \varphi(t), \ \textit{for all } t \in [a,b]$$

there exists $c_\varphi > 0$ such that

$$H(X(t), X^*(t)) \le c_\varphi \cdot \varphi(t), \text{ for all } t \in [a,b],$$

(denote by X^* the unique solution of the equation (3.1.2) which is obtained according to Theorem 3.1.7).

3.2 Cauchy problem for the differential equations in space of multivalued functions

In this paragraf we present an application of theorems the previous section the existence, uniqueness and approximating the Cauchy problem solution.

We consider the following Cauchy problem with respect to a differential equation in spaces of multivalued functions:

$$\begin{cases} D_H U = F(t,U), \ t \in J \\ \qquad\qquad U(t_0) = U^0 \end{cases} \tag{3.2.7}$$

where $U^0 \in P_{cp,cv}(\mathbb{R}^n), \ t_0 \ge 0, \ J = [t_0, t_0 + a], \ a > 0,$

$F \in C(J \times P_{cp,cv}(\mathbb{R}^n), P_{cp,cv}(\mathbb{R}^n))$ and D_H is the Hukuhara derivative of U.

Consider the following equations in spaces of multivalued functions:

$$U(t) = U^0 + \int_{t_0}^{t} D_H(U(s))ds, \ t \in J, \tag{3.2.8}$$

$$U(t) = U^0 + \int_{t_0}^{t} F(s, U(s))ds, \ t \in J. \tag{3.2.9}$$

DEFINITION 3.2.1 *(V. Lakshmikantham [49]) $U \in C^1(J, P_{cp,cv}(\mathbb{R}^n))$ is a solution of the problem (3.2.7)$\Longleftrightarrow$ U satisfies (3.2.7) for all $t \in J$.*

LEMMA 3.2.2 *(V. Lakshmikantham [49]) If $U \in C^1(J, P_{cp,cv}(\mathbb{R}^n))$, then* (3.2.7) $\Longleftrightarrow$ (3.2.8) $\Longleftrightarrow$ (3.2.9).

We consider on $C(J, P_{cp,cv}(\mathbb{R}^n))$ the metrics H_*^C and H_*^B defined by:

$$H_*^C(U,V) := \max_{t \in J} H(U(t), V(t)),$$
$$H_*^B(U,V) := \max_{t \in J}[H(U(t), V(t))e^{-\tau(t-t_0)}], \ \tau > 0.$$

The pairs $(C(J, P_{cp,cv}(\mathbb{R}^n)), H_*^C)$ and $(C(J, P_{cp,cv}(\mathbb{R}^n)), H_*^B)$ has the complete metric spaces and the metrics H_*^C and H_*^B are equivalent.

Our first result is a global existence theorem for a Cauchy problem associated to a set differential equation.

THEOREM 3.2.3 *(I.C. Tişe [99]) Consider the problem (3.2.7) where* $F : J \times P_{cp,cv}(\mathbb{R}^n) \longrightarrow P_{cp,cv}(\mathbb{R}^n)$ *is a continuous operator and* $U_0 \in P_{cp,cv}(\mathbb{R}^n)$.

Suppose that: $F(t, \cdot)$ *is Lipschitz, i.e. there exists* $L \geq 0$*, such that:*

$H(F(t, U), F(t, V)) \leq LH(U, V)$ *for all* $U, V \in P_{cp,cv}(\mathbb{R}^n)$ *and* $t \in J$.

Then the problem (3.2.7) has a unique solution U^* *and* $U^*(t) = \lim\limits_{n\to\infty} U_n(t)$ *for each* $t \in J$*, where* $(U_n)_{n\in\mathbb{N}} \in C(J, P_{cp,cv}(\mathbb{R}^n))$ *is recurrently defined by the relation:*

$$\begin{cases} U_{n+1}(t) = U^0 + \int_{t_0}^t F(s, U_n(s))ds, \; n \in \mathbb{N} \\ \qquad\qquad\qquad\qquad U^0 \in P_{cp,cv}(\mathbb{R}^n). \end{cases} \tag{3.2.10}$$

The main result is a local existence and uniqueness theorem for a Cauchy problem associated to a differential equation in spaces of multivalued functions is next.

