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Contributions to the Theory of Vector and Multifunction Equilibrium Problems

Summary of the Doctoral Thesis

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Contents

Introduction

The equilibrium theory, which is a part of the nonlinear analysis, provides a general, unified and natural framework for the study of a large variety of problems, such as: optimization problems, variational inequalities problems, saddle point problems, complementarity problems, Nash equilibria problems and fixed point problems. These problems often occur in economics, finance, network analysis, mechanics, physics, etc.

The first equilibrium problem studied in the literature, was the scalar equilibrium problem which consists in:

$$(EP) \qquad \text{find } \bar{a} \in A \text{ such that } \varphi(\bar{a}, b) \geq 0 \text{ for all } b \in B,$$

where A and B are two nonempty sets, and $\varphi : A \times B \to \mathbb{R}$ is a given function. The reference paper for the study of (EP) is considered to be the paper of E. Blum and W. Oettli [20]. They supposed that $B = A$ and $\varphi(a, a) = 0$ for all $a \in A$. Under the assumption $B = A$, A. N. Iusem and W. Sosa [77] presented six particular cases of (EP). They underlined the fact that the set of solutions of (EP) equals the set of solutions of convex minimization problems, of fixed point problems, of complementarity problems, of Nash equilibria problems in noncooperative games, of variational inequality problems, and of vector minimization problems, respectively.

In the last years, there has been an increasing interest in the study of existence results of solutions of (EP) and its particular cases, see for instance: M. Bianchi and S. Schaible [17], G. Bigi, M. Castellani and G. Kassay [19], D. Inoan and J. Kolumbán [78], A. N. Iusem and W. Sosa [77], G. Kassay and J. Kolumbán [82], and J. C. Yao [115].

The extension of the scalar equilibrium problem to vector equilibrium problems can be achieved in different ways. Given a real topological linear space Z, a convex cone $C \subseteq Z$ with $\operatorname{int} C \neq \emptyset$ (where $\operatorname{int} C$ denotes the interior of C), two nonempty sets A and B, and a bifunction $\varphi : A \times B \to Z$, there can be formulated the following vector equilibrium problems:

$$(WVEP) \qquad \text{find } \bar{a} \in A \text{ such that } \varphi(\bar{a}, b) \notin -\operatorname{int} C \text{ for all } b \in B;$$

$$(VEP) \qquad \text{find } \bar{a} \in A \text{ such that } \varphi(\bar{a}, b) \notin -C \setminus \{0\} \text{ for all } b \in B;$$

$$(SVEP) \qquad \text{find } \bar{a} \in A \text{ such that } \varphi(\bar{a}, b) \notin C \text{ for all } b \in B.$$

Let us denote by S_1, S_2 and S_3 the set of solutions of the vector equilibrium problems $(WVEP)$, (VEP) and $(SVEP)$, respectively. Then the following inclusion holds:

$$S_3 \subseteq S_2 \subseteq S_1.$$

These problems were introduced in the literature by Q. H. Ansari, W. Oettli and D. Schläger [6], in a more general framework, M. Bianchi, N. Hadjisavvas and S. Schaible [17] and W. Oettli [99].

In the last decade a large number of papers have been devoted to the study of existence results regarding these vector equilibrium problems and their particular cases. Q. H Ansari [3], Q. H.

Ansari, W. Oettli and D. Schläger [6], Q. H. Ansari and J. C. Yao [9], M. Bianchi, G. Kasay and R. Pini [16], G.-Y. Chen and Q. M. Cheng [41], Y. P. Fang and N. J. Huang [51], X. H. Gong [58], I. V. Konnov and S. Schaible [85], W. Oettli [99] and T. Tanaka [108] presented existence results for solutions of the above-mentioned vector equilibrium problems and their particular cases using separation theorems in infinite dimensional spaces, the partition of the unity, a fixed point theorem due to E. Tarafdar [109], a Fan-Browder fixed point theorem due to S. Park [101], Ky Fan's lemma, a generalized dual equilibrium problem, Ekeland's principle, etc.

The present thesis aims to extend the existence results obtained by G. Kassay and J. Kolumbán [82] for the scalar equilibrium problem to vector and multifunction equilibrium problems and to present new existence results for vector equilibrium problems.

The thesis consists of six chapters.

The mathematical notions and auxiliary results necessary for the study of vector equilibrium problems and multifunction equilibrium problems are recalled in Chapter 1. Section 1.1 contains properties concerning cones, convex sets, separation theorems in infinite dimensional spaces and different generalizations of the upper semicontinuity from the scalar case. Then, in Section 1.2 weakened convexity notions for vector-valued functions and multifunctions and their characterizations are presented. Section 1.3 focuses on specific notions from Fenchel's duality theory.

Chapter 2 is devoted to the presentation of some sufficient conditions for the existence of solutions for the weak vector equilibrium problem $(WVEP)$, which most of the time is studied under the assumptions $B = A$ and $\varphi(a, a) \in C$ for all $a \in A$. In Section 2.1, using Eidelheit's separation theorem in infinite dimensional spaces, there are obtained under certain assumptions, existence results for $(WVEP)$. Based on the definition of C-subconvexlikeness of a vector-valued function and its characterization, a new convexity notion for vector-valued bifunctions is introduced. Working in the scalar setting, the main theorem permits to recover an earlier existence result of G. Kassay and J. Kolumbán [82] concerning the scalar equilibrium problem (EP). Dealing with the same setting, in Section 2.2 existence results for a generalized equilibrium problem with composed functions are stated. A new convexity notion of a vector-valued bifunction which takes values in a product space is introduced. The section ends with an existence result given for classical assumptions on the sets and functions involved.

The Chapter 3 is the largest chapter of the thesis. The strong vector equilibrium problem (VEP) is studied, and existence results of solutions and proper solutions of (VEP) are obtained. Section 3.1 presents sufficient conditions for the existence of solutions of (VEP), by using Eidelheit's separation theorem in infinite dimensional spaces, under the hypothesis of a cone with nonempty interior. To see which assumptions satisfy the hypothesis of the main result of this section, a new upper semicontinuity notion for vector-valued functions is defined, and it seems that it is equivalent to the one introduced by W. W. Breckner and G. Orbán [31]. Moreover, using scalarization techniques, an existence result which generalizes Theorem 3.2 of X. H. Gong [58] is obtained. This improvement consists in: two different sets, A and B, are considered, a weaker convexity assumption is needed, and a supremum condition is considered instead of $\varphi(a, a) \in C$ for all $a \in A$. In Section 3.2, with the aid of a generalized dual strong vector equilibrium problem and Ky Fan's lemma, there are obtained existence results of solutions of (VEP), considering $B = A$ and $\varphi(a, a) \in C$ for all $a \in A$. Some of the results are given under pseudomonotonicity assumptions and some are given without pseudomonotonicity assumptions. The results allow to recover earlier existence results due to Ky Fan [49] and W. Oettli [99]. Section 3.3 focuses on existence results of proper solutions of (VEP). For a cone C, which has an empty interior, the concepts of Henig dilating cone for C and family of Henig dilating cones for C are given, and new proper efficient solutions are defined. In this way, the problem that the interior of C may be empty is overcome. Thus, existence results for K-Henig weakly efficient solutions, K-Henig efficient solutions, Henig weakly efficient solutions, Henig efficient solutions, superefficient solutions and globally efficient solutions of (VEP) are presented. Under a certain hypothesis, it is shown (see Theorem 3.3.18) that a net of K_i-Henig weakly efficient solutions of (VEP), where $(K_i)_{i\in I}$ is a net of Henig dilating cones for C, admits a subnet converging to a solution of (VEP).

Chapter 4 is devoted to the following generalization of the scalar equilibrium problem (EP), when the scalar function is replaced by a multifunction:

$$(WWMEP) \qquad \text{find } \bar{a} \in A \text{ such that } \varphi(\bar{a}, b) \nsubseteq -\operatorname{int} C \text{ for all } b \in B,$$

where $\varphi : A \times B \to 2^Z$ is a multifunction. In Section 4.1 sufficient conditions for the existence of solutions of $(WWMEP)$ are established, by using Eidelheit's separation theorem in infinite dimensional spaces. Further, a new convexity notion for multifunctions of two variables is defined. Since the gap multifunctions help to analyze whether a point is a solution of $(WWMEP)$, in Section 4.2 a gap multifunction and a gap function are constructed. For the gap function Fenchel's duality theory is used.

The last two chapters contain applications of the equilibrium problems considered in the previous chapters. In Section 5.1, with the aid of a scalar equilibrium problem, an existence result for a weak vector optimization problem $(WVMP)$ is obtained. Further, by an example it is shown that the semicontinuity assumption from this result can not be weakened. Section 5.2 and Section 5.3 deal with weak cone saddle points and strong cone saddle points, respectively. Each strong cone saddle point is a weak cone saddle point, but the viceversa does not hold, as Example 5.3.2 shows. To have a better view on the relation between vector equilibrium problems and cone saddle point problems, two examples are provided which show that not every cone saddle point is a solution of the corresponding vector equilibrium problem. By using scalarization and perturbation techniques, existence results are presented (see Theorem 5.2.4 and Theorem 5.3.5).

In Chapter 6, there are obtained existence results for different kinds of Minty and Stampacchia type vector variational inequalities and multifunction variational inequalities, respectively. The results are given under convexity, v-hemicontinuity, monotonicity and pseudomonotonicity assumptions. Some of the existence results are new, while one of them (namely Theorem 6.2.7) slightly generalizes a result established by Y. P. Fang and N. J. Huang [51]. The improvement consists in dealing with a coercivity condition instead of the compactness assumption. Section 6.2 and 6.3 are answers to the open problem proposed by G.-Y. Chen and S. H. Hou [42] about existence results for strong vector variational inequalities.

The author's original contributions are the following:

Chapter 2: Theorem 2.1.1, Definition 2.1.2, Proposition 2.1.3, Corollary 2.1.4, Corollary 2.1.6, Theorem 2.2.3, Definition 2.2.5, Theorem 2.2.6, Corollary 2.2.7.

Chapter 3: Theorem 3.1.1, Definition 3.1.3, Proposition 3.1.4, Proposition 3.1.5, Corollary 3.1.6, Corollary 3.1.7, Theorem 3.1.9, Corollary 3.1.10, Proposition 3.2.3, Theorem 3.2.4, Corollary 3.2.5, Remark 3.2.6, Corollary 3.2.9, Corollary 3.2.10, Definition 3.3.7, Theorem 3.3.11, Corollary 3.3.12, Definition 3.3.14, Theorem 3.3.15, Theorem 3.3.16, Theorem 3.3.18, Example 3.3.20, Theorem 3.3.24, Corollary 3.3.25, Theorem 3.3.26, Theorem 3.3.27, Corollary 3.3.28.

Chapter 4: Theorem 4.1.1, Definition 4.1.2, Theorem 4.1.3, Theorem 4.2.2, Corollary 4.2.4, Corollary 4.2.5, Proposition 4.2.6, Theorem 4.2.8.

Chapter 5: Proposition 5.1.1, Example 5.1.2, Proposition 5.1.3, Example 5.1.5, Proposition 5.1.6, Example 5.2.3, Theorem 5.2.4, Example 5.3.2, Example 5.3.4, Theorem 5.3.5.

Chapter 6: Theorem 6.1.1, Corollary 6.1.2, Theorem 6.1.5, Example 6.2.2, Proposition 6.2.4, Theorem 6.2.5, Example 6.2.6, Theorem 6.2.7, Theorem 6.3.4, Corollary 6.3.5, Corollary 6.3.6, Corollary 6.3.7, Proposition 6.3.8, Theorem 6.3.9, Theorem 6.3.10, Theorem 6.4.1, Theorem 6.4.3.

These results are partly included in the following papers: G. Bigi, A. Capătă and G. Kassay [18], R. I. Boţ and A. E. Capătă [27], A. Capătă [34], [35], [36], [37], A. Capătă and G. Kassay [38], and A. Capătă, G. Kassay and B. Mosoni [39].

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Key words: weak vector equilibrium problems, strong vector equilibrium problems, weak multifunction equilibrium problems, gap multifunction, weak cone saddle point, strong cone saddle point, Minty weak vector variational inequality, Minty strong vector variational inequality, Stampacchia weak vector variational inequality, Stampacchia strong vector variational inequality, multifunction variational inequalities.

Chapter 1

Preliminary notions and results

This Chapter contains the mathematical tools that we need in the present doctoral thesis.

1.1 Convex sets and convex cones

Section 1.1 recalls notions which regard to cones, separation theorems of convex sets, and different generalizations of the upper semicontinuity of a scalar function. Among them, there is the base of a cone, Eidelheit's separation theorem, Tukey's separation theorem, Ky Fan's lemma, the cone upper semicontinuity of a vector-valued function, and the cone upper semicontinuity of a multifunction.

1.2 Functions satisfying certain weakened convexity assumptions

Section 1.2 focuses on the definition of different weakened convexity assumptions, and their characterizations, for vector-valued functions and multifunctions. So, there are reminded the cone convexlikeness, the cone subconvexlikeness of a vector-valued function, and the cone convexlikeness, the cone subconvexlikeness, the cone λ-convexity of a multifunction, respectively.

1.3 Fenchel's duality theory

In the final part of this chapter notions regarding to Fenchel's duality theory are recalled. So, Section 1.3 contains the notion of Fenchel-Moreau conjugate and the infimal convolution of functions. Moreover, a recent theorem, concerning the existence of strong duality between a problem and its dual problem, is recalled.

Chapter 2

Existence results for weak vector equilibrium problems

Throughout this chapter we suppose that A is a nonempty subset of a topological space E, B is a nonempty set, Z is a real topological linear space, and $C \subseteq Z$ is a solid convex cone. Given vector-valued bifunction $\varphi : A \times B \to Z$, we study the so-called *weak vector equilibrium problem*:

$(WVEP)$ find $\bar{a} \in A$ such that $\varphi(\bar{a}, b) \notin -\operatorname{int} C$ for all $b \in B$.

2.1 Existence results established via Eidelheit's theorem

Our first result provides sufficient conditions for the existence of solutions of the weak vector equilibrium problem $(WVEP)$. In the proof of this result we use Eidelheit's separation theorem.

Theorem 2.1.1 (A. Capătă and G. Kassay [38]) *Let A be a compact set, and let the bifunction $\varphi : A \times B \to Z$ satisfy the following conditions:*

(i) *for each $b \in B$, the function $\varphi(\cdot, b) : A \to Z$ is C-upper semicontinuous on A;*

(ii) *for all $a_1, \ldots, a_m \in A$, all $\lambda_1, \ldots, \lambda_m \geq 0$ with $\lambda_1 + \cdots + \lambda_m = 1$, and all $b_1, \ldots, b_n \in B$, there exists $c^* \in C^* \setminus \{0\}$ such that*

$$\min_{1\leq j\leq n} \sum_{i=1}^{m} \lambda_i c^*\big(\varphi(a_i, b_j)\big) \leq \sup_{a\in A} \min_{1\leq j\leq n} c^*\big(\varphi(a, b_j)\big);$$

(iii) *for all $b_1, \ldots, b_n \in B$ and all $c_1^*, \ldots, c_n^* \in C^*$ not all zero, one has*

$$\sup_{a\in A} \sum_{j=1}^{n} c_j^*\big(\varphi(a, b_j)\big) \geq 0.$$

Then problem $(WVEP)$ admits a solution.

Assumption (ii) of Theorem 2.1.1 is a kind of generalized concavity of the bifunction φ in its first variable with respect to the cone C. In a similar way, as the C-subconvexlikeness of a vector-valued function was characterized in Section 1.2, we introduce a new convexity concept for vector-valued bifunctions.

Definition 2.1.2 (A. Capătă and G. Kassay [38]) A bifunction $\varphi : A \times B \to Z$ is said to be:

(i) C-subconcavelike in its first variable if, for all $c \in \operatorname{int} C$, all $a_1, a_2 \in A$ and all $\lambda \in [0,1]$, there exists $a \in A$ such that

$$\varphi(a,b) \geq_C \lambda\varphi(a_1,b) + (1-\lambda)\varphi(a_2,b) - c \text{ for all } b \in B;$$

(ii) C-subconvexlike in its second variable if, for all $c \in \operatorname{int} C$, all $b_1, b_2 \in B$ and all $\lambda \in [0,1]$, there exists $b \in B$ such that

$$\varphi(a,b) \leq_C \lambda\varphi(a,b_1) + (1-\lambda)\varphi(a,b_2) + c \text{ for all } a \in A;$$

(iii) C-subconcavelike – subconvexlike if it is C-subconcavelike in its first variable and C-subconvexlike in its second variable.

When $Z := \mathbb{R}$ and $C := \mathbb{R}_+$, then we use the terms subconcavelike, subconvexlike and subconcavelike – subconvexlike instead of $\mathbb{R}_+$-subconcavelike, $\mathbb{R}_+$-subconvexlike and $\mathbb{R}_+$-subconcavelike – subconvexlike, respectively.

The C-subconcavelikeness of a bifunction can be characterized as follows.

