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Contributions to the study of the subgroup lattice of a group

Ph.D. thesis summary

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Contents

Preface

The study of lattices, in general, has its origins in the late 19th century. While investigating the Boolean algebras, Charles S. Pierce and Ernst Schröder felt the need to introduce the lattice concept. Independently, studying the ideals of the algebraic numbers, Richard Dedekind reached to the same concept. Moreover, Dedekind concluded that the lattice of these ideals satisfies a certain law. This is what we now call the modular law (sometimes referred as Dedekind law as well).

Lattice theory was used as a tool in the development of some of the basic structure theorems for group theory, and for algebraic system, in general. For example, Øystein Ore provided in 1935 a purely theoretic lattice proof for the Krull-Schmidt theorem, of uniqueness of direct decompositions.

However, the study of the connection between the group and its subgroup lattice was triggered in 1928 by Ada Rottlaender's paper, ([45]), motivated by the Galois correspondence between a field extension and its Galois group. Subsequently, a large number of mathematicians studied the subgroup lattices. In particular, we mention Reinhold Baer, Øystein Ore, Kenkichi Iwasawa, Leonid Eftimovich Sadovskii, Michio Suzuki, Giovanni Zacher, Roland Schmidt and may others.

Until 20 years ago, Suzuki's monograph, [53], was the only adequate reference in the subject area. In 1994, Roland Schmidt wrote a thick monograph ([49]) dedicated to this subject.

If G is a group, we denote its subgroup lattice by $L(G)$. This is always complete and compactly generated. Among the first and most important problems which were raised in the study of subgroup lattices, we mention:

(A) *Given a class $\mathcal{X}$ of groups, what are the properties of the lattices isomorphic to subgroup lattices of groups from $\mathcal{X}$?* Conversely, *given a class of lattices $\mathcal{Y}$, what can we say about the class of groups (if such groups exist) with the subgroup lattice belonging to $\mathcal{Y}$?*

(B) *Which lattices are isomorphic to the subgroup lattices of (abelian) groups ?*

As we shall see, there are lattices which are not isomorphic to subgroup lattices. On the other hand, there are lattices which are isomorphic, to the subgroup lattice of exactly one or of (even infinitely) many groups. Two groups having isomorphic subgroup lattices are called projective. A new problem is raised:

(C) *Which groups G are determined by projectivities, that is, for any group G' and any projectivity from G to G', implies $G \cong G'$?*

Simultaneously, the study of the normal subgroup lattice also developed. This one is a modular sublattice of the subgroup lattice.

In what follows, we present the content of this thesis.

In the first Chapter, we listed some of the basic concepts and results concerning subgroup lattices (Sections 1.1, 1.2, 1.3, 1.6). In Section 1.4 we presented the normal subgroup lattice and its elementary properties. In Section 1.7 some concrete examples of subgroup lattices are provided, most of them inspired by Roland Schimdt 's monograph, [49]. Section 1.5 is dedicated to the study of projectivities and especially ro the study of problem (C), stated above. When a group is not determined by projectivities, there still remain some chances for the group to be determined by subgroup lattices. A first option would be to restrict the class of all groups to some specific class of groups, i.e., G, is determined by projectivities in $\mathcal{C}$, if $G \in \mathcal{C}$ and for any group $G' \in \mathcal{C}$ projective with G, we have $G \cong G'$. The second option would be to determine the group using the subgroup lattice of another group (built from the initial one). In Section 1.5.3 we list some results with this flavor, providing a generalization of this approach, as well. Some of the results presented within this section are original and obtained by S. Breaz and the author of this thesis in [9].

In the second Chapter we provide a solution for the problem (B), stated above, for the abelian case. We provide necessary and sufficient conditions under which a lattice is isomorphic to the subgroup lattice of an abelian group. The problem of finding necessary and sufficient conditions under which a lattice is isomorphic to the subgroup lattice of an arbitrary group was first raised by Suzuki in [**?**]. In Section 2.1 we listed the conditions provided by Benabdallah and Piché in [4]. These are necessary for a lattice to be isomorphic to the subgroup lattice of a torsion abelian group, but not sufficient. Yakovlev was the one who offered a complete solution for this problem, in his work [36] (1974). He provided a latticeal description of the elements and of the multiplication within a free group of rank ≥ 2. Moreover, Yakovlev succeeded to identify the normal subgroups within the subgroup lattice of such a group. Hence, the solution is a direct consequence of the fact that every group is the homomorphic image of a free group and of the correspondence theorem for groups. In the same manner, Scoppola managed to characterize the subgroup lattice of a torsion-free group of rand ≥ 2 and of an abelian group with the torsion-free rank ≥ 2, in [50] (1981), respectively [51] (1985). His results where synthesized in Section 2.4. Using the same techniques, we will provide a complete solution in what concerns the abelian groups. This solution relies on the latticeal characterization of the commutator subgroup of a free group and on the fact that every abelian group may be obtain by factorizing a free group by its commutator subgroup. In this purpose, in Section 2.2, we will recall the instruments that we shall need in order to present the mentioned solution. Most of them, were introduced by Yajkovlev in [54]. For the sake of completeness, in Section 2.3 we shall present the conditions under which a lattice is isomorphic to the subgroup lattice of an arbitrary group. In Section 2.5 we shall identify the commutator subgroup within the subgroup lattice of a free group. Like Yakovlev, we shall work in the general context of 2-free groups. In Section 2.6 we shall present the characterization of the subgroup lattice of a free abelian group of rank ≥ 2, respectively of an abelian group. In Section 2.7 we shall formulate conditions for a lattice to be isomorphic to the normal subgroup lattice of an arbitrary group, as a direct consequence of Yakovlev's results. The results presented in the 2.5, 2.6 and 2.7 sections are given by the author of this thesis and will appear in [15].

The conditions presented in Chapter 2, do not provide to much information on some (basic) properties of the subgroup lattice. Therefore, in Chapter 3 we provided some closure properties of the class $\mathcal{A}$, of lattices isomorphic to subgroup lattices of abelian groups. The same properties where studied for the complementary of $\mathcal{A}$ in the class of all lattices. We shall focus on sublattices, in Section 3.1, ideals in Section 3.2, direct products in Section 3.3 and homomorphic images in Section 3.4. As expected, we conclude that $\mathcal{A}$ is not closed, in most of the cases. However, we shall present conditions when complete sublattices or ideals of a lattice from $\mathcal{A}$ lay also $\mathcal{A}$. We shall also study the ideals and congruences lattices in 3.5 and 3.6 sections. Although simple, these properties cannot be found in the subgroup lattices literature.

The remarks from the previous chapter lead us to the conclusion that $\mathcal{A}$ is not a variety, that is , closed under sublattices, direct products and homomorphic images. Moreover, $\mathcal{A}$ is not even a quasi-variety (closed under isomorphisms, sublattices, direct products, ultraproducts and contains the trivial lattice). Given these circumstances, in Chapter 4, we focused on a more general class, $\mathcal{L}(\mathbf{Z})$, of lattices which embeed in subgroup lattices of abelian groups. In Section 4.1 we shall briefly recall the concepts of variety and quasi-variety. The class $\mathcal{T}_1$, of lattices with a type 1 representation, i.e. embeeding in permuting equivalence lattices, generalizes $\mathcal{L}(\mathbf{Z})$. The following inclusions hold

$$\mathcal{L}(\mathbf{Z}) \subset \mathcal{N}(\text{rep}) \subset \mathcal{T}_1$$

and none of them is an equality. We denoted by $\mathcal{N}(\text{rep})$ the class of lattices embeeding in normal subgroup lattices. Lattices belonging to any of the classes previously mentioned are arguesian. The arguesian law was discovered by Bjarni Jónsson in 1954 (see [33]). It represents the translation of the Desargues Theorem from projective geometry, in latticeal language. In Section 4.3 we outlined the properties of these lattices. In [33], Jónsson showed that in the presence of complementation, the type 1 representation and the arguesian law, become equivalent. Finally, in [16], the result identifying $\mathcal{L}(\mathbf{Z}) = \mathcal{T}_1$ with the class of arguesian lattices, is stated. In Section 4.4 we proved that for lattices with length less than four, these classes off lattices coincide, that is, the following equality holds

$$\mathcal{L}(\mathbf{Z}) = \mathcal{N}(\mathrm{rep}) = \mathcal{T}_1.$$

The results from this last section belong to G. Călugăreanu and the author and will be published in [14].