THEOREM 3.2.4 *(I.C. Tişe [99]) Consider the equation* $D_H U = F(t, U)$ *and* $\Omega \subset \mathbb{R} \times P_{cp,cv}(\mathbb{R}^n)$ *be an open set. Let* $F : \Omega \subset \mathbb{R} \times P_{cp,cv}(\mathbb{R}^n) \to P_{cp,cv}(\mathbb{R}^n)$ *be continuous. suppose that, for each t,the operator* $F(t, \cdot)$ *is L-Lipschitz with constant* $L > 0$.

Then for all $(t_0, U^0) \in \Omega$ *there exists a unique solution for the Cauchy problem (3.2.7), solution* $U^* : [t_0, t_0 + h] \to P_{cp,cv}(\mathbb{R}^n)$ *where* $h := min\{a, \frac{b}{M}\}$*, and* $a, b > 0$ *and* $M > 0$ *such that* $\overline{\Omega}_{a,b} := [t_0, t_0 + a] \times \overline{B}(U^0, b) \subset \Omega$ *and* $||F(t, U)||_H \leq M$*, for all* $(t, U) \in \overline{\Omega}_{a,b}$.

By the Characterization Theorem for the weakly Picard operator we have:

THEOREM 3.2.5 *(I.C. Tişe [99]) Consider the equation*

$$D_H U = F(t, U), \quad t \in [a, b] \tag{3.2.11}$$

where $F : [a, b] \times P_{cp,cv}(\mathbb{R}^n) \to P_{cp,cv}(\mathbb{R}^n)$ *is a continuous operator. Suppose that* $F(t, .)$ *is L-Lipschitz for each* $t \in [a, b]$.

Then:

(i) the operator $G : C([a, b], P_{cp,cv}(\mathbb{R}^n)) \to C([a, b], P_{cp,cv}(\mathbb{R}^n))$ *given by*

$$GU(t) = U(a) + \int_a^t F(s, U(s))ds$$

is a weakly Picard operator;

(ii) the solution set S of the equation (3.2.11) is infinite.

REMARK 3.2.6 *The results generalize the results of F.S. De Blasi [18].*

3.3 Functional-integral equations in spaces of multivalued functions

Let E be a Banach space and consider the following operators:
$Q : C([a,b], P_{cp,cv}(E)) \to C([a,b], P_{cp,cv}(E))$,
$G \in C([a,b] \times P_{cp,cv}(E)^3, P_{cp,cv}(E))$ and
$K \in C([a,b] \times [a,b] \times P_{cp,cv}(E), P_{cp,cv}(E))$.
We will study the following functional-integral equation:

$$X(t) = G(t, Q(X)(t), X(t), X(a)) + \int_a^t K(t,s,X(s))ds, t \in [a,b]. \tag{3.3.12}$$

By a solution of the above equation, we understand a function $X \in C([a,b], P_{cp,cv}(E))$ satisfying the relation (3.3.12), for each $t \in [a,b]$.

For our next considerations, we consider the Banach space $C([a,b], P_{cp,cv}(E))$ with the norm H_*^B.

With respect to the equation (3.3.12) we suppose that:

(i) there exists $L > 0$ such that

$$H(Q(X)(t), Q(Y)(t)) \leq LH(X(t), Y(t)),$$

for all $X, Y \in C([a,b], P_{cp,cv}(E))$, $t \in [a,b]$;

(ii) there exists $L_1 > 0, L_2 > 0$ such that
$H(G(t, U_1, V_1, W), G(t, U_2, V_2, W)) \leq L_1 H(U_1, U_2) + L_2 H(V_1, V_2)$,
for all $t \in [a,b], U_i, V_i, W \in P_{cp,cv}(E), i \in \{1,2\}$;

(iii) $L_1 L + L_2 < 1$;

(iv) there exists $L_3 > 0$ such that

$$H(K(t,s,U), K(t,s,V)) \leq L_3 H(U,V),$$

for all $t, s \in [a, b]$ and $U, V \in P_{cp,cv}(E)$;

(v) $G(a, Q(X)(a), X(a), X(a)) = X(a)$, for all $X \in C([a, b], P_{cp,cv}(E))$.