Proposition 2.1.3 (A. Capătă and G. Kassay [38]) *A bifunction $\varphi : A \times B \to Z$ is C-subconcavelike in its first variable if and only if, for all $c \in \operatorname{int} C$, all $a_1, \ldots, a_m \in A$, all $\lambda_1, \ldots, \lambda_m \geq 0$ with $\lambda_1 + \cdots + \lambda_m = 1$, there exists $a \in A$ such that*

$$\varphi(a,b) \geq_C \sum_{i=1}^{m} \lambda_i \varphi(a_i,b) - c \text{ for all } b \in B.$$

Using Proposition 2.1.3 we obtain by Theorem 2.1.1 the following result.

Corollary 2.1.4 (A. Capătă and G. Kassay [38]) *Let A be a compact set, and let the bifunction $\varphi : A \times B \to Z$ satisfy the following conditions:*

(i) *φ is C-upper semicontinuous on A and C-subconcavelike in its first variable;*

(ii) *for all $b_1, \ldots, b_n \in B$, all $c_1^*, \ldots, c_n^* \in C^*$ not all zero, one has*

$$\sup_{a \in A} \sum_{j=1}^{n} c_j^*\big(\varphi(a,b_j)\big) \geq 0.$$

Then problem $(WVEP)$ admits a solution.

In what follows we deal with the scalar case. Let $Z := \mathbb{R}$, and let $C := \mathbb{R}_+$. Then, our weak vector equilibrium problem $(WVEP)$ becomes the following *scalar equilibrium problem*:

$$(EP) \qquad \text{find } \bar{a} \in A \text{ such that } \varphi(\bar{a},b) \geq 0 \text{ for all } b \in B.$$

Theorem 2.1.1 permits us to reobtain a result of G. Kassay and J. Kolumbán [82] concerning the existence of solutions for (EP).

Corollary 2.1.5 (G. Kassay and J. Kolumbán [82]) *Let A be a compact set, and let $\varphi : A\times B \to \mathbb{R}$ satisfy the following conditions:*

(i) *for each $b \in B$, the function $\varphi(\cdot, b) : A \to \mathbb{R}$ is upper semicontinuous on A;*

(ii) *for all $a_1, \ldots, a_m \in A$, all $\lambda_1, \ldots, \lambda_m \geq 0$ with $\lambda_1 + \cdots + \lambda_m = 1$, and all $b_1, \ldots, b_n \in B$, one has*

$$\min_{1\leq j\leq n} \sum_{i=1}^{m} \lambda_i \varphi(a_i, b_j) \leq \sup_{a\in A} \min_{1\leq j\leq n} \varphi(a, b_j);$$

(iii) *for all $b_1, \ldots, b_n \in B$, all $\mu_1, \ldots, \mu_n \geq 0$ with $\mu_1 + \cdots + \mu_n = 1$, one has*

$$\sup_{a\in A} \sum_{j=1}^{n} \mu_j \varphi(a, b_j) \geq 0.$$

Then problem (EP) admits a solution.

Assumption (iii) of Corollary 2.1.5 is satisfied if the bifunction φ is subconvexlike in its second variable and an additional condition, namely $\sup_{a\in A} \varphi(a, b) \geq 0$ for each $b \in B$, is satisfied. This fact is illustrated in the next corollary. When we deal with $B = A$, then this additional condition can be replaced by a stronger one, namely that $\varphi(a, a) = 0$ for all $a \in A$.

Corollary 2.1.6 (A. Capătă and G. Kassay [38]) *Let A be a compact set, and let the bifunction $\varphi : A \times B \to \mathbb{R}$ satisfy the following conditions:*

(i) *for each $b \in B$, the function $\varphi(\cdot, b) : A \to \mathbb{R}$ is upper semicontinuous on A;*

(ii) *φ is subconcavelike – subconvexlike;*

(iii) $\sup_{a\in A} \varphi(a, b) \geq 0$ *for each $b \in B$.*

Then problem (EP) admits a solution.

2.2 Existence results for the generalized equilibrium problem with composed functions

Let E and Y be real topological linear spaces, the latter being partially ordered by a convex closed cone K, let A be a nonempty subset of E, and let $h : A \to Y$ and $g : Y \to \mathbb{R}$ be given functions. Further, let $Z := \mathbb{R}$, let $C := \mathbb{R}_+$, let $B := A$, and let $\varphi : A \times A \to \mathbb{R}$ satisfy the property

$$\varphi(a, a) = 0 \text{ for each } a \in A.$$

In this section we investigate the following *generalized equilibrium problem with composed functions*:

$(GEPC)$ find $\bar{a} \in A$ such that $\varphi(\bar{a}, b) + g \circ h(b) \geq g \circ h(\bar{a})$ for all $b \in A$.

Definition 2.2.1 (D. T. Luc [95]) The function $g : Y \to \mathbb{R}$ is said to be K-increasing if, for all $y_1, y_2 \in Y$ such that $y_1 \leq_K y_2$, one has $g(y_1) \leq g(y_2)$.

Proposition 2.2.2 (D. T. Luc [95]) *Let $h : E \to Y$ be K-lower semicontinuous at $x_0 \in E$, and let $g : Y \to \mathbb{R}$ be K-increasing and lower semicontinuous at $h(x_0)$. Then $g \circ h$ is lower semicontinuous at x_0.*

Theorem 2.2.3 (R. I. Boţ and A. E. Capătă [27]) *Let A be a compact set, and let the following conditions be fulfilled:*

(i) *for each $b \in A$, the function $\varphi(\cdot, b)$ is upper semicontinuous on A;*

(ii) *h is K-lower semicontinuous on A;*

(iii) *g is lower semicontinuous on Y;*

(iv) *the bifunction*
$$\forall\,(a,b) \in A \times A \mapsto \varphi(a,b) - g(h(a)) \in \mathbb{R}$$
is subconcavelike in its first variable;

(v) *the bifunction*
$$\forall\,(a,b) \in A \times A \mapsto \varphi(a,b) + g(h(b)) \in \mathbb{R}$$
is subconvexlike in its second variable.

Then problem $(GEPC)$ admits a solution.

Remark 2.2.4 It is an easy exercise to verify that the assumptions (iv) and (v) in Theorem 2.2.3 are consequences of:
(vi) the bifunction
$$\forall\,(a,b) \in A \times A \mapsto (\varphi(a,b), -g(h(a))) \in \mathbb{R}^2$$
is $\mathbb{R}^2_+$-subconcavelike in its first variable and, respectively,
(vii) the bifunction
$$\forall\,(a,b) \in A \times A \mapsto (\varphi(a,b), g(h(b))) \in \mathbb{R}^2$$
is $\mathbb{R}^2_+$-subconvexlike in its second variable. □

When the function $g : Y \to \mathbb{R}$ is convex and K-increasing one can give some sufficient conditions for the hypotheses (vi) and (vii) in the above remark which involve only the vector function h. To this end we consider two generalized convexity notions that are analogues of those introduced in Definition 2.1.2.

Definition 2.2.5 (R. I. Boţ and A. E. Capătă [27]) We say that:

(i) the bifunction
$$\forall\,(a,b) \in A \times A \mapsto (\varphi(a,b), -h(a)) \in \mathbb{R} \times Y$$
is subconcavelike – K-concavelike in its first variable whenever, for all $\varepsilon > 0$, all $\lambda \in [0,1]$ and all $a_1, a_2 \in A$, there exists $a \in A$ such that
$$(\varphi(a,b), -h(a)) \geq_{\mathbb{R}_+ \times K} \lambda(\varphi(a_1,b), -h(a_1)) +$$
$$+(1-\lambda)(\varphi(a_2,b), -h(a_2)) - (\varepsilon, 0) \text{ for all } b \in A;$$

(ii) the bifunction

$$\forall\,(a,b) \in A \times A \mapsto (\varphi(a,b), h(b)) \in \mathbb{R} \times Y$$

is subconvexlike – K-convexlike in its second variable whenever, for all $\varepsilon > 0$, all $\lambda \in [0,1]$ and all $b_1, b_2 \in A$, there exists $b \in A$ such that

$$(\varphi(a,b), h(b)) \leq_{\mathbb{R}_+ \times K} \lambda(\varphi(a,b_1), h(b_1)) +$$

$$+(1-\lambda)(\varphi(a,b_2), h(b_2)) + (\varepsilon, 0) \text{ for all } a \in A.$$

Now we can state a second result on the existence of solutions for $(GEPC)$.

Theorem 2.2.6 (R. I. Boţ and A. E. Capătă [27]) *Let A be a compact set, and let the following conditions be fulfilled:*

(i) *for each $b \in A$, the function $\varphi(\cdot, b)$ is upper semicontinuous on A;*

(ii) *h is C-lower semicontinuous on A;*

(iii) *g is convex, lower semicontinuous on Y and K-increasing;*

(iv) *the bifunction*

$$\forall\,(a,b) \in A \times A \mapsto (\varphi(a,b), -h(a)) \in \mathbb{R} \times Y$$

is subconcavelike – K-concavelike in its first variable;

(v) *the bifunction*

$$\forall\,(a,b) \in A \times A \mapsto (\varphi(a,b), h(b)) \in \mathbb{R} \times Y$$

is subconvexlike – K-convexlike in its second variable.

Then problem $(GEPC)$ admits a solution.

Imposing classical assumptions on the sets and functions involved in the formulation of problem $(GEPC)$, this problem has a solution, as the next corollary shows.

Corollary 2.2.7 (R. I. Boţ and A. E. Capătă [27]) *Let A be a convex and compact set, and let the following conditions be fulfilled:*

(i) *for each $b \in A$, the function $\varphi(\cdot, b)$ is upper semicontinuous on A;*

(ii) *h is K-convex and K-lower semicontinuous on A;*

(iii) *g is convex, lower semicontinuous on Y and K-increasing;*

(iv) *the bifunction φ is concave – convex.*

Then problem $(GEPC)$ admits a solution.

Chapter 3

Existence results for strong vector equilibrium problems

Let A be a nonempty subset of a topological space E, let B be a nonempty set, and let C be a nontrivial pointed convex cone of a real topological linear space Z, and let $\varphi : A \times B \to Z$ be a given bifunction. In Q. H. Ansari, W. Oettli and D. Schläger [6], the scalar equilibrium problem (EP) (see Section 2.1) was extended to vector-valued bifunctions in the following way:

$$(VEP) \qquad\qquad \text{find } \bar{a} \in A \text{ such that } \varphi(\bar{a}, b) \notin -C \setminus \{0\} \text{ for all } b \in B.$$

Throughout this chapter we deal with (VEP), which is called *the strong vector equilibrium problem.*

3.1 Existence results established via Eidelheit's theorem

In this section, the cone C is supposed to be solid. The next theorem states the existence of solutions of the strong vector equilibrium problem.

Theorem 3.1.1 (G. Bigi, A. Capătă and G. Kassay [18]) *Let $\varphi : A \times B \to Z$ satisfy the following conditions:*

(i) *if the family $(U_b)_{b\in B}$ covers A, then it contains a finite subcover, where*

$$U_b := \{a \in A \,|\, \varphi(a, b) \in -C\backslash\{0\}\};$$

(ii) *for all $a_1, \dots, a_m \in A$, all $\lambda_1, \dots, \lambda_m \geq 0$ with $\lambda_1 + \dots + \lambda_m = 1$, and all $b_1, \dots, b_n \in B$, there exists $c^* \in C^\sharp$ such that*

$$\min_{1\leq j\leq n} \sum_{i=1}^{m} \lambda_i c^*\big(\varphi(a_i, b_j)\big) \leq \sup_{a\in A} \min_{1\leq j\leq n} c^*\big(\varphi(a, b_j)\big);$$

(iii) *for all $b_1, \dots, b_n \in B$, and all $c_1^*, \dots, c_n^* \in C^*$ not all zero, one has*

$$\sup_{a\in A} \sum_{j=1}^{n} c_j^*\big(\varphi(a, b_j)\big) > 0.$$

Then problem (VEP) admits a solution.

The following generalization of the upper semicontinuity, extending the upper semicontinuity of real-valued functions, was also given by W. W. Breckner and G. Orbán [31].

Definition 3.1.2 (W. W. Breckner and G. Orbán [31]) Let C be a convex cone. A vector-valued function $f : A \to Z$ is said to be upper semicontinuous at a point $a_0 \in A$ if, for each $c \in C \setminus \{0\}$, there exists a neighbourhood U of a_0 such that

$$f(a) \in f(a_0) + c - C \text{ for all } a \in U \cap A.$$

Under the hypothesis that C is a pointed convex cone, the upper semicontinuity introduced in Definition 3.1.2 is equivalent to the next one, introduced by G. Bigi, A. Capătă and G. Kassay [18].

Definition 3.1.3 (G. Bigi, A. Capătă and G. Kassay [18]) A vector-valued function $f : A \to Z$ is said to be:

(i) properly C-upper semicontinuous at a point $a_0 \in A$ if, for each $c \in C \setminus \{0\}$, there exists a neighbourhood U of a_0 such that

$$f(a) \in f(a_0) + c - C \setminus \{0\} \text{ for all } a \in U \cap A;$$

(ii) properly C-upper semicontinuous on A if, it is properly C-upper semicontinuous at each point $a_0 \in A$;

(iii) properly C-lower semicontinuous at $a_0 \in A$ (respectively properly C-lower semicontinuous on A) if $-f$ is properly C-upper semicontinuous at $a_0 \in A$ (respectively properly C-upper semicontinuous on A).

Notice that every properly C-upper semicontinuous function $f : A \to Z$ is C-upper semicontinuous, but not the viceversa. For example, if we take $A := Z$ and $C \subseteq Z$ is any cone such that $C \setminus \{0\}$ is not open, then the identity function is not properly C-upper semicontinuous.

Proposition 3.1.4 (G. Bigi, A. Capătă and G. Kassay [18]) *Given a function $f : A \to Z$, the following properties are equivalent:*

(i) *f is properly C-upper semicontinuous on A;*

(ii) *For each $z \in Z$, the set $f^{-1}(z - C\setminus\{0\})$ is open with respect to the induced topology on A.*

Proposition 3.1.5 (G. Bigi, A. Capătă and G. Kassay [18]) *Suppose that A is a compact set, and that for each $b \in B$ the function $\varphi(\cdot, b) : A \to Z$ is properly C-upper semicontinuous on A. Then condition* (i) *in Theorem* 3.1.1 *is satisfied.*

By Proposition 3.1.5 and Definition 2.1.2 (i), we obtain the next corollary of Theorem 3.1.1.

Corollary 3.1.6 (G. Bigi, A. Capătă and G. Kassay [18]) *Let C be such that $C^\sharp \neq \emptyset$, and let $\varphi : A \times B \to Z$ satisfy the following conditions:*

(i) *for all $b \in B$, $\varphi(\cdot, b) : A \to Z$ is properly C-upper semicontinuous on A;*

(ii) *φ is C-subconcavelike in its first variable;*

(iii) *for all* $b_1, \ldots, b_n \in B$*, and all* $c_1^*, \ldots, c_n^* \in C^*$ *not all zero, the inequality*

$$\sup_{a \in A} \sum_{j=1}^{n} c_j^*\big(\varphi(a, b_j)\big) > 0$$

holds.

Then problem (VEP) *admits a solution.*

In the particular case when $Z := \mathbb{R}$ and $C := \mathbb{R}_+$, Corollary 3.1.6 yields the next result.

Corollary 3.1.7 (G. Bigi, A. Capătă and G. Kassay [18]) *Let* A *be a compact set, and let* $\varphi : A \times B \to \mathbb{R}$ *satisfy the following conditions:*

(i) *for all* $b \in B$*,* $\varphi(\cdot, b) : A \to \mathbb{R}$ *is upper semicontinuous on* A*;*

(ii) φ *is subconcavelike in its first variable;*

(iii) *for all* $b_1, \ldots, b_n \in B$ *and all* $\mu_1, \ldots, \mu_n \geq 0$ *with* $\mu_1 + \cdots + \mu_n = 1$*, one has*

$$\sup_{a \in A} \sum_{j=1}^{n} \mu_j \varphi(a, b_j) > 0.$$

Then problem (EP) *admits a solution.*

Theorem 3.1.8 (C. L. de Vito [112]) *Let* E *be a normed space, and let* A *be a subset of* E *that is weakly compact. Then every sequence of points of* A *has a subsequence that is weakly convergent to a point of* A*.*

The next result follows from Corollary 3.1.7.

Theorem 3.1.9 (G. Bigi, A. Capătă and G. Kassay [18]) *Let* E *be a normed space, let* A *be a weakly compact set, and let* $c^* \in C^\sharp$*. Suppose that* $\varphi : A \times B \to Z$ *satisfies the following conditions:*

(i) *for each* $b \in B$*, the function* $a \in A \mapsto c^*(\varphi(a, b)) \in \mathbb{R}$ *is weakly upper semicontinuous on* A*;*

(ii) φ *is* C*-subconcavelike – subconvexlike;*

(iii) *for all* $b \in B$*, one has*

$$\sup_{a \in A} c^*(\varphi(a, b)) \geq 0.$$

Then problem (VEP) *admits a solution.*

Theorem 3.1.9 allows us to get the following slight generalization of Theorem 3.2 established by X. H. Gong [58], in which convexlikeness is replaced by the weaker subconvexlikeness.

