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Fixed Point Theory in Kasahara Spaces

Ph.D. Thesis Summary

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Contents

Introduction

Fixed point theory becomes, in the last decades, not only a field with a huge development, but also a strong tool for solving various problems arising in different fields of pure and applied mathematics. A central element of the metric fixed point theory is the Banach-Caccioppoli Contraction Principle. Today we have many generalizations of this result, which were given in all kind of generalized metric spaces. If we carefully examine their proofs, one can see that the metric properties, in particular part of the axioms of the metric, are not all the time essential. Therefore the following problem arises: *In which general spaces contractive type fixed point theorems hold ?*

This problem has been studied since 1975 by a distinguished mathematician Shouro Kasahara, professor at the Kobe University. By following the work of Maurice Fréchet [42] which has introduced the structure of L-space, Kasahara has endowed this structure with a functional d which is not necessarily a metric. Therefore he has defined a more general space: the d-complete L-space. By using this notion, Kasahara has extended Maia's theorem, published in 1968 in [84], a well-known fixed point result given in a set endowed with two metrics. We mention here some other authors which have given fixed point theorems in a set with two metrics: V. Berinde [10], S. Iyer [57], A. Petruşel and I.A. Rus [102], R. Precup [105], I.A. Rus [118], I.A. Rus, A.S. Mureşan and V. Mureşan [122], B. Rzepecki [129], L.M. Saliga [130].

In a number of papers [66]-[70] Kasahara constructed a fixed point theory in d-complete L-spaces. T.L. Hicks [47] and T.L. Hicks - B.E. Rhoades [49] gave some fixed point theorems in a d-complete topological space. Other results in these directions were given by V.G. Angelov [3], J. Daneš [22], K. Iséki [55], L. Guran [45], P.Q. Khanh [75].

However, the notion of d-complete L-space was, in some sense, difficult to be used. Hence, by following the work of Kasahara and the results given by the mathematicians which have been already mentioned above, Ioan A. Rus has defined in 2010 the notions of *Kasahara space*, *generalized Kasahara space* and *large Kasahara space*. His work [121] contains also fixed point theorems and research problems concerning Kasahara spaces. Some solutions regarding the formulated research problems can be found in our thesis.

This thesis is divided into three chapters, each chapter containing several sections.

Chapter 1: Preliminaries.

In this chapter we present the basic notions and results regarding L-spaces, generalized metric spaces, partial metric spaces, w-distance and τ-distance on a metric space (X, d), Kasahara spaces and operators on Kasahara spaces, which are further considered in the next chapters of this work, allowing us to present the results of this thesis. Our contributions in this chapter are some solutions to the Problems 1.6.1, 1.6.2 and 1.6.3, posed by I.A. Rus in [121].

Chapter 2: Generalized contractions in Kasahara spaces.

$\diamond$ In the first section of this chapter we develop the theory of some well-known fixed point results as the Banach-Caccioppoli's Contraction Principle, the Graphic Contraction Principle, the Caristi-Browder and Matkowski type theorems. Our results are given for single-valued generalized contractions in the context of a Kasahara space $(X, \to, d)$, where $d : X \times X \to \mathbb{R}_+$ is a functional. We present also some extensions of our results in generalized and large Kasahara spaces. Our contributions in this section are: Theorem 2.1.1 which is a fixed point theory in Kasahara space extending and complementing the Banach-Caccioppoli's Contraction Principle; Theorem 2.1.2 which is a local fixed point result for Zamfirescu operators given in Kasahara spaces, extending and generalizing Krasnoselskii's local fixed point theorem; Theorem 2.1.3 which is a fixed point results in generalized Kasahara spaces ($d(x, y) \in \mathbb{R}_+ \cup \{+\infty\}$) for α-contractions; Theorem 2.1.5 which is a fixed point theory for the local variant of Banach-Caccioppoli's Contraction Principle, given in large Kasahara spaces (d is a w-distance); Theorem 2.1.6 which is given in large Kasahara spaces (d is perturbed by an increasing, subadditive and continuous function φ), extending and complementing Banach-Caccioppoli's Contraction Principle, the Graphic Contraction Principle, the Caristi-Browder and Matkowski type theorems; Theorem 2.1.2, extending Theorem 1 given by T. Zamfirescu in [150]; Lemma 2.1.2; Definitions 2.1.7, 2.1.8; Remark 2.1.2 and Examples 2.1.2, 2.1.3. Most of the results presented in the first section are included in the following papers: A.-D. Filip [35], [36]; A.-D. Filip and A. Petruşel [40].

$\diamond$ In the second section, the connexion between the Maia type theorems and the fixed point theorems in Kasahara spaces is presented. Some fixed point theorems of Maia type for single-valued operators in a set endowed with two metrics are also given. Our contributions in this section are: Theorem 2.2.2 which is a fixed point result given for almost contractions defined on a set endowed with two vector-valued metrics, extending and generalizing Maia's fixed point theorem; Remark 2.2.4 which express the connection between the fixed point result given in Kasahara spaces and the fixed point result of Maia type. Our Theorem 2.2.2 is included in the paper A.-D. Filip and A. Petruşel [39].

$\diamond$ In the third section, we introduce a new notion: *Kasahara space with respect to an operator* and we give in this setting several applications regarding the existence and uniqueness of solutions for integral and differential equations. Our contributions in this section are: Theorem 2.3.1 which is a fixed point theory in Kasahara spaces with respect to an operator, extending and complementing Banach-Caccioppoli's Contraction Principle; Theorem 2.3.2 which is a fixed point theory in Kasahara spaces with respect to an operator, extending and complementing the Graphic Contraction Principle; Theorem 2.3.3 which is an application of Theorem 2.3.1 regarding the existence and uniqueness of solution for integral equations; Theorems 2.3.4 which is also an application of Theorem 2.3.1 regarding the existence and uniqueness of solution for boundary value problems; Definition 2.3.1; Remarks 2.3.1, 2.3.2 and 2.3.3; Examples 2.3.1 and 2.3.2. All of the contributions are included in the paper A.-D. Filip [34].

Chapter 3: Multivalued generalized contractions in Kasahara spaces.

⋄ In the first section of this chapter, we present some fixed point theorems for multivalued generalized contractions in Kasahara spaces, generalized Kasahara spaces and large Kasahara spaces. Our contributions in this section are: Theorem 3.1.2 which extends Nadler's fixed point theorem (Nadler [94]) from complete metric spaces to Kasahara spaces; Theorem 3.1.3 given as a fixed point theory for Theorem 3.1.2; Theorem 3.1.4 which is a strict fixed point result, similar to Theorem 3.1.3; Theorem 3.1.5 which is a similar local fixed point result to Theorem 2.1.2, but for multivalued Zamfirescu operators; Theorem 3.1.6 which extends Theorem 3.1.5 to generalized Kasahara spaces ($d(x, y) \in \mathbb{R}^m_+$); Theorem 3.1.7 given as an application for multivalued Zamfirescu operators in generalized Kasahara spaces, concerning the existence of solutions for semi-linear inclusion systems; Theorem 3.1.8 which is a fixed point result for multivalued Zamfirescu operators in large Kasahara spaces; Theorem 3.1.9 which is a data dependence result for multivalued Zamfirescu operators in large Kasahara spaces; Corollaries 3.1.1, 3.1.2; Lemmas 3.1.2, 3.1.3; Definition 3.1.2 and Remarks 3.1.4, 3.1.5. Most of the results presented in the first section of this chapter are included in the following papers: A.-D. Filip [32], [33], [37].

⋄ In the second section of this chapter, we give some fixed point results of Maia type, in close connexion with the results given for multivalued generalized contractions in Kasahara spaces, presented in the first section of the third chapter. Our contributions in this section are: Theorem 3.2.2 which is a local fixed point result of Maia type in metric spaces; Theorem 3.2.3 which is a local fixed point result of Maia type in generalized metric spaces ($d(x, y) \in \mathbb{R}^m_+$); Corollaries 3.2.1 and 3.2.2; Remarks 3.2.1, 3.2.2. The results presented in this section are included in the following papers: A.-D. Filip [31], [32], [33]; A.-D. Filip and A. Petruşel [39].

⋄ In the third section of this chapter, we give the notion of *Kasahara space with respect to a multivalued operator* and we prove two fixed point theorems for multivalued α-contractions in the context of Kasahara spaces with respect to a multivalued operator. Our contributions in this section are: Theorems 3.3.1 and 3.3.2; Definition 3.3.1 and Example 3.3.1.

The author's contributions included in this thesis are also part of the following scientific papers:

- A.-D. Filip, *On the existence of fixed points for multivalued weak contractions*, Proceedings of the International Conference on Theory and Applications of Mathematics and Informatics, ICTAMI 2009, Alba Iulia, pp. 149-158.
- A.-D. Filip, *Fixed point theorems for multivalued contractions in Kasahara spaces*, Carpathian J. Math., submitted.
- A.-D. Filip, *Perov's fixed point theorem for multivalued mappings in generalized Kasahara spaces*, Studia Univ. Babeş-Bolyai Math., 56(2011), no. 3, 19-28.
- A.-D. Filip, *Fixed point theorems in Kasahara spaces with respect to an operator and applications*, Fixed Point Theory, 12(2011), no. 2, 329-340.
- A.-D. Filip, *Fixed point theory in large Kasahara spaces*, Anal. Univ. de Vest, Timişoara, submitted.

- A.-D. Filip, *A note on Zamfirescu's operators in Kasahara spaces*, General Mathematics, submitted.
- A.-D. Filip, *Several fixed point results for multivalued Zamfirescu operators in Kasahara spaces*, JP Journal of Fixed Point Theory and Applications, submitted.
- A.-D. Filip and P.T. Petra, *Fixed point theorems for multivalued weak contractions*, Studia Univ. Babeş-Bolyai Math., 54(2009), no. 3, 33-40.
- A.-D. Filip and A. Petruşel, *Fixed point theorems on spaces endowed with vector-valued metrics*, Fixed Point Theory and Applications, 2010, Art. ID 281381, 15 pp.
- A.-D. Filip and A. Petruşel, *Fixed point theorems for operators in generalized Kasahara spaces*, Sci. Math. Jpn., submitted.

A significant part of the original results proved in this thesis were also presented at the following scientific conferences:

- International Conference on Theory and Applications in Mathematics and Informatics (ICTAMI), September 3^{rd}-6^{th}, 2009, Alba Iulia, Romania;
- The 7^{th} International Conference on Applied Mathematics (ICAM7), September 1^{st}-4^{th}, 2010, North University of Baia Mare, Romania;
- International Conference on Nonlinear Operators, Differential Equations and Applications (ICNODEA), July 5^{th}-8^{th}, 2011, Babes-Bolyai University of Cluj-Napoca, Romania;
- The 13^{th} International Symposium on Symbolic and Numeric Algorithms for Scientific Computing (SYNASC), September 26^{th}-29^{th}, 2011, West University of Timişoara, Romania.

Keywords: fixed point, Kasahara space, generalized Kasahara space, large Kaahara space, Kasahara space with respect to an operator, L-space, w-distance, τ-distance, premetric, quasimetric, dislocated metric, partial metric, matrix convergent to zero, sequence of successive approximations, Picard operator, weakly Picard operator.

Chapter 1

Preliminaries

The purpose of this chapter is to present the basic notions and results which are further considered in the next chapters of this work, allowing us to present the results of this thesis. In this sense, we recall the notion of L-space, generalized metric, partial metric, w-distance, τ-distance, Kasahara space, generalized Kasahara space and large Kasahara space, giving also their properties and some illustrative examples. The second aim of this chapter is to give some solutions for the Problems 1.6.1, 1.6.2 and 1.6.3, posed by I.A. Rus in [121].

In order to develop the *Preliminaries*, we mention here the references which were taken in view: M. Fréchet [42]; L.M. Blumenthal [12]; M.M. Bonsangue, F. van Breugel and J.J.M.M. Rutten [13]; O. Kada, T. Suzuki and W. Takahashi [60]; S. Kasahara [62], [66]; I.A. Rus [117], [119], [121]; I.A. Rus, A. Petruşel and G. Petruşel [124]; T. Suzuki [139], [140].

1.1 L-spaces

In this section we recall the notion of L-space, an abstract space in which works one of the basic tools in the theory of operatorial equations, especially in the fixed point theory: the sequence of successive approximations method. On the other hand, the L-space plays a major role in the definition of Kasahara spaces. Some examples of L-spaces are also presented.

The notion of L-space was introduced in 1906 by M. Fréchet (see [42]) as follows:

Definition 1.1.1 (M. Fréchet [42], I.A. Rus [117]). *Let X be a nonempty set. Let*

$$s(X) := \{(x_n)_{n\in\mathbb{N}} \mid x_n \in X,\ n \in \mathbb{N}\}.$$

Let $c(X) \subset s(X)$ be a subset of $s(X)$ and $Lim : c(X) \to X$ be an operator. By definition, the triple $(X, c(X), Lim)$ is called an L-space if the following conditions are satisfied:

(i) If $x_n = x$, for all $n \in \mathbb{N}$, then $(x_n)_{n\in\mathbb{N}} \in c(X)$ and $Lim(x_n)_{n\in\mathbb{N}} = x$.

(ii) If $(x_n)_{n\in\mathbb{N}} \in c(X)$ and $Lim(x_n)_{n\in\mathbb{N}} = x$, then for all subsequences $(x_{n_i})_{i\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ we have that $(x_{n_i})_{i\in\mathbb{N}} \in c(X)$ and $Lim(x_{n_i})_{i\in\mathbb{N}} = x$.

By definition, an element $(x_n)_{n\in\mathbb{N}}$ of $c(X)$ is a convergent sequence and $x = Lim(x_n)_{n\in\mathbb{N}}$ is the limit of this sequence and we shall write

$$x_n \to x \text{ as } n \to \infty.$$

We denote an L-space by $(X, \to)$.

Example 1.1.1. *In general, an L-space is any set endowed with a structure implying a notion of convergence for sequences. For example, Hausdorff topological spaces, metric spaces, generalized metric spaces in Perov' sense (i.e. $d(x,y) \in \mathbb{R}^m_+$), generalized metric spaces in Luxemburg' sense (i.e. $d(x,y) \in \mathbb{R}_+ \cup \{+\infty\}$), K-metric spaces (i.e. $d(x,y) \in K$, where K is a cone in an ordered Banach space), gauge spaces, 2-metric spaces, D-R-spaces, probabilistic metric spaces, syntopogenous spaces, are such L-spaces. For more details in this sense, we have the paper of I.A. Rus [117] and the references therein.*

1.2 Generalized metric spaces

In this section we deal with the notions of *distance functional* and *G-metric* defined on a nonempty set X, both notions being used in the definition of generalized metric space. The connexion between L-spaces and generalized metric spaces is also discussed.

