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The theory of monotone operators with applications

Ph.D. Thesis Summary

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Introduction

The concept of monotonicity for operators defined on a Banach space into its dual has been introduced some fifty years ago by the celebrated works of Browder and Minty (see, for example, [22–24], [91, 92]. This notion (often called *Minty-Browder monotonicity*) have shown to be a cornerstone for the development of nonlinear analysis, especially in convex analysis, due to the fact that convexity of a proper, lower semicontinuous function can be characterized by monotonicity of its subdifferential (see, for instance, [34, 115]).

During the last decades, the concept of Minty-Browder monotonicity has imposed itself, due to its importance, and influenced some other branches of mathematics, such as differential equations, as well as economics, engineering, management science, probability theory and other applied sciences. Due to these interactions the concepts of monotonicity alongside with convexity were subjects of a dynamical evolution reflected in a number of new concepts - extensions of the classical assumption of monotonicity and convexity without the loss of valuable properties (see, for instance, [27], [55], [62], [94], [111] and the references therein).

This work is based on the original results of the author from 10 scientific papers, all submitted to prestigious journals, some of them accepted or already published, while the other ones are under review. The work is organized as follows. After a brief introduction, in **Chapter 1** the notions of monotone operator in the Minty-Browder sense and its most known generalizations are presented, such as the concepts of quasimonotonicity, strict quasimonotonicity, pseudomonotonicity and strict pseudomonotonicity. In many situations it is rather difficult to verify global monotonicity, respectively global generalized monotonicity of an operator. Instead, its counterpart the local monotonicity, respectively local generalized monotonicity, (i.e. every point admits a neighborhood on which the operator is monotone, respectively generalized monotone) seems to be easier to verify under some circumstances. In this chapter we prove that the local increasing monotonicity, respectively the local generalized monotonicity of an operator on the complement of a closed set having countable intersection with every segment implies its global increasing monotonicity, respectively its global generalized monotonicity. This particularly shows that locally convex, respectively locally generalized convex, differentiable functions are actually globally convex, respectively globally generalized convex. However, there is an exception: we will give an example of a continuous, locally quasimonotone real valued function of one real variable, defined on the whole space $\mathbb{R}$, which is not globally quasimonotone on $\mathbb{R}$. Hence, the quasiconvexity is also an exception: we will give an example of a continuously differentiable, locally quasiconvex real valued function of one real variable, defined on the whole space $\mathbb{R}$, which is not globally quasiconvex on $\mathbb{R}$. Further, we give an example of a continuous locally Minty-Browder monotone operator, defined on a connected but not convex subset of $\mathbb{R}^2$, which is not even quasimonotone globally. This shows that the convexity of the domain is essential when extending the local monotonicity to the global monotonicity. By an example we show, that our results cannot be extended for multivalued upper semicontinuous operators. This fact is surprising, since, as in Chapter 2. is shown, local (generalized) monotonicity of a multivalued operator, in general, implies its global counterpart without any continuity assumption imposed on operator, if the domain of the operator is convex.

On the hand, in [73], the authors proved that monotone operators have convex preimages, which shows that locally injective monotone operators are actually globally injective. Combining these facts, we are able to provide some global injectivity results for certain operators satisfying some analytical conditions of Gale-Nikaido type (see, for instance, [47]) which ensure both the local injectivity and the local increasing monotonicity. The authors's contributions with respect to these topics have been published in G. Kassay, C. Pintea, **S. László**: [72] and **S. László**: [77].