Corollary 3.1.10 *Let* E *be a normed space, let* A *be a weakly compact set, and let* $c^* \in C^\sharp$*. Suppose that* $\varphi : A \times B \to Z$ *satisfies the following conditions:*

(i) *for each* $b \in B$*, the function* $a \in A \mapsto c^*(\varphi(a, b)) \in \mathbb{R}$ *is weakly upper semicontinuous on* A*;*

(ii) φ *is* C*-subconcavelike – subconvexlike;*

(iii) $\varphi(a,a) \in C$ *for all* $a \in A$

Then problem (VEP) *admits a solution.*

It should be remarked that Theorem 3.1.9 extends the above-mentioned Theorem 3.2 established by X. H. Gong [58] also in two other directions: on the one hand, in Theorem 3.1.9 there are considered two different sets A and B, on the other hand, the equilibrium condition $\varphi(a,a) \in C$ is replaced by a weaker assumption involving an appropriate supremum over A.

3.2 Existence results established via Ky Fan's lemma

This section is devoted to the study of a special case of the strong vector equilibrium problem (VEP) by using a generalized dual problem.

Throughout this section, E is a real Hausdorff topological linear space, $A \subseteq E$ is a nonempty convex subset, $B = A$, and C is a pointed convex cone of the real topological linear space Z. So, problem (VEP) becomes:

$(PVEP)$ find $\bar{a} \in A$ such that $\varphi(\bar{a}, b) \notin -C \setminus \{0\}$ for all $b \in A$.

With the help of an operator, we attach to problem $(PVEP)$ a dual problem. Let $\mathcal{D}$ be an operator from $\mathcal{F}(A,Z) := \{\psi \mid \psi : A \times A \to Z\}$ into itself, which is called the duality operator. In fact, $\mathcal{D}$ is a set of fixed rules applied to problem $(PVEP)$. By means of $\mathcal{D}$ we introduce the following *generalized dual strong vector equilibrium problem*:

$(DVEP)$ find $\bar{a} \in A$ such that $\mathcal{D}(\varphi)(\bar{a}, b) \notin -C \setminus \{0\}$ for all $b \in A$.

The following proposition shows that, under a certain hypothesis, the generalized dual of this dual problem becomes the initial problem.

Proposition 3.2.1 *If*

$$\mathcal{D} \circ \mathcal{D}(\varphi) = \varphi,$$

then the generalized dual problem of $(DVEP)$ *is problem* $(PVEP)$*.*

Let $G : A \times A \to Z$ be defined by

$$G(a,b) := -\mathcal{D}(\varphi)(b,a) \text{ for all } a, b \in A.$$

In this framework, problem $(DVEP)$ can be written as:

$(GVEP)$ find $\bar{a} \in A$ such that $G(b, \bar{a}) \notin C \setminus \{0\}$ for all $b \in A$.

The next notions are generalizations of the g-monotonicity and maximal g-monotonicity, respectively, introduced by W. Oettli [99] in the scalar case.

Definition 3.2.2 The bifunction $\varphi : A \times A \to Z$ is said to be:

(i) G-pseudomonotone if, for all $a, b \in A$,

$$\varphi(a,b) \notin -C \setminus \{0\} \text{ implies } G(b,a) \notin C \setminus \{0\};$$

(ii) maximally G-pseudomonotone if it is G-pseudomonotone and, for all $a, b \in A$, the following implication holds:

$$G(x,a) \notin C \setminus \{0\} \text{ for all } x \in]a,b] \text{ implies } \varphi(a,b) \notin -C \setminus \{0\}.$$

Proposition 3.2.3 (A. Capătă [35]) *If $\varphi : A \times A \to Z$ is maximally G-pseudomonotone, then the sets of solutions of problems $(PVEP)$ and $(GVEP)$ coincide.*

By using the dual formulation $(GVEP)$ of problem $(PVEP)$ we obtain the following existence results for solutions of problem $(PVEP)$.

Theorem 3.2.4 (A. Capătă [35]) *Suppose that $\varphi : A \times A \to Z$ and $G : A \times A \to Z$ satisfy the following conditions:*

(i) *$\varphi(a,a) \in C$ for all $a \in A$;*

(ii) *φ is maximally G-pseudomonotone;*

(iii) *for each $b \in A$, the set $S(b) := \{a \in A \mid G(b,a) \notin C \setminus \{0\}\}$ is closed;*

(iv) *for each $a \in A$, the set $W(a) := \{b \in A \mid \varphi(a,b) \in -C \setminus \{0\}\}$ is convex;*

(v) *there exist a nonempty, compact and convex set $D \subseteq A$ as well as an element $\tilde{b} \in D$ such that*
$$\varphi(x,\tilde{b}) \in -C \setminus \{0\} \text{ for all } x \in A \setminus D.$$

Then problem $(PVEP)$ admits a solution.

Corollary 3.2.5 (A. Capătă [35]) *Suppose that $\varphi : A \times A \to Z$ and $G : A \times A \to Z$ satisfy the following conditions:*

(i) *$\varphi(a,a) \in C$ for all $a \in A$;*

(ii) *φ is maximally G-pseudomonotone;*

(iii) *for each $b \in A$, the set $S(b) := \{a \in A \mid G(b,a) \notin C \setminus \{0\}\}$ is closed;*

(iv) *for each $a \in A$, the function $\varphi(a,\cdot) : A \to Z$ is C-quasiconvex;*

(v) *there exist a nonempty, compact and convex set $D \subseteq A$ as well as an element $\tilde{b} \in D$ such that*
$$\varphi(x,\tilde{b}) \in -C \setminus \{0\} \text{ for all } x \in A \setminus D.$$

Then problem $(PVEP)$ admits a solution.

Remark 3.2.6 (A. Capătă [35]) Assumption (iv) in Theorem 3.2.4 does not imply assumption (iv) of Corollary 3.2.5. Indeed, let $E = Z$, let $C \subseteq Z$ be a pointed convex cone such that the ordering defined by C is not total on A, and let $\varphi : A \times A \to Z$ be defined by

$$\varphi(a,b) := b \text{ for all } a, b \in A.$$

In order to verify assumption (iv) of Theorem 3.2.4, fix $a \in A$ and take $b_1, b_2 \in W(a)$. Thus $b_1, b_2 \in -C \setminus \{0\}$. Because $-C \setminus \{0\}$ is convex, we have

$$\lambda b_1 + (1-\lambda) b_2 \in -C \setminus \{0\} \text{ for every } \lambda \in [0,1].$$

So, $W(a)$ is a convex set.

Now, let $b_1, b_2 \in A$ and $\lambda \in [0,1]$. Suppose that $\varphi(a, \cdot) : A \to Z$ is C-quasiconvex. Thus, we obtain

$$b_1 \in b_2 + C \text{ or } b_2 \in b_1 + C.$$

Since b_1 and b_2 were arbitrarily chosen and the ordering induced by C on A is not total, it follows that the function $\varphi(a, \cdot)$ is not C-quasiconvex. $\square$

In what follows we consider two particular cases of the operator $\mathcal{D}$. Firstly we define the operator $\mathcal{D} : \mathcal{F}(A, Z) \to \mathcal{F}(A, Z)$ by

$$\mathcal{D}(\psi)(a, b) := -\psi(b, a) \text{ for all } a, b \in A. \tag{3.1}$$

So, the generalized dual strong vector equilibrium problem becomes:

$$(DVEP1) \qquad \text{find } \bar{a} \in A \text{ such that } \varphi(b, \bar{a}) \notin C \setminus \{0\} \text{ for all } b \in A.$$

Under pseudomonotonicity assumptions we will give an existence result for the strong vector equilibrium problem $(PVEP)$. Taking into consideration that $G : A \times A \to Z$, associated with the operator $\mathcal{D} : \mathcal{F}(A, Z) \to \mathcal{F}(A, Z)$ defined by (3.1), coincides with φ, Definition 3.2.2 yields the following definition.

Definition 3.2.7 The bifunction $\varphi : A \times A \to Z$ is said to be:

(i) pseudomonotone if, for all $a, b \in A$,

$$\varphi(a, b) \notin -C \setminus \{0\} \text{ implies } \varphi(b, a) \notin C \setminus \{0\};$$

(ii) maximally pseudomonotone if it is pseudomonotone and, for all $a, b \in A$, the following implication holds:

$$\varphi(x, a) \notin C \setminus \{0\} \text{ for all } x \in]a, b] \text{ implies } \varphi(a, b) \notin -C \setminus \{0\}.$$

Proposition 3.2.8 *If $\varphi : A \times A \to Z$ is maximally pseudomonotone, then the sets of solutions of problems $(PVEP)$ and $(DVEP1)$ coincide.*

Theorem 3.2.4 provides the next existence result of solutions of $(PVEP)$ under a pseudomonotonicity assumption.

Corollary 3.2.9 (A. Capătă [35]) *Suppose that the bifunction $\varphi : A \times A \to Z$ satisfies the following conditions:*

(i) *$\varphi(a, a) \in C$ for all $a \in A$;*

(ii) *φ is maximally pseudomonotone;*

(iii) *for each* $b \in A$*, the set* $S(b) := \{a \in A \mid \varphi(b,a) \notin C \setminus \{0\}\}$ *is closed;*

(iv) *for each* $a \in A$*, the set* $W(a) := \{b \in A \mid \varphi(a,b) \in -C \setminus \{0\}\}$ *is convex;*

(v) *there exist a nonempty, compact and convex set* $D \subseteq A$ *as well as an element* $\tilde{b} \in D$ *such that*

$$\varphi(x,\tilde{b}) \in -C \setminus \{0\} \text{ for all } x \in A \setminus D.$$

Then problem $(PVEP)$ *admits a solution.*

Now, if we define $\mathcal{D} : \mathcal{F}(A,Z) \to \mathcal{F}(A,Z)$ by $\mathcal{D}(\psi) := \psi$, then we obtain an existence result for problem $(PVEP)$ without pseudomonotonicity assumptions. It is easy to verify that the assumption of φ to be maximally G-pseudomonotone is fulfilled.

In this case, the generalized dual problem of problem $(PVEP)$ is exactly $(PVEP)$:

$(DVEP2)$ find $\bar{a} \in A$ such that $\varphi(\bar{a},b) \notin -C \setminus \{0\}$ for all $b \in A$.

Corollary 3.2.10 (A. Capătă [35]) *Suppose that the bifunction* $\varphi : A \times A \to Z$ *satisfies the following conditions:*

(i) $\varphi(a,a) \in C$ *for all* $a \in A$*;*

(ii) *for each* $b \in A$*, the set* $S(b) := \{a \in A \mid \varphi(a,b) \notin -C \setminus \{0\}\}$ *is closed;*

(iii) *for each* $a \in A$*, the set* $W(a) := \{b \in A \mid \varphi(a,b) \in -C \setminus \{0\}\}$ *is convex;*

(iv) *there exist a nonempty, compact and convex set* $D \subseteq A$ *as well as an element* $\tilde{b} \in D$ *such that*

$$\varphi(x,\tilde{b}) \in -C \setminus \{0\} \text{ for all } x \in A \setminus D.$$

Then problem $(PVEP)$ *admits a solution.*

Theorem 3.2.4 and Corollary 3.2.10 allow us to reobtain Lemma 1 and Theorem 2 established by W. Oettli [99], which are existence results for scalar equilibrium problems. Indeed, in what follows assume that $Z := \mathbb{R}$ and $C := \mathbb{R}_+$.

Corollary 3.2.11 *Let the bifunctions* $\varphi : A \times A \to \mathbb{R}$ *and* $G : A \times A \to \mathbb{R}$ *satisfy the following conditions:*

(i) $\varphi(a,a) \geq 0$ *for all* $a \in A$*;*

(ii) φ *is maximally* G*-pseudomonotone;*

(iii) *for each* $b \in A$*, the set* $S(b) := \{a \in A \mid G(b,a) \leq 0\}$ *is closed;*

(iv) *for each* $a \in A$*, the set* $W(a) := \{b \in A \mid \varphi(a,b) < 0\}$ *is convex;*

(v) *there exist a nonempty, compact and convex set* $D \subseteq A$ *as well as an element* $\tilde{b} \in D$ *such that*

$$\varphi(x,\tilde{b}) < 0 \text{ for all } x \in A \setminus D.$$

Then problem (EP) *considered in Section 2.1 with* $B = A$ *admits a solution.*

Corollary 3.2.12 *Suppose that the bifunction $\varphi : A \times A \to \mathbb{R}$ satisfies the following conditions:*

(i) $\varphi(a, a) \geq 0$ *for all* $a \in A$;

(ii) *for each* $b \in A$, *the set* $S(b) := \{a \in A \mid \varphi(a, b) \geq 0\}$ *is closed;*

(iii) *for each* $a \in A$, *the set* $W(a) := \{b \in A \mid \varphi(a, b) < 0\}$ *is convex;*

(iv) *there exist a nonempty, compact and convex set* $D \subseteq A$ *as well as an element* $\tilde{b} \in D$ *such that*

$$\varphi(x, \tilde{b}) \in -C \setminus \{0\} \text{ for all } x \in A \setminus D.$$

Then problem (EP) *considered in Section 2.1 with* $B = A$ *admits a solution.*

Corollary 3.2.12 is a slight generalization of an existence result established by K. Fan [49] and recovered by L. J. Lin, Z. T. Yu and G. Kassay [94].

Corollary 3.2.13 (K. Fan [49]) *Let A be a compact set, and let $\varphi : A \times A \to \mathbb{R}$ satisfy the following conditions:*

(i) $\varphi(a, a) \geq 0$ *for all* $a \in A$;

(ii) $\varphi(\cdot, b) : A \to \mathbb{R}$ *is upper semicontinuous for all* $b \in A$;

(iii) $\varphi(a, \cdot) : A \to \mathbb{R}$ *is quasiconvex for all* $a \in A$.

Then problem (EP) *considered in Section 2.1 with* $B = A$ *admits a solution.*

3.3 Existence results for proper efficient solutions of strong vector equilibrium problems

Under the assumption that C is a solid convex cone, we have proved in Section 2.1 and Section 3.1 existence results for the vector equilibrium problems $(WVEP)$ and (VEP). But, there are important ordered topological linear spaces whose ordering cones have an empty interior. For example, when $Z := L^p(T, \mu)$, where (T, μ) is a σ-finite measure space and $p \in [1, +\infty[$, the cone

$$C := \{u \in L^p(T, \mu) \mid u(t) \geq 0 \text{ a.e. in } [0, T]\}$$

has an empty interior.

In what follows we state existence results proper solutions of the strong vector equilibrium problem (VEP).

Let Z be a real topological linear space, and let $C \subseteq Z$ be a nontrivial pointed convex cone.

Definition 3.3.1 A subset $K \subseteq Z$ is said to be a Henig dilating cone for C if it satisfies the following conditions:

(i) K is a pointed convex cone;

(ii) $C \setminus \{0\} \subseteq \operatorname{int} K$.

Remark 3.3.2 If $K \subseteq Z$ is a Henig dilating cone for C, then $K^* \setminus \{0\} \subseteq C^\sharp$. Indeed, let $c^* \in K^* \setminus \{0\}$. Since $c^*(c) \geq 0$ for all $c \in K$ and $\operatorname{int} K \neq \emptyset$, it follows (see, for instance, W. W. Breckner [30, pp. 352-353, Lemma 6.3.1]) that $c^*(c) > 0$ for all $c \in \operatorname{int} K$. By virtue of $C \setminus \{0\} \subseteq \operatorname{int} K$, we conclude that $c^* \in C^\sharp$. □

Definition 3.3.3 A family $(K_i)_{i \in I}$ of subsets of Z is said to be a family of Henig dilating cones for C if each $K_i\,(i \in I)$ is a Henig dilating cone for C.

To see that such families exist we give two examples.

Example 3.3.4 Let Z be a real normed space, and let $C \subseteq Z$ be a based cone. So, we can choose a subset $\mathcal{B} \subseteq C$ satisfying the following conditions: $\mathcal{B}$ is nonempty and convex; $C = \mathbb{R}_+\mathcal{B}$; and $0 \notin \operatorname{cl} \mathcal{B}$. Set

$$d := \inf \{\| \, b \, \| \mid \; b \in \mathcal{B}\}.$$

Further, let U_0 be the closed unit ball of Z. For every $\epsilon \in]0, d[$ put

$$K_\epsilon(\mathcal{B}) := \mathbb{R}_+(\mathcal{B} + \epsilon U_0).$$

We claim that $(K_\epsilon(\mathcal{B}))_{\epsilon \in]0,d[}$ is a family of Henig dilating cones for C.