By a generalized metric on a given nonempty set X, we mean:

1°. A functional $d : X \times X \to \mathbb{R}_+$ (also called *distance functional*) which satisfies some axioms.

2°. A functional $d : X \times X \to (G, +, \leq, \overset{G}{\to})$ (also called G-metric) satisfying the following axioms:

(i) $d(x,y) \geq 0$, for all $x, y \in X$ and $d(x,y) = 0$ if and only if $x = y$;

(ii) $d(x,y) = d(y,x)$, for all $x, y \in X$;

(iii) $d(x,y) \leq d(x,z) + d(z,y)$, for all $x, y, z \in X$,

where the structure $(G, +, \leq, \overset{G}{\to})$ is an ordered L-group [1].

In this section we analyze the following problem.

Problem 1.2.1. *Which of the distance functionals $d : X \times X \to \mathbb{R}_+$ induces an L-space structure on X?*

[1]Let $(G, +)$ be a group, $\leq$ be a partial order relation on G and $\overset{G}{\to}$ be an L-space structure on G. By definition, $(G, +, \leq, \overset{G}{\to})$ is an ordered L-group if the following axioms are satisfied:

(1) $x_n \to x$ and $y_n \to y$ as $n \to \infty$ imply $x_n + y_n \to x + y$ as $n \to \infty$;

(2) $x_n \to x$, $y_n \to y$ as $n \to \infty$ and $x_n \leq y_n$ for all $n \in \mathbb{N}$ imply $x \leq y$;

(3) $x \leq y$ and $u \leq v$ imply $x + u \leq y + v$.

More consideration on ordered L-groups can be found in the work of I.A. Rus, A. Petruşel and G. Petruşel [124], p.79 .

1.3 Partial metric spaces

In this section we recall the notion of partial metric as a particular case of generalized metric. Several examples of partial metric spaces are also presented. We give also the notions regarding the convergence induced by the quasimetric q_p and the metric d_p, both these functionals being obtained from a partial metric p.

Definition 1.3.1 (S.G. Matthews in [87]). *Let X be a nonempty set. A functional $p : X \times X \to \mathbb{R}_+$ is a partial metric on X if p satisfies the following conditions:*

(p_1) $p(x, x) = p(y, y) = p(x, y)$ *if and only if* $x = y$*;*

(p_2) $p(x, x) \le p(x, y)$*, for all* $x, y \in X$*;*

(p_3) $p(x, y) = p(y, x)$*, for all* $x, y \in X$*;*

(p_4) $p(x, y) \le p(x, z) + p(z, y) - p(z, z)$*, for all* $x, y, z \in X$*.*

The couple (X, p), where X is a nonempty set and p is a partial metric on X, is called a partial metric space.

Example 1.3.1 (I.A. Rus [119]). *Let (X, d) be a metric space. Then (X, d) is a partial metric space.*

Example 1.3.2 (S.G. Matthews [87]). *Let $X := \{[a, b] \mid a, b \in \mathbb{R}_+,\ a \le b\}$ and $p : X \times X \to \mathbb{R}_+$ be the functional defined by*

$$p([a, b], [c, d]) := \max\{b, d\} - \min\{a, c\},\ \textit{for all } [a, b], [c, d] \in X,\ \textit{with } [c, d] \subseteq [a, b]..$$

Then (X, p) is a partial metric space.

Remark 1.3.1. *For more considerations on partial metric spaces and applications, see S.G. Matthews [87], [88], H.-P. A. Künzi and V. Vajner [81], M. Fitting [41], R. Kopperman, S. Matthews and H. Pajoohesh [79], S.J. O'Neill [97], S. Romaguera and M. Schellekens [109], A.K. Seda [134], S. Oltra and O. Valero [96], I.A. Rus [119].*

1.4 w-distance on a metric space (X, d)

Another generalized metric is the so called w-distance. We present in this section its definition, properties and some examples.

Definition 1.4.1 (O. Kada, T. Suzuki and W. Takahashi [60]). *Let (X, d) be a metric space. Then a function $p : X \times X \to \mathbb{R}_+$ is called a w-distance on X if the following conditions are satisfied:*

(w_1) $p(x, z) \le p(x, y) + p(y, z)$*, for all* $x, y, z \in X$*;*

(w_2) for any $x \in X$, $p(x, \cdot) : X \to \mathbb{R}_+$ is lower semicontinuous

(w_3) for any $\varepsilon > 0$, there exists $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ imply $d(x, y) \leq \varepsilon$.

Example 1.4.1 (L. Guran [45])**.** *Let (X, d) be a metric space. Then the metric d is a w-distance on (X, d).*

Example 1.4.2 (L. Guran [45])**.** *Let X be a normed linear space with norm $\|\cdot\|$. Then the functional $p : X \times X \to \mathbb{R}_+$, defined by $p(x, y) = \|x\| + \|y\|$, for all $x, y \in X$, is a w-distance on X.*

Remark 1.4.1. *More considerations on w-distances can be found in the papers of O. Kada, T. Suzuki and W. Takahashi [60], T. Suzuki [138], L. Guran [45] and the references therein.*

1.5 τ-distance on a metric space (X, d)

In [139], T. Suzuki introduces the concept of τ-distance on a metric space, which is a generalized concept of both w-distance and Tataru's distance (see D. Tataru [144]). He also give generalizations for Banach's contraction principle, Caristi's fixed point theorem, Ekeland's variational principle and the nonconvex minimization theorem of Takahashi.

Definition 1.5.1 (T. Suzuki [139])**.** *Let (X, d) be a metric space. A functional $p : X \times X \to \mathbb{R}_+$ is called a τ-distance on X if there exists an operator $\eta : X \times \mathbb{R}_+ \to \mathbb{R}_+$ and the following are satisfied:*

(τ_1) *$p(x, z) \leq p(x, y) + p(y, z)$, for all $x, y, z \in X$;*

(τ_2) *$\eta(x, 0) = 0$ and $\eta(x, t) \geq t$ for all $x \in X$ and $t \in \mathbb{R}_+$, and η is concave and continuous in its second variable;*

(τ_3) *$\lim\limits_{n\to\infty} x_n = x$ and $\lim\limits_{n\to\infty} \sup\limits_{m\geq n} \eta(z_n, p(z_n, x_m)) = 0$ imply $p(w, x) \leq \liminf\limits_{n\to\infty} p(w, x_n)$, for all $w \in X$;*

(τ_4) *$\lim\limits_{n\to\infty} \sup\limits_{m\geq n} p(x_n, y_m) = 0$ and $\lim\limits_{n\to\infty} \eta(x_n, t_n) = 0$ imply $\lim\limits_{n\to\infty} \eta(y_n, t_n) = 0$;*

(τ_5) *$\lim\limits_{n\to\infty} \eta(z_n, p(z_n, x_n)) = 0$ and $\lim\limits_{n\to\infty} \eta(z_n, p(z_n, y_n)) = 0$ imply $\lim\limits_{n\to\infty} d(x_n, y_n) = 0$.*

Example 1.5.1 (T. Suzuki [139])**.** *Let p be a w-distance on a metric space (X, d). Then p is a τ-distance on (X, d).*

Example 1.5.2 (T. Suzuki [139])**.** *Let p be a τ-distance on a metric space X and let c be a positive real number. Then a functional $q : X \times X \to \mathbb{R}_+$, defined by $q(x, y) = c \cdot p(x, y)$, for all $x, y \in X$, is also a τ-distance on X.*

Remark 1.5.1. *More considerations on τ-distances and fixed point results, can be found in the work of T. Suzuki [139], [140] and L. Guran [45].*

1.6 Kasahara spaces

Let X be a nonempty set and $d : X \times X \to \mathbb{R}_+$ be a functional. Let $\to$ be a convergence structure on X. By following S. Kasahara [66], the L-space $(X, \to)$ is called d-complete if any sequence $(x_n)_{n\in\mathbb{N}}$ in X, with $\sum_{n\in\mathbb{N}} d(x_n, x_{n+1}) < \infty$, converges in $(X, \to)$.

In a number of papers [66]-[70] S. Kasahara constructs a fixed point theory in such spaces. T.L. Hicks [47] and T.L. Hicks - B.E. Rhoades [49] give some fixed point theorems in a d-complete topological space. Other results in these directions were given by V.G. Angelov [3], J. Daneš [22], K. Iséki [55], L. Guran [45], P.Q. Khanh [75]. On the other hand, some authors give some fixed point theorems in a set with two metrics: M.G. Maia [84], V. Berinde [10], R. Precup [105], A. Petruşel and I.A. Rus [102], I.A. Rus [118], B. Rzepecki [129], L.M. Saliga [130], S. Iyer [57], I.A. Rus, A. Petruşel and G. Petruşel ([124], pp. 39-40).

We recall the notions of Kasahara space, generalized Kasahara space and large Kasahara space which were introduced by I.A. Rus in [121]:

Definition 1.6.1 (*Kasahara space*, I.A. Rus [121]). *Let $(X, \to)$ be an L-space and $d : X\times X \to \mathbb{R}_+$ be a functional. The triple $(X, \to, d)$ is a Kasahara space if and only if we have the following compatibility condition between $\to$ and d:*

$$x_n \in X,\ \sum_{n\in\mathbb{N}} d(x_n, x_{n+1}) < +\infty \ \Rightarrow (x_n)_{n\in\mathbb{N}} \text{ converges in } (X, \to). \tag{1.6.1}$$

Definition 1.6.2 (*Generalized Kasahara space*, I.A. Rus [121]). *Let $(X, \to)$ be an L-space, $(G, +, \leq, \stackrel{G}{\to})$ be an L-space ordered semigroup with unity, 0 be the least element in $(G, \leq)$ and $d_G : X \times X \to G$ be an operator. The triple $(X, \to, d_G)$ is a generalized Kasahara space if and only if we have the following compatibility condition between $\to$ and d_G:*

$$x_n \in X,\ \sum_{n\in\mathbb{N}} d_G(x_n, x_{n+1}) < +\infty \ \Rightarrow (x_n)_{n\in\mathbb{N}} \text{ converges in } (X, \to). \tag{1.6.2}$$

Notice that by the inequality with the symbol $+\infty$ in the compatibility condition (1.6.2), we mean that the series $\sum\limits_{n\in\mathbb{N}} d_G(x_n, x_{n+1})$ is bounded in $(G, \leq)$.

Definition 1.6.3 (*Large Kasahara space*, I.A. Rus [121]). *Let $(X, \to)$ be an L-space, $(G, +, \leq, \stackrel{G}{\to})$ be an L-space ordered semigroup with unity, 0 be the least element in $(G, \leq)$ and $d_G : X \times X \to G$ be an operator. The triple $(X, \to, d_G)$ is a large Kasahara space if and only if we have the following compatibility condition between $\to$ and d_G:*

$$\begin{aligned} x_n \in X,\ (x_n)_{n\in\mathbb{N}} &\text{ a Cauchy sequence (in a certain sense) with respect to } d_G \\ &\text{implies that } (x_n)_{n\in\mathbb{N}} \text{ converges in } (X, \to). \end{aligned} \tag{1.6.3}$$

Some examples of Kasahara spaces are presented in the sequel.

Example 1.6.1 (The trivial Kasahara space). *Let (X, d) be a complete metric space. Let $\stackrel{d}{\to}$ be the convergence structure induced by the metric d on X. Then $(X, \stackrel{d}{\to}, d)$ is a Kasahara space.*

Example 1.6.2 (I.A. Rus [121]). *Let (X, ρ) be a complete semimetric space, where $\rho : X \times X \to \mathbb{R}_+$ is continuous. Let $d : X \times X \to \mathbb{R}_+$ be a functional such that there exists $c > 0$ with $\rho(x, y) \leq c \cdot d(x, y)$, for all $x, y \in X$. Then $(X, \overset{\rho}{\to}, d)$ is a Kasahara space.*

Example 1.6.3 (I.A. Rus [121]). *Let (X, ρ) be a complete quasimetric space where $\rho : X \times X \to \mathbb{R}_+$. Let $d : X \times X \to \mathbb{R}_+$ be a functional such that there exists $c > 0$ with $\rho(x, y) \leq c \cdot d(x, y)$, for all $x, y \in X$. Then $(X, \overset{\rho}{\to}, d)$ is a Kasahara space.*

Example 1.6.4 (I.A. Rus [121]). *Let $\rho : X \times X \to \mathbb{R}^m_+$ be a generalized complete metric on a set X. Let $x_0 \in X$ and $\lambda \in \mathbb{R}^m_+$ with $\lambda \neq 0$. Let $d_\lambda : X \times X \to \mathbb{R}^m_+$ be defined by*

$$d_\lambda(x, y) := \begin{cases} \rho(x, y), & \text{if } x \neq x_0 \text{ and } y \neq x_0 \\ \lambda, & \text{if } x = x_0 \text{ or } y = x_0. \end{cases}$$

Then $(X, \overset{\rho}{\to}, d_\lambda)$ is a generalized Kasahara space.

Example 1.6.5 (I.A. Rus [121]). *Let (X, ρ) be a complete partial metric space. Then $(X, \overset{\rho}{\to}, d_\rho)$ is a large Kasahara space, where $d_\rho : X \times X \to \mathbb{R}_+$ is defined by*

$$d_\rho(x, y) := \rho(x, y) + \rho(y, x) - \rho(x, x) - \rho(y, y), \text{ for all } x, y \in X.$$

We present in our thesis some solutions for the following problems, formulated by I.A. Rus in [121]:

Problem 1.6.1. *Give relevant examples of Kasahara spaces.*

Problem 1.6.2. *Let p be a w-distance on a complete metric space (X, d). In which conditions $(X, \overset{d}{\to}, p)$ is a large Kasahara space?*

Problem 1.6.3. *Let p be a τ-distance on the complete metric space (X, d). In which conditions $(X, \overset{d}{\to}, p)$ is a large Kasahara space?*

1.7 Operators on Kasahara spaces

In this section we consider the Kasahara space $(X, \to, d)$, where $d : X \times X \to \mathbb{R}_+$ is a functional. We define the continuity and the closeness properties for self-mappings $f : X \to X$ with respect to $\to$ and we give metric conditions for f with respect to d, by presenting some generalized contractions in this sense. Finally, we define the well-posed fixed point problem and the limit shadowing property for f with respect to d. By a similar way, the case of multivalued operators defined on Kasahara spaces is also presented.