Finally, I would like to thank my scientifical advisor, Prof. Grigore Călugăreanu, for his support, advice and supervision, while elaborating this thesis. Also, many thanks go to the members of the Chair of Algebra, especially to Conf. Simion Breaz.

Chapter 1

Fundamental Concepts. Examples

In this chapter we briefly recalled some basic lattice-theoretic concepts, which allowed us to present some of the elementary properties of subgroup lattices. The lattice of normal subgroups of a group is also presented, along with its fundamental properties. We presented a short inventory of the results that provide an answer to ones of the most notable questions: *Which groups are determined (up to an isomorphism) by their subgroup lattice? How does the structure of the group reflect into the structure of its subgroup lattice? Which lattices are subgroup lattice?*

In Section 1.5.3, we presented an original approach, related to the first question raised above. Some of the results mention in this section were obtained by S. Breaz, in collaboration with the author of this thesis (see [9]).

Definition 1.0.1 Let G be a group. The set of its subgroups, partially ordered by set inclusion, forms a complete lattice, called *the subgroup lattice of* G and denoted by $L(G)$.

1.1 Basic concepts of lattice theory

In this sectin we recalled some basic concepts of lattice theory. Moreover, the section was structured in five paragraphs. We recalled concepts as *inteval*, *chain*, *antichain*, *atom*, *coatom* and *compact element* of a lattice. The *length* and the *width* of a lattice were also mentioned.

We presented the notion of *algebraic* (*compactly generated*) lattice, given the fact that G. Birkhoff and O. Frink characterized the subgroup lattices as being of this kind. In Chapter 3, we studied closure properties with respect to sublattices and direct products, hence we found it useful to recall this concepts.

1.2 Embeddings in subgroup lattices

In this section we outlined the results related to embeddings in subgroup lattices. We saw that *every lattice embeds in a certai n subgroup lattice* (Whitman, 1946), and even more, *every algebraic lattice is isomorphic to an interval of a subgroup lattice* (Tuma, 1989).

1.3 Subgroup lattices properties

In this section we recalled some of the most important properties of subgroup lattice, determined by the structure and properties of the group. Firstly, we focused our attention on the modular law, discovered by Richard Dedekind in 1877. Dedekind, also showed that subgroup lattices of abelian groups are modular. The converse, does not hold, while the class of non-abelian groups with modular subgroup lattice was completely determined (see Iwasawa in [31] and Schmidt in [49]).

Secondly, we focused on the *distributivity*, which is stronger then modularity. It was also discovered by Dedekind. We outlined the Ore's (1937-1938) results ,which characterize the class of groups with distributive subgroup lattice. Moreover, we recalled Baer's results, from 1939, which provide an image of the subgroup lattices of cyclic groups. Finally, we listed some properties of finite groups which may be deduced from the properties (structure) of their subgroup lattice and vice-versa, Most of them were retrieved from [21].

1.4 Normal subgroup lattice

In this section we recalled the concept of normal subgroup lattice of a group. If G is a group, its normal subgroup lattice was denoted by $\mathcal{N}(G)$. We also provided some basic properties and examples, retrieved from [49].

1.5 Projectivities

In this section we focused on projectivities. A *projectivity* from G to G' is a latticeal isomorphism between their subgroup lattices (see [49]). Two such groups were called *projective*. In general, projective groups are not isomorphic. We recalled some examples with this flavor.

In what followed, we recalled a special class of lattices, which played an important role in the study of subgroup lattices (we made use of this class especially in

Chapter 4), i.e., the $\mathcal{M}_n$ lattices, where $n \in \mathbb{N}$. Such a lattice consists of a smallest, respectively a greatest element and n distinct atoms (see [43]).

1.5.1 Projectivities of abelian groups

In this section we provided an inventory of the results related to abelian groups projectivities. Frequently, two projective abelian groups are also isomorphic.

The case of torsion groups was solved by Baer in [2]. Two projective abelian groups with the torsion-free rank ≥ 2 will be isomorphic, as well, also according to Baer's results from 1939. For the torsion-fee abelian groups of rank 1, Fuchs established the conditions under which, two such projective groups are also isomorphic (see [21]). Recently, Călugăreanu and Rangaswamy solved the case of mixte abelian groups with torsion-free rank 1 (see [13]).

1.5.2 Classes of groups invariant under projectivities

In this section we recalled the *invariance of a class of groups with respect to projectivities*. As in [49], a class $\mathcal{C}$ of groups is invariant under projectivities if for every projectivity between two groups G and G', $G \in \mathcal{C}$ implies $G' \in \mathcal{C}$.

Afterwards, we listed some nice classes of groups having this property. Most of the example were taken from [43].

1.5.3 Groups determined by projectivities

In this section we presented some conditions under which a group is determined by the (normal) subgroup lattice of another group. As in [49], a group G is *determined by projectivities* , if for every group H and every projectivity $\varphi : L(G) \to L(H)$, we have $G \cong H$. Some of the concepts and results from this section are original and obtained in [9].

We denoted by *Grp* the class of all groups, by *Ab* the class of all abelian groups, by Ab_p the class of all p-abelian groups, while by *Lat* the class of all lattices.

As a first option to determine the group using its subgroup lattice, would be to restrict the class of all groups to a smaller class. This way, the lattice of subgroups may determine some groups. For example, R Baer showed in [2] that a p-abelian group A it is determined by $L(A)$ in Ab_p. In general, this does not hold, not even in the class of p-groups, having a modular subgroup lattice (see [3]).

As a second option, we could use the (normal) subgroup lattice of another lattice to determine the initial group. For example, if $A \in Ab$, while $G \in Grp$ such that $L(\mathbb{Z} \times A) \cong L(\mathbb{Z} \times G)$ (or $\mathcal{N}(\mathbb{Z} \times A) \cong \mathcal{N}(\mathbb{Z} \times G)$) then $A \cong G$.

In the next paragraphs, we presented some results with this flavor, for the subgroup lattice and for normal subgroup lattice as well.

Formalizing the approach

Definition 1.5.1 [9] Let $\mathcal{S} : Grp \to Lat$ such that $\mathcal{S}(G)$ is a sublattice of $L(G)$, for all $G \in Grp$. If $V : Grp \to Grp$ is a map and $\mathcal{C}$ is a class of groups, we say that a group $G \in \mathcal{C}$ is *determined by V and $\mathcal{S}$-projectivities in $\mathcal{C}$* if

$$H \in \mathcal{C} \text{ and } \mathcal{S}(V(G)) \cong \mathcal{S}(V(H)) \text{ implies } G \cong H.$$

If $\mathcal{C}$ is the class of all groups we say that G is *determined by V and $\mathcal{S}$-projectivities*. A group G is *determined by $\mathcal{S}$-projectivities* if it is determined by 1_{Grp} and $\mathcal{S}$-projectivities, i.e. if $G \cong H$ whenever $\mathcal{S}(G) \cong \mathcal{S}(H)$.