In **Chapter 2** we introduce the concept of $\theta-$*monotonicity* for operators and the concept of $\theta-$*convexity* for real valued functions. These concepts contain as particular cases several monotonicity, respectively, convexity notions already known in literature. We also establish some fundamental properties of operators having this monotonicity property. The concept of a maximal $\theta-$monotone operator is also introduced, and it is shown that such an operator has convex and closed values. Further we are going to analyze some conditions which ensure that the local $\theta-$monotonicity property of an operator provides the global $\theta-$monotonicity property for that operator. Via some examples it is shown that the $\theta-$monotonicity is more general than most of monotonicity properties known in literature, while an example of a $\theta-$monotone operator is given, which is not even quasimonotone. An analytical condition on the function θ that ensures, beside some extra requirements, the $\theta-$convexity of a real valued function is also established. It is shown that the $\theta-$convexity property of a function is more general than the majority of the convexity properties known in literature, while an example of a $\theta-$convex function is given, which is not even quasiconvex. First we establish some fundamental properties of the operators having the $\theta-$monotonicity property. We provide some conditions that ensure their local boundedness. We show that under some circumstances the inverse of a $\theta-$monotone operator is also $\theta-$monotone. We introduce the concept of a maximal $\theta-$monotone operator and we show that these operators have as values closed and convex subsets of X^*, and that, if the function θ is continuous, then their graph is demi-closed. Further, we introduce the concept of a locally $\theta-$monotone operator, and we give a sufficient condition involving θ, guaranteeing that the local $\theta-$monotonicity property of an operator provides the global $\theta-$monotonicity property for that operator. Further, an analytical condition involving the function θ is given which ensures the local $\theta-$monotonicity of an operator. Via some examples it is shown that the concept of $\theta-$monotonicity is larger then most of the monotonicity notions known in literature. In the next section we introduce the notion of a $\theta-$convex, respectively, weak $\theta-$convex function and we show, that under some circumstances, a differentiable function is a $\theta-$convex if and only if its differential is a $2\theta-$monotone operator. Also here an example of $\theta-$convex function that is not even quasiconvex, is given. We end this chapter by presenting some applications of our results and obtaining some surjectivity results in finite dimensional spaces. Finally, we underly some possible further related research. The authors's contributions with respect to these topics have been published in the work **S. László**: [79].

The variational inequality theory, which is mainly due to Stampacchia (see [124]) and Fichera (see [45]) provides very powerful techniques for studying problems arising in mechanics, optimization, transportation, economics equilibrium, contact problems in elasticity, and other branches of mathematics. For instance the free boundary value problem can be studied effectively in the framework of variational inequalities, the moving boundary value problem can be characterized by a class of variational inequalities, the traffic assignment problem is a variational inequality problem (see [5, 10, 38]). However, the variational inequality theory so far developed is applicable for studying free and moving boundary value problems of even order.

In recent years, many generalizations of variational inequalities have been considered, studied and applied in various directions (see [9]). General variational inequalities, which were introduced and studied by Noor [99], are an important and useful generalization of variational inequalities. It was realized that the general variational inequality can be used to study both the odd- and even-order free and moving boundary value problems. It has been shown that general variational inequalities provide us with an unified, simple, and natural framework to study a wide class of problems including unilateral, moving, obstacle, free, equilibrium, and economics arising in various areas of pure and applied sciences.

In **Chapter 3,** we give some existence results of the solutions for several general variational inequalities. We introduce a new class of operators, that contain in particular the class of linear operators, and we extend some results already established for general variational inequalities involving linear operators.

First, the notion of general variational inequality is presented, which was introduced by Noor. We introduce a similar variational inequality that will be called *general variational inequality of Stampacchia type*, and we introduce its counterpart, that will be called *general variational inequality of Minty type*. Also here some generalization for variational inequalities involving multivalued operators are presented. In the next section we introduce a new type of operator, the so called *operator of type ql*, which on one hand may be viewed as the generalization of the monotonicity of real valued functions of one real variable, one the other hand may be viewed as the generalization of the notion of linear operator. Also here are established some fundamental properties of the operators of this type. It is shown that for operators having their range a subset of real numbers, this class coincides with the class of quasilinear functions. At the end of the section is established a property for these type of operators, that will be intensively used in the next section to provide some existence results of the solutions for several generalized variational inequalities. The next section deals with general variational inequalities of Stampacchia type and of Minty type respectively. Relying on the concept of operators that were introduced previously, some sufficient conditions that ensure the existence of the solutions for the general variational inequalities of Stampacchia type are provided. Also here some sufficient conditions are provided, which ensure that the solutions of the general variational inequalities of Stampacchia type and the solutions the general variational inequalities of Minty type coincide. We give some sufficient conditions for the existence of solutions of so called *inverted problems*. By examples are shown, that the existence results for these problems fail outside the class of operators that where introduced in this chapter. As consequences are obtained some well known classic results. Further, we give some existence results of the solution for some generalized variational inequalities involving multivalued operators. Also here, the fact that one of the operators involved is of type ql, is intensively used via Fan's minimax theorem. As applications we obtain some easy proofs of Brouwer's, respectively Kakutani's fixed point theorem. The authors's contributions with respect to these topics have been published in **S. László**: [78, 80, 81].