Chapter 2

Generalized contractions in Kasahara spaces

In this chapter we develop the theory of some important fixed point results as the Banach-Caccioppoli's Contraction Principle, the Graphic Contraction Principle, Caristi-Browder and Matkowski type theorems. Our results are given for single-valued generalized contractions in the context of a Kasahara space $(X, \to, d)$, where $d : X \times X \to \mathbb{R}_+$ is a functional. We present also some extensions of our results in generalized and large Kasahara spaces.

In the sequel, we present the connexion between the Maia type theorems and the fixed point theorems in Kasahara spaces, we introduce a new notion: Kasahara space with respect to an operator and we give in this setting several applications regarding the existence and uniqueness of solutions for integral and differential equations.

The references which were used to develop this chapter are: A.-D. Filip [34], [35], [36]; A.-D. Filip and A. Petruşel [39], [40]; S. Kasahara [66]; M.G. Maia [84]; I.A. Rus [110], [115], [117], [119], [121]; I.A. Rus, A.S. Mureşan and V. Mureşan [122]; I.A. Rus, A. Petruşel and G. Petruşel [124]; M.-A. Şerban [142]; T. Zamfirescu [150].

2.1 Fixed point theorems in Kasahara spaces

The aim of this section is to present the theory of some well-known fixed point results in the context of Kasahara spaces. Some of these results are also given in generalized and large Kasahara spaces as follows:

- fixed point theorems for generalized contractions in generalized Kasahara spaces $(X, \to, d)$, where $d : X \times X \to \mathbb{R}_+ \cup \{+\infty\}$ is a functional;
- a fixed point theory for the local variant of Banach-Caccioppoli's Contraction Principle in large Kasahara spaces $(X, \overset{d}{\to}, p)$, where $d : X \times X \to \mathbb{R}_+$ is a complete metric on X and $p : X \times X \to \mathbb{R}_+$ is a w-distance on X;

- fixed point theorems for generalized contractions in large Kasahara spaces $(X, \xrightarrow{d}, \varphi \circ d)$ which are obtained from complete metric spaces (X, d), by perturbing the metric with an increasing, subadditive and continuous function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$.

We consider first the Kasahara space $(X, \to, d)$, where $d : X \times X \to \mathbb{R}_+$ is a functional. In our results we will use the following notions and notations:

Definition 2.1.1. *Let $(X, \to, d)$ be a Kasahara space, where $d : X \times X \to \mathbb{R}_+$ is a functional. Let $f : X \to X$ be an operator. Then*

(i) f is a Picard operator if and only if $F_f = \{x^*\}$ and $f^n(x) \to x^*$ as $n \to \infty$, for all $x \in X$;

(ii) f is a weakly Picard operator if and only if the sequence $(f^n(x))_{n\in\mathbb{N}}$ converges for all $x \in X$ and the limit (which may depend on x) is a fixed point of f;

(iii) if f is a weakly Picard operator, then we define the operator

$$f^\infty : X \to X \text{ by } f^\infty(x) := Lim(f^n(x))_{n\in\mathbb{N}};$$

Remark 2.1.1. *More considerations on Picard operators and weakly Picard operators can be found in the work of I.A. Rus [117], [115], I.A. Rus, A. Petruşel and M.A. Şerban [127].*

We recall also a very useful tool which will help us to prove the uniqueness of a fixed point for a single-valued operator defined on a Kasahara space.

Lemma 2.1.1 (Kasahara's lemma [66])**.** *Let $(X, \to, d)$ be a Kasahara space, where $d : X \times X \to \mathbb{R}_+$ is a functional. Then*

$$\text{for all } x, y \in X,\ d(x, y) = d(y, x) = 0 \Rightarrow x = y.$$

We present next one of our fixed point results and its theory.

Theorem 2.1.1 (The Contraction Principle). *Let $(X, \to, d)$ be a Kasahara space, where $d : X \times X \to \mathbb{R}_+$ is a functional. Let $f : X \to X$ be an operator. We assume that*

(i) $f : (X, \to) \to (X, \to)$ has closed graph;

(ii) $f : (X, d) \to (X, d)$ is an α-contraction, i.e., there exists $\alpha \in [0, 1[$ such that

$$d(f(x), f(y)) \le \alpha d(x, y),\ \text{for all } x, y \in X.$$

Then the following statements hold:

(1) *$F_f = F_{f^n} = \{x_f^*\}$, for all $n \in \mathbb{N}^*$ and $d(x_f^*, x_f^*) = 0$;*

(2) *$f^n(x) \to x_f^*$ as $n \to \infty$, for all $x \in X$, i.e., $f : (X, \to) \to (X, \to)$ is a Picard operator;*

(3) *for all $x \in X$ we have,*

(3.1) $d(f^n(x), x_f^*) \overset{\mathbb{R}}{\to} 0$ *as* $n \to \infty$*;*

(3.2) $d(x_f^*, f^n(x)) \overset{\mathbb{R}}{\to} 0$ *as* $n \to \infty$*;*

(4) *if the functional* d *is a quasimetric (i.e.,* $d(x, y) = d(y, x) = 0 \Leftrightarrow x = y$ *for all* $x, y \in X$ *and* d *satisfies the triangle inequality), then*

(4.1) $d(x, x_f^*) \leq \frac{1}{1-\alpha} d(x, f(x))$*, for all* $x \in X$*;*

(4.2) $d(x_f^*, x) \leq \frac{1}{1-\alpha} d(f(x), x)$*, for all* $x \in X$*;*

(4.3) $d(f^n(x), x_f^*) \leq \frac{\alpha^n}{1-\alpha} d(x, f(x))$*, for all* $x \in X$*;*

(4.4) $d(x_f^*, f^n(x)) \leq \frac{\alpha^n}{1-\alpha} d(f(x), x)$*, for all* $x \in X$*;*

(4.5) *if* $(z_n)_{n\in\mathbb{N}} \subset X$ *is such that* $d(z_n, f(z_n)) \overset{\mathbb{R}}{\to} 0$ *as* $n \to \infty$ *then* $d(z_n, x_f^*) \overset{\mathbb{R}}{\to} 0$ *as* $n \to \infty$*, i.e., the fixed point problem for the operator* f *is well-posed with respect to* d*;*

(4.6) *if* $(z_n)_{n\in\mathbb{N}} \subset X$ *is such that* $d(z_{n+1}, f(z_n)) \overset{\mathbb{R}}{\to} 0$ *as* $n \to \infty$ *then* $d(z_{n+1}, f^{n+1}(z)) \overset{\mathbb{R}}{\to} 0$ *as* $n \to \infty$*, for all* $z \in X$*, i.e., the operator* f *has the limit shadowing property with respect to* d*;*

(4.7) *if* $g : X \to X$ *has the property that there exists* $\eta > 0$ *for which* $d(g(x), f(x)) \leq \eta$*, for all* $x \in X$*, then*

$$x_g^* \in F_g \text{ implies } d(x_g^*, x_f^*) \leq \frac{\eta}{1-\alpha}.$$

Remark 2.1.2. *Theorem 2.1.1 extends Banach-Caccioppoli's Contraction Principle in the sense that instead of the metric space* (X, d) *it can be considered the Kasahara space* $(X, \to, d)$*. The functional* $d : X \times X \to \mathbb{R}_+$ *need not to satisfy all of the axioms of the metric. On the other hand, Theorem 2.1.1 complements the conclusions of Banach-Caccioppoli's Contraction Principle in the sense that some fixed point problems are considered: well-possedness (item (4.5)), limit shadowing property (item (4.6)), data dependence (item (4.7)).*

- We give next one of the fixed point results concerning single-valued Zamfirescu operators.

In 1972, T. Zamfirescu gives in [150] several fixed point theorems for single-valued mappings of contractive type in metric spaces, obtaining generalizations for Banach-Caccioppoli's contraction principle, Kannan's, Edelstein's and Singh's theorems. We give local and global similar results for Zamfirescu operators in Kasahara spaces. Since the domain invariance for Zamfirescu's operators is not always satisfied, we use in our proofs the successive approximations method. Our local results extend and generalize Krasnoselskii's local fixed point theorem by replacing the context of metric space with a Kasahara space. On the other hand, instead of contractions we use Zamfirescu's operators.

Definition 2.1.2. *Let* $(X, \to, d)$ *be a Kasahara space, where* $d : X \times X \to \mathbb{R}_+$ *is a functional. The mapping* $f : X \to X$ *is called Zamfirescu operator if there exist* α*,* β*,* $\gamma \in \mathbb{R}_+$ *with* $\alpha < 1$*,* $\beta < \frac{1}{2}$ *and* $\gamma < \frac{1}{2}$ *such that for each* $x, y \in X$ *at least one of the following conditions is true:*

(1_z) *$d(f(x), f(y)) \leq \alpha d(x, y)$;*

(2_z) *$d(f(x), f(y)) \leq \beta[d(x, f(x)) + d(y, f(y))]$;*

(3_z) *$d(f(x), f(y)) \leq \gamma[d(x, f(y)) + d(y, f(x))]$.*

Remark 2.1.3. *In our fixed point results we will consider the Kasahara space $(X, \to, d)$, where $d : X \times X \to \mathbb{R}_+$ is a premetric, i.e.,*

(1) *$d(x, x) = 0$, for all $x \in X$;*

(2) *$d(x, z) \leq d(x, y) + d(y, z)$, for all $x, y, z \in X$.*

We also will consider the following notion and notation.

Definition 2.1.3. *Let $(X, \to, d)$ be a Kasahara space, where $d : X \times X \to \mathbb{R}_+$ is a premetric. Then*

$$\tilde{B}(x_0, r) := \{x \in X \mid d(x_0, x) \leq r\}$$

is the right closed ball centered in $x_0 \in X$ with radius $r \in \mathbb{R}_+$.

Remark 2.1.4. *Let $(X, \to, d)$ be a Kasahara space, where $d : X \times X \to \mathbb{R}_+$ is a premetric. Let $x_0 \in X$ and $r \in \mathbb{R}_+$. If d is continuous on X with respect to the second argument, then the right closed ball $\tilde{B}(x_0, r)$ is a closed set in X with respect to $\to$, i.e., for any sequence $(z_n)_{n\in\mathbb{N}} \subset \tilde{B}(x_0, r)$, with $z_n \to z \in X$, as $n \to \infty$, we get that $z \in \tilde{B}(x_0, r)$.*

Our main local fixed point result which extends and generalizes Krasnoselskii's theorem (see e.g. [44]) is the following:

Theorem 2.1.2 (A.-D. Filip [36])**.** *Let $(X, \to, d)$ be a Kasahara space, where $d : X \times X \to \mathbb{R}_+$ is a premetric. Let $x_0 \in X$, $r \in \mathbb{R}_+$ and $f : \tilde{B}(x_0, r) \to X$ be a Zamfirescu operator. We assume that:*

(i) $Graph(f)$ is closed in $X \times X$ with respect to $\to$;

(ii) $d(x_0, f(x_0)) \leq (1 - \delta)r$, where $\delta = \max\left\{\alpha, \frac{\beta}{1-\beta}, \frac{\gamma}{1-\gamma}\right\}$;

(iii) d is continuous with respect to the second argument.

Then:

(1°) *f has at least one fixed point in $\tilde{B}(x_0, r)$ and $f^n(x_0) \to x^* \in F_f$, as $n \to \infty$.*

(2°) *the following estimation holds:*

$$d(x_n, x^*) \leq \delta^n r, \text{ for all } n \in \mathbb{N}, \tag{2.1.1}$$

where $x^* \in F_f$ and $(x_n)_{n\in\mathbb{N}}$ is the sequence of successive approximations for f starting from x_0.

Remark 2.1.5. *An extension of our fixed point result to large Kasahara spaces can be made. In order to obtain a large Kasahara space from the Kasahara space $(X, \to, d)$, where $d : X \times X \to \mathbb{R}_+$ is a premetric, we need to define a certain notion of Cauchy sequence with respect to the premetric d. We must take also into account the fact that d is not symmetric.*

Definition 2.1.4. *Let (X, d) be a premetric space with $d : X \times X \to \mathbb{R}_+$ and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. Then $(x_n)_{n\in\mathbb{N}}$ is a right-Cauchy sequence with respect to d if and only if*

$$\lim_{\substack{n\to\infty \\ m\to\infty}} d(x_n, x_m) = 0,$$

i.e., for any $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$, for every $m, n \in \mathbb{N}$ with $m \geq n \geq k$.

The following notion of large Kasahara space arises.

Definition 2.1.5 (A.-D. Filip [36])**.** *Let $(X, \to)$ be an L-space. Let $d : X \times X \to \mathbb{R}_+$ be a premetric on X. The triple $(X, \to, d)$ is a large Kasahara space if and only if the following compatibility condition between $\to$ and d holds:*

$$\text{if } (x_n)_{n\in\mathbb{N}} \subset X \text{ with } \lim_{\substack{n\to\infty \\ m\to\infty}} d(x_n, x_m) = 0 \text{ then } (x_n)_{n\in\mathbb{N}} \text{ converges in } (X, \to).$$

Remark 2.1.6 (A.-D. Filip [36])**.** *Let $(X, \to, d)$ be a large Kasahara space in the sense of Definition 2.1.5. Then $(X, \to, d)$ is a Kasahara space.*

Remark 2.1.7. *Let $(X, \to, d)$ be a large Kasahara space in the sense of Definition 2.1.5. In this context, Theorem 2.1.2 holds.*

- We present in the sequel one of the fixed point results given in generalized Kasahara spaces $(X, \to, d)$, where $d : X \times X \to \mathbb{R}_+ \cup \{+\infty\}$ is a functional. An example of such generalized Kasahara space is given bellow.