We focused on the two cases: $\mathcal{S}(G) = L(G)$ and $\mathcal{S}(G) = \mathcal{N}(G)$.

$\mathcal{N}$-Proiectivităţi

We briefly recalled the results which establish the conditions under which an abelian group is determined by its (normal) lattice of subgroups (see Brandl, [5], and Curzio, [17]).

In what concerns the correspondence V from Definition 1.5.1, we focused on two cases:

$$V = B \times - : Grp \to Grp,$$

where B torsion-free abelian group, and secondly

$$V = (-)^n : Grp \to Grp,$$

where n is a positive integer.

This approach, of determining the group using the subgroup lattice of another group , was used by Lukács and Pálfy in [38], for $V(G) = G^2$, and by de Călugăreanu in [12] for $V(G) = G^n$. The case $V(G) = B \times G$, where B is fixed, was handled by Călugăreanu and Breaz in [8]. We generalized the approach used within those papers in the following metatheorem:

Teorema 1.5.2 [9] *Let $V : Grp \to Grp$ be a map and $\mathcal{S} : Grp \to Lat$ such that $\mathcal{S}(G)$ is a sublattice of $L(G)$ for all $G \in Grp$. Suppose that G is a group such that there exists a class $\mathcal{C}$ of groups with the following properties:*

(i) $V(G) \in \mathcal{C}$;

(ii) $V(G)$ *is determined by $\mathcal{S}$-projectivities in $\mathcal{C}$;*

(iii) If $\mathcal{S}(V(G)) \cong \mathcal{S}(V(H))$ then $V(H) \in \mathcal{C}$,

Then G is determined by V and $\mathcal{S}$-projectivities if and only if G is determined by V, i.e., the following implication holds

$$V(G) \cong V(H) \Rightarrow G \cong H.$$

This metatheorem was used in the mentioned papers, for $\mathcal{C}$ being the class of all abelian groups. Therefore, in order to apply this metatheorem in our case, we should establish sufficient conditions such that $V(H)$ is Abelian, whenever $V(G)$ is.

The cancellation property. The n-root property

We recalled that a group B has the *cancellation property (with respect a class $\mathcal{C}$)* if every group $G \in \mathcal{C}$ is determined by $V = B \times -$ in $\mathcal{C}$, while for a integer $n > 0$, the group A has the *n-root property*, if A is determined by $V = (-)^n$.

We also listed some groups possessing the properties mentioned above. It is known that countable torsion Abelian groups and countable mixed Abelian groups of torsion-free rank 1 share the square-root property.Moreover, the Abelian groups with semilocal endomorphism rings have the n-th root property (see [20, Proposition 4.8]), for any positive integer $n \geq 2$. These groups were studied by Călugăreanu in [11]. Other mixed groups with the n-root property were studied Breaz in [7].

The case $\mathcal{S} = L$

In the beginning of this paragraph we recall some commutativity criterions , for a group G, which make use of the subgroup lattice of $V(G)$. For $V = K \times -$, the criteria was offered by Breaz and Călugăreanu in [8] and refers to the situation in which G is an arbitrary group, while K is an abelian group which is not a torsion one, respectively to the situation in which G is a p-group, while K a p-abelian unbounded group. For $V = (-)^n$ Lukács, E. and Pálfy, P. in [38] established the criteria for an arbitrary group. Using theses criterions, we were able to provide some direct consequence of the Metatheorem 1.5.2.

Corrolary 1.5.3 [9] *Let B be an abelian group. The following statements hold:*

(a) If B is not a torsion group, then for every abelian group A and every group G, the implication

$$L(B \times A) \cong L(B \times G) \Rightarrow A \cong G$$

holds if and only if B has the cancellation property with respect to Ab.

(b) *If B is an unbounded p-group, then for every p-abelian group A and every p-group G, the implication*

$$L(B \times A) \cong L(B \times G) \Rightarrow A \cong G$$

holds if and only if B has the cancellation property with respect to Ab.

(c) *If $n > 1$ is an integer, then for every group G, the implication*

$$L(B^n) \cong L(G^n) \Rightarrow B \cong G$$

holds if and only if B has the n-root property.

The case $\mathcal{S} = \mathcal{N}$

For the normal subgroup lattice we also recalled a commutativity criterion, proved by S. Breaz in [36], which uses $\mathcal{N}(V(G))$, when $V = B \times -$, where $B \neq 0$ torsion-free abelian group. For $\mathcal{S} = \mathcal{N}$, we presented another direct consequence of the Metatheorem 1.5.2.

Corrolary 1.5.4 [9] *Let $B \neq 0$ be an abelian group. The following statements hold:*

(a) *If B is torsion-free, then for every abelian group A and every group G, the implication*

$$\mathcal{N}(B \times A) \cong \mathcal{N}(B \times G) \Rightarrow A \cong G$$

holds, if and only if B has the cancellation property with respect to Ab.

(b) *If B is a p-group, $A \neq 0$ a p-abelian group and G a group, the implication*

$$\mathcal{N}(B \times A) \cong \mathcal{N}(B \times G) \Rightarrow A \cong G$$

holds , if and only if B has the cancellation property with respect to Ab.

(c) *Dacă $n > 1$ este un întreg, atunci pentru un grup G implicaţia*

$$\mathcal{N}(B^n) \cong \mathcal{N}(G^n) \Rightarrow B \cong G$$

are loc, if and only if B has the n-root property.

Corrolary 1.5.5 *Let A be an Abelian group. If G is a group and B is a finite rank torsion-free Abelian group such that $L(B \times A) \cong L(B \times G)$ (or $\mathcal{N}(B \times A) \cong \mathcal{N}(B \times G)$) then there exists a positive integer n such that $A^n \cong G^n$.*

An open problem

In this paragraph we outlined a conjecture regarding the groups B, with the property that $B \times A \cong B \times G$ (and $A, G \in \mathcal{C}$) implies $A^n \cong G^n$, for a positive integer n. $\mathbb{Z}$ poses this property (Hirshon, [27, Teorema 1]), while torsion free abelian groups of finite rank , as well (Goodearl, [23, Teorema 5.1]).

The question *Does the property* $L(\mathbb{Z} \times G_1) \cong L(\mathbb{Z} \times G_2)$ *(or* $\mathcal{N}(\mathbb{Z} \times G_1) \cong \mathcal{N}(\mathbb{Z} \times G_2)$*) imply* $G_1^n \cong G_2^n$*, for some integer* $n > 0$*?* seems to be natural. The answer is negative one, see [6], and it uses some classes of groups constructed in [52] and [30]. We finally recalled, the conjecture state in [9].

Conjecture: If B is a finite rank torsion-free abelian group and G_1, G_2 are groups (non-necessarily abelian) such that $L(B \times G_1) \cong L(B \times G_2)$ (or $\mathcal{N}(B \times G_1) \cong \mathcal{N}(B \times G_2)$) then there exists a positive integer n such that $L(G_1^n) \cong L(G_2^n)$ (respectively $\mathcal{N}(G_1^n) \cong \mathcal{N}(G_2^n)$).

1.6 Lattices isomorphic to subgroup lattices

In this section we presented some remarks regarding the problem (B) stated in the preface of this thesis, of *characterizing lattices which are (not) isomorphic to lattices of subgroups of (abelian) groups.*

B.V. Yakovlev formulated in [54], necessary and sufficient conditions under for a lattice to be isomorphic to the subgroup lattice of an arbitrary group. His conditions are briefly exposed in Chapter 2. However, taking into account the complexity of these conditions, for concrete cases, the problem of deciding if a lattice is or not a subgroup lattice is still a hard or impossible one. We solved [49, pag. 10, Exercise 2], deciding which of the lattices with at most 5 elements are isomorphic with the subgroup lattice of a group.