Using again the Characterization Theorem, we have the following result.

THEOREM 3.3.1 *(I.C. Tişe [99]) Consider the equation (3.3.12) under the conditions (i)-(v). If $S \subset C([a, b], P_{cp,cv}(E))$ is the solution set of this equation, then $card(S) = card(P_{cp,cv}(E))$ and hence the solution set S of the equation (3.3.12) is infinite.*

A data dependence theorem for equation (3.3.12).

THEOREM 3.3.2 *(I.C. Tişe [99]) Consider the equations*

$$X(t) = G_1(t, Q_1(X)(t), X(t), X(a)) + \int_a^t K_1(t, s, X(s))ds, t \in [a, b], \tag{3.3.13}$$

$$X(t) = G_2(t, Q_2(X)(t), X(t), X(a)) + \int_a^t K_2(t, s, X(s))ds, t \in [a, b] \tag{3.3.14}$$

where the operators $G_1, G_2, Q_1, Q_2, K_1, K_2$ satisfy the conditions (i)-(v).

Let S_1 be the solution set of the equation (3.3.13) and S_2 be the solutions set of the equation (3.3.14). We suppose that there exist $\eta_1, \eta_2, \eta_3 > 0$, such that:

(a) $H(G_1(t, U_1, U_2, U_3), G_2(t, U_1, U_2, U_3)) \leq \eta_1$ *for all* $t \in [a, b]$, $U_1, U_2, U_3 \in P_{cp,cv}(E)$,

(b) $H_*^C(Q_1(X), Q_2(X)) \leq \eta_2$, *for all* $X \in C([a, b], P_{cp,cv}(E))$

(c) $H(K_1(t, s, U), K_2(t, s, U)) \leq \eta_3$, *for all* $t, s \in [a, b]$, $U \in P_{cp,cv}(E)$.

Then

$$H_*^B(S_1, S_2) \leq [\eta_1 + \eta_2 L_1 + (b - a)\eta_3] \cdot max\{c_1, c_2\}$$

where $c_i := \frac{1}{1 - L_{A_i}}$ *with* $L_{A_i} == L_1^i L^i + L_2^i + \frac{L_3^i}{\tau}$, $i \in \{1, 2\}$.

REMARK 3.3.3 *The above results extend the results of the unequivocal case given by I.A. Rus [76].*

For the final part of this section, we will move our attention to a functional-integral Cauchy problem arising in biomathematics.

Let

$$\begin{cases} X(t) = \int_{t-\tau}^{t} F(s, X(s))ds, \ t \in [0,T] \\ \qquad\qquad X(t) = \varphi(t), \ t \in [-\tau, 0] \end{cases} \tag{3.3.15}$$

where $F : [-\tau, T] \times P_{cp,cv}(\mathbb{R}_+) \to P_{cp,cv}(\mathbb{R}_+)$, $\varphi : [-\tau, 0] \to P_{cp,cv}(\mathbb{R}_+)$ are continuous operators. We also suppose that $\varphi(0) = \int_{-\tau}^{0} F(s, \varphi(s))ds$.

As a conclusion, we have the following result:

THEOREM 3.3.4 *(I.C. Tişe [99]) Consider the Cauchy problem (3.3.15).*
Suppose that:

(i) $F : [-\tau, T] \times P_{cp,cv}(\mathbb{R}_+) \to P_{cp,cv}(\mathbb{R}_+)$, $\varphi : [-\tau, 0] \to P_{cp,cv}(\mathbb{R}_+)$ *are continuous;*

(ii) $\varphi(0) = \int_{-\tau}^{0} F(s, \varphi(s))ds$*;*

(iii) there exists $k \in L^1[-\tau, T]$ *such that* $H(F(s, A), F(s, B) \leq k(s)H(A, B)$*, for all* $A, B \in P_{cp,cv}(\mathbb{R}_+)$ *and* $s \in [-\tau, T]$.

Then the problem (3.3.15) has a unique solution.

We present the result of generalized Ulam-Hyers-Rassias stability.