Example 2.1.1 (A.-D. Filip and A. Petruşel [40])**.** *Let $a > 0$ and $I := [t_0 - a, t_0 + a] \subset \mathbb{R}$. Denote*

$$X := C(I) := \big\{x : I \to \mathbb{R} \mid x \text{ is a continuous function on } I\big\}.$$

Let $\lambda > 0$ and consider $d_\lambda : C(I) \times C(I) \to \mathbb{R}_+ \cup \{+\infty\}$ defined by

$$d_\lambda(x, y) := \max\left\{\frac{1}{|t - t_0|^\lambda}|x(t) - y(t)| : t \in I\right\}, \text{ for } x, y \in C(I). \tag{2.1.2}$$

Notice that d_λ is not necessarily finite for every pair of functions $x, y \in C(I)$. Thus, by following W.A.J. Luxemburg [82], we have that d_λ is a generalized metric on $C(I)$ and

$$\lim_{\substack{n\to\infty \\ m\to\infty}} d_\lambda(x_n, x_m) = 0 \;\Rightarrow\; \text{there exists } x \in C(I) \text{ such that } \lim_{n\to\infty} d_\lambda(x_n, x) = 0. \tag{2.1.3}$$

We also denote by $\rho = \max\{|x(t) - y(t)| : t \in I\}$ the metric of uniform convergence on $C(I)$ and by $\overset{\rho}{\to}$ the convergence structure induced by ρ on $C(I)$.

The triple $(C(I), \overset{\rho}{\to}, d_\lambda)$ is a generalized Kasahara space.

In our results, we will also use the following notions.

Definition 2.1.6 (A.-D. Filip and A. Petruşel [40])**.** *Let $(X, \to, d)$ be a generalized Kasahara space, where $d : X \times X \to \mathbb{R}_+ \cup \{+\infty\}$ is a functional. Let $f : X \to X$ be an operator. We say that f is a*

- *Picard operator if*
  1) *$F_f = \{x^*\}$;*
  2) *$f^n(x_0) \to x^*$ as $n \to \infty$, for each $x_0 \in X$ with the property $d(x_0, f(x_0)) < +\infty$.*
- *weakly Picard operator if*
  1) *$F_f \neq \emptyset$;*
  2) *the sequence $(f^n(x_0))_{n\in\mathbb{N}}$ converges for each $x_0 \in X$ with $d(x_0, f(x_0)) < +\infty$ and the limit is a fixed point of f.*

Remark 2.1.8. *Kasahara's Lemma 2.1.1 also holds in the case when $(X, \to, d)$ is a generalized Kasahara space, where $d : X \times X \to \mathbb{R}_+ \cup \{+\infty\}$ is a functional. The lemma is proved in the work of S. Kasahara [66].*

Theorem 2.1.3 (A.-D. Filip and A. Petruşel [40])**.** *Let $(X, \to, d)$ be a generalized Kasahara space, where $d : X \times X \to \mathbb{R}_+ \cup \{+\infty\}$ is a functional. Let $f : X \to X$ be an operator. We assume that:*

i) $f : (X, \to) \to (X, \to)$ has closed graph;

ii) there exists $\alpha \in [0, 1[$ such that

$$d(f(x), f(y)) \le \alpha d(x, y), \text{ for all } x, y \in X, \text{ with } d(x, y) < +\infty;$$

iii) there exists $x_0 \in X$ such that $d(x_0, f(x_0)) < +\infty$.

Then we have:

1) f is a weakly Picard operator;

2) if $d(x^*, y^*) < +\infty$, for all $x^*, y^* \in F_f$ then f is a Picard operator;

3) if $d(x, x) = 0$, for all $x \in X$ then $d(x^*, f(x^*)) < +\infty$, for all $x^* \in F_f$;

4) if $x \in X$ and $x^* \in F_f$ such that $d(x, x^*) < +\infty$, then

$$d(f^n(x), x^*) \to 0 \text{ as } n \to \infty;$$

5) if $d(x_0, x^*) < +\infty$, for all $x^* \in F_f$ and

$$d(f^k(x_0), x^*) \le d(f^k(x_0), f^{k+1}(x_0)) + d(f^{k+1}(x_0), x^*), \text{ for all } k \in \mathbb{N},$$

then

$$d(x_0, x^*) \le \frac{1}{1-\alpha} d(x_0, f(x_0)).$$

- We consider next the generalized Kasahara space $(X, \to, d)$, where d is a real vector-valued functional, i.e., $d : X \times X \to \mathbb{R}^n_+$. In this setting, we have some fixed point results given by I.A. Rus in [121]. One of them is the following one.

Theorem 2.1.4 (I.A. Rus [121]). *Let $(X, \to, d)$ be a generalized Kasahara space, where $d : X \times X \to \mathbb{R}^n_+$ is a functional. Let $f : X \to X$ be an operator. We suppose that:*

(i) *$f : (X, \to) \to (X, \to)$ has closed graph;*

(ii) *$f : (X, d) \to (X, d)$ is a S-contraction, i.e. $d(f(x), f(y)) \leq Sd(x, y)$, for all $x, y \in X$, with S a matrix convergent to zero.*

Then:

(1) *$F_f = \{x^*\}$; $d(x^*, x^*) = 0$;*

(2) *$f^n(x) \to x^*$ as $n \to +\infty$, for all $x \in X$;*

(3) $\diamond$ *$d(f^n(x), x^*) \overset{\mathbb{R}^n}{\to} 0$, as $n \to \infty$, for all $x \in X$;*
$\diamond$ *$d(x^*, f^n(x)) \overset{\mathbb{R}^n}{\to} 0$, as $n \to \infty$, for all $x \in X$;*

(4) *If d is a quasimetric (i.e., $d(x, y) = d(y, x) = 0 \Leftrightarrow x = y$ for all $x, y \in X$ and d satisfies the triangle inequality), then:*

(a) $\diamond$ *$d(x, x^*) \leq (I - S)^{-1} d(x, f(x))$, for all $x \in X$;*
$\diamond$ *$d(x^*, x) \leq (I - S)^{-1} d(f(x), x)$, for all $x \in X$;*

(b) *If $g : X \to X$ is such that*

$$d(f(x), g(x)) \leq \eta, \text{ for all } x \in X,$$

then $d(x^*, y^*) \leq (I - S)^{-1}\eta$, for all $y^* \in F_g$.

- We present in the sequel a theory for the local variant of Banach-Caccioppoli's Contraction Principle in the context of large Kasahara spaces. To achieve this purpose, some auxiliary notions need to be defined.

Definition 2.1.7. *Let X be a nonempty set and $p : X \times X \to \mathbb{R}_+$ be a w-distance (see Definition 1.4.1) on X. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. Then*

(1) *the convergence structure induced by p on X is denoted by $\overset{p}{\to}$ and it is defined as follows*

$$x_n \overset{p}{\to} x \text{ as } n \to \infty \text{ if and only if } \lim_{n\to\infty} p(x_n, x) = 0.$$

(2) *$(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to p if and only if there exists a sequence $(\alpha_n)_{n\in\mathbb{N}}$ in $\mathbb{R}_+$ such that*

(2_a) $\lim_{n\to\infty} \alpha_n = 0$;

(2_b) $p(x_n, x_m) \leq \alpha_n$ *for all* $n, m \in \mathbb{N}$ *with* $m > n$.

By Definition 2.1.7 the following notion of large Kasahara space arises.

Definition 2.1.8. *Let* $(X, \to)$ *be an L-space. Let* $p : X \times X \to \mathbb{R}_+$ *be a w-distance on* X. *The triple* $(X, \to, p)$ *is a large Kasahara space if and only if the following compatibility condition between* $\to$ *and* p *holds:*

if $(x_n)_{n\in\mathbb{N}} \subset X$ *is a Cauchy sequence with resepct to* p *in the sense of Definition 2.1.7 then* $(x_n)_{n\in\mathbb{N}}$ *converges in* $(X, \to)$.

Example 2.1.2. *Let* (X, d) *be a complete metric space and* p *be a w-distance on* X. *Then* $(X, \overset{d}{\to}, p)$ *is a large Kasahara space in the sense of Definition 2.1.8.*

Lemma 2.1.2. *Let* (X, d) *be a metric space and* $p : X \times X \to \mathbb{R}_+$ *be a w-distance on* X. *Let* $x_0 \in X$, $r \in \mathbb{R}_+$ *and*

$$\tilde{B}_p(x_0, r) := \{x \in X \mid p(x_0, x) \leq r\}$$

be the right closed ball centered in x_0 *with radius* r. *Then*

(1) $\tilde{B}_p(x_0, r)$ *is a closed set in* (X, d);

(2) *If* (X, d) *is complete, then* $\left(\tilde{B}_p(x_0, r), \overset{d}{\to}, p\right)$ *is a large Kasahara space in the sense of Definition 2.1.8.*

Theorem 2.1.5. *Let* $(X, \overset{d}{\to}, p)$ *be a large Kasahara space in the sense of Definition 2.1.8, where* $\overset{d}{\to}$ *is the convergence structure induced by the complete metric* $d : X \times X \to \mathbb{R}_+$ *on* X *and* $p : X \times X \to \mathbb{R}_+$ *is a w-distance on* X. *Let* $x_0 \in X$, $r \in \mathbb{R}_+$ *and* $f : \tilde{B}_p(x_0, r) \to X$ *be an operator such that:*

(i) $f : (\tilde{B}_p(x_0, r), d) \to (X, d)$ *has closed graph;*

(ii) $f : (\tilde{B}_p(x_0, r), p) \to (X, p)$ *is an* α*-contraction on* $\tilde{B}_p(x_0, r)$, *i.e., there exists* $\alpha \in [0, 1[$ *such that*

$$p(f(x), f(y)) \leq \alpha p(x, y) \text{ for all } x, y \in \tilde{B}_p(x_0, r);$$

(iii) $p(x_0, f(x_0)) \leq (1 - \alpha)r$.

Then the following statements hold

(1) $F_f = F_{f^n} = \{x_f^*\}$, *for all* $n \in \mathbb{N}^*$ *and* $p(x_f^*, x_f^*) = 0$;

(2) $f^n(x_0) \overset{d}{\to} x_f^* \in \tilde{B}_p(x_0, r)$ *as* $n \to \infty$, *for all* $x \in \tilde{B}_p(x_0, r)$, *i.e.,* $f : (\tilde{B}_p(x_0, r), \overset{d}{\to}) \to (X, \overset{d}{\to})$ *is a Picard operator;*

(3) $\lim_{n\to\infty} p(f^n(x), x_f^*) = 0$, *for all* $x \in \tilde{B}_p(x_0, r)$;

(4) *for all* $x \in \tilde{B}_p(x_0, r)$ *we have:*

(4.1) $p(x, x_f^*) \leq \frac{1}{1-\alpha} p(x, f(x))$;

(4.2) $p(x_f^*, x) \leq \frac{1}{1-\alpha} p(f(x), x)$;

(4.3) $p(f^n(x), x_f^*) \leq \frac{\alpha^n}{1-\alpha} p(x, f(x))$;

(4.4) $p(x_f^*, f^n(x)) \leq \frac{\alpha^n}{1-\alpha} p(f(x), x)$;

(4.5) *if* $g : \tilde{B}_p(x_0, r) \to X$ *has the property that there exists* $\mu > 0$ *for which*

$$p(g(x), f(x)) \leq \mu, \text{ for all } x \in \tilde{B}_p(x_0, r)$$

then

$$x_g^* \in F_g \text{ and } x_g^* \in \tilde{B}_p(x_0, r) \text{ implies } p(x_g^*, x_f^*) \leq \frac{\mu}{1-\alpha}.$$

- We give next one of the fixed point theorems in large Kasahara spaces that are obtained from complete metric spaces by perturbing the metric.

Several fixed point theorems were proved in metric spaces with perturbed metric. In this sense we have the works of M.S. Khan, M. Swaleh and S. Sessa [74] ,K.P.R. Sastry and G.V.R. Babu [131], [132], K.P.R. Sastry, G.V.R. Babu and D.N. Rao [133], M.A. Şerban [142].

Theorem 2.1.6 (A.-D. Filip [35]). *Let* $(X, \xrightarrow{d}, \rho)$ *be a large Kasahara space with* $d : X \times X \to \mathbb{R}_+$ *a complete metric on* X *and* $\rho : X \times X \to \mathbb{R}_+$ *a distance functional defined by* $\rho = \varphi \circ d$, *where* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *is an increasing, subadditive and continuous function. Let* $f : X \to X$ *be an operator. We assume that:*

(i) $f : (X, \xrightarrow{d}) \to (X, \xrightarrow{d})$ *has closed graph;*

(ii) $f : (X, \rho) \to (X, \rho)$ *is an* α*-contraction, i.e., there exists* $\alpha \in [0, 1[$ *such that*

$$\rho(f(x), f(y)) \leq \alpha \rho(x, y), \text{ for all } x, y \in X;$$

(iii) $\varphi(t) = 0 \Rightarrow t = 0$, *for all* $t \in \mathbb{R}_+$.

Then the following statements hold:

(1) $F_f = F_{f^n} = \{x_f^*\}$, *for all* $n \in \mathbb{N}^*$ *and* $\rho(x_f^*, x_f^*) = 0$;

(2) $f^n(x) \xrightarrow{d} x_f^*$ *as* $n \to \infty$, *for all* $x \in X$, *i.e.,* $f : (X, \xrightarrow{d}) \to (X, \xrightarrow{d})$ *is a Picard operator;*

(3) *for all* $x \in X$ *we have:*

(3_a) $\rho(f^n(x), x_f^*) \xrightarrow{\mathbb{R}} 0$ *as* $n \to \infty$;

(3_b) $\rho(x, x_f^*) \leq \frac{1}{1-\alpha} \rho(x, f(x))$;

(3_c) $\rho(f^n(x), x_f^*) \leq \frac{\alpha^n}{1-\alpha}\rho(x, f(x))$, *for all* $n \in \mathbb{N}$;

(4) $(z_n)_{n\in\mathbb{N}} \subset X$, $\rho(z_n, f(z_n)) \overset{\mathbb{R}}{\to} 0$ *as* $n \to \infty \Rightarrow \rho(z_n, x_f^*) \overset{\mathbb{R}}{\to} 0$ *as* $n \to \infty$, *i.e., the fixed point problem for the operator* f *is well-posed with respect to* ρ;

(5) $(z_n)_{n\in\mathbb{N}} \subset X$, $\rho(z_{n+1}, f(z_n)) \overset{\mathbb{R}}{\to} 0$ *as* $n \to \infty \Rightarrow \rho(z_{n+1}, f^{n+1}(z)) \overset{\mathbb{R}}{\to} 0$ *as* $n \to \infty$, *for all* $z \in X$, *i.e., the operator* f *has the limit shadowing property with respect to* ρ;

(6) *if* $g : X \to X$ *has the property that there exists* $\eta > 0$ *for which*

$$\rho(g(x), f(x)) \leq \eta, \text{ for all } x \in X,$$

then

$$x_g^* \in F_g \text{ implies } \rho(x_g^*, x_f^*) \leq \frac{\eta}{1-\alpha}.$$

Remark 2.1.9. *Particular cases of large Kasahara spaces can be obtained for a given perturbing function* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$. *The following example is relevant in this sense.*

Example 2.1.3 (A.-D. Filip [35])**.** *Let* (X, d) *be a complete metric space and* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *be a function defined by*

$$\varphi(t) = t + \theta(t, u(t)), \text{ for all } t \in \mathbb{R}_+$$

where $\theta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$ *is a symmetric function satisfying the triangle inequality and* $u : \mathbb{R} \to \mathbb{R}_+$ *is a function.*

Then $(X, \overset{d}{\to}, \varphi \circ d)$ *is a large Kasahara space.*

2.2 Maia type fixed point theorems

The aim of this section is to recall the Maia fixed point theorem and some of its versions in order to establish a connexion with fixed point theorems in Kasahara spaces.