1.7 Examples

In this section we briefly recalled some remarkable subgroup lattices examples. We focused on groups of form $\mathbb{Z}(p^n) \oplus \mathbb{Z}(q^m)$, since we used the subgroup lattice of these ones in Chapter 4. Also we presented the subgroup lattice of the p-quasi-cyclic group, frequently used in the constructions from Chapter 3. The other examples outlined within this section are *the elementary abelian groups*, *groups of order* pq, for p and q primes, *the alternating group* A_4, *dihedral and quaternion groups* , *the Tarski groups.* These examples were retrieved from [49].

Chapter 2

Conditions under which a lattice is isomorphic to the subgroup lattice of an abelian group

This chapter is dedicated to the problem of finding necessary and sufficient conditions for a lattice to be isomorphic to the subgroup lattice of an abelian group.

In sections 2.1 and 2.4 we outlined some partial solutions of this problem, belonging to Benabdallah and Piché, respectively to Scoppola. In sections 2.5 and 2.6 we provided a complete solution for the subgroup lattice of an abelian group. The solution provided by Yakovlev in [54], for the subgroup lattice of an arbitrary group, inspired us in formulating the conditions earlier mentioned. Moreover, the instruments and techniques are the same as Yakovlev's ones and presented in Section 2.2. The germ idea of our solution is to identify the commutator subgroup in the subgroup lattice of a 2-free group and to characterize the subgroup lattice of a free abelian group.

In Section 2.7 we formulated conditions for a lattice to be isomorphic to the normal subgroup lattice of a arbitrary group. The results stated within the sections 2.5, 2.6 and 2.7 are original and obtained by the author of this thesis în [15].

2.1 Necessary conditions for a lattice to be isomorphic with the subgroup lattice of a torsion abelian group

In this section, we briefly presented some necessary conditions for a complete modular lattice in order to be the subgroup lattice of a torsion abelian group, provided by Benabdallah and Piché, in [4]. This study deals with complete modular lattices, satisfying some additional conditions, and generalize concepts from the theory of abelian groups.

2.2 Subgroup lattice of a free group

In this section we outlined the instruments that will be used in order to provide the conditions from Section 2.6. In order to obtain the desired result, we shall need the characterization of the subgroup lattice of a free group. This was already done, by Yakovlev. His idea, was to localize some latticeal structure in the set of cyclic elements of a subgroup lattice, that could provide enough information about the generators and the relations of a group.

2.2.1 Cyclic elements. Complexes

Almost all concepts presented in this paragraph are due to Yakovlev (see [54]). From no on, we denoted by $L = (L, \leq) = (L, \vee, \wedge)$ a complete lattice, while by 0 its least element.

Cyclic elements

An element $a \in L$ is cyclic if the interval $a/0$ is a distributive lattice satisfying the ascending chain condition. By $C(L)$ or simply C, when there was no risk of confusion, we denoted the set of cyclic elements of a lattice L. We also recalled the following subsets of C, used by Yakovlev, which play an essential role in the latticeal description of the elements of a free group and their multiplication.

Definition 2.2.1 If $a, b \in C$, $A, B \subseteq C$ we denote by

$$a \circ b = \{x \in C \mid x \vee a = x \vee b = a \vee b\}$$

$$b \uparrow a = \{c \in C(L) \mid c \in (a \circ b) \circ a, c \notin (a \circ a) \circ b, c \circ c \subseteq (a \circ (b \circ b)) \circ a\}.$$

Complexes

In this paragraph, we briefly recalled the concept of *a-complex with respect* to a system $E = (e_1, \ldots, e_n)$, of cyclic elements, as it was defined in [54]. By $K(a, E)$ we denoted the set of a-complexes with respect to a system E, while by $K(E)$ we denote the set of all complexes with respect to E. By convention, $\varepsilon = (\{e_1\}, \ldots, \{e_n\})$ is 0-complex with respect to E, and hence, $K(0, E) = \{\varepsilon\}$. We also recalled how the equality and the product of two complexes were defined in [54], since these play an important role when spotting the product of two elements in the subgroup lattice of a free group.

2.2.2 Subgroup lattice of a group

The result presented in this section makes use of complexes and their multiplication. In certain conditions complexes multiplication becomes a binary operation on the set $K(E)$. Moreover, this operation defines a group structure on $K(E)$ whose subgroup lattice is isomorphic to the initial lattice. In this manner, Yakovlev defined the sufficient conditions for a lattice in order to be isomorphic to a subgroup lattice.

Teorema 2.2.2 [54, Teorema 1], [49, Teorema 7.1.6] *Let L be a complete lattice in which every element is the join of cyclic elements and suppose there exists a system $E = (e_1, \ldots, e_n)$ of elements $e_i \in C(L)$ with the following properties:*

(a) For each $a \in C \setminus \{0\}$, $|K(a, E)| = 2$.

(b) If $a \in C$, $\alpha = (A_1, \ldots, A_n)$, $\alpha' = (A'_1, \ldots, A'_n) \in K(a, E)$ and $\alpha \neq \alpha'$, then

$$e_i \circ A'_j \cap A_i \circ e_j \neq \emptyset, \text{ for all } i, j \in \{1, \ldots, n\}.$$

(c) If $a, b \in C$, $\alpha \in K(a, E)$ and $\beta \in K(b, E)$ such that $\alpha = \beta$, then $a = b$.

(d) For all $\alpha, \beta \in K(E)$, the product $\alpha\beta$ consists of a unique complex $\alpha * \beta$.

(e) For all $\alpha, \beta, \gamma \in K(E)$, $(\alpha\beta)\gamma = \alpha(\beta\gamma)$.

(f) Let $a \in C$ and $X \subseteq C$ such that $a \leq \bigvee X$ and let $\alpha \in K(a, E)$. Then there exist finitely many elements $b_i \in X$ and $\beta_i \in K_i(b_i, E)$ such that $\alpha \in ((\ldots(\beta_1\beta_2)\beta_3\ldots)\beta_{m-1})\beta_m$.

In these conditions $G = K(E)$ with the operation $* : G \times G \to G$ given by d), $(\alpha, \beta) \mapsto \alpha * \beta$, $\alpha, \beta \in G$, is a group whose subgroup lattice is isomorphic to L.

2.2.3 2-Free groups

In this paragraph, we recalled the notion of 2-free group and the main properties of such a group. As in [54], by a 2-*free* group, G, we understood a non-abelian group with the property that every two elements of G, generate a free group. Every free group of rank ≥ 2 is in particular 2-free.

Basic Properties

We listed the essential properties of the subgroup generated by two elements of a 2-free group. If $a, b \in G$, where G is 2-free, we have:

i) If $\langle a \rangle \cap \langle b \rangle \neq 1$, the rank of $\langle a, b \rangle$ is 1, hence $ab = ba$.

ii) If $\langle a\rangle \cap \langle b\rangle = 1$, $\langle a, b\rangle$ is free on $\{a, b\}$.

(iii) If $a \neq 1 \neq b$ and $\langle a\rangle \cap \langle b\rangle = 1$ then

$$\langle a\rangle \circ \langle b\rangle = \{\langle ab\rangle, \langle a^{-1}b\rangle, \langle ab^{-1}\rangle, \langle a^{-1}b^{-1}\rangle\}$$

and these four groups are distinct (see [49, Lema 7.1.7]).

Latticeal description of the product of two elements

In this paragraph we presented the latticeal description of the product of two elements of a 2-free group, as it was provided by Yakovlev in [54].