It is worthwhile to stress that several open questions are still unanswered even within the theory of classical Minty-Browder monotone operators. One of the most interesting is the so called sum problem. It is well known that in a reflexive Banach space the sum of two (set-valued) maximal monotone operators is still maximal monotone, provided the domain of one of them intersects the interior of the domain of the other (cf. Rockafellar see [114]), but in the nonreflexive case it is still unknown whether this condition is sufficient. However, there are several results, that in particular validate this conjecture. Motivated by a study of parallel connection of electrical multiports, Anderson and Duffin (see [2]) introduced the concept of parallel addition of matrices. Passty (see [104]) approached the parallel sum of arbitrary nonlinear monotone operators starting from the following situation arising from electricity: if two resistors having resistance S and T are connected in parallel, Kirchhoff's law and Ohm's law can be combined to show that their joint resistance is $(S^{-1} + T^{-1})^{-1}$. The same considerations apply to parallel connections of electrical multiports. Instead of resistances which are positive real numbers, however, one must work with impedance operators which map a finite- or infinite dimensional space into itself. There then arises the issue of proper extension of the joint resistance formula given above. Motivated from above, but also inspired from the significant number of results concerning on the problem of maximality of the sum of two maximal monotone operators, Penot and Zălinescu in [109] introduced the concepts of *generalized parallel sums*. A related problem - that was still open till now - is the following: due to our best knowledge in the literature did not exist any regularity condition that ensures the maximal monotonicity of the generalized parallel sums. Nevertheless, there exists some interior point regularity conditions that ensure the maximal monotonicity of the parallel sum.

In **Chapter 4** we give a closdness type regularity condition concerning on this problem, and, by an example, we show that our condition is the weakest among those already known in the literature. Concerning on the generalized parallel sums, we will give several regularity conditions, both closedness and interior point type, and we show that our results cannot be deduced from the results known in the literature. Nevertheless, many known results, concerning on the sum of two maximal monotone operator, $S+T$, respectively $S+A^*TA$, where S and T are maximal monotone operators, A is a linear, continuous operator and A^* its adjoint operator, are easy consequences of ours. Our results are based on the concepts of representative function and Fenchel conjugate, while the technique, used to establish closedness type, respectively interior-point type regularity conditions, that ensure the maximal monotonicity of these sums, is stable strong duality. In this chapter we deal with the sum problems involving strongly representable operators in nonreflexive Banach spaces, hence, according to a recent result of Marques Alves and Svaiter, our results also hold for operators of negative infimum type (see [119]) and of Gossez type (D) in arbitrary Banach spaces, (see Remark 4.1.1). As particular cases, beside new results, we will establish some well known ones in the setting of reflexive Banach spaces. We give an useful application for the stable strong duality, involving the function $f \circ A + g \circ B$, where f and g are proper, convex and lower semicontinuous functions, and A and B, respectively are linear operators. Let us mention, that due to our best knowledge, this problem was not considered till now in the literature, since it has had no field of applicability. We also introduce some new generalized inf-convolution formulas, and establish some result concerning on their Fenchel conjugate. Collaterally with our new results, we obtain some known ones, using the same technique, even more, we show that many of our results can not be obtained using the known techniques from the literature. The authors's contributions with respect to these topics have been published in the works R.I. Boţ, **S. László**: [20] and **S. László**: [82], [83], [84].