Theorem 2.2.1 (M.G. Maia, [84])**.** *Let* X *be a nonempty set,* d *and* ρ *be two metrics on* X *and* $f : X \to X$ *be a mapping. Suppose that:*

(*i*) $\rho(x, y) \leq d(x, y)$, *for all* $x, y \in X$;

(*ii*) (X, ρ) *is a complete metric space;*

(*iii*) $f : (X, \rho) \to (X, \rho)$ *is continuous;*

(*iv*) $f : (X, d) \to (X, d)$ *is an* α*-contraction, i.e., there exists* $\alpha \in [0, 1[$ *such that*

$$d(f(x), f(y)) \leq \alpha \cdot d(x, y), \text{ for all } x, y \in X.$$

Then

(1) *$F_f = \{x^*\}$;*

(2) *$(f^n(x_0))_{n\in\mathbb{N}}$ converges in (X,ρ) to x^*, for all $x_0 \in X$.*

In applications we usually use the Rus variant of Maia's Theorem 2.2.1. In this sense, a very useful remark was made by I.A. Rus in [110] (see also [115]).

Remark 2.2.1. *Theorem 2.2.1 remains true if condition (i) is replaced by*

(i') *there exists $c > 0$ such that $\rho(f(x), f(y)) \le c \cdot d(x,y)$, for all $x, y \in X$;*

Remark 2.2.2. *Some other Maia type results are the fixed point theorems given on a set endowed with two metrics. We recall one of them bellow.*

Theorem 2.2.2 (A.-D. Filip and A. Petruşel [39])**.** *Let X be a nonempty set and $d, \rho : X \times X \to \mathbb{R}^m_+$ be two generalized metrics on X. Let $f : X \to X$ be an operator. We assume that*

1) there exists $C \in M_{m,m}(\mathbb{R}_+)$ such that $\rho(f(x), f(y)) \le C \cdot d(x,y)$, for all $x, y \in X$;

2) (X,ρ) is a complete generalized metric space;

3) $f : (X,\rho) \to (X,\rho)$ is continuous;

4) $f : (X,d) \to (X,d)$ is an almost contraction, i.e., there exist $A, B \in M_{m,m}(\mathbb{R}_+)$ such that for all $x, y \in X$ one has

$$d(f(x), f(y)) \le Ad(x,y) + Bd(y, f(x)).$$

If the matrix A converges towards zero, then $F_f \neq \emptyset$.

In addition, if the matrix $A + B$ converges to zero, then $F_f = \{x^*\}$.

Remark 2.2.3. *Other fixed point theorems on a set endowed with two metrics can be found in the work of M. Albu [1], V. Berinde [9], B.C. Dhage [24], A.S. Mureşan [92], [90], A.S. Mureşan and V. Mureşan [91], V. Mureşan [93], R. Precup [105], B.K. Ray [107], I.A. Rus [110], [111], [113], B. Rzepecki [129], I.A. Rus, A.S. Mureşan and V. Mureşan [122].*

Remark 2.2.4. *The fixed point theorems in Kasahara spaces are natural generalizations of Maia type fixed point theorems.*

Remark 2.2.5. *In order to include Rus' variant of Maia's Theorem 2.2.1 in the field of fixed point theory in Kasahara spaces, a special construction is imposed, which will be presented in the next section.*

2.3 Fixed point theorems in Kasahara spaces with respect to an operator

The aim of this section is to introduce a new notion: Kasahara space with respect to an operator. In this setting, some fixed point results are given. We study also the existence and uniqueness of solutions for integral equations and boundary value problems.

Definition 2.3.1 (A.-D. Filip [34]). *Let $(X, \to)$ be an L-space, $d : X \times X \to \mathbb{R}_+$ be a functional and $f : X \to X$ be an operator. The triple $(X, \to, d)$ is a Kasahara space with respect to the operator f if and only if*

$$\sum_{n \in \mathbb{N}} d(f^n(x), f^{n+1}(x)) < +\infty, \text{ for all } x \in X$$

implies that

$$(f^n(x))_{n \in \mathbb{N}} \text{ is convergent in } (X, \to), \text{ for all } x \in X.$$

Remark 2.3.1. *The notion of Kasahara space with respect to an operator generalizes the notion of orbital-completeness and the notion of completeness with respect to an operator.*

Remark 2.3.2. *The applications concerning w-distances and τ-distances are also generalized in the context of Kasahara spaces with respect to an operator.*

Remark 2.3.3 (A.-D. Filip [34]). *Notice that, in a Kasahara space with respect to an operator, Kasahara's Lemma 2.1.1 need not to be satisfied. Notice also that a Kasahara space is a Kasahara space with respect to an operator, but the reverse implication is false.*

Example 2.3.1 (A.-D. Filip [34]). *Let X be a nonempty set, $f : X \to X$ be an operator and $d, \rho : X \times X \to \mathbb{R}_+$ be two functionals. We suppose that:*

(i) (X, ρ) is a complete metric space;

(ii) there exists $c > 0$ such that $\rho(f(x), f(y)) \leq c d(x, y)$, for all $x, y \in X$.

Then $(X, \overset{\rho}{\to}, d)$ is a Kasahara space with respect to f.

Example 2.3.2 (A.-D. Filip [34]). *Let*

$$X := C(\overline{\Omega}) := \{x : \overline{\Omega} \to \mathbb{R} \mid x \text{ is a continuous function on } \overline{\Omega}\},$$

where $\Omega \subseteq \mathbb{R}^m$ is a bounded domain.

Let $\overset{\rho}{\longrightarrow}$ be the convergence structure induced by $\rho : C(\overline{\Omega}) \times C(\overline{\Omega}) \to \mathbb{R}_+$, where

$$\rho(x, y) := \|x - y\|_\infty := \sup_{t \in \overline{\Omega}} |x(t) - y(t)|, \text{ for all } x, y \in C(\overline{\Omega}).$$

Let $d : C(\overline{\Omega}) \times C(\overline{\Omega}) \to \mathbb{R}_+$ be the functional defined by

$$d(x, y) := \|x - y\|_{L^2(\Omega)} := \left(\int_\Omega |x(t) - y(t)|^2 dt \right)^{\frac{1}{2}}, \text{ for all } x, y \in C(\overline{\Omega}).$$

We consider the operator $f : C(\overline{\Omega}) \to C(\overline{\Omega})$, defined by

$$f(x)(t) := \int_{\Omega} K(t, s, x(s))ds$$

where $K \in C(\overline{\Omega} \times \overline{\Omega} \times \mathbb{R})$.

We assume that there exists $L \in C(\overline{\Omega} \times \overline{\Omega})$ such that

$$|K(t, s, u) - K(t, s, v)| \leq L(t, s)|u - v|,$$

for all $t, s \in \overline{\Omega}$ and $u, v \in \mathbb{R}$.

Then the triple $(X, \xrightarrow{\rho}, d)$, i.e., $\big(C(\overline{\Omega}), \xrightarrow{\|\cdot\|_\infty}, \|\cdot\|_{L^2(\Omega)}\big)$ is a Kasahara space with respect to the operator f.

Theorem 2.3.1 (A.-D. Filip [34])**.** *Let X be a nonempty set and $f : X \to X$ be an operator. Suppose that $(X, \to, d)$ is a Kasahara space with respect to f. We assume that:*

(i) $f : (X, \to) \to (X, \to)$ *has closed graph;*

(ii) $f : (X, d) \to (X, d)$ *is an α-contraction;*

(iii) $d(x, y) = d(y, x) = 0 \ \Rightarrow \ x = y$.

Then

(1) $F_f = F_{f^n} = \{x^*\}$ *for all $n \in \mathbb{N}^*$ and $d(x^*, x^*) = 0$.*

(2) $f^n(x) \to x^*$ *as $n \to \infty$, for all $x \in X$, i.e., f is a Picard operator.*

(3) *We have:*

- (3_a) $d(f^n(x), x^*) \xrightarrow{\mathbb{R}} 0$ *as $n \to \infty$, for all $x \in X$;*
- (3_b) $d(x^*, f^n(x)) \xrightarrow{\mathbb{R}} 0$*, as $n \to \infty$, for all $x \in X$.*

(4) If d is a quasimetric (i.e., $d(x, y) = d(y, x) = 0 \Leftrightarrow x = y$ for all $x, y \in X$ and d satisfies the triangle inequality), then:

- (4_a) $d(x, x^*) \leq \frac{1}{1-\alpha} d(x, f(x))$*, for all $x \in X$;*
- (4_b) $d(x^*, x) \leq \frac{1}{1-\alpha} d(f(x), x)$*, for all $x \in X$;*
- (4_c) $d(f^n(x), x^*) \leq \frac{\alpha^n}{1-\alpha} d(x, f(x))$*, for all $x \in X$ and all $n \in \mathbb{N}$;*
- (4_d) $d(x^*, f^n(x)) \leq \frac{\alpha^n}{1-\alpha} d(f(x), x)$*, for all $x \in X$ and all $n \in \mathbb{N}$;*
- (4_e) *if $(z_n)_{n\in\mathbb{N}} \subset X$ is such that $d(z_n, f(z_n)) \xrightarrow{\mathbb{R}} 0$ as $n \to \infty$ then $d(z_n, x^*) \xrightarrow{\mathbb{R}} 0$ as $n \to \infty$, i.e., the fixed point problem for the operator f is well-posed with respect to d;*

(4_f) *if* $(z_n)_{n\in\mathbb{N}} \subset X$ *is such that* $d(z_{n+1}, f(z_n)) \overset{\mathbb{R}}{\to} 0$ *as* $n \to \infty$ *then* $d(z_{n+1}, f^{n+1}(z)) \overset{\mathbb{R}}{\to} 0$ *as* $n \to \infty$*, for all* $z \in X$*, i.e., the operator* f *has the limit shadowing property with respect to* d*;*

(4_g) *If* $g : X \to X$ *is an operator such that*

$$d(f(x), g(x)) \leq \eta, \text{ for all } x \in X,$$

then

$$d(x^*, y^*) \leq \frac{\eta}{1-\alpha}, \text{ for all } y^* \in F_g.$$

Theorem 2.3.2 (A.-D. Filip [34])**.** *Let* X *be a nonempty set and* $f : X \to X$ *be an operator. Suppose that* $(X, \to, d)$ *is a Kasahara space with respect to* f*. We assume that:*

(*i*) $f : (X, \to) \to (X, \to)$ *has closed graph;*

(*ii*) $f : (X, d) \to (X, d)$ *is an* α*-graphic contraction, i.e., there exists* $\alpha \in [0, 1[$ *such that* $d(f(x), f^2(x)) \leq \alpha d(x, f(x))$*, for all* $x \in X$*.*

Then the following statements hold:

(1) $F_f \neq \emptyset$.

(2) $f^n(x) \to f^\infty(x) \in F_f$ *as* $n \to \infty$*, for all* $x \in X$*, i.e.,* $f : (X, \to) \to (X, \to)$ *is a weakly Picard operator.*

(3) $d(x^*, x^*) = 0$*, for all* $x^* \in F_f$*.*

(4) *if* d *satisfies the triangle inequality and* d *is continuous with respect to* $\to$*, then*

(4_a) $d(x, f^\infty(x)) \leq \frac{1}{1-\alpha} d(x, f(x))$*, for all* $x \in X$*,*

(4_b) *Let* $g : X \to X$ *be an operator. If there exists* $c > 0$ *such that*

$$d(x, g^\infty(x)) \leq c \cdot d(x, g(x)), \text{ for all } x \in X \tag{2.3.1}$$

and for each $x \in X$*, there exists* $\eta > 0$ *such that*

$$\max\{d(g(x), f(x)), d(f(x), g(x))\} \leq \eta, \tag{2.3.2}$$

then

$$H_d(F_f, F_g) \leq \max\left\{\frac{1}{1-\alpha}, c\right\}\eta,$$

where H_d *stands for the Pompeiu-Hausdorff functional generated by* d *(see [51]).*

In what follows, we study the existence and uniqueness for integral equations and boundary value problems.

Theorem 2.3.3 (A.-D. Filip [34])**.** *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded domain,* $K \in C(\overline{\Omega} \times \overline{\Omega} \times \mathbb{R})$ *and* $g \in C(\overline{\Omega})$*. We suppose that:*

(i) $K(t,s,\cdot):\mathbb{R}\to\mathbb{R}$ *is increasing, for all* $t,s\in\overline{\Omega}$.

(ii) *there exists* $L\in C(\overline{\Omega}\times\overline{\Omega})$ *such that*

$$|K(t,s,u)-K(t,s,v)|\le L(t,s)|u-v|,$$

for all $t,s\in\overline{\Omega}$ *and* $u,v\in\mathbb{R}$.

(iii) $\displaystyle\int_{\Omega\times\Omega}L(t,s)^2dsdt<1.$

Then the integral equation

$$x(t)=\int_{\Omega}K(t,s,x(s))ds+g(t),\ \ t\in\Omega \tag{2.3.3}$$

has a unique solution $x^*\in C(\overline{\Omega})$.

We consider next the following boundary value problem

$$\begin{cases} y''(t)=f(t,y(t)),\ \ for\ all\ t\in[a,b]\\ a_1y(a)+a_2y(b)+a_3y'(a)+a_4y'(b)=0\\ b_1y(a)+b_2y(b)+b_3y'(a)+b_4y'(b)=0\end{cases} \tag{2.3.4}$$

where $a_i,b_i\in\mathbb{R}$, $i=\overline{1,4}$ and $f:[a,b]\times\mathbb{R}\to\mathbb{R}$ is a continuous function.