Basic systems

In this paragraph we recalled the concept of *basic system* (as in [54]). For such a system, the conditions (a)-(f), from Theorem 2.2.2 become necessary for the subgroup lattice of a 2-free group.

As expected, the subgroup lattice of a 2-free group (and in particular, of a free group) possesses basic systems, and more, these satisfy the conditions (a)-(f) from Theorem 2.2.2, as it was shown in Theorem 7.1.11 from [49].

Subgroup lattice of a free group

This paragraph was dedicated to the characterization of the subgroup lattice of a free group. This was provided by Yakovlev in [54].

Teorema 2.2.3 [49, Teorema 7.1.12] *Let $r \geq 2$ be a cardinal number. The lattice L is isomorphic to the subgroup lattice of a free group of rank r if and only if L is complete, any of its elements is the join of cyclic elements, and L has the following properties:*

a) For each $c \in C(L) \setminus \{0\}$, the interval $c/0$ is infinite.

b) If $a, b \in C(L)$ such that $a \vee b \notin C(L)$ and if $d \in a \circ b$, then $d \wedge a = d \wedge b = 0$.

c) There exists a basic system E of L and a subset S of $C(L)$ such that $|S| = r$, $\bigvee S = \bigvee L$ and for every finite sequence $b_1, \ldots, b_s$, where $b_i \in S$, with $b_i \neq b_{i+1}$ $(i = 1, \ldots, s-1)$ and $a_i \in C(L)$ with $0 \neq a_i \leq b_i$ and $\alpha_i \in K(a_i, E)$, the trivial complex ε is not contained in $(\ldots((\alpha_1\alpha_2)\alpha_3)\ldots)\alpha_s$,

where the basic system, E, satisfies a)-f) from 2.2.2.

2.2.4 Normal subgroups

In this paragraph we recalled the latticeal description of normal subgroups of a 2-free group. Yakovlev provided the latticeal description of the conjugate of an element within a 2-free group. We recall here this result.

Lemma 2.2.4 [49, Lema 7.1.15] *Let G be a 2-free group, while $a, b \in G$ such that $a \neq 1 \neq b$ and $\langle a \rangle \cap \langle b \rangle = 1$. Then*

$$\langle b \rangle \uparrow \langle a \rangle = \{\langle aba^{-1} \rangle, \langle a^{-1}ba \rangle\}.$$

2.3 Subgroup lattice of a group

For the sake of completeness, in this section, we recalled the characterization of the subgroup lattice of an arbitrary group. This theorem is a direct consequence of the previous results and of the fact that every group is the homomorphic image of a free group.

2.4 Conditions for a lattice to be isomorphic to the subgroup lattice of a torsion-free abelian group of rank > 1

In this section we briefly presented the conditions provided by Scoppola, in [50], which characterize the subgroup lattice of a torsion-free group of rank ≥ 2. Scoppola used techniques to those of Yakovlev. Starting from a lattice satisfying certain conditions, he built the only group whose subgroup lattice is isomorphic to the initial one.

2.5 The commutator subgroup

In this section we spotted the commutator subgroup in the subgroup lattice of a free group. The final purpose was to formulate sufficient and necessary under which a lattice is isomorphic to the subgroup lattice of a free abelian group. As in the previous sections, we worked in a more general context, the one of 2-free groups.

The first step for identifying the commutator subgroup in the subgroup lattice of a 2-free group, was to to provide a latticeal description for the commutator of two elements. If $a, b \in G$, by the *commutator* of these elements we understood $a^{-1}b^{-1}ab$, and denoted this element by $[a, b]$. If G is a group, denoted by G' his commutator subgroup. Note that $G' = \langle [a,b] \mid a, b \in G \rangle$. In what followed, we constructed the following subset of the set of cyclic elements of a complete lattice.

Definition 2.5.1 [15]Let L be a complete lattice. If $x, y \in C(L)$,

$$y \updownarrow x = \{z \in C(L) \mid z \in (y \uparrow x) \circ y \text{ and } \exists t_1, t_2 \in C(L), t_1 \neq t_2, \text{ such that } \\ t_1, t_2 \in x \circ y, z \in t_1 \circ t_2, x \circ x \cap t_1 \circ t_2 = \emptyset\}.$$

Moreover, we proved the following result.

Lemma 2.5.2 [15, Lema 2.4] *If G is a 2-free group, while $a, b \in G$ such that $a \neq 1 \neq b$ and $\langle a \rangle \cap \langle b \rangle = 1$, then*

$$\langle b \rangle \updownarrow \langle a \rangle = \{\langle [a,b] \rangle, \langle [a^{-1}, b] \rangle, \langle [a, b^{-1}] \rangle, \langle [a^{-1}, b^{-1}] \rangle\}.$$

Once the commutator of two elements was spotted in the subgroup lattice of such a group, we provided the following lemmas regarding the commutator subgroup.

Lemma 2.5.3 [15] *Let G be a 2-free group and let $H \leq G$. Then H contains the commutator subgroup of G, if and only if, $\langle b \rangle \updownarrow \langle a \rangle \subseteq H/1$ for all $a, b \in G$, such that $a \neq 1 \neq b$ and $\langle a \rangle \cap \langle b \rangle = 1$.*

Lemma 2.5.4 [15] *Let G be a 2-free group and let $H \leq G$. Then H is the commutator subgroup of G if and only if*

$$H = \bigvee \Big(\bigcup_{a,b \in G, a \neq 1 \neq b, \langle a \rangle \cap \langle b \rangle = 1} \langle b \rangle \updownarrow \langle a \rangle \Big).$$

2.6 Conditions for the subgroup lattice of an abelian group

This section is dedicated to the problem of providing conditions for a lattice under which a lattice is isomorphic to the subgroup lattice of an abelian group. In this purpose, we provided conditions for the subgroup lattice of a free abelian group. We made use of the fact that every free abelian group may be obtained by factorizing a free group by his commutator subgroup.

Teorema 2.6.1 [15] *Fie $r \geq 2$ un număr cardinal. Let $r \geq 2$ be a cardinal number. The lattice L is isomorphic to the subgroup lattice of a free abelian group of rank r if and only if there exist a lattice L^* and an element $d \in L^*$ with the following properties:*

a) L^* *is a complete lattice in which every element is the join of cyclic elements. Furthermore,* L^* *satisfies a)-c) from Theorem 2.2.3, for the cardinal number* r*, where the basic system* E *satisfies in addition a)-f) from Theorem 2.2.2.*

b) $d = \bigvee(\bigcup_{a,b\in C(L^*)\setminus\{0\}, a\wedge b=0} b \updownarrow a)$.

c) $L \cong 1^*/d$*, where* 1^* *is the greatest element of* L^*.

The previous theorem provides conditions for the subgroup lattice of a free abelian group of rank ≥ 2, where r is finite or infinite. The subgroup lattice of the free abelian of rank 1 is well known. This is the T_∞ lattice, i.e. the set of natural numbers ordered by the realtion

$$a \leq' b \Leftrightarrow b \text{ divides } a.$$

In what followed we provided the central result of this section.

Teorema 2.6.2 [15] *The lattice* L *is isomorphic to the subgroup lattice of some abelian group if and only if* L *is isomorphic to a principal filter of the* T_∞ *lattice or there exists a lattice* L^* *and two elements* $d, e \in L^*$ *such that:*

a) L^* *and* $d \in L^*$ *satisfies a),b) from Theorem 2.6.1.*

b) $e \in 1^*/d$*, where* 1^* *is the greatest element of* L^* *and* $L \cong 1^*/e$.