THEOREM 3.3.5 *(I.C. Tişe [100]) Let be the equation*

$$X(t) = \int_{t-\tau}^{t} F(s, \varphi(s))ds, \ \textit{where } \tau > 1, \quad t, s \in [-\tau, T]. \tag{3.3.16}$$

Suppose that:

(i) $F : [-\tau, T] \times P_{cp,cv}(\mathbb{R}_+) \to P_{cp,cv}(\mathbb{R}_+)$*, are continuous;*

(ii) there exists $k \in L^1[-\tau, T]$ *such that* $H(F(s, A), F(s, B) \leq k(s)H(A, B)$*, for all* $A, B \in P_{cp,cv}(\mathbb{R}_+)$ *and* $s \in [-\tau, T]$*;*

(iii) $\varphi \in C((-\tau, T), P_{cp,cv}(\mathbb{R}_+))$*;*

(iv) there exists $\lambda_\varphi > 0$ *such that:* $\int_{t-\tau}^{t} \varphi(s)ds \leq \lambda_\varphi \cdot \varphi(t)$.

Then the integral equation (3.3.16) is a generalized Ulam-Hyers-Rassias stability for φ*, i.e., there exists* $c_{F,\varphi} > 0$ *such that for each solution* $Y \in C^1([-\tau, T], P_{cp,cv}(\mathbb{R}_+))$ *of a inequality*

$$H(Y(t), \int_{t-\tau}^{t} F(s, Y(s))ds) \leq \varphi(t), \ \textit{for all } t \in [-\tau, T]$$

with have the properties $Y(0) = \int_{-\tau}^{0} F(s, Y(s))ds$, *there exists a solution* $X^* \in C^1([-\tau, T], P_{cp,cv}(\mathbb{R}_+))$ *of equation (3.3.16) such that:*

$$H(Y(t), X^*(t)) \leq c_{F,\varphi} \cdot \varphi(t), \text{ for all } t \in [0, T].$$

REMARK 3.3.6 *The above results extend the results of the unequivocal case given by: R. Precup [74], J. Vasundhara Devi, A.S Vatsala [102].*

4

Qualitative properties of solutions of differential equations in space of multivalued functions

The aim of this chapter is to present some properties for a set solutions of differential equations in space of multivalued functions.

The first paragraph of this chapter is dedicated to Gronwall type Lemmas and comparison theorems. In following section we discuss data dependence of the solution of differential equations in spaces of multivalued functions.

The contributions from this chapter are Theorem 4.1.3, Theorem 4.1.7, Theorem 4.1.9, Theorem 4.1.11, Theorem 4.2.1, results contained in the papers I.C Tişe [95], [97].

The results of this chapter extend and generalize some theorems from works by: J.P. Aubin, H. Frankovska [5], A. J. Brandao Lopes Pinto, F. S. De Blasi, F. Iervillino [12], C. Chifu, G. Petruşel [15], A. Filippov [25], G. N. Galanis, T. G. Bhaskar, V. Lakshmikantham [32], M. Hukuhara [42], N.D. Phua, L.T. Quang, T.T. Tung [62], D. O'Regan, R. Precup [63], A. Petruşel [65], I.A. Rus [76], [79], [80], M.A. Şerban [92], N.N. Tu, T.T. Tung [101].

4.1 The properties obtained by Gronwall type Lemmas

We consider the following Cauchy problem with respect to a differential equation in space of multivalued functions:

$$\begin{cases} D_H U = F(t,U), \ t \in J \\ U(t_0) = U^0 \end{cases} \tag{4.1.1}$$

where $U^0 \in P_{cp,cv}(\mathbb{R}^n)$, $t_0 \geq 0$, $J = [t_0, t_0 + a]$, $a > 0$, D_H is the Hukuhara derivative of U and $F : J \times P_{cp,cv}(\mathbb{R}^n) \to P_{cp,cv}(\mathbb{R}^n)$ continuous multivalued operator.

$U : J \to P_{cp,cv}(\mathbb{R}^n)$ is a solution of the problem (4.1.1), it is equivalent whit U satisfies (4.1.1) for all $t \in J$.

For a Cauchy problem (4.1.1) associate to a integral equation:

$$U(t) = U^0 + \int_{t_0}^{t} F(s, U(s))ds, \ \ t \in J \tag{4.1.2}$$

where integral is Hukuhara (see [42]).