We consider also the following linear mappings:

(1) $L:C^2([a,b])\to C([a,b])$, $L(y)=y''(t)$;

(2) $l_1:C^2([a,b])\to\mathbb{R}$, $l_1(y)=a_1y(a)+a_2y(b)+a_3y'(a)+a_4y'(b)$

(3) $l_2:C^2([a,b])\to\mathbb{R}$, $l_2(y)=b_1y(a)+b_2y(b)+b_3y'(a)+b_4y'(b)$

Then the boundary value problem (2.3.4) can be written as follows:

$$L(y)=f(\cdot,y),\ \ l_1(y)=0,\ \ l_2(y)=0. \tag{2.3.5}$$

We recall that the Green's function associated to the boundary value problem (2.3.5) is the mapping

$$G:[a,b]\times[a,b]\to\mathbb{R};\ \ (t,s)\mapsto G(t,s)$$

which satisfies the following conditions:

(i) $G\in C([a,b]\times[a,b])$;

(ii) For any $s\in[a,b]$, $G(\cdot,s)\in C^2([a,s[\cup]s,b])$ and

$$\frac{\partial}{\partial t}G(s+0,s)-\frac{\partial}{\partial t}G(s-0,s)=-\frac{1}{p(s)},$$

where $p\in C([a,b])$ and $p(s)\neq 0$ for any $s\in[a,b]$;

(iii) $G(\cdot, s)$ is a solution for $L(y) = 0$ on $[a, b] \setminus \{s\}$ and satisfies the boundary conditions $l_1(y) = l_2(y) = 0$.

We have the following result:

Theorem 2.3.4 (A.-D. Filip [34]). *Let $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ be a continuous function and consider the boundary value problem* (2.3.5)*. We assume that:*

(i) there exists $L_f > 0$ such that

$$|f(s, u) - f(s, v)| \leq L_f |u - v|,$$

for all $s \in [a, b]$ and $u, v \in \mathbb{R}$;

(ii) $\displaystyle\int_a^b \int_a^b G(t, s)^2 ds dt < 1$, *where G is the Green's function associated to the boundary value problem* (2.3.5)*.*

If the homogeneous boundary value problem

$$\begin{cases} L(y) = 0 \\ l_1(y) = l_2(y) = 0 \end{cases} \tag{2.3.6}$$

admits only the trivial solution $y \equiv 0$, then the boundary value problem (2.3.5) *has a unique solution in $C([a, b])$.*

Chapter 3

Multivalued generalized contractions in Kasahara spaces

The aim of this chapter is to present some fixed point results for multivalued generalized contractions in Kasahara spaces, generalized Kasahara spaces and large Kasahara spaces. We give also several Maia type theorems in close connexion with the results given in the first section of this chapter. The case of Kasahara spaces with respect to a multivalued operator is also studied.

The references which were followed in order to obtain the fixed point results presented in this chapter are: M. Berinde and V. Berinde [8]; A.-D. Filip [39], [31], [32], [33], [37]; S. Kasahara [65]; A. Petruşel and I.A. Rus, [102], [103]; I.A. Rus [112], [115]; I.A. Rus, A. Petruşel and G. Petruşel [123].

3.1 Fixed point theorems in Kasahara spaces

In this section we give corresponding results to Nadler's fixed point theorem, multivalued φ-contractions, multivalued Caristi operators, multivalued (θ, L)-weak contractions, multivalued Kannan and Reich operators which were given in complete metric spaces. We shall adapt these results in order to hold in Kasahara spaces $(X, \to, d)$, where $d : X \times X \to \mathbb{R}_+$ is a functional, satisfying some properties.

We also present some fixed point theorems in generalized Kasahara spaces and large Kasahara spaces, more precisely:

- fixed point theorems for multivalued generalized contractions in generalized Kasahara spaces $(X, \to, d)$, where $d : X \times X \to \mathbb{R}^m_+$ is a functional, satisfying some properties.
- fixed point theorems for multivalued Zamfirescu operators in large Kasahara spaces $(X, \xrightarrow{d}, p)$, where $d : X \times X \to \mathbb{R}_+$ is a complete metric on X and $p : X \times X \to \mathbb{R}_+$ is a w-distance on X.

Definition 3.1.1 (S. Kasahara [65]). *Let $(X, \to, d)$ be a Kasahara space, where $d : X \times X \to$*

$\mathbb{R}_+$ is a functional. Let $x \in X$. Then a set $A \in P(X)$ is said to be d-closed if and only if

$$D(x, A) = 0 \Rightarrow x \in A$$

We define the set

$$P_d(X) := \{A \in P(X) \mid A \text{ is } d\text{-closed } \}.$$

Concerning d-closed sets in Kasahara spaces, we have the following result.

Lemma 3.1.1 (Kasahara, [65])**.** *Let $(X, \to, d)$ be a Kasahara space, where $d : X \times X \to \mathbb{R}_+$ is a functional, satisfying the property $d(x, x) = 0$ for every $x \in X$. If $A, B \in P_d(X)$ then $H_d(A, B) = 0$ if and only if $A = B$.*

In the following fixed point results, we consider the Kasahara space $(X, \to, d)$, where $d : X \times X \to \mathbb{R}_+$ is a functional satisfying the properties:

- $\diamond$ $d(x, x) = 0$, for all $x \in X$;
- $\diamond$ $d(x, y) = 0 \Rightarrow x = y$, for all $x, y \in X$.

The study of fixed point theorems for multivalued mappings has been initiated by Markin [85] and Nadler [94]. The following result, usually referred as Nadler's fixed point theorem, extends Banach-Caccioppoli's Contraction Principle from single-valued maps to set-valued contractive maps.

Theorem 3.1.1 (S.B. Nadler Jr. [94])**.** *Let (X, d) be a complete metric space and $T : X \to P_{b,cl}(X)$ be a set-valued α-contraction, i.e., a mapping for which there exists a constant $\alpha \in]0, 1[$ such that $H(Tx, Ty) \le \alpha \cdot d(x, y)$, for all $x, y \in X$. Then T has at least one fixed point.*

In the above result, $P_{b,cl}(X)$ stands for the set of all bounded and closed subsets of X. In addition, H is the Pompeiu-Hausdorff functional (see [8], [15]).

We remark also that the Nadler's fixed point theorem is given in the context of metric spaces. We adapt this result into the context of Kasahara spaces.

Lemma 3.1.2 (A.-D. Filip [32])**.** *Let $(X, \to, d)$ be a Kasahara space, where $d : X \times X \to \mathbb{R}_+$ is a functional satisfying $d(x, x) = 0$ and $d(x, y) = 0 \Rightarrow x = y$, for all $x, y \in X$. Let $A, B \in P_d(X)$ and a real number $q > 1$. Then for every $a \in A$, there exists $b \in B$ such that*

$$d(a, b) \le q \cdot H_d(A, B).$$

Theorem 3.1.2 (A.-D. Filip [32])**.** *Let $(X, \to, d)$ be a Kasahara space, where $d : X \times X \to \mathbb{R}_+$ is a functional satisfying $d(x, x) = 0$ and $d(x, y) = 0 \Rightarrow x = y$, for all $x, y \in X$. Let $T : X \to P_d(X)$ be a multivalued operator. We assume that:*

i) $Graph(T)$ is closed in $(X, \to)$;

ii) T is a multivalued α-contraction, i.e.,

$$\text{there exists } \alpha \in [0, 1[\text{ such that } H_d(Tx, Ty) \le \alpha \cdot d(x, y), \text{ for all } x, y \in X.$$

Then T has at least one fixed point in X.

Remark 3.1.1. *Theorem 3.1.2 extends Nadler's fixed point Theorem 3.1.1 in the sense that the context of the complete metric space is replaced by the context of a Kasahara space, where the functional $d : X \times X \to \mathbb{R}_+$ is not necessarily a metric.*

A. Petruşel and I.A. Rus introduced in [103] the concept of *theory of a metric fixed point theorem* and used this theory for the case of multivalued contractions. By following [103], we present next a fixed point theory for Theorem 3.1.2.

Theorem 3.1.3. *Let $(X, \to, d)$ be a Kasahara space, where $d : X \times X \to \mathbb{R}_+$ is a functional satisfying $d(x, x) = 0$ and $d(x, y) = 0 \Rightarrow x = y$, for all $x, y \in X$. Let $T : X \to P_d(X)$ be a multivalued operator. We assume that:*

(i) Graph(T) is closed in $(X, \to)$;

(ii) T is a multivalued α-contraction, i.e.,

$$\text{there exists } \alpha \in [0, 1[\text{ such that } H_d(Tx, Ty) \leq \alpha \cdot d(x, y), \text{ for all } x, y \in X;$$

(iii) d satisfies the triangle inequality and it is continuous with respect to the second argument.

Then

(1) T is a multivalued weakly Picard operator and for every $x^* \in F_T$, $x_0 \in X$ and $x_1 \in Tx_0$ we have

$$d(x_0, x^*) \leq \frac{1}{1-\alpha} d(x_0, x_1) \tag{3.1.1}$$

(2) Let $S : X \to P_d(X)$ be a multivalued α-contraction and $\eta > 0$ such that for each $x \in X$, $H_d(Sx, Tx) \leq \eta$. Then $H_d(F_S, F_T) \leq \frac{\eta}{1-\alpha}$.

(3) Let $T_n : X \to P_d(X)$, $n \in \mathbb{N}$ be a sequence of multivalued α-contractions such that $T_n x \xrightarrow{H_d} Tx$ as $n \to \infty$, uniformly with respect to $x \in X$. Then $F_{T_n} \xrightarrow{H_d} F_T$ as $n \to \infty$.

(4) If in addition, Tx is a compact set in X for each $x \in X$, then we have

- $\diamond$ *(Ulam-Hyers stability of the inclusion $x \in Tx$)*
  Let $\varepsilon > 0$ and $x \in X$ be such that $D(x, Tx) \leq \varepsilon$. Then there exists $x^* \in F_T$ such that $d(x, x^*) \leq \frac{\varepsilon}{1-\alpha}$.

In addition, we have the following result:

Theorem 3.1.4. *Let $(X, \to, d)$ be a Kasahara space, where $d : X \times X \to \mathbb{R}_+$ is a functional satisfying $d(x, x) = 0$ and $d(x, y) = 0 \Rightarrow x = y$, for all $x, y \in X$. Let $T : X \to P_d(X)$ be a multivalued operator. We assume that*

(i) Graph(T) is closed in $(X, \to)$;

(ii) *T is a multivalued α-contraction, i.e.,*

$$\text{there exists } \alpha \in [0,1[\text{ such that } H_d(Tx,Ty) \le \alpha \cdot d(x,y), \text{ for all } x,y \in X;$$

(iii) $(SF)_T \neq \emptyset$.

Then, the following assertions hold:

(1) $F_T = (SF)_T = \{x^*\}$;

(2) $F_{T^n} = (SF)_{T^n} = \{x^*\}$ *for each* $n \in \mathbb{N}^*$;

(3) $H_d(T^n x, x^*) \overset{\mathbb{R}}{\to} 0$ *as* $n \to \infty$, *for each* $x \in X$;

(4) *If d satisfies the triangle inequality, then*

  (4_a) *Let $S : X \to P_d(X)$ be a multivalued operator and $\eta > 0$ such that $F_S \neq \emptyset$ and $H_d(Sx,Tx) \le \eta$, for each $x \in X$. Then $H_d(F_S, F_T) \le \frac{\eta}{1-\alpha}$;*

  (4_b) *Let $T_n : X \to P_d(X)$, $n \in \mathbb{N}$ be a sequence of multivalued operators such that $F_{T_n} \neq \emptyset$ for each $n \in \mathbb{N}$ and $H_d(T_n x, Tx) \to 0$ as $n \to \infty$, uniformly with respect to $x \in X$. Then $H_d(F_{T_n}, F_T) \to 0$ as $n \to \infty$;*

(5) *If $(x_n)_{n\in\mathbb{N}}$ is a sequence in X such that $D(x_n, Tx_n) \to 0$ as $n \to \infty$, then $d(x_n, x^*) \to 0$ as $n \to \infty$;*

(6) *If $(x_n)_{n\in\mathbb{N}}$ is a sequence in X such that $H_d(x_n, Tx_n) \to 0$ as $n \to \infty$, then $d(x_n, x^*) \to 0$ as $n \to \infty$;*

(7) *Assuming that d satisfies the triangle inequality, the limit shadowing property for T holds, i.e. if $(y_n)_{n\in\mathbb{N}}$ is a sequence in X such that $D(Ty_n, y_{n+1}) \to 0$ as $n \to \infty$, then there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ of successive approximations for T, such that $d(x_n, y_{n+1}) \to 0$ as $n \to \infty$.*

Remark 3.1.2. *Theorems 3.1.3 and 3.1.4 extend Theorems 3.1 and 3.2 given by A. Petruşel and I.A. Rus in [103] in the sense that Kasahara spaces are considered instead of complete metric spaces.*

- We present next a local fixed point results for multivalued Zamfirescu operators in Kasahara spaces, by extending the results given for single-valued Zamfirescu operators in A.-D. Filip [36].

Let us recall first the notion of multivalued Zamfirescu operator.

Definition 3.1.2 (A.-D. Filip, [37]). *Let $(X, \to, d)$ be a Kasahara space. The mapping $T : X \to P(X)$ is called multivalued Zamfirescu operator if there exist α, β, $\gamma \in \mathbb{R}_+$ with $\alpha < 1$, $\beta < \frac{1}{2}$ and $\gamma < \frac{1}{2}$ such that for each $x, y \in X$ and $u \in Tx$, there exists $v \in Ty$ such that at least one of the following conditions is true:*

(1_m) $d(u, v) \leq \alpha d(x, y)$;

(2_m) $d(u, v) \leq \beta[d(x, u) + d(y, v)]$;

(3_m) $d(u, v) \leq \gamma[d(x, v) + d(y, u)]$.

In our following results, we consider the Kasahara space $(X, \to, d)$ and assume that $d : X \times X \to \mathbb{R}_+$ is a premetric, i.e. the functional d satisfies the following conditions:

(d_1) $d(x, x) = 0$, for all $x \in X$;

(d_2) $d(x, z) \leq d(x, y) + d(y, z)$, for all $x, y, z \in X$.