Fix any $\epsilon \in]0, d[$. Then the set $\mathcal{B} + \epsilon U_0$ is nonempty and convex. Moreover, we have

$$\|b + \epsilon y\| \geq \|b\| - \epsilon\|y\| \geq d - \epsilon \text{ for all } b \in \mathcal{B} \text{ and all } y \in U_0,$$

whence

$$\inf \{\|z\| \mid \; z \in \mathcal{B} + \epsilon U_0\} \geq d - \epsilon > 0.$$

This inequality implies that $0 \notin \operatorname{cl}(\mathcal{B} + \epsilon U_0)$. Thus, $K_\epsilon(\mathcal{B})$ is a based cone, hence a pointed convex cone. Finally, we prove that

$$C \setminus \{0\} \subseteq \operatorname{int} K_\epsilon(\mathcal{B}).$$

Let $z \in C \setminus \{0\}$. Then there exists $\lambda \in]0, \infty[$ such that $z \in \lambda\mathcal{B}$. Consequently, it follows that

$$z + \lambda\epsilon U_0 \subseteq \lambda(\mathcal{B} + \epsilon U_0) \subseteq K_\epsilon(\mathcal{B}),$$

whence $z \in \operatorname{int} K_\epsilon(\mathcal{B})$. Summing up, $K_\epsilon(\mathcal{B})$ is a Henig dilating cone for C. □

Example 3.3.5 Let Z be a real locally convex space, and let $C \subseteq Z$ be a based cone. So, there exists a nonempty and convex subset $\mathcal{B} \subseteq C$ such that $C = \mathbb{R}_+\mathcal{B}$ and $0 \notin \operatorname{cl} \mathcal{B}$. By Tukey's separation theorem, there exists a functional $z^* \in Z^*$ such that

$$r := \inf \{z^*(b) \mid \; b \in \; \mathcal{B}\} > 0.$$

The set

$$V_{\mathcal{B}}(z^*) := \{z \in Z \mid \; |z^*(z)| < \frac{r}{2}\}$$

is a convex and balanced neighbourhood of the origin of Z. Further, set

$$\mathcal{U} := \{U \mid \; U \text{ is a convex neighbourhood of the origin of } Z \text{ with } U \subseteq V_{\mathcal{B}}(z^*)\}.$$

For every $U \in \mathcal{U}$ put

$$K_U(\mathcal{B}) := \mathbb{R}_+(\mathcal{B} + U).$$

We claim that $(K_U(\mathcal{B}))_{U\in\mathcal{U}}$ is a family of Henig dilating cones for C.

Fix any $U \in \mathcal{U}$. Then the set $\mathcal{B}+U$ is nonempty and convex. Moreover, we have

$$|z^*(b+y)| \geq |z^*(b)| - |z^*(y)| \geq r - \frac{r}{2} \text{ for all } b \in \mathcal{B} \text{ and all } y \in U_0,$$

whence

$$\inf\{|z^*(z)| \mid z \in \mathcal{B}+U\} \geq \frac{r}{2} > 0.$$

This inequality implies that $0 \notin \operatorname{cl}(\mathcal{B}+U)$. Thus, $K_U(\mathcal{B})$ is a based cone, hence a pointed convex cone. Finally, we prove that

$$C \setminus \{0\} \subseteq \operatorname{int} K_U(\mathcal{B}).$$

Let $z \in C \setminus \{0\}$. Then there exists $\lambda \in]0,\infty[$ such that $z \in \lambda\mathcal{B}$. Consequently, it follows that

$$z + \lambda U \subseteq \lambda(\mathcal{B}+U) \subseteq K_U(\mathcal{B}),$$

whence $z \in \operatorname{int} K_U(\mathcal{B})$. Summing up, $K_U(\mathcal{B})$ is a Henig dilating cone for C. □

Definition 3.3.6 Let $K \subseteq Z$ be a Henig dilating cone for C. A point $\bar{a} \in A$ is said to be:

(i) a K-Henig weakly efficient solution of (VEP) if

$$\varphi(\bar{a}, B) \cap (-\operatorname{int} K) = \emptyset.$$

(ii) a K-Henig efficient solution of (VEP) if

$$\varphi(\bar{a}, B) \cap (-K) = \{0\}.$$

The next definition generalizes the Henig efficient solutions introduced by X. H. Gong, W. T. Fu and W. Liu [65].

Definition 3.3.7 (A. Capătă [37]) Let $(K_i)_{i\in I}$ (where $K_i \subseteq Z$ for each $i \in I$) be a family of Henig dilating cones for C. A point $\bar{a} \in A$ is said to be:

(i) a Henig weakly efficient solution of (VEP) if there exists $i_0 \in I$ such that $\bar{a}$ is a K_{i_0}-Henig weakly efficient solution of (VEP);

(ii) a Henig efficient solution of (VEP) if there exists $i_0 \in I$ such that $\bar{a}$ is a K_{i_0}-Henig efficient solution of (VEP).

Theorem 3.3.8 *Let A be a compact set, let $K \subseteq Z$ be a Henig dilating cone for C, and let $\varphi : A \times B \to Z$ satisfy the following conditions:*

(i) *for each $b \in B$, the function $\varphi(\cdot, b) : A \to Z$ is K-upper semicontinuous on A;*

(ii) *for all $a_1, \dots, a_m \in A$, all $\lambda_1, \dots, \lambda_m \geq 0$ with $\lambda_1 + \dots + \lambda_m = 1$, and all $b_1, \dots, b_n \in B$, there exists $k^* \in K^* \setminus \{0\}$ such that*

$$\min_{1\leq j\leq n} \sum_{i=1}^{m} \lambda_i k^*\big(\varphi(a_i, b_j)\big) \leq \sup_{a\in A} \min_{1\leq j\leq n} k^*\big(\varphi(a, b_j)\big);$$

(iii) *for all* $b_1, \ldots, b_n \in B$ *and all* $k_1^*, \ldots, k_n^* \in K^*$ *not all zero, one has*

$$\sup_{a \in A} \sum_{j=1}^{n} k_j^*\big(\varphi(a, b_j)\big) \geq 0.$$

Then problem (VEP) *admits a* K*-Henig weakly efficient solution.*

Theorem 3.3.9 *Let* A *be a compact set, let* $K \subseteq Z$ *be a Henig dilating cone for* C, *and let* $\varphi : A \times B \to Z$ *satisfy the following conditions:*

(i) *for each* $b \in B$, *the function* $\varphi(\cdot, b) : A \to Z$ *is properly* K*-upper semicontinuous on* A*;*

(ii) *for all* $a_1, \ldots, a_m \in A$, *all* $\lambda_1, \ldots, \lambda_m \geq 0$ *with* $\lambda_1 + \cdots + \lambda_m = 1$, *and all* $b_1, \ldots, b_n \in B$, *there exists* $k^* \in K^\sharp$ *such that*

$$\min_{1 \leq j \leq n} \sum_{i=1}^{m} \lambda_i k^*\big(\varphi(a_i, b_j)\big) \leq \sup_{a \in A} \min_{1 \leq j \leq n} k^*\big(\varphi(a, b_j)\big);$$

(iii) *for all* $b_1, \ldots, b_n \in B$ *and all* $k_1^*, \ldots, k_n^* \in K^*$ *not all zero, one has*

$$\sup_{a \in A} \sum_{j=1}^{n} k_j^*\big(\varphi(a, b_j)\big) > 0.$$

Then problem (VEP) *admits a* K*-Henig efficient solution.*

Let $K := (K_i)_{i \in I}$ (where $K_i \subseteq Z$ for each $i \in I$) be a family of Henig dilating cones for C. In the sequel we give existence results for Henig weakly efficient solutions of (VEP) by using the following set:

$$K^\triangle := \{c^* \in E^* \mid \exists\, i \in I \,:\, c^* \in K_i^* \setminus \{0\}\}.$$

By virtue of Remark 3.3.2 it follows that $K^\triangle \subseteq C^\sharp$.

Proposition 3.3.10 (X. H. Gong [58], [59]) *If* K *is the family* $(K_\epsilon(\mathcal{B}))_{\epsilon \in]0,d[}$ *of Henig dilating cones for* C *constructed in Example 3.3.4 or the family* $(K_U(\mathcal{B}))_{U \in \mathcal{U}}$ *of Henig dilating cones for* C *constructed in Example 3.3.5, then*

$$K^\triangle = \{c^* \in C^\sharp \mid \inf c^*(\mathcal{B}) > 0\}.$$

Theorem 3.3.11 (A. Capătă [37]) *Let* A *be a compact set, let* $K := (K_i)_{i \in I}$ *be a family of Henig dilating cones for* C, *and let* $\varphi : A \times B \to Z$ *satisfy the following conditions:*

(i) *for each* $b \in B$ *and each* $i \in I$, *the function* $\varphi(\cdot, b) : A \to Z$ *is* K_i*-upper semicontinuous on* A*;*

(ii) *there exists* $c^* \in K^\triangle$ *such that, for all* $a_1, \ldots, a_m \in A$, *all the numbers* $\lambda_1, \ldots, \lambda_m \geq 0$ *with* $\lambda_1 + \cdots + \lambda_m = 1$, *and all* $b_1, \ldots, b_n \in B$, *the functional* c^* *satisfies*

$$\min_{1 \leq j \leq n} \sum_{i=1}^{m} \lambda_i c^*\big(\varphi(a_i, b_j)\big) \leq \sup_{a \in A} \min_{1 \leq j \leq n} c^*\big(\varphi(a, b_j)\big);$$

(iii) *for all* $b_1, \ldots, b_n \in B$, *each* $i \in I$, *and all* $k_1^*, \ldots, k_n^* \in K_i^*$ *not all zero, one has*

$$\sup_{a \in A} \sum_{j=1}^{n} k_j^*\big(\varphi(a, b_j)\big) \geq 0.$$

Then problem (VEP) *admits a Henig weakly efficient solution.*

We observe that the assumptions (i) and (iii) of Theorem 3.3.11 are stronger than the assumptions (i) and (iii) of Theorem 3.3.8, while assumption (ii) of Theorem 3.3.8 has not to be satisfied for all cones K_i $(i \in I)$.

The next corollary is stated under stronger assumptions than those of Theorem 3.3.11.

Corollary 3.3.12 (A. Capătă [37]) *Let* A *be a compact set, let* $K := (K_i)_{i \in I}$ *be a family of Henig dilating cones for* C, *and let* $\varphi : A \times B \to Z$ *satisfy the following conditions:*

(i) *for each* $b \in B$ *and each* $i \in I$, *the function* $\varphi(\cdot, b) : A \to Z$ *is* K_i*-upper semicontinuous on* A*;*

(ii) *there exists* $c^* \in K^{\triangle}$ *such that* $c^* \circ \varphi$ *is subconcavelike in its first variable;*

(iii) *for all* $b_1, \ldots, b_n \in B$, *each* $i \in I$, *and all* $k_1^*, \ldots, k_n^* \in K_i^*$ *not all zero, one has*

$$\sup_{a \in A} \sum_{j=1}^{n} k_j^*\big(\varphi(a, b_j)\big) \geq 0.$$

Then problem (VEP) *admits a Henig weakly efficient solution.*

Remark 3.3.13 When φ is K_i-subconcavelike for each $i \in I$, then assumption (ii) of Corollary 3.3.12 is satisfied. □

In order to give another existence result for Henig weakly efficient solutions of problem (VEP), we need the following notion.

Definition 3.3.14 (A. Capătă [37]) Let $(K_i)_{i \in I}$ (where $K_i \subseteq Z$ for each $i \in I$) be a family of Henig dilating cones for C. A pair (K_{i_1}, K_{i_2}), where $i_1, i_2 \in I$, is said to be admissible if

$$K_{i_1} + K_{i_2} = K_{i_0} \text{ for some } i_0 \in I.$$

It is easy to see that the family of Henig dilating cones for C constructed in Example 3.3.4 admits such kind of pairs. This remark is also true for the family of Henig efficient cones for C constructed in Example 3.3.5. Indeed, when $U_1, U_2 \in \mathcal{U}$, then $U_3 := \mathrm{co}(U_1 \cup U_2)$ belongs to $\mathcal{U}$ and $K_{U_1}(\mathcal{B}) + K_{U_2}(\mathcal{B}) = K_{U_3}(\mathcal{B})$.

Theorem 3.3.15 (A. Capătă [37]) *Let* A *be a compact set, let* $K := (K_i)_{i \in I}$ *be a family of Henig dilating cones for* C, *let* (K_{i_1}, K_{i_2}) *be an admissible pair, and let* $\varphi : A \times B \to Z$ *satisfy the following conditions:*

(i) *for each* $b \in B$, *the function* $\varphi(\cdot, b) : A \to Z$ *is* K_{i_1}*-upper semicontinuous on* A*;*

(ii) *for each $i \in I$, all $a_1, \ldots, a_m \in A$, all $\lambda_1, \ldots, \lambda_m \geq 0$ with $\lambda_1 + \cdots + \lambda_m = 1$, and all $b_1, \ldots, b_n \in B$, there exists $k^* \in K_i^* \setminus \{0\}$ such that*

$$\min_{1 \leq j \leq n} \sum_{i=1}^{m} \lambda_i k^*\bigl(\varphi(a_i, b_j)\bigr) \leq \sup_{a \in A} \min_{1 \leq j \leq n} k^*\bigl(\varphi(a, b_j)\bigr);$$

(iii) *for all $b_1, \ldots, b_n \in B$ and all $k_1^*, \ldots, k_n^* \in K_{i_2}^*$ not all zero, one has*

$$\sup_{a \in A} \sum_{j=1}^{n} k_j^*\bigl(\varphi(a, b_j)\bigr) \geq 0.$$

Then problem (VEP) admits a Henig weakly efficient solution.

Let $K := (K_i)_{i \in I}$ (where $K_i \subseteq Z$ for each $i \in I$) be a family of Henig dilating cones for C. Put

$$K^{\blacktriangle} := \{c^* \in E^* \mid \exists\, i \in I \,:\, c^* \in K_i^\sharp\}.$$

Obviously, we have $K^{\blacktriangle} \subseteq K^{\triangle}$.

Theorem 3.3.16 *Let A be a compact set, let $K := (K_i)_{i \in I}$ be a family of Henig dilating cones for C, and let $\varphi : A \times B \to Z$ satisfy the following conditions:*

(i) *for each $b \in B$ and each $i \in I$, the function $\varphi(\cdot, b) : A \to Z$ is properly K_i-upper semicontinuous on A;*

(ii) *there exists $c^* \in K^{\blacktriangle}$ such that, for all $a_1, \ldots, a_m \in A$, all the numbers $\lambda_1, \ldots, \lambda_m \geq 0$ with $\lambda_1 + \cdots + \lambda_m = 1$, and all $b_1, \ldots, b_n \in B$, the following inequality is satisfied:*

$$\min_{1 \leq j \leq n} \sum_{i=1}^{m} \lambda_i c^*\bigl(\varphi(a_i, b_j)\bigr) \leq \sup_{a \in A} \min_{1 \leq j \leq n} c^*\bigl(\varphi(a, b_j)\bigr);$$

(iii) *for all $b_1, \ldots, b_n \in B$, each $i \in I$, and all $k_1^*, \ldots, k_n^* \in K_i^*$ not all zero, one has*

$$\sup_{a \in A} \sum_{j=1}^{n} k_j^*\bigl(\varphi(a, b_j)\bigr) > 0.$$

Then problem (VEP) admits a Henig efficient solution.

Let $K := (K_i)_{i \in I}$ be a family of Henig dilating cones for C. If, for each $i \in I$ there exists $i_0 \in I$ such that $K_{i_0} \setminus \{0\} \subseteq \operatorname{int} K_i$, then each Henig weakly efficient solution is a Henig efficient solution and $K^{\triangle} = K^{\blacktriangle}$. Obviously, the family of Henig dilating cones $K := (K_\epsilon(\mathcal{B}))_{\epsilon \in]0,d[}$, constructed in Example 3.3.4 admits cones with this property. This remark is also true for the family of Henig dilating cones constructed in Example 3.3.5, as J. H. Qiu and Y. Hao noticed in [102, Lemma 3.3].

Let $(K_i)_{i \in I}$ be a family of Henig dilating cones for C. For any $i \in I$ and any $k_i^* \in K_i^* \setminus \{0\}$, let us consider the following scalar equilibrium problem:

$(EP_{k_i^*})$ find $\bar{a} \in A$ such that $k_i^*(\varphi(\bar{a}, b)) \geq 0$ for all $b \in B$.

Proposition 3.3.17 *Let $(K_i)_{i\in I}$ be a family of Henig dilating cones for C, and take any $i \in I$ and any $k_i^* \in K_i^* \setminus \{0\}$. Then, each solution of the scalar equilibrium problem $(EP_{k_i^*})$ is a K_i-Henig weakly efficient solution of problem (VEP).*

The next statement proves that under certain assumptions each net of K_i-weakly efficient solutions of (VEP) (where $(K_i)_{i\in I}$ is a family of Henig dilating cones for C), obtained by scalarization, admits a subnet converging to a solution of (VEP).

Theorem 3.3.18 (A. Capătă [37]) *Let A be a compact subset of a Hausdorff topological space E, let Z be a real Hausdorff locally convex space, let $(K_i)_{i\in I}$ be a family of Henig dilating cones for C, and let $(k_i^*)_{i\in I}$ be a net of functionals with $k_i^* \in K_i \setminus \{0\}$ for all $i \in I$ such that the following conditions are satisfied:*

(i) *for each $b \in B$, the function $\varphi(\cdot, b)$ is C-upper semicontinuous on A;*

(ii) *the set $\varphi(A \times B)$ is weakly bounded;*

(iii) *the net $(k_i^*)_{i\in I}$ converges with respect to $\beta(Z^*, Z)$ to a functional $k^* \in C^\sharp$.*

Then any net $(\bar{a}_i)_{i\in I}$ in A, where $\bar{a}_i \in A$ is a solution of problem $(EP_{k_i^*})$, admits a subnet converging to a solution of problem (VEP).