Keywords
Minty-Browder monotonicity; generalized monotonicity; generalized convexity; locally monotone operator; generalized variational inequality; KKM mapping; Minty's lemma; Ky Fan's lemma; fixed point theorem; conjugate functions; strong quasi-relative interior; conjugate duality; regularity conditions; maximal monotone operators; Fitzpatrick function; representative function; parallel sum;

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Chapter 1

Monotone operators, convex functions and closed countable sets

1.1 The monotonicity of real-valued functions of one real variable

1.1.1 Locally increasing real-valued functions of one real variable

In this section we prove that the locally increasing monotonicity property of real-valued functions of one real variable, on the complement of a closed countable set, implies the global increasing monotonicity property for that function.

Let be $I \subseteq \mathbb{R}$ and let $f : I \longrightarrow \mathbb{R}$ be a function. One says that f is (monotone) increasing (respectively decreasing) on I, if for every $x, y \in I$, $x \leq y$ one has $f(x) \leq f(y)$, (respectively $f(x) \geq f(y)$).

It can be easily observed that the monotonicity property of the real valued function f is equivalent with one of the following inequalities

(1. 1) $$(f(x) - f(y))(x - y) \geq 0, \text{ for all } x, y \in I,$$

respectively

(1. 2) $$(f(x) - f(y))(x - y) \leq 0, \text{ for all } x, y \in I.$$

The first inequality is satisfied if, and only if, f is increasing, while the second one is satisfied if, and only if, f is decreasing.

One says that f is locally increasing if for all $t \in I$ there exists an open interval $J_t \subseteq \mathbb{R}$, with $t \in J_t$, such that the restriction $f|_{J_t \cap I}$ is increasing.

The next result provides a sufficient condition for global monotonicity.

Theorem 1.1.1. *Let $J \subseteq \mathbb{R}$ be an open interval and $f : J \longrightarrow \mathbb{R}$ be a continuous function. If $Y \subseteq J$ is a countable set, closed relative to J, such that f is locally increasing on $J \setminus Y$, then f is increasing on J.*

1.1.2 Local generalized monotonicity of the real valued functions of one real variable

In this section we prove that most of the local generalized monotonicity of real-valued functions of one real variable, on the complement of a closed countable set, provide their global counterpart.

However the case of quasimonotonicity is an exception, for which a counterexample is given.

We recall that the function $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is called pseudomonotone (see [36, 53, 55, 67, 69]), if for all $x, y \in I$,

$$f(x)(y-x) \geq 0 \implies f(y)(y-x) \geq 0,$$

or equivalently, for all $x, y \in I$,

$$f(x)(y-x) > 0 \implies f(y)(y-x) > 0.$$

f is called strictly pseudomonotone (see [55, 68, 69]), if for all $x, y \in I$, $x \neq y$,

$$f(x)(y-x) \geq 0 \implies f(y)(y-x) > 0.$$

The function f is called quasimonotone (see [36, 53, 55, 58, 68, 69]), if for all $x, y \in I$,

$$f(x)(y-x) > 0 \implies f(y)(y-x) \geq 0.$$

Let I be an interval. f is called strictly quasimonotone (see [36, 55, 56]), if f is quasimonotone, and for all $x, y \in I$, $x \neq y$ there exists $z \in (x, y)$ such that $f(z)(y-x) \neq 0$.

Using the previous definitions, we are able to define the notion of local generalized monotonicity of one dimensional maps. Let $I \subseteq \mathbb{R}$ and let $f : I \longrightarrow \mathbb{R}$ be a function.

One says that f is locally quasimonotone, (respectively locally strictly quasimonotone, locally pseudmonotone, locally strictly pseudmonotone) if for all $t \in I$ there exists an open interval $J_t \subseteq \mathbb{R}$, with $t \in J_t$, such that the restriction $f|_{J_t \cap I}$ is quasimonotone, (respectively strictly quasimonotone, pseudmonotone, strictly pseudmonotone).

The next result provides a sufficient condition for global strict quasimonotonicity, (respectively global pseudomonotonicity, global strict pseudomonotonicity).