We assume in addition that

(d_3) d is continuous with respect to the second argument.

Remark 3.1.3. *Under the above assumptions on $(X, \to, d)$, the right closed ball*

$$\tilde{B}_d(x_0, r) := \{x \in X \mid d(x_0, x) \leq r\}$$

where $x_0 \in X$ and $r \in \mathbb{R}_+$, is a closed set with respect to $\to$, in the sense that for any sequence $(z_n)_{n\in\mathbb{N}} \subset \tilde{B}_d(x_0, r)$, with $z_n \to z \in X$ as $n \to \infty$, we get that $z \in \tilde{B}_d(x_0, r)$.

We give next our local fixed point results in Kasahara spaces.

Theorem 3.1.5 (A.-D. Filip, [37])**.** *Let $(X, \to, d)$ be a Kasahara space and $T : \tilde{B}_d(x_0, r) \to P(X)$ be a multivalued Zamfirescu operator. We assume that:*

(i) T has closed graph with respect to $\to$;

(ii)
$$d(x_0, z) \leq (1 - \delta)r; \tag{3.1.2}$$
where $z \in Tx_0$ and $\delta := \max\left\{\alpha, \frac{\beta}{1-\beta}, \frac{\gamma}{1-\gamma}\right\}$;

(iii) $d : X \times X \to \mathbb{R}_+$ is a premetric, which is continuous with respect to the second argument.

Then the following statements hold:

(1) T has at least one fixed point in $\tilde{B}_d(x_0, r)$.

(2) there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset \tilde{B}_d(x_0, r)$ such that

(2.a) $x_{n+1} \in Tx_n$, for all $n \in \mathbb{N}$;

(2.b) $x_n \to x^* \in F_T$ as $n \to +\infty$;

(2.c) we have
$$d(x_n, x^*) \leq \delta^n r, \text{ for all } n \in \mathbb{N}, \tag{3.1.3}$$
where $x^* \in F_T$ and $(x_n)_{n\in\mathbb{N}}$ is the sequence of successive approximations for T starting from $(x_0, x_1) \in Graph(T)$.

- The following fixed point results are given for multivalued operators in the context of generalized Kasahara spaces $(X, \to, d)$, where $d : X \times X \to \mathbb{R}^m_+$ is a functional.

We consider the following set

$$\mathcal{M}^{\Delta}_{m,m}(\mathbb{R}_+) := \left\{ Q = \begin{pmatrix} q_{11} & q_{12} & \dots & q_{1m} \\ 0 & q_{22} & \dots & q_{2m} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & q_{mm} \end{pmatrix} \in \mathcal{M}_{m,m}(\mathbb{R}_+) \;\middle|\; \max_{i=\overline{1,m}} q_{ii} < \frac{1}{2} \right\}.$$

Then the following lemma holds.

Lemma 3.1.3 (A.-D. Filip, [37]). *If $Q \in \mathcal{M}^{\Delta}_{m,m}(\mathbb{R}_+)$ then*

(1) the matrix Q is convergent to zero;

(2) the matrix $(I_m - Q)^{-1}Q$ is convergent to zero.

We give next our local and global fixed point results for multivalued operators in generalized Kasahara spaces.

Theorem 3.1.6 (A.-D. Filip, [37]). *Let $(X, \to, d)$ be a generalized Kasahara space and $T : \tilde{B}_d(x_0, r) \to P(X)$ be a multivalued operator. We assume that:*

(i) T has closed graph with respect to $\to$;

(ii) one of the following conditions holds:

(ii_1) there exists a matrix $A \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ convergent to zero such that for all $x, y \in X$ and $u \in Tx$ there exists $v \in Ty$ such that

$$d(u, v) \le Ad(x, y);$$

(ii_2) there exists a matrix $B \in \mathcal{M}^{\Delta}_{m,m}(\mathbb{R}_+)$ such that for all $x, y \in X$ and $u \in Tx$ there exists $v \in Ty$ such that

$$d(u, v) \le B[d(x, u) + d(y, v)];$$

(ii_3) there exists a matrix $C \in \mathcal{M}^{\Delta}_{m,m}(\mathbb{R}_+)$ such that for all $x, y \in X$ and $u \in Tx$ there exists $v \in Ty$ such that

$$d(u, v) \le C[d(x, v) + d(y, u)];$$

(iii) if $u \in \mathbb{R}^m_+$ is such that $u(I_m - M)^{-1} \le (I_m - M)^{-1}r$ then $u \le r$, for all $M \in \mathcal{M}_{m,m}(\mathbb{R}_+)$;

(iv)

$$d(x_0, z)(I_m - W)^{-1} \le r \tag{3.1.4}$$

where $z \in Tx_0$ and $W := \max\left\{A, (I_m - B)^{-1}B, (I_m - C)^{-1}C\right\} \in \mathcal{M}_{m,m}(\mathbb{R}_+)$;

(*v*) $d : X \times X \to \mathbb{R}^m_+$ *is a premetric, which is continuous with respect to the second argument on* X.

Then the following statements hold:

(1) T *has at least one fixed point in* $\tilde{B}_d(x_0, r)$.

(2) *there exists a sequence* $(x_n)_{n\in\mathbb{N}} \subset \tilde{B}_d(x_0, r)$ *such that*

(2.*a*) $x_{n+1} \in Tx_n$, *for all* $n \in \mathbb{N}$;

(2.*b*) $x_n \to x^* \in F_T$ *as* $n \to +\infty$;

(2.*c*) *we have*

$$d(x_n, x^*) \le W^n(I_m - W)^{-1}d(x_0, x_1), \text{ for all } n \in \mathbb{N}, \tag{3.1.5}$$

where $x^* \in F_T$ *and* $(x_n)_{n\in\mathbb{N}}$ *is the sequence of successive approximations for* T *starting from* $(x_0, x_1) \in Graph(T)$.

Remark 3.1.4. *Any matrix* $M = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, *with* $a, b \in \mathbb{R}_+$ *and* $\max\{a, b\} < 1$, *is convergent towards zero and satisfies the assumption* (*iii*) *of Theorem 3.1.6.*

Remark 3.1.5. *Theorem 3.1.6 holds even if the assumption* (ii_1) *is replaced by the following one:*

(ii_1') *there exists a matrix* $A \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ *convergent to zero and a matrix* $B \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ *such that for all* $x, y \in X$ *and* $u \in Tx$ *there exists* $v \in Ty$ *such that*

$$d(u, v) \le Ad(x, y) + Bd(y, u).$$

The corresponding global result for Theorem 3.1.6 is the following:

Corollary 3.1.1 (A.-D. Filip, [37]). *Let* $(X, \to, d)$ *be a generalized Kasahara space and* $T : X \to P(X)$ *be a multivalued operator. We assume that:*

(*i*) T *has closed graph with respect to* $\to$;

(*ii*) *one of the conditions* (ii_1), (ii_2), (ii_3) *of Theorem 3.1.6 holds;*

(*iii*) $d : X \times X \to \mathbb{R}^m_+$ *is a premetric, which is continuous with respect to the second argument.*

Then the following statements hold:

(1) T *has at least one fixed point in* X.

(2) *there exists a sequence* $(x_n)_{n\in\mathbb{N}} \subset X$ *such that* (2.*a*), (2.*b*) *and* (2.*c*) *of Theorem 3.1.6 hold.*

As an application of the previous results, we present a fixed point theorem concerning the existence of solutions for semi-linear inclusion systems.

Theorem 3.1.7 (A.-D. Filip, [37]). *Let $\varphi, \psi : [0,1]^2 \to]0, \frac{1}{2}]$ be two functions and $T_1, T_2 : [0,1]^2 \to P([0,1])$ be two multivalued operators defined as follows:*

$$T_1(x_1,x_2) = [\varphi(x_1,x_2), \tfrac{1}{2} + \varphi(x_1,x_2)] \text{ and}$$
$$T_2(x_1,x_2) = [\psi(x_1,x_2), \tfrac{1}{2} + \psi(x_1,x_2)].$$

We assume that for each (x_1,x_2), $(y_1,y_2) \in [0,1]^2$ and each $u_1 \in T_1(x_1,x_2)$ and $u_2 \in T_2(x_1,x_2)$, there exist $v_1 \in T_1(y_1,y_2)$ and $v_2 \in T_2(y_1,y_2)$ such that one of the following couples of conditions holds:

(I) for all $a,b,c,d \in \mathbb{R}_+$ with $|a+d \pm \sqrt{(a-d)^2+4bc}| < 2$,

$$|u_1 - v_1| \le a|x_1 - y_1| + b|x_2 - y_2|,$$
$$|u_2 - v_2| \le c|x_1 - y_1| + d|x_2 - y_2|,$$

(II) for all $a,b,c \in \mathbb{R}_+$ with $a,c < \frac{1}{2}$,

$$|u_1 - v_1| \le a\big(|x_1 - u_1| + |y_1 - v_1|\big) + b\big(|x_2 - u_2| + |y_2 - v_2|\big),$$
$$|u_2 - v_2| \le c\big(|x_2 - u_2| + |y_2 - v_2|\big),$$

(III) for all $a,b,c \in \mathbb{R}_+$ with $a,c < \frac{1}{2}$,

$$|u_1 - v_1| \le a\big(|x_1 - v_1| + |y_1 - u_1|\big) + b\big(|x_2 - v_2| + |y_2 - u_2|\big),$$
$$|u_2 - v_2| \le c\big(|x_2 - v_2| + |y_2 - u_2|\big).$$

Then the system

$$\begin{cases} x_1 \in T_1(x_1,x_2) \\ x_2 \in T_2(x_1,x_2), \end{cases}$$

has at least one solution in $[0,1]^2$.

- We give next some fixed point results for multivalued Zamfirescu operators in large Kasahara spaces in the sense of Definition 2.1.8.

Theorem 3.1.8 (A.-D. Filip, [37]). *Let $(X, \xrightarrow{d}, p)$ be a large Kasahara space in the sense of Definition 2.1.8, where $d : X \times X \to \mathbb{R}_+$ is a complete metric on X and $p : X \times X \to \mathbb{R}_+$ is a w-distance on X. Let $x_0 \in X$, $r > 0$ and $T : \tilde{B}_p(x_0, r) \to P(X)$ be a multivalued Zamfirescu operator w.r.t. p. We assume that:*

(i) T has closed graph with respect to $\xrightarrow{d}$;

(ii) $p(x_0, z) < (1-\delta)r$, where $z \in Tx_0$ and $\delta := \max\left\{\alpha, \frac{\beta}{1-\beta}, \frac{\gamma}{1-\gamma}\right\}$;

(iii) $p(x,x) = 0$, for all $x \in X$.

Then the following statements hold:

(1) *T has at least one fixed point in $\tilde{B}_p(x_0, r)$.*

(2) *there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset \tilde{B}_p(x_0, r)$ such that*

(2.a) *$x_{n+1} \in Tx_n$, for all $n \in \mathbb{N}$;*

(2.b) *$x_n \to x^* \in F_T$ as $n \to +\infty$;*

(2.c) *the following estimation holds*

$$p(x_n, x^*) \leq \delta^n r, \text{ for all } n \in \mathbb{N}, \tag{3.1.6}$$

where $x^* \in F_T$ and $(x_n)_{n\in\mathbb{N}}$ is the sequence of successive approximations for T starting from $(x_0, x_1) \in Graph(T)$.

The global version of Theorem 3.1.8 is the following

Corollary 3.1.2 (A.-D. Filip, [37])**.** *Let $(X, \overset{d}{\to}, p)$ be a large Kasahara space in the sense of Definition 2.1.8, where $d : X \times X \to \mathbb{R}_+$ is a complete metric on X and $p : X \times X \to \mathbb{R}_+$ is a w-distance on X. Let $T : X \to P(X)$ be a multivalued Zamfirescu operator w.r.t. p. We assume that T has closed graph with respect to $\overset{d}{\to}$ and $p(x, x) = 0$, for all $x \in X$. Then the following statements hold:*

(1) T has at least one fixed point in X;

(2) the sequence $(x_n)_{n\in\mathbb{N}} \subset X$ of successive approximations for T starting from $(x_0, x_1) \in Graph(T)$ converges to an element $x^* \in F_T$ as $n \to \infty$;

(3) the following estimation holds

$$p(x_n, x^*) \leq \frac{\delta^n}{1-\delta} p(x_0, x_1), \text{ for all } n \in \mathbb{N},$$

where $\delta := \max\left\{\alpha, \frac{\beta}{1-\beta}, \frac{\gamma}{1-\gamma}\right\}$, $x^* \in F_T$ and $(x_n)_{n\in\mathbb{N}}$ is the sequence of successive approximations for T starting from $(x_0, x_1) \in Graph(T)$.

We give next a data dependence result for multivalued Zamfirescu operators.

Theorem 3.1.9 (A.-D. Filip, [37])**.** *Let $(X, \overset{d}{\to}, p)$ be a large Kasahara space in the sense of Definition 2.1.8, where $d : X \times X \to \mathbb{R}_+$ is a complete metric on X and $p : X \times X \to \mathbb{R}_+$ is a w-distance on X with $p(x, x) = 0$, for all $x \in X$. Let $T_1, T_2 : X \to P(X)$ be two multivalued Zamfirescu operators w.r.t. p, having closed graph w.r.t $\overset{d}{\to}$. Then*

(i) T_1 and T_2 have at least one fixed point in X;

(ii) If we assume that there exists $\eta > 0$ such that for all $x \in X$ and $u \in T_1x$, there exists $v \in T_2x$ such that $p(u, v) \leq \eta$, then for all $u^* \in F_{T_1}$, there exists $v^* \in F_{T_2}$ such that

$$p(u^*, v^*) \leq \frac{\eta}{1-\delta_2}, \text{ where } \delta_2 = \max\left\{\alpha_2, \frac{\beta_2}{1-\beta_2}, \frac{\gamma_2}{1-\gamma_2}\right\} \tag{3.1.7}$$

respectively, if we assume that there exists $\eta > 0$ such that for all $x \in X$ and $v \in T_2x$, there exists $u \in T_1x$ such that $p(v,u) \leq \eta$, then for all $v^* \in F_{T_2}$, there exists $u^* \in F_{T_1}$ such that

$$p(v^*,u^*) \leq \frac{\eta}{1-\delta_1}, \text{ where } \delta_1 = \max\left\{\alpha_1, \frac{\beta_1}{1-\beta_1}, \frac{\gamma_1}{1-\gamma_1}\right\}. \tag{3.1.8}$$

3.2 Maia type fixed point theorems

The aim of this section is to present several Maia type theorems for multivalued generalized contractions in close connexion with the results given in Kasahara spaces.