Now, we turn our attention on existence results of superefficient solutions of problem (VEP). For being able to give the following results, we assume, in addition, that Z is a Hausdorff locally convex space and work with the Henig family of dilating cones for C from Example 3.3.5. By Example 3.3.5 it follows that $C^\triangle \neq \emptyset$.

The next concepts of proper efficient solutions were introduced in locally convex spaces by X. H. Gong, W. T. Fu and W. Liu [65] and X. H. Gong [59].

Definition 3.3.19 A point $a \in A$ is said to be:

(i) a superefficient solution to (VEP) if, for each neighbourhood V of the origin of Z, there exists a neighbourhood U of the origin of Z such that

$$\operatorname{cone}(\varphi(a, B)) \cap (U - C) \subseteq V;$$

(ii) a globally efficient solution to (VEP) if there exists a Henig dilating cone $K \subseteq Z$ for C such that

$$\varphi(a, B) \cap (-K \setminus \{0\}) = \emptyset.$$

The sets of Henig weakly efficient solutions, superefficient solutions and globally efficient solutions, respectively, are denoted by $V_H(\varphi)$, $V_S(\varphi)$ and $V_G(\varphi)$, respectively.

To see that the set of Henig weakly efficient solutions, defined by the family of Henig dilating cones from Example 3.3.5, is larger than the set of superefficient solutions we give an example.

Example 3.3.20 Let $Z := \mathbb{R}^2$, $C := \mathbb{R}^2_+$, $A := [-2, -1]$, $B := [1, 2]$, and let the bifunction $\varphi : [-2, -1] \times [1, 2] \to \mathbb{R}^2$ be defined by

$$\varphi(x, y) := \begin{cases} (2, -2) & \text{if } (x, y) = (-2, 1) \\ (x, y) & \text{otherwise.} \end{cases}$$

Take the base $\mathcal{B}$ to be the set
$$\{(x,y) \in \mathbb{R}^2_+ \mid x+y=2\}.$$
We observe that this base is a closed and convex subset of $\mathbb{R}^2$. Let

$$z^*(b) := \langle (1,1),(b_1,b_2)\rangle = b_1+b_2 \text{ for all } b := (b_1,b_2) \in \mathcal{B}.$$

So, we get $r=2$ and
$$V_{\mathcal{B}}(z^*) := \{z \in Z \mid\, \mid z^*(z) \mid < 1\} = B(0,1).$$
For each $a \in [-2,-1]$ there exists a convex neighbourhood U_a of the origin of $\mathbb{R}^2$, with $U_a \subseteq B(0,1)$, such that
$$\varphi(a,B) \cap (-\operatorname{int} C_{U_a}(\mathcal{B})) = \emptyset$$
This means that all the points $a \in [-2,-1]$ are Henig weakly efficient solutions of the vector equilibrium problem (VEP).

On the other part, each point $a \in]-2,-1]$ is a superefficient solution of (VEP). Hence, we have the following inclusion

$$V_S(\varphi) =]-2,-1] \subseteq [-2,-1] = V_H(\varphi),$$

which assures that the set of Henig weakly efficient solutions is larger than the set of superefficient solutions. □

Definition 3.3.21 Given a functional $c^* \in C^* \setminus \{0\}$, a point $\bar{a} \in A$ is said to be a c^*-efficient solution to (VEP) if
$$c^*(\varphi(\bar{a},b)) \geq 0 \text{ for all } b \in B.$$

By $V_{c^*}(\varphi)$ we denote the set of all c^*-efficient solutions to (VEP).

Lemma 3.3.22 (X. H. Gong [59]) *If C is closed and has a closed and bounded base $\mathcal{B}$, then*

$$\operatorname{int} C^* = C^{\triangle}(\mathcal{B}),$$

where $\operatorname{int} C^*$ *is the interior of C^* with respect to the strong topology $\beta(Z^*,Z)$.*

In normed spaces, this property was proved by X. Y. Zheng [119].

Theorem 3.3.23 (X. H. Gong [61]) *Assume that, for each $a \in A$, $\varphi(a,A)$ is a C-convex set. If C is based, then the following properties hold:*

(i) $V_G(\varphi) = \bigcup_{c^* \in C^\sharp} V_{c^*}(\varphi)$.

(ii) $V_H(\varphi) = \bigcup_{c^* \in C^\triangle} V_{c^*}(\varphi)$.

(iii) *If C is closed and has a closed and bounded base, then*

$$V_S(\varphi) = \bigcup_{c^* \in \operatorname{int} C^*} V_{c^*}(\varphi).$$

So, by Lemma 3.3.22 and Theorem 3.3.23, a point $\bar{a} \in A$ is a superefficient solution of (VEP) if and only if $\bar{a}$ is a Henig efficient solution.

By Theorem 3.3.11, Corollary 3.3.12, and Theorem 3.3.15 we have the following results.

Theorem 3.3.24 *Let E be a real Hausdorff topological linear space, let A be a compact set, let $B = A$, let $\varphi(a, A)$ be a C-convex set for each $a \in A$, let C be a closed cone with a closed and bounded base $\mathcal{B}$, and let $\varphi : A \times B \to Z$ satisfy the following conditions:*

(i) *for each $b \in A$ and for each convex neighbourhood U of the origin of Z satisfying $U \subseteq V_{\mathcal{B}}(z^*)$, the function $\varphi(\cdot, b) : A \to Z$ is $K_U(\mathcal{B})$-upper semicontinuous on A;*

(ii) *there exists $c^* \in K^{\triangle}$ such that, for all $a_1, \ldots, a_m \in A$, all the numbers $\lambda_1, \ldots, \lambda_m \geq 0$ with $\lambda_1 + \cdots + \lambda_m = 1$, and all $b_1, \ldots, b_n \in A$, the following inequality is satisfied:*

$$\min_{1 \leq j \leq n} \sum_{i=1}^{m} \lambda_i c^*\big(\varphi(a_i, b_j)\big) \leq \sup_{a \in A} \min_{1 \leq j \leq n} c^*\big(\varphi(a, b_j)\big);$$

(iii) *for all $b_1, \ldots, b_n \in A$, each convex neighbourhood U of the origin of Z satisfying $U \subseteq V_{\mathcal{B}}(z^*)$, and all $k_1^*, \ldots, k_n^* \in K_U^*(\mathcal{B})$ not all zero, one has*

$$\sup_{a \in A} \sum_{j=1}^{n} k_j^*\big(\varphi(a, b_j)\big) \geq 0.$$

Then problem (VEP) admits a superefficient solution.

Corollary 3.3.25 *Let E be a real Hausdorff topological linear space, let A be a compact set, let $B = A$, let $\varphi(a, A)$ be a C-convex set for each $a \in A$, let C be a closed cone with a closed and bounded base $\mathcal{B}$, and let $\varphi : A \times B \to Z$ satisfy the following conditions:*

(i) *for each $b \in A$ and each convex neighbourhood U of the origin of Z with $U \subseteq V_{\mathcal{B}}(z^*)$, the function $\varphi(\cdot, b) : A \to Z$ is $K_U(\mathcal{B})$-upper semicontinuous on A;*

(ii) *there exists $c^* \in K^{\triangle}$ such that $c^* \circ \varphi$ is subconcavelike in its first variable;*

(iii) *for all $b_1, \ldots, b_n \in A$, each convex neighbourhood U of the origin of Z satisfying $U \subseteq V_{\mathcal{B}}(z^*)$, and all $k_1^*, \ldots, k_n^* \in K_U^*(\mathcal{B})$ not all zero, one has*

$$\sup_{a \in A} \sum_{j=1}^{n} k_j^*\big(\varphi(a, b_j)\big) \geq 0.$$

Then problem (VEP) admits a superefficient solution.

Theorem 3.3.26 *Let E be a real Hausdorff topological linear space, let A be a compact set, let $B = A$, let $\varphi(a, A)$ be a C-convex set for each $a \in A$, let C be a closed cone with a closed and bounded base $\mathcal{B}$, let $(K_{U_1}(\mathcal{B}), K_{U_2}(\mathcal{B}))$ be an admissible pair, and let $\varphi : A \times B \to Z$ satisfy the following conditions:*

(i) *for each $b \in B$, the function $\varphi(\cdot, b) : A \to Z$ is $K_{U_1}(\mathcal{B})$-upper semicontinuous on A;*

(ii) *for each convex neighbourhood U of the origin of Z satisfying $U \subseteq V_{\mathcal{B}}(z^*)$, and all the points $a_1, \ldots, a_m \in A$, all $\lambda_1, \ldots, \lambda_m \geq 0$ with $\lambda_1 + \cdots + \lambda_m = 1$, and all $b_1, \ldots, b_n \in B$, there exists $k^* \in K_U^*(\mathcal{B}) \setminus \{0\}$ such that*

$$\min_{1\le j\le n}\sum_{i=1}^{m}\lambda_i k^*\big(\varphi(a_i,b_j)\big)\le \sup_{a\in A}\min_{1\le j\le n} k^*\big(\varphi(a,b_j)\big);$$

(iii) *for all* $b_1,\dots,b_n\in B$ *and all* $k_1^*,\dots,k_n^*\in K_{U_2}^*(\mathcal{B})$ *not all zero, one has*

$$\sup_{a\in A}\sum_{j=1}^{n}k_j^*\big(\varphi(a,b_j)\big)\ge 0.$$

Then problem (VEP) *admits a superefficient solution.*

Finally, we present sufficient conditions for the existence of globally efficient solutions of the vector equilibrium problem (VEP).

Theorem 3.3.27 *Let* A *be a compact set, and let* $\varphi : A\times B\to Z$ *satisfy the following conditions:*

(i) *for each* $b\in B$*, the function* $\varphi(\cdot,b):A\to Z$ *is properly* C*-upper semicontinuous on the set* A*;*

(ii) *there exists* $c^*\in C^\sharp$ *such that, for all* $a_1,\dots,a_m\in A$*, all numbers* $\lambda_1,\dots,\lambda_m\ge 0$ *with* $\lambda_1+\dots+\lambda_m=1$*, and all* $b_1,\dots,b_n\in B$*, the following inequality is satisfied:*

$$\min_{1\le j\le n}\sum_{i=1}^{m}\lambda_i c^*\big(\varphi(a_i,b_j)\big)\le \sup_{a\in A}\min_{1\le j\le n} c^*\big(\varphi(a,b_j)\big);$$

(iii) *for all* $b_1,\dots,b_n\in B$*, and all* $c_1^*,\dots,c_n^*\in C^*$ *not all zero, one has*

$$\sup_{a\in A}\sum_{j=1}^{n}c_j^*\big(\varphi(a,b_j)\big)>0.$$

Then problem (VEP) *admits a globally efficient solution.*

Corollary 3.3.28 *Let* A *be a compact set, and let* $\varphi : A\times B\to Z$ *satisfy the following conditions:*

(i) *for each* $b\in B$*, the function* $\varphi(\cdot,b):A\to Z$ *is properly* C*-upper semicontinuous on the set* A*;*

(ii) *there exists* $c^*\in C^\sharp$ *such that* $c^*\circ\varphi$ *is concavelike in its first variable;*

(iii) *for all* $b_1,\dots,b_n\in B$*, and all* $c_1^*,\dots,c_n^*\in C^*$ *not all zero, one has*

$$\sup_{a\in A}\sum_{j=1}^{n}c_j^*\big(\varphi(a,b_j)\big)>0.$$

Then problem (VEP) *admits a globally efficient solution.*

When $Z:=\mathbb{R}$ and $C:=\mathbb{R}_+$, then this corollary reduces to Corollary 3.1.7.

Chapter 4

Existence results and gap multifunctions for weak multifunction equilibrium problems

Throughout this chapter E and Z are real topological linear spaces. Further, suppose that A is a nonempty subset of E, B is a nonempty set, $C \subseteq Z$ is a convex and solid cone, and $\varphi : A \times B \to 2^Z$ is a multifunction.

The weak vector equilibrium problem $(WVEP)$, studied in Section 2.1, can be extended to multifunctions in two ways:

$(WWMEP)$ find $\bar{a} \in A$ such that $\varphi(\bar{a}, b) \nsubseteq -\operatorname{int} C$ for all $b \in B$;

$(SWMEP)$ find $\bar{a} \in A$ such that $\varphi(\bar{a}, b) \cap (-\operatorname{int} C) = \emptyset$ for all $b \in B$.

In what follows we present existence results for the problem $(WWMEP)$, which is called *the weak multifunction equilibrium problem*. Furthermore, there are constructed two gap functions associated with the studied problem, one of them by means of Fenchel's duality theory.

4.1 Existence results established via Eidelheit's theorem

Our first result is a technical result. Its proof is based on Eidelheit's separation theorem. In what follows, we denote by $\mathcal{C}(Z)$ the set of all compact subsets of the space Z.

Theorem 4.1.1 (A. Capătă, G. Kassay and B. Mosoni [39]) *Let $\varphi : A \times B \to 2^Z$ satisfy the following conditions:*

(i) *$\varphi(a, b) \in \mathcal{C}(Z)$ for every $(a, b) \in A \times B$;*

(ii) *if the family $(U_{b,c})$ covers A, then it contains a finite subcover, where $U_{b,c}$ is defined by*

$$U_{b,c} := \{a \in A \mid \varphi(a, b) + c \subseteq -\operatorname{int} C\} \text{ for all } b \in B,\ c \in \operatorname{int} C;$$

(iii) *for all $a_1, \ldots, a_m \in A$, all $\lambda_1, \ldots, \lambda_m \geq 0$ with $\lambda_1 + \cdots + \lambda_m = 1$, all $b_1, \ldots, b_n \in B$, and all $d_j^i \in \varphi(a_i, b_j)$, where $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$, there exists $c^* \in C^* \setminus \{0\}$ such that*

$$\min_{1\le j\le n}\sum_{i=1}^{m}\lambda_i c^*(d_j^i) \le \sup_{a\in A}\min_{1\le j\le n}\ \max c^*(\varphi(a,b_j));$$

(iv) *for all $b_1,\ldots,b_n \in B$ and all $c_1^*,\ldots,c_n^* \in C^*$ not all zero, one has*

$$\sup_{a\in A}\sum_{j=1}^{n}\ \max c_j^*(\varphi(a,b_j)) \ge 0.$$

Then problem $(WWMEP)$ admits a solution.

Taking into consideration the characterization of C-subconvexlikeness of a multifunction presented in Section 1.2, we introduce a new convexity notion for multifunctions of two variables.

Definition 4.1.2 (A. Capătă, G. Kassay and B. Mosoni [39]) We say that the multifunction $\varphi : A \times B \to 2^Z$ is:

(i) C-subconvexlike in its first variable if, for all $c \in \operatorname{int} C$, all $a_1,\ a_2 \in A$ and all $\lambda \in [0,1]$, there exists an $a_3 \in A$ such that

$$c + \lambda\varphi(a_1,b) + (1-\lambda)\varphi(a_2,b) \subseteq \varphi(a_3,b) + \operatorname{int} C \text{ for all } b \in B.$$

(ii) C-subconcavelike in its first variable if $-\varphi$ is C-subconvexlike in its first variable.

The next result provides sufficient conditions for the existence of solutions of $(WWMEP)$ by means of convexity and continuity assumptions.

Theorem 4.1.3 (A. Capătă, G. Kassay and B. Mosoni [39]) *Let A be a compact set, and let $\varphi : A \times B \to 2^Z$ satisfy the following conditions:*

(i) *$\varphi(a,b) \in \mathcal{C}(Z)$ for every $(a,b) \in A \times B$;*

(ii) *for all $b \in B$, $\varphi(\cdot,b) : A \to \mathcal{C}(Z)$ is C-upper semicontinuous on A;*

(iii) *φ is C-subconcavelike in its first variable;*

(iv) *for all $b_1,\ldots,b_n \in B$ and all $c_1^*,\ldots,c_n^* \in C^*$ not all zero, one has*

$$\sup_{a\in A}\sum_{j=1}^{n}\max c_j^*(\varphi(a,b_j)) \ge 0.$$

Then problem $(WWMEP)$ admits a solution.

4.2 Gap multifunctions

In connection with scalar equilibrium problems and their particular cases the so-called gap functions play an important role. They help to analyze whether a point is a solution of these problems.

4.2.1 A gap multifunction

First let us recall the definition of a gap multifunction.

Definition 4.2.1 (N. J. Huang, J. Li and S. Y. Wu [76]) A multifunction $T : A \to 2^Z$ is said to be a gap multifunction for $(WWMEP)$ if:

(i) $T(a) \subseteq -C$ for all $a \in A$;

(ii) $0 \in T(a)$ if and only if $a \in A$ is a solution of $(WWMEP)$.

In what follows we give an example of a gap multifunction for the problem $(WWMEP)$. In this way we extend a result from N. J. Huang, J. Li and S. Y. Wu [76] to a multifunction that takes values in a real topological linear space.