2.7 Normal subgroup lattice

In this section we focused on solving the problem of finding conditions for the normal subgroup lattice of a group. The solution was a direct consequence of Yakovlev's results and of the correspondence theorems for groups. In order to simplify the thing we introduced the following definition.

Definition 2.7.1 Let L be a complete lattice. We say that an element $d \in L$ is *normal* in L and write $d \trianglelefteq L$, every time $b \uparrow a \subseteq d/0$ holds for each $a, b \in C(L) \setminus \{0\}$ such that $a \wedge b = 0$ and $b \leq d$.

In [54] Yakovlev proved that the normal elements in the subgroup lattice of a 2-free group are exactly the normal subgroups of that group.

Finally, we provide the result representing the solution of the problem mentioned above.

Teorema 2.7.2 [15] *The lattice* L *is isomorphic to the normal subgroup lattice of a group if and only if there is a lattice* L^* *and an element* $d \in L^*$ *such that:*

a) L^* *is a complete lattice in which every element is the join of cyclic elements. Furthermore,* L^* *satisfies a)-c) from Theorem 2.2.3, for some cardinal number* $r \geq 2$ *where the basic system* E *satisfies in addition a)-f) from Theorem 2.2.2.*

b) $d \trianglelefteq L^*$.

c) $\{d' \in L^* \mid d' \trianglelefteq L^*, d \leq d'\}$ *is a complete sublattice of* $1^*/d$*, isomorphic with* L.

Chapter 3

Closure properties of the subgroup lattice of an abelian group

In this chapter we studied closure properties of the class $\mathcal{A}$, of lattices isomorphic with subgroup lattices of abelian groups. Although simple, these properties cannot be found in the existing bibliography. It is well-known that lattice from this class are compactly generated and modular.

We studied the complementary of $\mathcal{A}$ (in the class of all lattices), as well. We investigated closure properties with respect to sublattice, ideals, direct products, homomorphic images. We also studied ideals and congruences lattices. The majority of the result from this chapter belong to the author of this thesis.

3.1 Sublattices

In this section we study closure properties with respect to sublattices. We constructed examples which led to the conclusions: *If $L \in \mathcal{A}$ and U is a non-trivial sublattice, in general $U \notin \mathcal{A}$.* Similarly, *If $L \notin \mathcal{A}$, it is possible that all its non-trivial sublattices belong to $\mathcal{A}$.*

In what followed, we investigated the conditions under which a lattice from $\mathcal{A}$ has the property that all its complete sublattices are in $\mathcal{A}$. Inspired by [32], we introduced the following definition.

Definition 3.1.1 We say that a complete lattice L is *the disjoint union of chains* $C_1, \ldots, C_n \subseteq L$, where $n \in \mathbb{N}^*$, if the following conditions are satisfied:

(i) $L = \bigcup_{i=1}^{n} C_i$,

(ii) for any $i, j \in \{1, \ldots, n\}, i \neq j$ we have $C_i \cap C_j = \{0, 1\}$,

(iii) if $x \leq y$, there exists $i \in \{1, \ldots, n\}$ such that $x, y \in C_i$.

For example, if $n \in \mathbb{N}$, the lattice $\mathcal{M}_n$ is the disjoint union of n of length 2. Also, if a lattice L can be written as the disjoint union of chains, then all its complete sublattices can be written as disjoint union of chains too. We obtained the following intermediary result.

Proposition 3.1.2 *Let L be a complete lattice. If L does not contain any sublattice isomorphic with $\mathbf{C}_5$, $\mathbf{D}_5$ or $\mathcal{M}_5$, then L is the disjoint union of at most 4 chains.*

In what followed obtained the following intermediary result.

Proposition 3.1.3 *Let $L \in \mathcal{A}$. Then every complete sublattice U of L is in $\mathfrak{L}$ if and only if L does not contain any sublattice isomorphic with $\mathbf{C}_5$, $\mathbf{D}_5$ or $\mathcal{M}_5$.*

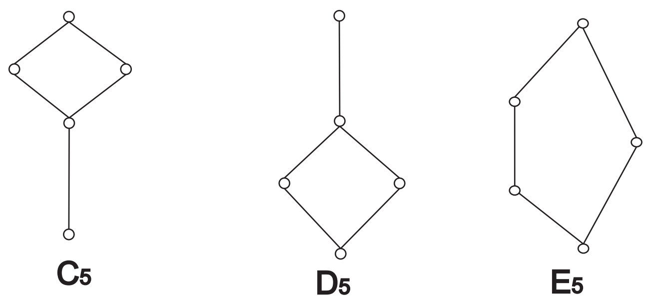

Figure 3.1: Latici cu 5 elemente care nu sunt în $\mathcal{A}$

The previous result provides those lattices in $\mathcal{A}$ whose complete sublattices are also in $\mathcal{A}$. These lattices are isomorphic with either $L(\mathbb{Z}(p^n))$, $n \in \mathbb{N} \cup \{\infty\}$ or $L(\mathbb{Z}(pq))$, for two distinct primes p and q, or $L(\mathbb{Z}(2) \oplus \mathbb{Z}(2))$ or $L(\mathbb{Z}(3) \oplus \mathbb{Z}(3))$.

3.2 Ideals

In this section we focused on closure properties of $\mathcal{A}$ and its complemetary with respect to ideals. As in [16], by an *ideal* of a lattice, we understood a subset closed under finite joins which is also a lower set. An ideal I is said to be *principal* if $I = x/0$, for a $x \in L$.

We concluded that *if $L \in \mathcal{A}$ and I is a principal ideal of L, then $I \in \mathcal{A}$.* The conclusion, does not hold if the ideal is not principal.

Proposition 3.2.1 *Let $L \in \mathcal{A}$ and I an ideal of L. Then $I \in \mathcal{A}$ if and only if I is principal.*

As a direct consequence of 3.2.1 we provided the following result.

Teorema 3.2.2 *Let $L \in \mathcal{A}$. Then for every non-empty ideal I of L, we have $I \in \mathcal{A}$ if and only if L satisfies the ascending chain condition.*

It is not hard to notice that if $L \notin \mathcal{A}$, we may find I, a principal ideal of L, such that $I \in \mathcal{A}$.

3.3 Direct products

In this section we focused on the behavior of lattices from $\mathcal{A}$ with respect to *direct products* (of lattices). Suzuki provided a fundamental result concerning the decomposition of the subgroup lattice as a direct product coprime groups (see [49, Teorema 1.6.5]). As a consequence, we provided the following proposition.

Proposition 3.3.1 *Let L_1, L_2 such that $L_1 \times L_2 \in \mathcal{A}$. Then L_1, $L_2 \in \mathcal{A}$.*

Moreover, we showed that the reverse implication does not hold.

3.4 Homomorphic images

In this section we provided some examples which led to the conclusion that neither $\mathcal{A}$ nor its complementary in the class of all lattices are not closed under homomorphic images.

3.5 Ideals lattice

In this section we studied the ideals lattice, respectively the nonempty ideals lattice of a lattices from $\mathcal{A}$. Recall that the collection of all the ideals of a lattice L, together with the relation of set-inclusion, forms a lattice which we will denote by $\mathfrak{I}(L)$. We denote by $\mathfrak{I}_0(L)$ the set of all nonempty ideals of the lattice L. If L has a least element $\mathfrak{I}_0(L)$ is a complete sublattice of $\mathfrak{I}(L)$. This does happens when $L \in \mathcal{A}$.

We proved that *if $L \in \mathcal{A}$, it is possible that* $\mathfrak{I}_0(L) \notin \mathcal{A}$. Similarly, *if $L \notin \mathcal{A}$, it is possible that* $\mathfrak{I}_0(L) \in \mathcal{A}$. Moreover, we provided sufficient conditions for a lattice such that $L \in \mathcal{A}$ to imply $\mathfrak{I}_0(L) \in \mathcal{A}$.