LEMMA 4.1.1 *(V. Lakshmikantham [49]) If* $U : J \to P_{cp,cv}(\mathbb{R}^n)$ *is differentiable continuous, then we have:*

$$U(t) = U^0 + \int_{t_o}^{t} D_H U(s)ds, \ \ t \in [a,b].$$

LEMMA 4.1.2 *(V.Lakshmikantham [49]) The problem (4.1.1) and equation (4.1.2) are equivalent.*

On $C(J, P_{cp,cv}(\mathbb{R}^n))$ consider the metric H_*^B defined by:

$$H_*^B(U,V) := \max_{t\in[t_0,t_0+a]}[H(U(t),V(t))e^{-\tau(t-t_0)}], \ \tau > 0.$$

The pair $(C(J, P_{cp,cv}(\mathbb{R}^n)), H_*^B)$ has the complete metric spaces.

Existence Theorem for a Cauchy problem.

THEOREM 4.1.3 *(I.C. Tişe [97]) Consider the problem (4.1.1) and*
$F : J \times P_{cp,cv}(\mathbb{R}^n) \longrightarrow P_{cp,cv}(\mathbb{R}^n)$ *is a continuous operator multivalued.*
Supose that:

(i) F *is continuous on* $J \times P_{cp,cv}(\mathbb{R}^n)$ *and* $U^0 \in P_{cp,cv}(\mathbb{R}^n)$;

(ii) $F(t,\cdot)$ *is Lipschitz, i.e. there exists* $L \geq 0$ *such that*

$$H(F(t,U),F(t,V)) \leq LH(U,V)$$

for all $U,V \in P_{cp,cv}(\mathbb{R}^n)$ *and* $t \in J$.

Then the problem (4.1.1) has a unique solution U^* *and* $U^*(t) = \lim_{n\to\infty} U_n(t)$, *where* $U_n \in C(J,P_{cp,cv}(\mathbb{R}^n))$ *is recurrently defined by the relation:*

$$\begin{cases} U_{n+1}(t) = U^0 + \int_{t_0}^{t} F(s,U_n(s))ds, \ n \in \mathbb{N} \\ \qquad\qquad\qquad U^0 \in P_{cp,cv}(\mathbb{R}^n). \end{cases}$$

Let us consider the following integral equations:

$$U(t) = U^0 + \int_{t_0}^{t} D_H(U(s))ds, \ t \in J \tag{4.1.3}$$

$$U(t) = U^0 + \int_{t_0}^{t} F(s,U(s))ds, \ t \in J. \tag{4.1.4}$$

LEMMA 4.1.4 *(V.Lakshmikantham [49]) If* $U \in C^1(J,P_{cp,cv}(\mathbb{R}^n))$, *then* (4.1.1) $\Longleftrightarrow$ (4.1.3) $\Longleftrightarrow$ (4.1.4).

We consider on $P_{cp,cv}(\mathbb{R}^n)$ the order relation " $\leq_m$ " defined by:

$$U,V \in P_{cp,cv}(\mathbb{R}^n) : U \leq_m V \Longleftrightarrow U \subseteq V.$$

DEFINITION 4.1.5 *The operator* $F(t,\cdot) : J \times P_{cp,cv}(\mathbb{R}^n) \to P_{cp,cv}(\mathbb{R}^n)$, *is called increasing if:*

$$A,B \in P_{cp,cv}(\mathbb{R}^n), \ A \leq_m B \Rightarrow F(t,A) \leq_m F(t,B), \ \textit{for all } t \in J.$$

Define on $C(J,P_{cp,cv}(\mathbb{R}^n))$ an order relation " $\leq$ " defined by:

$$X,Y \in C(J,P_{cp,cv}(\mathbb{R}^n)), \ X \leq Y \Leftrightarrow X(t) \leq_m Y(t), \ \text{for all } t \in J.$$

The space $(C(J,P_{cp,cv}(\mathbb{R}^n)),H_*^B,\leq)$ being an order an complete metric space is also an ordered L-space.

Let $(X,d,\leq)$ be an order metric space and $T : X \to X$ an operator.