Consider the following assumption:

Assumption A.

$$\text{Let } B = A. \text{ If } a \in A \text{ is a solution of } (WWMEP), \text{ then } \bigcap_{b \in A} \{\varphi(a,b) \cap C\} \neq \emptyset.$$

Theorem 4.2.2 (A. Capătă, G. Kassay and B. Mosoni [39]) *Suppose that the following conditions are satisfied:*

(i) *C is a pointed cone;*

(ii) *$\varphi(a,a) \subseteq -C$ for all $a \in A$;*

(iii) *Assumption* **A** *holds.*

Then the multifunction $T : A \to 2^Z$, defined by

$$T(a) := \bigcap_{b \in A} \varphi(a,b) \textit{ for each } a \in A,$$

is a gap multifunction for $(WWMEP)$.

Now let us consider the particular case of $(WWMEP)$ which has been studied by N. J. Huang, J. Li and S. Y. Wu [76]. For $n \in \mathbb{N}$, $N := \{1, \ldots, n\}$ and $F_l : A \times A \to 2^{\mathbb{R}}$ $(l \in N)$, consider the problem $(WWMEP)$ for the multifunction defined by

$$F(a,b) := F_1(a,b) \times \cdots \times F_n(a,b),$$

i.e.

$$(GFVEP1) \qquad \text{find } \bar{a} \in A \text{ such that } F(\bar{a},b) \nsubseteq -\operatorname{int} \mathbb{R}^n_+ \text{ for all } b \in A.$$

Define a multifunction $T_1 : A \to 2^{\mathbb{R}}$ as follows:

$$T_1(a) := \bigcap_{b \in A} \bigcup_{l \in \mathbb{N}} F_l(a,b) \text{ for all } a \in A. \tag{4.1}$$

Further, consider the following assumption used by N. J. Huang, J. Li and S. Y. Wu [76]:
Assumption B. Let $B = A$. If $a \in A$ and $\bigcup_{l\in N} F_l(a,b) \cap \mathbb{R}_+ \neq \emptyset$ for all $b \in A$, then

$$\bigcap_{b\in A} \bigcup_{l\in N} \{F_l(a,b) \cap \mathbb{R}_+\} \neq \emptyset.$$

Corollary 4.2.3 (N. J. Huang, J. Li and S. Y. Wu [76], Theorem 4.4) *If*

$$F_l(a,a) \subseteq -\mathbb{R}_+ \text{ for each } a \in A \text{ and each } l \in N,$$

and Assumption **B** *holds, then the multifunction T_1 defined by (4.1) is a gap multifunction for* $(GFVEP1)$ *in the sense of Definition 4.2.1, where $Z := \mathbb{R}$ and $C := \mathbb{R}_+$.*

4.2.2 A gap function using Fenchel's duality

Throughout this section let $Z := \mathbb{R}$ and $C := \mathbb{R}_+$. Then $\varphi : A \times B \to 2^{\mathbb{R}}$ and $(WWMEP)$ becomes:

(MEP) find $\bar{a} \in A$ such that $\varphi(\bar{a}, b) \nsubseteq -\operatorname{int} \mathbb{R}_+$ for all $b \in B$.

For the equilibrium problem (MEP), we establish the following results by means of the existence results given for $(WWMEP)$ in Section 4.1.

Corollary 4.2.4 (A. Capătă, G. Kassay and B. Mosoni [39]) *Let $\varphi : A \times B \to 2^{\mathbb{R}}$ satisfy the following conditions:*

(i) *$\varphi(a,b) \in \mathcal{C}(\mathbb{R})$ for every $(a,b) \in A \times B$;*

(ii) *if the family $(U_{b,c})$ covers A, then it contains a finite subcover, where $U_{b,c}$ is defined by*

$$U_{b,c} = \{a \in A \,|\, \varphi(a,b) + c \subseteq]-\infty, 0[\,\} \text{ for all } b \in B \text{ and } c \in]-\infty, 0[;$$

(iii) *for all $a_1, \ldots, a_m \in A$, all $\lambda_1, \ldots, \lambda_m \geq 0$ with $\lambda_1 + \cdots + \lambda_m = 1$, all $b_1, \ldots, b_n \in B$, and all $d^i_j \in \varphi(a_i, b_j)$ where $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$, one has*

$$\min_{1\leq j\leq n} \sum_{i=1}^{m} \lambda_i d^i_j \leq \sup_{a\in A} \min_{1\leq j\leq n} \max \varphi(a, b_j);$$

(iv) *for all $b_1, \ldots, b_n \in B$ and all $\mu_1, \ldots, \mu_n \geq 0$ with $\mu_1 + \cdots + \mu_n = 1$, one has*

$$\sup_{a\in A} \sum_{j=1}^{n} \max \mu_j \varphi(a, b_j) \geq 0.$$

Then problem (MEP) admits a solution.

Corollary 4.2.5 (A. Capătă, G. Kassay and B. Mosoni [39]) *Let A be a compact set, and let $\varphi : A \times B \to 2^{\mathbb{R}}$ satisfy the following conditions:*

(i) *$\varphi(a,b) \in \mathcal{C}(\mathbb{R})$ for every $(a,b) \in A \times B$;*

(ii) *for all* $b \in B$, $\varphi(\cdot, b)$ *is* $-\mathbb{R}_+$*-upper semicontinuous on* A*;*

(iii) φ *is* $\mathbb{R}_+$*-subconcavelike in its first variable;*

(iv) *for all* $b_1, \ldots, b_n \in B$ *and all* $\mu_1, \ldots, \mu_n \geq 0$ *with* $\mu_1 + \cdots + \mu_n = 1$*, one has*

$$\sup_{a \in A} \sum_{j=1}^{n} \max \mu_j \varphi(a, b_j) \geq 0.$$

Then problem (MEP) *admits a solution.*

In the final part of this section we suppose that A is a closed and convex subset of a real locally convex space E, $B = A$, and that $\varphi(a, b) \in \mathcal{C}(\mathbb{R})$ for every $(a, b) \in A \times A$. We observe that (MEP) is equivalent to the following problem:

$$\text{find } \bar{a} \in A \text{ such that } \max \varphi(\bar{a}, b) \geq 0 \ \text{ for all } b \in A,$$

or, equivalently:

$$(EP_\psi) \qquad \text{find } \bar{a} \in A \text{ such that } \psi(\bar{a}, b) \geq 0 \ \text{ for all } b \in A,$$

where $\psi : E \times E \to \mathbb{R} \cup \{+\infty\}$, with $A \times A \subseteq \operatorname{dom} \psi$, is defined by

$$\psi(a, b) := \max \varphi(a, b) \text{ for all } a, b \in A.$$

Further, suppose that

$$\max \varphi(a, a) = 0 \text{ for all } a \in A.$$

Let $a \in E$. According to L. Altangerel, R. I. Boţ and G. Wanka [2], (EP_ψ) can be reduced to the following scalar minimization problem:

$$(P_a) \qquad \inf_{b \in A} \psi(a, b).$$

Indeed, it is easy to check that $\bar{a} \in A$ is a solution of (EP_ψ) if and only if it is a solution of $(P_{\bar{a}})$.

The next definition is a particular case of Definition 4.2.1, when $C := -\mathbb{R}_+$. A function $\gamma : E \to \mathbb{R} \cup \{-\infty, +\infty\}$ is said to be a gap function for (EP_ψ) (see G. Mastroeni [96]) if it satisfies the following conditions:

(i) $\gamma(a) \geq 0$ for all $a \in A$;

(ii) $\gamma(a) = 0$ and $a \in A$ if and only if a is a solution for (EP_ψ).

By means of the indicator function δ_A, we can rewrite problem (P_a) as follows:

$$(P_a) \qquad \inf_{b \in E} \{\psi(a, b) + \delta_A(b)\}.$$

Proposition 4.2.6 (A. Capătă, G. Kassay and B. Mosoni [39]) *Let* $a \in A$*. If the multifunction* $b \in A \mapsto \varphi(a, b) \in 2^{\mathbb{R}}$ *is* λ*-concave for each* $\lambda \in]0, 1[$ *and* d*-upper semicontinuous on* A*, where* d *is the absolute value on* $\mathbb{R}$*, then the function*

$$(4.2) \qquad b \in A \mapsto \psi(a, b) \in \mathbb{R} \cup \{+\infty\}$$

is convex and lower semicontinuous on A.

Let the assumptions of the previous proposition be satisfied. The Fenchel dual of (P_a) is

$$(D_a) \qquad \sup_{x^* \in E^*} \{-\psi_b^*(a, x^*) - \sigma_A(-x^*)\},$$

where

$$\psi_b^*(a, x^*) := \sup_{b \in E} [x^*(b) - \psi(a, b)].$$

For problem (P_a), the regularity condition (FRC) introduced in Section 1.3 becomes

$(FRC; a)$ $\qquad$ $\psi_b^* \Box \sigma_A$ is a lower semicontinuous function and exact at 0,

where

$$(\psi_b^* \Box \sigma_A)(x^*) := \inf \{\psi_b^*(x_1^*) + \sigma_A(x_2^*) \mid x_1^* + x_2^* = x^*\}.$$

Theorem 4.2.7 (R. I. Boţ and G. Wanka [29]) *Assume that, for all $a \in A$, the following conditions are satisfied:*

(i) *the regularity condition $(FRC; a)$ is fulfilled;*

(ii) *the function (4.2) is convex and lower semicontinuous on A.*

Then the function $\gamma : E \to \mathbb{R} \cup \{-\infty, +\infty\}$, defined by

$$\gamma(a) := -v(D_a),$$

is a gap function for (EP_ψ).

By Proposition 4.2.6 and Theorem 4.2.7 we have the next result.

Theorem 4.2.8 (A. Capătă, G. Kassay and B. Mosoni [39]) *Assume that, for all $a \in A$, the following conditions are astisfied:*

(i) *the regularity condition $(FRC; a)$ is fulfilled;*

(ii) *the multifunction (4.2) is λ-concave for each $\lambda \in]0, 1[$ and d-upper semicontinuous on A.*

Then γ, defined in Theorem 4.2.7, is a gap function for (MEP).

Chapter 5

A vector optimization problem and cone saddle points

5.1 The vector optimization problem

The weak vector equilibrium problem $(WWEP)$ studied in Section 2.1 contains as particular cases, vector optimization problems, vector variational inequalities and cone saddle point problems (see e.g. Q. H. Ansari [3]). The vector optimization problems consist in finding the subset of weak minima (or weak maxima) of a set in a real topological linear space Z with respect to a solid, convex cone C of Z.

Let $S \subseteq Z$. A point $z_0 \in S$ is said to be a weak minimum of S with respect to the cone C if

$$S \cap (z_0 - \operatorname{int} C) = \emptyset.$$

By $\operatorname{Min}_w S$ we denote the set of weak minima of S with respect to the cone C.

Let A be a nonempty subset of a topological space E, let $F : A \to Z$ be a given function, and let $c^* \in C^* \setminus \{0\}$. We consider the scalar equilibrium problem:

$$(EP_{c^*}) \qquad \text{find } \bar{a} \in A \text{ such that } f(\bar{a}, b) \geq 0 \text{ for all } b \in A,$$

where $f : A \times A \to \mathbb{R}$ is given by $f(a, b) := c^*(F(b) - F(a))$.

Denoting by $\varphi : A \times A \to Z$ the vector bifunction defined by

$$\varphi(a, b) := F(b) - F(a) \text{ for all } a, b \in A,$$

it is easy to see that problem $(WVEP)$, considered in Chapter 2, becomes for this φ *the weak vector minimization problem*:

$$(WVMP) \qquad \text{find } \bar{a} \in A \text{ such that } F(b) - F(\bar{a}) \notin -\operatorname{int} C \text{ for all } b \in A.$$

A point $\bar{a} \in A$ is a solution of $(WVMP)$ if and only if $F(\bar{a}) \in \operatorname{Min}_w F(A)$.

Proposition 5.1.1 (A. Capătă and G. Kassay [38]) *For each $c^* \in C^* \setminus \{0\}$, the set of solutions of (EP_{c^*}) is contained in the set of solutions of (WVMP).*

Observe that the reverse implication in Proposition 5.1.1 is not true in general, as the following example shows.

Example 5.1.2 (A. Capătă and G. Kassay [38]) Let $F : \mathbb{R} \to \mathbb{R}^2$ be the function defined by

$$F(a) := \begin{cases} (-1, \dfrac{1}{\mid a \mid}) & \text{if } a \neq 0 \\ (0,0) & \text{if } a = 0. \end{cases}$$

Take $C := \mathbb{R}^2_+$, and define $c^* : \mathbb{R}^2 \to \mathbb{R}$ by

$$c^*(x_1, x_2) := \langle (1,0), (x_1, x_2) \rangle = x_1 \text{ for all } (x_1, x_2) \in \mathbb{R}^2.$$

Thus, we have

$$f(0,b) = \langle (1,0), F(b) - F(0) \rangle = \begin{cases} -1 & \text{if } b \neq 0 \\ 0 & \text{if } b = 0, \end{cases}$$

thus $\bar{a} := 0$ is not a solution of the scalar equilibrium problem (EP_{c^*}).

We verify whether $\bar{a}$ is a solution of $(WVMP)$. Indeed,

$$\varphi(0,b) = F(b) - F(0) = \begin{cases} (-1, \dfrac{1}{\mid b \mid}) & \text{if } b \neq 0 \\ (0,0) & \text{if } b = 0. \end{cases}$$

This relation shows that $\varphi(0,b) \notin -\operatorname{int} \mathbb{R}^2_+$ for each $b \in \mathbb{R}$, which implies that $\bar{a}$ is a solution of $(WVMP)$. □

It is interesting to notice that in the previous example every $a \neq 0$ is a solution of (EP_{c^*}). Indeed,

$$f(a,b) = \langle (1,0), F(b) - F(a) \rangle = \begin{cases} 0 & \text{if } b \neq 0 \\ 1 & \text{if } b = 0, \end{cases}$$

hence $f(a,b) \geq 0$ for all $b \in \mathbb{R}$, i.e. $a \neq 0$ is a solution of (EP_{c^*}). By Proposition 5.1.1 it follows that each real number is a solution of $(WVMP)$.

The next result is a consequence of Corollary 2.1.5.

Proposition 5.1.3 (A. Capătă and G. Kassay [38]) *If A is a compact set and the function $F : A \to Z$ is C-lower semicontinuous on A, then the scalar equilibrium problem (EP_{c^*}) admits a solution for every $c^* \in C^* \setminus \{0\}$.*

In vector optimization different concepts of lower semicontinuity have been used. The next concept (considered in J. Borwein, J. Penot and M. Théra [25] and M. Théra [110]) is a slight relaxation of lower semicontinuity.

Definition 5.1.4 Let A be a nonempty subset of a topological space. A vector-valued function $f : A \to Z$ is said to be quasi-lower semicontinuous at $a \in A$ if, for each $z \in Z$ such that $z \not\geq_C f(a)$, there exists a neighbourhood U of a such that

$$z \not\geq_C f(u) \text{ for each } u \in U \cap A.$$

The question whether the C-lower semicontinuity in Proposition 5.1.3 can be weakened to quasi-lower semicontinuity arises naturally. The next example shows that the answer is negative.

Example 5.1.5 Let $Z := \mathbb{R}^3$, $C := \mathbb{R}^3_+$, $A := [0,1]$, and define $F : \mathbb{R} \to \mathbb{R}^3$ by

$$F(t) := \begin{cases} \left(-1, -\frac{2}{t}, \frac{1}{t}\right) & \text{if } t \neq 0 \\ (0,0,0) & \text{if } t = 0. \end{cases}$$

It is easy to verify that F is quasi-lower semicontinuous at 0, but it is not $\mathbb{R}^3_+$-lower semicontinuous at 0.

Let the functional $c^* : \mathbb{R}^3 \to \mathbb{R}$ be defined by

$$c^*(x_1, x_2, x_3) := \langle \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), (x_1, x_2, x_3) \rangle = \frac{1}{\sqrt{3}}(x_1 + x_2 + x_3) \text{ for all } (x_1, x_2, x_3) \in \mathbb{R}^3.$$

Further, define $f : A \times A \to \mathbb{R}$ by $f(a,b) := c^*(F(b) - F(a))$. We claim that (EP_{c^*}) does not admit any solution.

Indeed, if $a \neq 0$, we obtain

$$f(a,b) = \langle \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), F(b) - F(a) \rangle = \begin{cases} \frac{1}{\sqrt{3}}\left(\frac{1}{a} - \frac{1}{b}\right) & \text{if } b \neq 0 \\ \frac{1}{\sqrt{3}}\left(1 + \frac{1}{a}\right) & \text{if } b = 0, \end{cases}$$

Therefore we have $f(a,b) < 0$ for $b < a$. This shows that $a \neq 0$ is not a solution of (EP_{c^*}).

For $a := 0$ the bifunction f becomes

$$f(0,b) = \langle \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), F(b) - F(0) \rangle = \begin{cases} \frac{1}{\sqrt{3}}\left(-\frac{1}{b} - 1\right) & \text{if } b \neq 0 \\ 0 & \text{if } b = 0\ . \end{cases}$$

We observe that $f(0,b) < 0$ for each $b \in]0,1]$. This means that a is not a solution of (EP_{c^*}). By consequence problem (EP_{c^*}) has no solutions, although the set of solutions of $(WVMP)$ is $[0,1]$. □

The Propositions 5.1.1 and 5.1.3 provide the following existence result for $(WVMP)$.