Proposition 3.5.1 *Let $L \in \mathcal{A}$. If L satisfies the ascending chain condition, then*

$$\mathfrak{I}_0(L) \in \mathcal{A}.$$

We constructed examples which proved that *if $L \in \mathcal{A}$, it is possible that* $\mathfrak{I}(L) \notin \mathcal{A}$. Also, we proved that *if $L \notin \mathcal{A}$, it is possible that* $\mathfrak{I}(L) \in \mathcal{A}$.

The following result holds.

Lemma 3.5.2 *Let $L \in \mathcal{A}$, such that $L \cong L(G)$. If $\mathfrak{I}(L) \in \mathcal{A}$ then G is cociclic.*

In what concerns the lemma above, we proved that the converse implication does not hold, in general. In what followed, we provided conditions for $\mathfrak{I}(L)$ to belong to $\mathcal{A}$.

Proposition 3.5.3 *Let $L \in \mathcal{A}$, such that $L \cong L(G)$. $\mathfrak{I}(L) \in \mathcal{A}$ if and only if G is cociclic and of prime power order.*

3.6 Congruences lattice

In this section we studies the congruences lattice. We briefly recalled the concept and its basic properties (see [16]). An equivalence relation on a lattice L is a *congruence* if it is compatible with both meets and joins.

The set of all congruence relations in a lattice L, denoted by $\mathrm{Con}(L)$, is partially ordered by the relation

$$\theta \leq \psi \text{ if } a\theta b \text{ implies } a\psi b.$$

We proved the following result.

Proposition 3.6.1 *If $L \notin \mathcal{A}$, we might have* $\mathrm{Con}(L) \in \mathcal{A}$.

Chapter 4

Lattices representable by abelian groups

In this chapter we focused on the class $\mathcal{L}(\mathbf{Z})$, of lattices representable by lattices in $\mathcal{A}$. This class is larger than $\mathcal{A}$, studied in the previous chapter. We denoted by $\mathcal{N}$, respectively $\mathcal{N}(\text{rep})$, the class of lattices isomorphic with normal subgroup lattices, respectively the class of lattices embedding in lattices from $\mathcal{N}$.

In Section 4.1 we listed the existing results concerning the class $\mathcal{L}(\mathbf{Z})$. In Section 4.2 we briefly recalled the properties of the class of lattices with a type 1 representation, denoted by $\mathcal{T}_1$. Since lattices belonging to any of the previously mentioned classes are arguesian, in Section 4.3 we recalled the concept of arguesian lattice, along with its basic properties

The central result of this chapter is presented in Section 4.4. We proved that for (modular) lattices of length ≤ 4, we have

$$\mathcal{L}(\mathbf{Z}) = \mathcal{N}(\text{rep}) = \mathcal{T}_1.$$

This result was obtained by G. Călugăreanu in collaboration with the author of this thesis în [14].

4.1 Varieties of lattices. Quasi-varieties of lattices

In this section we briefly recalled some notions like *variety*, respectively *quasi-variety* of lattices. In Chapter 3 we saw that $\mathcal{A}$ is not closed nighter under direct products, sublattices nor homomorphic images. Hence, $\mathcal{A}$ is not a variety, i.e. a class of lattices satisfying every equation from a given set Σ, or equivalently (by Garrett Birkhoff result from 1934), a class of lattices closed under direct products, sublattices and homomorphic images.

The concept of quasi-variety generalizes the of variety. However, $\mathcal{A}$ is not even a quasi-variety. We provided a short inventory of the results (proved using different approaches) which state that $\mathcal{L}(\mathbf{Z})$ is a quasi-variety.

It still remains an open question whether *the quasi-variety generated by* $\mathcal{L}(\mathbf{Z})$ *is a variety? In other words, if a lattice L may be embedded in the subgroup lattice of an abelian group, does the same thing happen with its factor lattices?.*

4.2 Type 1 representable lattices

In this section we briefly recalled the lattices of type 1, respectively with a type 1 representation, introduced by Jónsson. Lattices of *type* 1 (also called *linear*) are isomorphic to lattices of permuting equivalences. The class of these lattices was denoted by $\mathcal{L}$. A lattice with a type 1 representation, embeds in a lattice from $\mathcal{L}$ and their class was denoted by $\mathcal{T}_1$.

Jónsson proved that every lattice with a type 1 representation is modular. Congruences induces by normal subgroups commute, hence normal subgroup lattices are linear, i.e. $\mathcal{N} \subseteq \mathcal{L}$.

It still remains an open question whether $\mathcal{T}_1$ is a variety.

4.3 Arguesian lattices

in this section we focused on *arguesian lattices*, introduced by Jónsson in 1954. In 36 it is shown that this identity is equivalent to an implication which naturally reflects the Desargues theorem from projective geometry.

It is well-known that $\mathcal{A} \subset \mathcal{N} \subset \mathcal{L}$ şi $\mathcal{L}(\mathbf{Z}) \subset \mathcal{N}(\text{rep}) \subset \mathcal{T}_1$. Moreover, lattices from these classes are arguesian. None of these inclusion is an equality. In [33], Jónsson, showed that $\mathcal{N} \subsetneq \mathcal{L}$, while Pálfy and Csaba Szabo constructed in [42] an example, which proves that $\mathcal{A} \subsetneq \mathcal{N}$. Combining the results of Birkhoff, Frink, Schutzenberger and Jónsson we may state the following result.

Teorema 4.3.1 *If L is a geomodular lattice, the following statements are equivalent: (i)$L \in \mathcal{A}$; (ii)$L \in \mathcal{N}$; (iii)$L \in \mathcal{L}$; (iv)L is arguesian.*

In [33], Jónsson extended the previous result and proved that in the presence of complementation, the two concepts, i.e. type 1 representation and the arguesian identity, become equivalent.

Teorema 4.3.2 [16] *If L is a complemented (modular) lattice, then the following statements are equivalent: (i)$L \in \mathcal{L}(\mathbf{Z})$; (ii) $L \in \mathcal{T}_1$; (iii) L is arguesian.*

In what followed, we focused on lattices with a relatively small length (≤ 4).

4.4 Type 1 representabile of length ≤ 4

In this section we proved that the equality $\mathcal{L}(\mathbf{Z}) = \mathcal{N}(\text{rep}) = \mathcal{T}_1$ holds for (modular) lattices of length ≤ 4. This is the last original contribution of the author, presented within this thesis, and obtain in collaboration with G. Călugăreanu (see [14]).

This study can also be related to the following (frequently hard) open question: *when is a given quasivariety, actually a variety?* For classes like $\mathcal{L}(\mathbb{Z})$, $\mathcal{N}(\text{rep})$, şi $\mathcal{T}_1$ am văzut că răspunsul nu este cunoscut. Since our study shows that for length ≤ 4 all these classes coincide with the arguesian lattices and arguesian lattices form a variety, our results somehow encourage to conjecture a positive answer.

Modular lattices of length ≤ 4

In [35], Jónsson represented (using diagrams), all modular lattices of length ≤ 4. Since this is also our environment (as it was for Arguesian lattices), [35]. we briefly remind the reader the results.

A lattice of length 0 consists of just one element $0 = 1$, and a lattice of length one consists of exactly one chain with two elements 0 and 1. A lattice of length 2 is isomorphic with $\mathcal{M}_n$, if it has n distinct atoms. Since the join of two distinct atoms is always 1, and the meet is always 0, such a lattice is completely determined up to isomorphism by the number of atoms.