We note:

$$(UF)_T := \{x \in X | Tx \leq x\} \text{ the upper fixed point set for T;}$$

$$(LF)_T := \{x \in X | Tx \geq x\} \text{ the lower fixed point set for T.}$$

In what follows we will present the Abstract Gronwall Lemma:

LEMMA 4.1.6 *(I. A. Rus [78]) Let* $(X, d, \leq)$ *be an ordered L-space and* $T : X \to X$ *an operator. We suppose that:*

(i) T *is Picard operator;*

(ii) T *is increasing.*

Then $(LF)_T \leq x_T^* \leq (UF)_T$*, where* x_T^* *is the unique fixed point of the operator* T.

We will apply this abstract lemma to the Cauchy problem (4.1.1).

THEOREM 4.1.7 *(I.C. Tişe [97]) Let Cauchy problem (4.1.1).*
Suppose that:

(i) $F(t, \cdot) : J \times P_{cp,cv}(\mathbb{R}^n) \to P_{cp,cv}(\mathbb{R}^n)$ *is L-Lipschitz for all* $t \in J$*;*

(ii) $F(t, \cdot) : J \times P_{cp,cv}(\mathbb{R}^n) \to P_{cp,cv}(\mathbb{R}^n)$ *is increasing monotone operator for all* $t \in J$;

(iii) F *is continuous on* $J \times P_{cp,cv}(\mathbb{R}^n)$ *and* $U^0 \in P_{cp,cv}(\mathbb{R}^n)$.

Then we have:

$$(LS)_{(1)} \leq U^* \leq (US)_{(1)}$$

where U^* *is the unique solution for problem (4.1.1) and* $(LS)_{(1)}$ *respectively* $(US)_{(1)}$ *represents the set of lower solution respectively the set of upper solution for the problem (4.1.1).*

In what follows an abstract comparison lemma will be presented

THEOREM 4.1.8 *(I. A. Rus [78]) Let* $(X, d, \leq)$ *be an ordered L-space and* $T_1, T_2 : X \to X$ *two operators. We suppose that:*

(i) T_1 *and* T_2 *are Picard operators;*

(ii) T_1 *is increasing;*

(iii) $T_1 \leq T_2$.

Then $x \leq T_1 x \Rightarrow x \leq x_{T_2}^*$.

We have the following theorem.

THEOREM 4.1.9 *(I.C. Tişe [97]) Let* $F, G : J \times P_{cp,cv}(\mathbb{R}^n) \to P_{cp,cv}(\mathbb{R}^n)$. *Consider the following tow Cauchy problem:*

$$\begin{cases} D_H U = F(t, U), \ t \in J \\ U(t_0) = U^0 \end{cases} \tag{4.1.5}$$

$$\begin{cases} D_H V = G(t, V), \ t \in J \\ V(t_0) = V^0 \end{cases} \tag{4.1.6}$$

where $U^0, V^0 \in P_{cp,cv}(\mathbb{R}^n)$, $t_0 \geq 0$, $J = [t_0, t_0 + a]$, $a > 0$.
Supose that:

(i) F *is continuous on* $J \times P_{cp,cv}(\mathbb{R}^n)$ *and* $F(t, \cdot)$ *is Lipschitz;*

(ii) G *is continuous on* $J \times P_{cp,cv}(\mathbb{R}^n)$, $V^0 \in P_{cp,cv}(\mathbb{R}^n)$ *and* $G(t, \cdot)$ *is Lipschitz;*

(iii) $F(t, \cdot)$ *is increasing for all* $t \in J$;

(iv) $F \subseteq G$.

Then $D_H U \leq_m F(t, U) \Longrightarrow U \leq V^*$ *where* V^* *is the unique solution for the problem (4.1.6).*

We recall the following abstract Gronwall lemma for the case of weakly Picard operators.

LEMMA 4.1.10 *(I. A. Rus [78]) Let* $(X, d, \leq)$ *be an ordered L-space and* $T : X \to X$ *an operator. We suppose that:*

(i) T *is weakly Picard operator;*

(ii) T *is increasing.*

Then

(a) $x \leq Tx \Rightarrow x \leq T^\infty x$;

(b) $x \geq Tx \Rightarrow x \geq T^\infty x$;

(b) if there exists $x \in (LF)_T$ *and* $y \in (UF)_T$ *such that* $x \leq y$ *then*
$x \leq T(x) \leq ... \leq T^n(x) \leq ... \leq T^\infty(x) \leq T^\infty(y) \leq ... \leq T^n(y) \leq ... \leq T(y) \leq y$.