Proposition 5.1.6 (A. Capătă and G. Kassay [38]) *If A is a compact set and $F : A \to Z$ is C-lower semicontinuous on A, then problem (WVMP) admits a solution.*

5.2 Existence results for weak cone saddle points

By using the existence results for the weak vector equilibrium problem $(WVEP)$, presented in Section 2.1, in what follows we give existence results for weak cone saddle points of a bifunction. For this, let X and Y to be nonempty subsets of topological spaces, let Z be a real topological linear space, and let $f : X \times Y \to Z$ be a bifunction. For $x \in X$ and $y \in Y$ we introduce the sets

$$f(x, Y) := \{f(x,y) |\, y \in Y\} \text{ and } f(X, y) := \{f(x,y) |\, x \in X\}.$$

Let $S \subseteq Z$ be a nonempty set, and let $C \subseteq Z$ be a solid, convex cone. By $\mathrm{Max}_w\, S$ the set of weak maxima of S with respect to the cone C, i.e. $z_0 \in \mathrm{Max}_w S$ means that

$$z_0 \in S \text{ and } S \cap (z_0 + \operatorname{int} C) = \emptyset.$$

We recall the following concept which extends the classical definition of a saddle point of a scalar functions.

Definition 5.2.1 (T. Tanaka [108]) A point $(x_0, y_0) \in X \times Y$ is said to be a weak C-saddle point of f if

$$f(x_0, y_0) \in \mathrm{Max}_w\, f(X, y_0) \cap \mathrm{Min}_w\, f(x_0, Y).$$

Proposition 5.2.2 *Let $A := X \times Y$, $B := X \times Y$, and let the bifunction $\varphi : A \times B \to Z$ be defined by*

$$\varphi(a, b) := f(x, v) - f(u, y) \text{ for all } a := (x, y) \text{ and } b := (u, v) \in X \times Y.$$

If $\bar{a} \in A$ is a solution for (WVEP), then $\bar{a}$ is a weak C-saddle point of f.

We notice that the converse of Proposition 5.2.2 does not hold. To show this, we give an example.

Example 5.2.3 Let $X := [-1, 1]$, $Y := [-1, 1]$, $Z := \mathbb{R}^2$, $C := \mathbb{R}^2_+$. Further, let the bifunction $f : [-1, 1] \times [-1, 1] \to \mathbb{R}^2$ be defined by

$$f(x, y) := \begin{cases} (x, y) & \text{if } x \geq 0 \text{ and } y \leq 0 \text{ or, } x \leq 0 \text{ and } y \geq 0 \\ (0, 0) & \text{otherwise.} \end{cases}$$

It is easy to check that $\bar{a} := (0, 0)$ is a weak $\mathbb{R}^2_+$-saddle point of the bifunction f.

In order to verify whether the point $\bar{a}$ is a solution for $(WVEP)$, let the set $A := [-1, 1] \times [-1, 1]$, and let $B := A$. So, we have to check whether

$$\varphi(\bar{a}, b) = f(0, v) - f(u, 0) \notin -\operatorname{int} \mathbb{R}^2_+ \text{ for all } b := (u, v) \in A.$$

Taking $b := (1, -1) \in A$, we have

$$\varphi(\bar{a}, b) = f(0, -1) - f(1, 0) = (-1, -1) \in -\operatorname{int} \mathbb{R}^2_+.$$

Hence, $\bar{a}$ is a weak $\mathbb{R}^2_+$-saddle point for f, but is not a solution of the weak vector equilibrium problem $(WVEP)$. □

Theorem 5.2.4 (A. Capătă and G. Kassay [38]) *Let X and Y be compact sets, and let the bifunction $f : X \times Y \to Z$ satisfy the following conditions:*

(i) *f is C-upper semicontinuous with respect to its first variable on X and C-lower semicontinuous with respect to its second variable on Y;*

(ii) *f is C-subconcavelike – subconvexlike.*

Then f admits a weak C-saddle point.

5.3 Existence results for strong cone saddle points

Let S be a nonempty subset of a real topological linear space Z, which is ordered by a convex cone C. In what follows we shall denote by $\operatorname{Min} S$, the set of minima of S with respect to the cone C, i.e. $z_0 \in \operatorname{Min} S$ means that

$$z_0 \in S \text{ and } (S - z_0) \cap (-C) = \{0\}.$$

Similarly, $\operatorname{Max} S$ denotes the set of maxima of S with respect to the cone C, i.e. $z_0 \in \operatorname{Max} S$ means that

$$z_0 \in S \text{ and } (S - z_0) \cap C = \{0\}.$$

Further, denote by $\operatorname{IMin} S$ the set of ideal minima of S with respect to the cone C, i.e. $z_0 \in \operatorname{IMin} S$ means that

$$z_0 \in S \text{ and } z \geq_C z_0 \text{ for all } z \in S.$$

Similarly, $\operatorname{IMax} S$ denotes the set of ideal maxima of S with respect to the cone C, i.e. $z_0 \in \operatorname{IMax} S$ means that

$$z_0 \in S \text{ and } z_0 \geq_C z \text{ for all } z \in S.$$

Definition 5.3.1 Let X and Y be nonempty subsets of topological spaces, and let the bifunction $f : X \times Y \to Z$. A point $(x_0, y_0) \in X \times Y$ is said to be:

(i) a strong C-saddle point of f if

$$f(x_0, y_0) \in \operatorname{Max} f(X, y_0) \cap \operatorname{Min} f(x_0, Y).$$

(ii) an ideal strong C-saddle point of f if

$$f(x_0, y_0) \in \operatorname{IMax} f(X, y_0) \cap \operatorname{IMin} f(x_0, Y).$$

X. H. Gong [63] gave existence results for ideal strong C-saddle points for vector-valued bifunctions. It is well-known (see D. T. Luc [95], Proposition 2.2, page 41), that the set of ideal minima of a set (if it is nonempty) is equal to the set of Pareto minima, and if the cone is pointed, then the set of ideal minima is a singleton. Hence, X. H. Gong stated existence and uniqueness for strong C-saddle points.

Obviously, whenever $\operatorname{int} C \neq \emptyset$, each strong C-saddle point is a weak C-saddle point, but the viceversa is not true. To show this, we give the following example.

Example 5.3.2 Let $f : [-1, 1] \times [-1, 1] \to \mathbb{R}^2$ be defined by $f(x) := x$, $Z := \mathbb{R}^2$, and the cone $C := \mathbb{R}^2_+$. It is easy to see that $(0, 0)$ is a weak C-saddle point of f, but it is not a strong C-saddle point of f. □

Strong C-saddle points can be obtained as particular cases of the solutions of strong vector equilibrium problems, as the next proposition shows. Indeed, let us consider $A := X \times Y$, $B := A$, and define $\varphi : A \times A \to \mathbb{R}$ by

$$\varphi(a, b) := f(x, v) - f(u, y),$$

where $a := (x, y) \in A$ and $b := (u, v) \in A$.

Proposition 5.3.3 *If $\bar{a} \in A$ is a solution of (VEP), then $\bar{a}$ is a strong C-saddle point of the bifunction f.*

The converse of Proposition 5.3.3 does not hold, as the next example shows.

Example 5.3.4 (G. Bigi, A. Capătă and G. Kassay [18]) Let $X := [-1,0]$, $Y := X$, $A := [-1,0] \times [-1,0]$, $Z := \mathbb{R}^2$, $C := \mathbb{R}^2_+$. Define $f : [-1,0] \times [-1,0] \to \mathbb{R}^2$ by

$$f(x,y) := \begin{cases} (0,0) & \text{if } x = 0,\ y = -1 \\ (-\frac{1}{2}, \frac{1}{2}) & \text{if } x = 0 \text{ and } y \neq -1 \\ (-\frac{1}{4}, 1) & \text{if } x \neq 0 \text{ and } y = -1 \\ (x, y+1) & \text{otherwise.} \end{cases}$$

Then, $(0,-1)$ is a strong C-saddle point of the bifunction f. Taking $\bar{a} := (0,-1) \in A$ and $b := (u,v) \in A$ with $u \neq 0$ and $v := -1$, we get

$$\varphi(a,b) = f(0,v) - f(u,-1) = (-\frac{1}{2}, \frac{1}{2}) - (-\frac{1}{4}, 1) = (-\frac{1}{4}, -\frac{1}{2}).$$

Hence

$$\varphi(a,b) \in -\mathbb{R}^2_+ \setminus \{0\},$$

which implies that $(0,-1)$ is not a solution of (VEP). $\square$

Theorem 5.2.4 assures the existence of weak C-saddle points. Next we show that under a further assumption, not very demanding, namely, that $C^\sharp \neq \emptyset$, we are able to obtain a better result: the existence of strong C-saddle points.

Theorem 5.3.5 (G. Bigi, A. Capătă and G. Kassay [18]) *Suppose that* $\operatorname{int} C \neq \emptyset$, *let* $C^\sharp \neq \emptyset$, *let* X *and* Y *are compact subsets of two metrizable topological linear spaces, and* $f : X \times Y \to Z$ *satisfies the following conditions:*

(i) f *is* C*-upper semicontinuous with respect to its first variable on* X *and* C*-lower semicontinuous with respect to its second variable on* Y*;*

(ii) f *is* C*-subconcavelike – subconvexlike.*

Then f *admits a strong* C*-saddle point.*

Note that our result is not related to the existence result of strong C-saddle points given by X. H. Gong [63]. Our continuity assumptions are weaker, but the convexity assumptions are stronger than those in Theorem 2.1 of X. H. Gong [63].

Chapter 6

Minty and Stampacchia type variational inequalities

The domain of vector variational inequalities received a great interest in the academic and professional communities since the paper by F. Giannessi [56] appeared and the first existence results for vector variational inequalities were published in G.-Y. Chen and Q. M. Cheng [41]. G.-Y. Chen and S. H. Hou [42] presented some of the most fundamental existence results for vector variational inequalities. Most of the research results in this area deal with a weak version of vector variational inequalities and their generalizations. Thus, the authors of [42] proposed a study of the existence of solutions for strong vector variational inequalities. Recently, Y. P. Fang and N. J. Huang [51] and B. S. Lee, M. F. Khan and Salahuddin [89] obtained some results of this kind.

6.1 Minty and Stampacchia type weak vector variational inequalities

The weak vector variational inequality problems are particular cases of the weak vector equilibrium problem ($WVEP$) considered in Chapter 2. Let E and Z be real topological linear spaces, $A \subseteq E$ be a nonempty subset, and let $F : A \to L(E, Z)$ be an operator, where $L(E, Z)$ denotes the set of all continuous linear mappings from E to Z. Further, let $C \subseteq Z$ be a solid convex cone. Using these notations, in this section we will study the following variational inequalities:

($WMVI$) find $\bar{a} \in A$ such that $\langle F(b), b - \bar{a}\rangle \notin -\operatorname{int} C$ for all $b \in A$;

and

($WSVI$) find $\bar{a} \in A$ such that $\langle F(\bar{a}), b - \bar{a}\rangle \notin -\operatorname{int} C$ for all $b \in A$.

Here, $\langle F(b), b-a\rangle$ denotes the value of the function $F(b)$ at the point $b-a$ for all $a, b \in A$. Problem ($WMVI$) is called *the weak Minty vector variational inequality*, while ($WSVI$) is called *the weak Stampacchia vector variational inequality*.

By Theorem 2.1.1 we have the following existence result for the weak Minty vector variational inequality.

Theorem 6.1.1 (A. Capătă [34]) *Let A be a compact set, and let the following conditions be satisfied:*

(i) *for all* $a_1, \ldots, a_m \in A$, *all* $\lambda_1, \ldots, \lambda_m \geq 0$ *with* $\lambda_1 + \cdots + \lambda_m = 1$, *and all* $b_1, \ldots, b_n \in A$, *there exists* $c^* \in C^* \setminus \{0\}$ *such that*

$$\min_{1 \leq j \leq n} c^*\big(\langle F(b_j), b_j - \sum_{i=1}^{m} \lambda_i a_i \rangle\big) \leq \sup_{a \in A} \min_{1 \leq j \leq n} c^*\big(\langle F(b_j), b_j - a \rangle\big);$$

(ii) *for all* $b_1, \ldots, b_n \in A$, *and all* $c_1^*, \ldots, c_n^* \in C^*$ *not all zero, one has*

$$\sup_{a \in A} \sum_{j=1}^{n} c_j^*\big(\langle F(b_j), b_j - a \rangle\big) \geq 0. \tag{6.1}$$

Then problem $(WMVI)$ *admits a solution.*

The first assumption of Theorem 6.1.1, which is a generalized concavity condition, is satisfied if we assume that the set A is convex.

Corollary 6.1.2 (A. Capătă [34]) *Let* A *be a compact and convex set, let* $C^* \neq \{0\}$, *and suppose that, for all* $b_1, \ldots, b_n \in A$ *and all* $c_1^*, \ldots, c_n^* \in C^*$ *not all zero, one has (6.1). Then problem* $(WMVI)$ *admits a solution.*

In order to establish existence results for the weak Stampacchia vector variational inequality, we need the following continuity notion.

Definition 6.1.3 (X. H. Gong [58]) Let A be a convex set. We say that the operator F is v-hemicontinuous if, for all $a, b \in A$, the function

$$\forall\, \lambda \in [0,1] \mapsto \langle F(\lambda b + (1-\lambda)a), b - a \rangle \in Z$$

is continuous at 0.

Proposition 6.1.4 *If* A *is a convex set, and* F *is* v*-hemicontinuous, then each solution of* $(WMVI)$ *is a solution of* $(WSVI)$.

Theorem 6.1.5 (A. Capătă [34]) *Let* A *be a compact and convex set, let* $C^* \neq \{0\}$, *let* F *be* v*-hemicontinuous, and let the following condition be satisfied: for all* $b_1, \ldots, b_n \in A$ *and all* $c_1^*, \ldots, c_n^* \in C^*$ *not all zero, one has (6.1). Then problem* $(WSVI)$ *admits a solution.*

6.2 Minty and Stampacchia type strong vector variational inequalities

The strong vector variational inequalities are particular cases of the strong vector equilibrium problem (VEP) investigated in Chapter 3. Let A be a nonempty convex subset of a real topological linear space E, and let $F : A \to L(E, Z)$ be an operator, where $L(E, Z)$ denotes the set of all continuous linear functions from E to a real Hausdorff topological linear space Z. Further, let $C \subseteq Z$ be a nontrivial pointed convex cone. Using these notations, in this section we will study the following variational inequalities:

(MVI) find $\bar{a} \in A$ such that $\langle F(b), b - \bar{a} \rangle \notin -C \setminus \{0\}$ for all $b \in A$;

$$(SVI) \qquad \text{find } \bar{a} \in A \text{ such that } \langle F(\bar{a}), b - \bar{a} \rangle \notin -C \setminus \{0\} \text{ for all } b \in A.$$

Problem (MVI) is called *the strong Minty vector variational inequality*, while (SVI) is called *the strong Stampacchia vector variational inequality.*

Using the generalized duality theory presented in Section 3.2 we deduce that the strong Stampacchia vector variational inequality (SVI) admits as a generalized dual the strong Minty vector variational inequality (MVI). We notice that the vice-versa also holds, i.e. the generalized dual problem of (MVI) is (SVI).

In Y. P. Fang and N. J. Huang [51] there is presented an existence result for (SVI) under the following monotonicity property. The operator $F : A \to L(E, Z)$ is said to be strongly pseudomonotone if, for all $a, b \in A$, the following property holds:

$$\langle F(a), b - a \rangle \notin -C \setminus \{0\} \text{ implies } \langle F(b), b - a \rangle \in C.$$

In what follows we work with the notion of pseudomonotonicity, which is weaker than the above one. To see this, we will give an example.

Definition 6.2.1 (Y. P. Fang and N. J. Huang [51]) The operator $F : A \to L(E, Z)$ is said to be pseudomonotone if, for all $a, b \in A$, the following property holds:

$$\langle F(a), b - a \rangle \notin -C \setminus \{0\} \text{ implies } \langle F(b), b - a \rangle \notin -C \setminus \{0\}.$$

Example 6.2.2 (A. Capătă [35]) Let $E := \mathbb{R}^2$, $A := [0, 1] \times [0, 1]$, $Z := \mathbb{R}^2$, $C := \mathbb{R}^2_+$, and define $F : A \to L(\mathbb{R}^2, \mathbb{R}^2)$ by

$$\langle F(a), x \rangle := (x_1 + x_2)(a_1 - 2, a_2 + 2) \text{ for all } a := (a_1, a_2) \in A \text{ and all } x := (x_1, x_2) \in \mathbb{R}^2.$$

Let $a := (a_1, a_2)$ and $b := (b_1, b_2)$ be points from A. Since $a_1 - 2 < 0$ and $a_2 + 2 > 0$, it follows from

$$\langle F(a), b - a \rangle = (b_1 + b_2 - a_1 - a_2)(a_1 - 2, a_2 + 2)$$

that $\langle F(a), b - a \rangle \notin -\mathbb{R}^2_+ \setminus \{0\}$. Similarly, taking into consideration that $b_1 - 2 < 0$ and $b_2 + 2 > 0$, we obtain from

$$\langle F(b), b - a \rangle = (b_1 + b_2 - a_1 - a_2)(b_1 - 2, b_2 + 2) \tag{6.2}$$

that $\langle F(b), b - a \rangle \notin -\mathbb{R}^2_+ \setminus \{0\}$. Consequently, F is pseudomonotone. On the other hand, when $b_1 + b_2 - a_1 - a_2 \neq 0$, then (6.2) implies that

$$\langle F(b), b - a \rangle \notin \mathbb{R}^2_+.$$

Thus F is not strongly pseudomonotone. □

The following notion, is a particular case of Definition 3.2.2.