First, we recall the multivalued version of Maia's fixed point theorem 2.2.1.

Theorem 3.2.1 (A. Petruşel and I.A. Rus, [102]). *Let X be a nonempty set, d and ρ be two metrics on X and $T : X \to P(X)$ be a multivalued operator. We suppose that:*

(i) *(X,ρ) is a complete metric space;*

(ii) *there exists $c > 0$ such that $\rho(x,y) \leq c \cdot d(x,y)$, for each $x,y \in X$;*

(iii) *$T : (X,\rho) \to (P(X), H_\rho)$ has closed graph (here H_ρ stands for the Pompeiu-Hausdorff functional generated by ρ (see [51]));*

(iv) *there exists $\alpha \in [0,1[$ such that $H_d(Tx,Ty) \leq \alpha d(x,y)$, for each $x,y \in X$.*

Then we have:

(a) *$F_T \neq \emptyset$;*

(b) *for each $x \in X$ and each $y \in Tx$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that:*

  (1) *$x_0 = x$, $x_1 = y$;*

  (2) *$x_{n+1} \in Tx_n$, for each $n \in \mathbb{N}$;*

  (3) *$x_n \xrightarrow{\rho} x^* \in Tx^*$, as $n \to \infty$.*

We mention here another two local fixed point results of Maia type.

Theorem 3.2.2 (A.-D. Filip, [31]). *Let X be a nonempty set, ρ and d be two metrics on X, $x_0 \in X$, $r > 0$ and $T : \tilde{B}_d(x_0,r) \to P(X)$ be a multivalued operator. We suppose that:*

(i) *(X,ρ) is a complete metric space;*

(ii) *there exists $c > 0$ such that $\rho(x,y) \leq c \cdot d(x,y)$, for each $x,y \in \tilde{B}_d(x_0,r)$;*

(iii) *$T : (\tilde{B}_d(x_0,r),\rho) \to (P(X), H_\rho)$ has closed graph (here H_ρ stands for the Pompeiu-Hausdorff functional generated by ρ (see [51]));*

(*iv*) *there exists* $L \geq 0$ *such that for all* $x \in \tilde{B}_d(x_0, r)$, *there exists* $y \in I^x_{b,d}$ *such that*

$$H_d(Tx, Ty) \leq \Lambda(d(x,y)) \cdot d(x,y) + L \cdot D_d(y, Tx)$$

where

- $\diamond$ $I^x_{b,d} := \{y \in Tx \mid b \cdot d(x,y) \leq D_d(x, Tx)\}$, *where* $b \in]0,1[$ *and* $D_d(x, Tx) = \inf_{z \in Tx} d(x,z)$.
- $\diamond$ $\Lambda : \mathbb{R}_+ \to [0,1[$ *is a function defined by* $\Lambda(t) = b \cdot \alpha(t)$, *for all* $t \in \mathbb{R}_+$, *where* $b \in]0,1[$ *is the same number used in the definition of the set* $I^x_{b,d}$ *and* $\alpha : \mathbb{R}_+ \to [0,1[$ *is a function with the property* $\limsup_{s \to t^+} \alpha(s) < 1$, *for all* $t \in \mathbb{R}_+$.

(*v*) $D_d(x_0, Tx_0) < b(1-\theta)r$, *where* $\theta \in [0,1[$ *satisfies* $\Lambda(t) < b\theta$, *for all* $t \in \mathbb{R}_+$.

Then we have:

(*a*) $F_T \neq \emptyset$;

(*b*) *there exists a sequence* $(x_n)_{n \in \mathbb{N}}$ *in* $\tilde{B}_d(x_0, r)$ *such that:*

(*b*1) $x_{n+1} \in Tx_n$, *for all* $n \in \mathbb{N}$;

(*b*2) $x_n \xrightarrow{\rho} x^* \in F_T$, *as* $n \to \infty$;

(*b*3) $\rho(x_n, x^*) \leq c \cdot \theta^n \cdot r$, *for each* $n \in \mathbb{N}$.

Remark 3.2.1. *In Theorem 3.2.2, by taking* $n = 0$ *in the conclusion* (*b*3), *it follows that* $x^* \in \tilde{B}_\rho(x_0, cr)$.

We consider now the case of generalized metric spaces (X, d), where $d : X \times X \to \mathbb{R}^m_+$. The following Maia type theorem holds.

Theorem 3.2.3 (A.-D. Filip and A. Petruşel [39])**.** *Let* X *be a nonempty set and* $d, \rho : X \times X \to \mathbb{R}^m_+$ *be two generalized metrics on* X. *Let* $x_0 \in X$, $r := (r_1, r_2, \ldots, r_m) \in \mathbb{R}^m_+$ *and let* $T : \tilde{B}_d(x_0, r) \to P(X)$ *be a multivalued operator. Suppose that:*

(*i*) (X, ρ) *is a complete generalized metric space;*

(*ii*) *there exists* $C \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ *such that* $\rho(x,y) \leq C \cdot d(x,y)$, *for all* $x, y \in X$;

(*iii*) $T : (\tilde{B}_d(x_0, r), \rho) \to (P(X), H_\rho)$ *has closed graph (here* H_ρ *stands for the Pompeiu-Hausdorff functional generated by* ρ *(see [51]));*

(*iv*) *there exist* $A, B \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ *such that* A *is a matrix that converges to zero and for all* $x, y \in \tilde{B}_d(x_0, r)$ *and* $u \in Tx$ *there exists* $v \in Ty$ *such that*

$$d(u,v) \leq Ad(x,y) + Bd(y,u);$$

(*v*) *if* $u \in \mathbb{R}^m_+$ *is such that* $u(I_m - A)^{-1} \leq (I_m - A)^{-1}r$, *then* $u \leq r$;

(vi) $d(x_0, x_1)(I_m - A)^{-1} \le r$.

Then $F_T \neq \emptyset$.

Remark 3.2.2. *Notice that in Theorem 3.2.3, the fixed point* $x^* \in \tilde{B}_\rho(x_0, Cr)$.

Indeed, we have proved that the sequence of successive approximations for T *starting from* $x_0 \in X$ *is* $(x_n)_{n\in\mathbb{N}}$ *with* $x_n \in \tilde{B}_d(x_0, r)$*, for all* $n \in \mathbb{N}$ *and there exists* $x^* \in X$ *such that* $x_n \xrightarrow{\rho} x^*$ *as* $n \to \infty$.

By (ii), there exists $C \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ *such that*

$$\rho(x_0, x_n) \le C \cdot d(x_0, x_n) \le Cr, \text{ for all } n \in \mathbb{N}. \tag{3.2.1}$$

Hence $x_n \in \tilde{B}_\rho(x_0, Cr)$*, for all* $n \in \mathbb{N}$.

By letting $n \to \infty$ *in* (3.2.1)*, we get that* $x^* \in \tilde{B}_\rho(x_0, Cr)$.

Remark 3.2.3. *Some other Maia type fixed point results can be obtained in the case when d is not necessarily a metric.*

Let X *be a nonempty set and* $\rho : X \times X \to \mathbb{R}_+$ *be a complete metric on* X*. Let* $(x_n)_{n\in\mathbb{N}}$ *be a sequence in* X *and let* $x \in X$*. We consider the convergence structure* $\xrightarrow{\rho}$ *induced by* ρ *on* X *and defined by*

$$x_n \xrightarrow{\rho} x \;\Leftrightarrow\; \rho(x_n, x) \to 0, \text{ as } n \to \infty.$$

We have the following Maia type results:

Corollary 3.2.1 (A.-D. Filip [32])**.** *Let* X *be a nonempty set and* $\rho : X \times X \to \mathbb{R}_+$ *be a complete metric on* X*. Let* $d : X \times X \to \mathbb{R}_+$ *be a functional with the property that for all* $x, y \in X$, $d(x, y) = 0 \Rightarrow x = y$*. Let* $T : X \to P_d(X)$ *be a multivalued operator. We assume that:*

i) *there exists* $\alpha \in [0, 1[$ *such that* $H_d(Tx, Ty) \le \alpha \cdot d(x, y)$*, for all* $x, y \in X$*;*

ii) $Graph(T)$ *is closed in* $(X, \xrightarrow{\rho})$*;*

iii) *there exists* $c > 0$ *such that* $\rho(x, y) \le c \cdot d(x, y)$.

Then the following statements hold:

1) $F_T \neq \emptyset$*;*

2) *there exists* $\theta \in [0, 1[$ *such that*

$$\rho(x_n, x^*) \le c\frac{\theta^n}{1-\theta} d(x_0, x_1), \text{ for all } n \in \mathbb{N},$$

where $x^* \in F_T$ *and* $(x_n)_{n\in\mathbb{N}}$ *is the sequence of successive approximations for* T *starting from* $(x_0, x_1) \in Graph(T)$.

Corollary 3.2.2 (A.-D. Filip, [33])**.** *Let* X *be a nonempty set and* $\rho : X \times X \to \mathbb{R}^m_+$ *be a complete generalized metric on* X*. Let* $d : X \times X \to \mathbb{R}^m_+$ *be a functional and* $T : X \to P(X)$ *be a multivalued operator. We assume that:*

i) there exists $A \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ *and for all* $x, y \in X$ *and* $u \in Tx$, *there exists* $v \in Ty$ *such that*

$$d(u, v) \leq Ad(x, y);$$

ii) $Graph(T)$ *is closed in* $X \times X$;

iii) there exists $c > 0$ *such that* $\rho(x, y) \leq c \cdot d(x, y)$.

Then the following statements hold:

1) if A converges to zero, then $F_T \neq \emptyset$. *If, in addition,* $(I_m - A)$ *is non-singular,* $(I_m - A)^{-1} \in \mathcal{M}_{m\times m}(\mathbb{R}_+)$ *and*

$$\max\{d(u, v) \mid u \in Tx,\ v \in Ty\} \leq Ad(x, y),\ \textit{for all } x, y \in X$$

then T has a unique fixed point in X.

2) $\rho(x_n, x^*) \leq c \cdot A^n(I_m - A)^{-1}d(x_0, x_1)$, *for all* $n \in \mathbb{N}$, *where* $x^* \in F_T$ *and* $(x_n)_{n\in\mathbb{N}}$ *is the sequence of successive approximations for T starting from* $(x_0, x_1) \in Graph(T)$.

3.3 Fixed point theorems in Kasahara spaces with respect to an operator

We introduce in this section a new notion: Kasahara space with respect to a multivalued operator. Two fixed point results for multivalued α-contractions defined on Kasahara spaces with respect to a multivalued operator are presented.

Definition 3.3.1. *Let* $(X, \to)$ *be an L-space,* $d : X \times X \to \mathbb{R}_+$ *be a functional and* $T : X \to P(X)$ *be a multivalued operator. The triple* $(X, \to, d)$ *is called Kasahara space with respect to the operator T if and only if for any sequence* $(x_n)_{n\in\mathbb{N}} \subset X$ *satisfying:*

(i) $x_{n+1} \in Tx_n$, *for all* $n \in \mathbb{N}$;

(ii) $\sum_{n\in\mathbb{N}} H_d(Tx_n, Tx_{n+1}) < \infty$

we have that $(x_n)_{n\in\mathbb{N}}$ *is convergent in* $(X, \to)$.

Example 3.3.1. *Let X be a nonempty set,* $T : X \to P_d(X)$ *be a multivalued operator and* $d, \rho : X \times X \to \mathbb{R}_+$ *be two functionals. We suppose that:*

(i) (X, ρ) *is a complete metric space;*

(ii) for all $x \in X$ *and* $y \in Tx$, *there exist* $z \in Ty$ *and* $c > 0$ *such that* $H_\rho(Tx, Ty) \leq c\cdot d(y, z)$;

(iii) $d(x, x) = 0$, *for all* $x \in X$;

(iv) $d(x,y)=0 \Rightarrow x=y$, *for all* $x,y \in X$.

Then $(X,\to,d)$ *is a Kasahara space with respect to the operator* T.

Theorem 3.3.1. *Let* $(X,\to,d)$ *be a Kasahara space with respect to a multivalued operator* $T: X \to P_d(X)$, *where* $d: X\times X \to \mathbb{R}_+$ *is a functional satisfying* $d(x,x)=0$ *and* $d(x,y)=0 \Rightarrow x=y$, *for all* $x,y\in X$. *We assume that:*

(i) *Graph*(T) *is closed with respect to* $\to$*;*

(ii) T *is a multivalued* α*-contraction with respect to* d.

Then we have:

(1) $F_T \neq \emptyset$*;*

(2) *for each* $x\in X$ *and each* $y\in Tx$, *there exists a sequence* $(x_n)_{n\in\mathbb{N}} \subset X$ *such that*

  (2_a) $x_0=x$, $x_1=y$*;*

  (2_b) $x_{n+1}\in Tx_n$, *for each* $n\in\mathbb{N}$*;*

  (2_c) $x_n \to x^* \in F_T$ *as* $n\to\infty$.

Theorem 3.3.2. *Let* $(X,\to,d)$ *be a Kasahara space with respect to a multivalued operator* $T: X\to P_d(X)$, *where* $d: X\times X\to\mathbb{R}_+$ *is a functional satisfying* $d(x,x)=0$, *for all* $x\in X$. *We assume that:*

(i) *Graph*(T) *is closed with respect to* $\to$*;*

(ii) T *is a multivalued* α*-contraction with respect to* d*;*

(iii) $(SF)_T \neq \emptyset$*;*

(iv) $d(x,y)=0\Rightarrow x=y$, *for all* $x,y\in X$.

Then we have:

(1) $F_T=(SF)_T=\{x^*\}$*;*

(2) $F_{T^n}=(SF)_{T^n}=\{x^*\}$*;*

(3) $H_d(T^n x, x^*) \le \alpha^n d(x,x^*)$, *for each* $n\in\mathbb{N}$ *and each* $x\in X$*;*

(4) *if* d *satisfies the triangle inequality, then*

  (4_a) $d(x,x^*) \le \frac{1}{1-\alpha} H_d(x,Tx)$ *for each* $x\in X$*;*

  (4_b) *the fixed point problem for* T *is well-posed with respect to* D.

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