We will apply the above lemma to the Cauchy problem (4.1.1).

THEOREM 4.1.11 *(I.C. Tişe [97]) Let us consider the equation*

$$D_H U = F(t, U), \quad t \in J \tag{4.1.7}$$

We suppose that:

(i) $F(t, .) : J \times P_{cp,cv}(\mathbb{R}^n) \to P_{cp,cv}(\mathbb{R}^n)$ *is Lipschitz, for all* $t \in J$;

(ii) $F(t, .) : J \times P_{cp,cv}(\mathbb{R}^n) \to P_{cp,cv}(\mathbb{R}^n)$ *is increasing, for all* $t \in J$;

(iii) F *is continuous on* $J \times P_{cp,cv}(\mathbb{R}^n)$.

Then

(i) if V *is a lower solution of the equation (4.1.7)* $\Rightarrow V \leq U_V^*$;

(ii) if V *is a upper solution of the equation (4.1.7)* $\Rightarrow V \geq U_V^*$,

where U_V^* *is the uniform limited of the subset and recurrently defined by the relation*

$$\begin{cases} U_{n+1}(t) = V(t_0) + \int_{t_0}^t F(s, U_n(s))ds, \ t \in J \\ U(t_0) = V; \end{cases}$$

(iii) if $U_1, U_2 \in C^1(J, P_{cp,cv}(\mathbb{R}^n))$ *are two spaces for (4.1.7) such that* $U_1(t_0) \leq_m U_2(t_0)$ *then* $U_1 \leq U_2$.

4.2 Data dependence of the solutions for differential equations in spaces of multivalued functions

We consider the following Cauchy problems:

$$\begin{cases} D_H U = F(t, U) \\ U(a) = U^0 \end{cases} \tag{4.2.8}$$

$$\begin{cases} D_H U = G(t, U) \\ U(a) = V^0 \end{cases} \tag{4.2.9}$$

where $F : [a, b] \times P_{cp,cv}(\mathbb{R}^n) \to P_{cp,cv}(\mathbb{R}^n)$ is a continuous multivalued operator, $U^0, V^0 \in P_{cp,cv}(\mathbb{R}^n)$. We consider on $C([a, b], P_{cp,cv}(\mathbb{R}^n))$ the metric H_*^C defined by:

$$H_*^C(U, V) := \max_{t \in [a,b]} H(U(t), V(t)).$$

The pairs $(C([a,b], P_{cp,cv}(\mathbb{R}^n)), H_*^C)$ has the Banach space.

A data dependence result is:

THEOREM 4.2.1 *(I.C. Tişe [95]) Let* $F, G : [a,b] \times P_{cp,cv}(\mathbb{R}^n) \to P_{cp,cv}(\mathbb{R}^n)$*, continuous. Consider the following problems (4.2.8) and (4.2.9). Suppose:*

(i) there exists $k_1 > 0$ *such that* $H(F(t,U), F(t,V)) \leq k_1 H(U,V)$*, for all* $U, V \in P_{cp,cv}(\mathbb{R}^n)$*, for all* $t \in [a,b]$*. Denote by* U_F^* *the unique solution of the problem* (4.2.8)*;*

(ii) there exists $\eta_i > 0$*,* $i = 1, 2$ *such that:*
$H(F(t,U), G(t,U)) \leq \eta_1$*, for all* $(t,U) \in [a,b] \times P_{cp,cv}(\mathbb{R}^n)$
and $H(U^0, V^0) \leq \eta_2$*;*

(iii) there exists U_G^* *a solution of the problem* (4.2.9).

Then

$$H_*^C(U_F^*, U_G^*) \leq \frac{\eta_2 + \eta_1(b-a)}{1 - k_1(b-a)}.$$

REMARK 4.2.2 *Similar results in case of impulsive differential equations in spaces of multivalued functions appear in the article of F.A. McRae, J Vasundhara Devi [56].*

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