Remark 4.4.1 If A and A' are lattices of length 2 and A' has at least as many elements as A, then A is isomorphic to a sublattice of A'. In fact given atoms p in A and p' in A', there exists an isomorphism from A to A' such that $f(p) = p'$.

All these lattices are complemented, so the equalities

$$\mathcal{A} = \mathcal{N} = \mathcal{L} \text{ and } \mathcal{L}(\mathbf{Z}) = \mathcal{N}(\text{rep}) = \mathcal{T}_1$$

hold according to the theorems 4.3.1 and 4.3.2. Thus, we can discard at once, the case of lattices of length ≤ 2.

We denoted by s the socle (the join of all atoms) and by r the radical (the meet of all dual atoms). Since *a finite length modular lattice is complemented iff* 1 *is the socle (join of all its atoms) iff* 0 *is the radical (meet of all its dual atoms),* the conditions $\delta(s) = n$ and $\delta(r) = 0$ are equivalent and imply that A is complemented.

If $\delta(s) = 1$, then s is an atom of A, and in fact s is the only atom of A. In this case A is completely determined by its sublattice $1/s$, of length $n-1$. Similarly, if $\delta(r) = n-1$, then the study of A reduces to the study of its sublattice $r/0$, of length $n-1$. We shall therefore be concerned here with the cases in which $1 < \delta(s) < n$ and $0 < \delta(r) < n-1$.

Thus if $n = 3$ then only $\delta(s) = 2$ and $\delta(r) = 1$ has to be considered and if $n = 4$, $\delta(s) \in \{2, 3\}$ and $\delta(r) \in \{1, 2\}$. Therefore we distinguish only the following two cases:

Teorema 4.4.2 [35] *For $n = 3, 4$, if $0 < \delta(r) < \delta(s) < n$ then $r < s$ and $A = s/0 \cup 1/r$.*

Teorema 4.4.3 [35] *For $n = 4$, if $\delta(s) = 2$ and $\delta(r) = 2$ then $s/0 \cup 1/r = A - X$, where X is the set of irreducible elements $x \in A$, with $\delta(x) = 2$. Moreover, every element of X covers exactly one atom and it is covered by exactly one coatom. Two elements cover the same atom if and only if they are covered by the same coatom. Finally, if $s \neq r$ then $s \wedge r$ is an atom covered by r, $s \vee r$ is a coatom which covers s, while $s \wedge r \prec x \prec s \vee r$, for every element $x \in X$.*

4.4.1 Lattices of length 3 and 4

We already saw the a modular lattice of length 3, which is not complemented, is isomorphic to a lattice as in Figure 4.1 (with s not an atom, while r not a coatom).

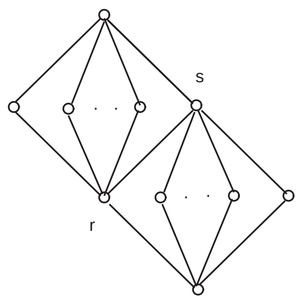

Figure 4.1: Family of lattices of length 3

This diagram represents the lattice $\mathcal{M}_n$ glued to $\mathcal{M}_m$ with a prime ideal of the top lattice being identified with a prime filter of the bottom lattice. We denoted this lattice by $\mathcal{M}_n / \mathcal{M}_m$.

Since finite Abelian groups are self-dual (Baer 1937), if $m \neq n$, then $\mathcal{M}_n / \mathcal{M}_m \notin \mathcal{A}$. Moreover, it can be proved that only $\mathcal{M}_{p+1} / \mathcal{M}_{p+1} = L(\mathbb{Z}(p) \oplus \mathbb{Z}(p^2)) \in \mathcal{A}$, for a prime p.

Thus we obtained the following results for lattices of length 3.

Teorema 4.4.4 [14] *Every modular non complemented lattice of length 3 belongs to $\mathcal{L}(\mathbf{Z})$.*

Corrolary 4.4.5 [14] *For lattices of length ≤ 3, $\mathcal{L}(\mathbf{Z}) = \mathcal{N}(\text{rep}) = \mathcal{T}_1$ holds.*

To conclude our study, a length 4 modular non complemented lattice is isomorphic (see previous Section) either to one of the four types of lattices represented in Figure 4.2 (if $r \neq s$) or to a lattice of the form represented in Figure 4.3 (if $r = s$).

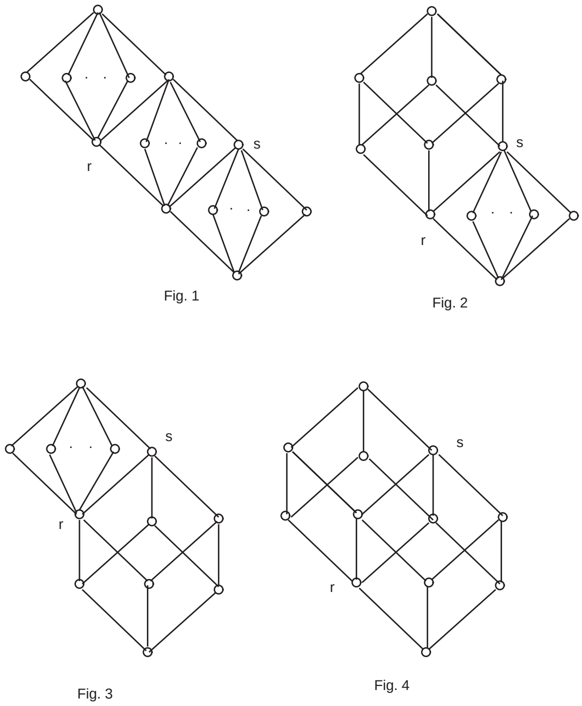

Figure 4.2: Family of lattices of length 4 with $r \neq s$

These figures improve Jónsson's illustration from ([35], pag 168). The figure presented there is incomplete. Using again the fact that for Abelian finite groups, the subgroup lattice is self-dual, one checks that most of the lattices above do not belong to $\mathcal{A}$.

We obtained the following result for lattices of length 4.

Teorema 4.4.6 [14] *Every modular non complemented lattice of length* 4 *belongs to* $\mathcal{L}(\mathbf{Z})$.

Finally, we provided the desired result.

Corrolary 4.4.7 [14] *For a lattice L, of length ≤ 4, the following properties are equivalent:*

(i) L is representable by abelian groups;

(ii) L is representable by lattices of normal subgroups of arbitrary groups;

(iii) L is representable by linear lattices;

(iv) L is arguesian.

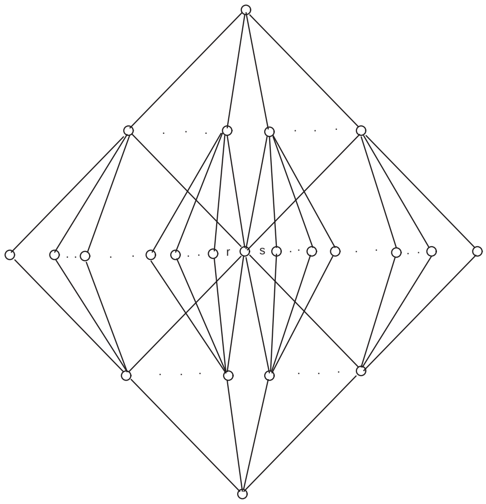

Figure 4.3: Family of lattices of length 4 with $r = s$

Remark 4.4.8 We don't have equality $\mathcal{A} = \mathcal{N}$ for dimension ≤ 4 lattices. The subgroup lattice of the (8 element) quaternion group, which is dimension 3, is a simple counterexample.

Whether we have equality $\mathcal{N} = \mathcal{L}$ for dimension ≤ 4 (modular) lattices, remains an open question.

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