Definition 6.2.3 The operator $F : A \to L(E, Z)$ is said to be maximally pseudomonotone if the following conditions are satisfied:

(i) F is pseudomonotone;

(ii) for all $a, b \in A$ the following implication holds: if $\langle F(x), a - x \rangle \notin C \setminus \{0\}$ for all $x \in]a, b]$, then $\langle F(a), a - b \rangle \notin C \setminus \{0\}$.

The next statement follows by Proposition 3.2.8.

Proposition 6.2.4 (A. Capătă [35]) *If F is maximally pseudomonotone, then the solution sets of problems (SVI) and (MVI) coincide.*

Using Corollary 3.2.9, we obtain the following existence result for (SVI).

Theorem 6.2.5 (A. Capătă [35]) *Suppose that the following conditions are satisfied:*

(i) *F is maximally pseudomonotone;*

(ii) *the set $S(b) := \{a \in A \mid \langle F(b), b-a\rangle \notin -C \setminus \{0\}\}$ is closed for all $b \in A$;*

(iii) *there exist a nonempty, compact and convex set $D \subseteq A$ as well as an element $\tilde{b} \in D$ such that*
$$\langle F(x), \tilde{b} - x\rangle \in -C \setminus \{0\} \text{ for all } x \in A \setminus D.$$

Then problem (SVI) admits a solution.

Example 6.2.6 (A. Capătă [35]) To show that there exist operators which satisfy the assumptions of Theorem 6.2.5, let $E := \mathbb{R}^2$, $A := [0,1]\times[0,1]$, $Z := \mathbb{R}^2$, $C := \mathbb{R}^2_+$, and define $F : A \to L(\mathbb{R}^2, \mathbb{R}^2)$ by

$$\langle F(a), x\rangle := (x_1 + x_2)(a_1 + 1, a_2 + 1) \tag{6.3}$$

for all $a := (a_1, a_2) \in A$ and all $x := (x_1, x_2) \in \mathbb{R}^2$.

Since
$$(a_1 + 1, a_2 + 1) \in \mathbb{R}^2_+ \setminus \{0\} \text{ for each } a := (a_1, a_2) \in A,$$
it results from (6.3) that

$$\forall\, a \in A \,:\, \{x \in \mathbb{R}^2 \mid \langle F(a), x\rangle \notin \mathbb{R}^2_+ \setminus \{0\}\} = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 +_2 \le 0\}. \tag{6.4}$$

This inequality implies that

$$\forall\, a, b \in A \,:\, \{x \in \mathbb{R}^2 \mid \langle F(a), x\rangle \notin \mathbb{R}^2_+ \setminus \{0\}\} = \{x \in \mathbb{R}^2 \mid \langle F(b), x\rangle \notin \mathbb{R}^2_+ \setminus \{0\}\}. \tag{6.5}$$

Let $a := (a_1, a_2)$ and $b := (b_1, b_2)$ be points from A. Suppose that
$$\langle F(a), b - a\rangle \notin -\mathbb{R}^2_+ \setminus \{0\}.$$

Then we have $\langle F(a), a - b\rangle \notin \mathbb{R}^2_+ \setminus \{0\}$. By virtue of (6.5) we obtain
$$\langle F(b), a - b\rangle \notin \mathbb{R}^2_+ \setminus \{0\}, \text{ whence } \langle F(b), b - a\rangle \notin -\mathbb{R}^2_+ \setminus \{0\}.$$

Thus F is a pseudomonotone operator.

Next suppose that
$$\langle F(x), a - x\rangle \notin \mathbb{R}^2_+ \setminus \{0\} \text{ for all } x \in]a, b].$$

In particular, we have
$$\langle F(b), a - b\rangle \notin \mathbb{R}^2_+ \setminus \{0\}.$$

By virtue of (6.5) we get $\langle F(a), a-b\rangle \notin \mathbb{R}^2_+ \setminus \{0\}$. Hence the operator F is maximally pseudomonotone. In other words, condition (i) in Theorem 6.2.5 is satisfied.

From (6.4) it follows that

$$S(b) = \{a \in A \mid \langle F(b), a-b\rangle \notin \mathbb{R}^2_+ \setminus \{0\}\} = \{(a_1, a_2) \in A \mid a_1 + a_2 \leq b_1 + b_2\}$$

for each $b := (b_1, b_2) \in A$. Consequently, condition (ii) in Theorem 6.2.5 is also satisfied.

Finally, it is obvious that condition (iii) in Theorem 6.2.5 is satisfied for $D := A$. $\square$

By Corollary 3.2.10 we obtain an existence result for the strong Stampacchia vector variational inequality without monotonicity assumptions. This new existence result is a slight generalization of Theorem 2.1 of Y. P. Fang and N. J. Huang [51].

Theorem 6.2.7 (A. Capătă [35]) *Suppose that the following conditions are satisfied:*

(i) *for all $b \in A$ the set $S(b) := \{a \in A \mid \langle F(a), b-a\rangle \notin -C \setminus \{0\}\}$ is closed;*

(ii) *there exist a nonempty, compact and convex set $D \subseteq A$ as well as an element $\tilde{b} \in D$ such that*

$$\langle F(x), \tilde{b} - x\rangle \in -C \setminus \{0\} \text{ for all } x \in A \setminus D.$$

Then problem (SVI) admits a solution.

6.3 Proper solutions of some generalized vector variational inequalities

In this section we present existence results for proper efficient solutions of some generalized vector variational inequalities. Let A be a nonempty subset of a metrizable topological linear space E, let Z be a real topological linear space, let $C \subseteq Z$ be a nontrivial pointed convex cone, let $q : A \to Z$ be a given function, and let $F : A \to L(E, Z)$ be an operator.

The next existence results are devoted to the study of the following vector variational inequalities:

$(GMVI)$ find $\bar{a} \in A$ such that $\langle F(b), b-\bar{a}\rangle + q(b) - q(\bar{a}) \notin -C \setminus \{0\}$ for all $b \in A$,

and

$(GSVI)$ find $\bar{a} \in A$ such that $\langle F(\bar{a}), b-\bar{a}\rangle + q(b) - q(\bar{a}) \notin -C \setminus \{0\}$ for all $b \in A$.

By $\langle F(a), b-a\rangle$ we understand the value of $F(a)$ at $b-a$, for all $a, b \in A$. We refer to these problems as *the generalized strong Minty vector variational inequality*, and *the generalized strong Stampacchia vector variational inequality*, respectively. We state existence results for proper solutions of them.

First, let us recall some definitions concerning vector variational inequalities (see [58], [107] and [117]).

Definition 6.3.1 Let $c^* \in C^* \setminus \{0\}$. The operator F is said to be:

(i) c^*-monotone if, for all $a, b \in A$, we have

$$c^*(\langle F(b) - F(a), b-a\rangle) \geq 0.$$

(ii) c^*-upper hemicontinuous if A is a convex set and, for all $a, b \in A$, the function

$$\forall\, \lambda \in [0,1] \mapsto c^*(\langle F(\lambda b + (1-\lambda)a), b-a\rangle) \in \mathbb{R}$$

is upper semicontinuous at 0.

(iii) C-monotone if, for all $a, b \in A$, we have

$$\langle F(b) - F(a), b-a\rangle) \in C.$$

(iv) v-hemicontinuous if A is a convex set and, for all $a, b \in A$, the function

$$\forall\, \lambda \in [0,1] \mapsto \langle F(\lambda b + (1-\lambda)a), b-a\rangle \in Z$$

is upper semicontinuous at 0.

Remark 6.3.2 It is clear that, if F is C-monotone and v-hemicontinuous, then, for each functional $c^* \in C^* \setminus \{0\}$, F is c^*-monotone and c^*-upper hemicontinuous. □

Definition 6.3.3 (X. H. Gong [61]) A point $\bar{a} \in A$ is said to be:

(i) a globally efficient solution of $(GMVI)$ if there exists a Henig dilating cone $K \subseteq Z$ for C such that
$$\langle F(b), b-\bar{a}\rangle + q(b) - q(\bar{a}) \notin -K \setminus \{0\} \text{ for all } b \in A;$$

(ii) a Henig weakly efficient solution of $(GMVI)$ if Z is a real locally convex space and there exists a convex neighbourhood U of the origin of Z satisfying $U \subseteq V_{\mathcal{B}}$ (see Section 3.3) such that
$$\langle F(b), b-\bar{a}\rangle + q(b) - q(\bar{a}) \notin -\operatorname{int} C_U(\mathcal{B}) \text{ for all } b \in A;$$

(iii) a globally efficient solution of $(GSVI)$ if there exists a Henig dilating cone $K \subseteq Z$ for C such that
$$\langle F(a), b-\bar{a}\rangle + q(b) - q(\bar{a}) \notin -K \setminus \{0\} \text{ for all } b \in A;$$

(iv) a Henig weakly efficient solution of $(GSVI)$ if Z is a real locally convex space and there exists a convex neighbourhood U of the origin of Z satisfying $U \subseteq V_{\mathcal{B}}$ (see Section 3.3) such that
$$\langle F(a), b-\bar{a}\rangle + q(b) - q(\bar{a}) \notin -\operatorname{int} C_U(\mathcal{B}) \text{ for all } b \in A.$$

Theorem 6.3.4 (A. Capătă [36]) *Let A be a compact and convex set, let $K \subseteq Z$ be a Henig dilating cone for C, let $k^* \in K^\sharp$, and let the following conditions be satisfied:*

(i) *$k^* \circ q$ is lower semicontinuous on A;*

(ii) *q is K-convex;*

(iii) *F is k^*-monotone.*

Then problem $(GMVI)$ admits a globally efficient solution.

The next theorem gives existence results for globally efficient solutions of $(GSVI)$, under additional assumptions than those of Theorem 3.1 established by X. H. Gong [58], where the author states existence results for solution of $(GSVI)$, namely that there exists a Henig dilating cone with $K^\sharp \neq \emptyset$. Such an hypothesis is not very demanding, since such a cone always exists if we suppose the cone C to be based.

Theorem 6.3.5 (A. Capătă [36]) *Let A be a compact and convex set, let $K \subseteq Z$ be a Henig dilating cone for C, let $k^* \in K^\sharp$, and let the following conditions be satisfied:*

(i) *$k^* \circ q$ is lower semicontinuous on A;*

(ii) *q is K-convex;*

(iii) *F is k^*-monotone;*

(iv) *F is k^*-upper hemicontinuous.*

Then problem $(GSVI)$ admits a globally efficient solution.

In the final part of this section, we give existence results for the generalized strong Stampacchia vector variational inequality, under stronger assumptions than those of Theorem 6.3.5.

Corollary 6.3.6 (A. Capătă [36]) *Let A be a compact and convex set, let $K \subseteq Z$ be a Henig dilating cone for C with $K^\sharp \neq \emptyset$, and let the following assumptions be satisfied:*

(i) *q is K-lower semicontinuous on A;*

(ii) *q is K-convex;*

(iii) *F is K-monotone;*

(iv) *F is v-hemicontinuous.*

Then problem $(GSVI)$ admits a globally efficient solution.

It is worth to notice that each globally efficient solution of $(GSVI)$ is a solution of $(GSVI)$, due to the inclusion $C \setminus \{0\} \subseteq \operatorname{int} K$, where K is a Henig dilating cone for C.

Assumptions (i), (ii) and (iii) of Corollary 6.3.6 are satisfied, if we consider the function q to be C-lower semicontinuous, C-convex and the operator F to be C-monotone on A.

Corollary 6.3.7 (A. Capătă [36]) *Let Z be a real locally convex topological linear space, let C be a based cone, let A be a compact and convex set, and let the following conditions be satisfied:*

(i) *q is C-lower semicontinuous on A;*

(ii) *q is C-convex;*

(iii) *F is C-monotone;*

(iv) *F is v-hemicontinuous.*

Then problem $(GSVI)$ admits a Henig weakly efficient solution.

Taking $q := 0$ in the definition of $(GMVI)$ and $(GSVI)$, by the above results, we obtain existence results for solutions of (SVI).

Proposition 6.3.8 (G. Bigi, A. Capătă and G. Kassay [18]) *Let A be a convex set, and let F be v-hemicontinuous. If $\bar{a} \in A$ is a globally efficient solution of (MVI), then $\bar{a}$ is a solution of (SVI).*

By Theorem 6.3.4, Theorem 6.3.5 and Remark 6.3.2 we have the next results.

Theorem 6.3.9 (G. Bigi, A. Capătă and G. Kassay [18]) *Let A be a compact and convex set, and let $K \subseteq Z$ be a Henig dilating cone for C with $K^\sharp \neq \emptyset$, and let F be K-monotone. Then problem (MVI) admits a globally efficient solution.*

Theorem 6.3.10 (G. Bigi, A. Capătă and G. Kassay [18]) *Let A be a compact and convex set, let $K \subseteq Z$ be a Henig dilating cone for C with $K^\sharp \neq \emptyset$. If the operator F is K-monotone and v-hemicontinuous, then problem (SVI) admits a solution.*

6.4 Minty and Stampacchia type multifunction variational inequalities

Let A be a nonempty convex subset of a reflexive Banach space E, and let $F : A \to \mathcal{F}(E^*)$, where $\mathcal{F}(E^*)$ denotes the set of all nonempty and finite subsets of E^*. We study the following variational inequalities, also considered by L. J. Lin, Z. T. Yu and G. Kassay [94]:

$$(MMVI) \qquad \text{find } \bar{a} \in A \text{ such that } \inf_{v \in F(b)} \langle v, b - \bar{a} \rangle \geq 0 \text{ for all } b \in A,$$

and

$$(SMVI) \qquad \text{find } \bar{a} \in A \text{ such that } \sup_{u \in F(\bar{a})} \langle u, b - \bar{a} \rangle \geq 0 \text{ for all } b \in A.$$

Because F takes values in $\mathcal{F}(E^*)$, these multifunction variational inequalities become:

$$(MMVI) \qquad \text{find } \bar{a} \in A \text{ such that } \min_{v \in F(b)} \langle v, b - \bar{a} \rangle \geq 0 \text{ for all } b \in A,$$

and

$$(SMVI) \qquad \text{find } \bar{a} \in A \text{ such that } \max_{u \in F(\bar{a})} \langle u, b - \bar{a} \rangle \geq 0 \text{ for all } b \in A,$$

respectively. These problems are called *the Minty multifunction variational inequality*, and *the Stampacchia multifunction variational inequality*, respectively. We notice that $(MMVI)$ is equivalent to the following scalar equilibrium problem:

$$(EP_1) \qquad \text{find } \bar{a} \in A \text{ such that } h(\bar{a}, b) \geq 0 \text{ for all } b \in A,$$

where $h : A \times A \to \mathbb{R}$ is defined by

$$h(a, b) := \min_{v \in F(b)} \langle v, b - a \rangle \text{ for all } a, b \in A.$$

Taking into consideration this remark, we deduce from Corollary 2.1.5 the following existence theorem regarding $(MMVI)$.

Theorem 6.4.1 (A. Capătă [35]) *Let A be a compact set, and let the following condition be satisfied: for all $b_1, \ldots, b_n \in A$, and all $\mu_1, \ldots, \mu_n \geq 0$ with $\mu_1 + \cdots + \mu_n = 1$, one has*

$$\sup_{a\in A} \sum_{j=1}^{n} \mu_j \min_{v\in F(b_j)} \langle v, b_j - a\rangle \geq 0.$$

Then problem $(MMVI)$ admits a solution.

In order to give an existence result for $(SMVI)$, we need the following notion.

Definition 6.4.2 (L. J. Lin, Z. T. Yu and G. Kassay [94]) Let X and Y be real topological linear spaces, and let A be a nonempty convex subset of X. A multifunction $T : A \to 2^Y$ is said to be upper semicontinuous along lines at 0 if, for any $a, b \in A$, the multifunction

$$\forall\, \lambda \in [0,1] \mapsto T(\lambda b + (1-\lambda)a) \in 2^Y$$

is upper semicontinuous at 0.

Theorem 6.4.3 (A. Capătă [34]) *Let A be compact, let F be upper semicontinuous along lines at 0, and let the following condition be satisfied: for all $b_1, \ldots, b_n \in A$, and all $\mu_1, \ldots, \mu_n \geq 0$ with $\mu_1 + \cdots + \mu_n = 1$, one has*

$$\sup_{a\in A} \sum_{j=1}^{n} \mu_j \min_{v\in F(b_j)} \langle v, b_j - a\rangle \geq 0.$$

Then problem $(SMVI)$ admits a solution.

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