Theorem 1.1.2. *Let $J \subseteq \mathbb{R}$ be an open interval and $f : J \longrightarrow \mathbb{R}$ be a continuous function. If $Y \subseteq J$ is a closed countable set such that f is locally strictly quasimonotone, (respectively locally pseudomonotone, locally strictly pseudomonotone) on $J \setminus Y$, and $f(x) \neq 0$ for all $x \in Y$, then f is strictly quasimonotone, (respectively pseudomonotone, strictly pseudomonotone) on J.*

However local quasimonotonicity does not imply global quasimonotonicity even if the function f is continuous, as the next example shows.

Example 1.1.1. Let us consider the function $f : \mathbb{R} \longrightarrow \mathbb{R},\ f(x) = \begin{cases} -x-1, \text{ if } x < -1, \\ 0, \text{ if } x \in [-1, 1], \\ -x+1, \text{ if } x > 1. \end{cases}$

It is easy to check that f is locally quasimonotone on $\mathbb{R}$. On the other hand for $x = -2$ and $y = 2$ we have $\min\{f(x)(y-x), f(y)(x-y)\} = 4$ which shows that f is not globally quasimonotone.

1.2 Locally monotone operators

1.2.1 Locally increasing operators on the complement of a closed countable set

Let X be a Banach space and let X^* be its topological dual. For $x \in X$ and $x^* \in X^*$ denote by $\langle x^*, x\rangle$ the scalar $x^*(x)$. Recall that an operator $T : D \longrightarrow X^*$, where D is a subset of X is said to be *Minty-Browder monotone* if either $\langle Tx - Ty, x - y\rangle \geq 0$ for all $x, y \in D$ or $\langle Tx - Ty, x - y\rangle \leq 0$ for all $x, y \in D$. The inequality symbol "$\geq$" corresponds to Minty-Browder *increasing* operators, while the inequality symbol "$\leq$" corresponds to Minty-Browder *decreasing* operators.

Definition 1.2.1. *Let X be a Banach space and $D \subseteq X$ be an open subset. An operator $T : D \longrightarrow X^*$ is said to be* locally Minty-Browder increasing *if each $x \in D$ has an open neighborhood $U_x \subseteq D$, such that the restriction $T|_{U_x} : U_x \longrightarrow X^*$ is a Minty-Browder increasing operator.*

Definition 1.2.2. *Let X be a Banach space and $\emptyset \neq D \subseteq X$. An operator $T : D \longrightarrow X^*$ is said to be quasi-monotone, if $\langle Ty, x - y\rangle \geq 0$ implies $\langle Tx, x - y\rangle \geq 0$, for all $x, y \in D$.*

Remark 1.2.1. If $T : D \longrightarrow X^*$ is a Minty-Browder increasing operator, then T is quasi-monotone. However, the property does not hold if we replace the global increasing monotonicity with the local increasing monotonicity and remove the convexity assumption on D.

Example 1.2.1. We provide an example of an open connected set $D \subseteq \mathbb{R}^2$, which is not convex, and a locally increasing, continuous operator $T : D \longrightarrow \mathbb{R}^2$ which is not globally increasing. Our operator is, in fact, not even quasi-monotone. Indeed, let

$$D = (-1, 1) \times (-1, 1) \setminus \left\{(-1, 0] \times \left\{-\frac{1}{2}\right\} \cup \{0\} \times \left(-\frac{1}{2}, 0\right]\right\} \subseteq \mathbb{R}^2$$

and

$$U_1 = \left\{(x, y) \,:\, x \in \left(-1, -\frac{1}{2}\right),\ x \leq y < -\frac{1}{2}\right\} \subset D,$$

$$U_2 = \left\{(x, y) \,:\, x \in (-1, 0),\ -\frac{1}{2} < y \leq -x\right\} \subset D.$$

We now consider the operator $T : D \to \mathbb{R}^2$, $T(x, y) = \big(p(x, y), q(x, y)\big)$, where

$$p(x,y) = \begin{cases} x + y, & \text{if } (x, y) \in D \setminus (U_1 \cup U_2), \\ 2x, & \text{if } (x, y) \in U_1, \\ 0, & \text{if } (x, y) \in U_2, \end{cases} \qquad q(x,y) = \begin{cases} -x + y, & \text{if } (x, y) \in D \setminus (U_1 \cup U_2), \\ 0, & \text{if } (x, y) \in U_1, \\ 2y, & \text{if } (x, y) \in U_2. \end{cases}$$

It is easy to check that T is locally increasing and it is continuous. On the other hand

$$\langle T(x, y), (u, v) - (x, y)\rangle = 2x(u - x) = \frac{3}{40} > 0$$

and

$$\langle T(u, v), (u, v) - (x, y)\rangle = 2v(v - y) = -\frac{5}{24} < 0,$$

with $(x, y) = \big(-\frac{3}{4}, -\frac{2}{3}\big) \in U_1$, $(u, v) = \big(-\frac{4}{5}, -\frac{1}{4}\big) \in U_2$. This shows that T is not quasi-monotone.

In what follows, let X be a Banach space and $C \subseteq D \subseteq X$ with D open and convex and C closed relative to D with empty interior, such that the intersection $[x, y] \cap C$ is countable, possibly empty, for all $x, y \in D \setminus C$. Note that such a C is actually a proper subset of D.

Remark 1.2.2. Examples of subsets $C \subset D \subseteq \mathbb{R}^n$ which satisfy the above requirements, consist in finite families of spheres $S(p, r) := \{x \in \mathbb{R}^n : ||x - p|| = r\}$ in D, since spheres do not contain segments. However there are sets C containing segments still satisfying these requirements. Indeed, every real algebraic variety has the mentioned properties. In particular let $D = \{x \in \mathbb{R}^n \,:\, ||x|| < 1\}$ and

$$C = \left\{x = (x_1, \ldots, x_n) \in D \,:\, \sum_{i=1}^{n-1} x_i^2 = \frac{1}{4}\right\}.$$

The figure bellow shows an example of such set C.

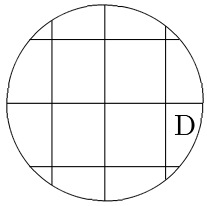

Fig. 3

Here D is an open disk from $\mathbb{R}^2$, and C is the union of a finite number of open line segments having their endpoints on the boundary of D.

Definition 1.2.3. *Let $T : X \longrightarrow X^*$ be an operator. We say that T is hemicontinuous at $x \in X$, if for all $(t_n)_{n\in\mathbb{N}} \subset \mathbb{R}$, $t_n \longrightarrow 0$, $n \longrightarrow \infty$ and $y \in X$, we have $T(x + t_n y) \rightharpoonup^* Tx$, $n \longrightarrow \infty$, where " $\rightharpoonup^*$ " denotes the convergence with respect to the weak* topology of X^*.*

The next result provides, in a Banach space context, a sufficient condition for global monotonicity.

Theorem 1.2.1. *If $T : D \longrightarrow X^*$ is a hemicontinuous operator whose restriction $T|_{D\setminus C}$ is locally Minty-Browder increasing, then T is Minty-Browder increasing on D.*

1.2.2 Generalized monotone operators on the complement of a closed countable set

In this section we extend the results from the previous section for generalized monotone operators defined on an open and convex subset of a real Banach space. In previous section we proved, that for an operator defined on the open and convex subset D of the real Banach space X, the local Minty-Browder monotonicity of the operator on the complement of a closed countable set implies its global Minty-Browder monotonicity. However, for generalized monotone maps this implication is no more true in the absence of further conditions.

Let X be a real Banach space, X^* its dual, $D \subseteq X$ a subset of X, and $T : D \longrightarrow X^*$ an operator. We denote by int Y the interior of the set $Y \subseteq X$, and by (x, y) the open line segment in X with the endpoints x and y, i.e. $(x, y) = \{z \in X : z = x + t(y - x),\ t \in (0, 1)\}$. The closed segment $[x, y]$ with the endpoints $x, y \in X$ is defined as usual, i.e. $[x, y] = \{z \in X : z = x + t(y - x),\ t \in [0, 1]\}$.

We recall that the operator T is called pseudomonotone (see [36,53,55,67,69]), if for all $x, y \in D$, $\langle Tx, y - x\rangle \geq 0$ implies $\langle Ty, y - x\rangle \geq 0$, or equivalently, for all $x, y \in D$, $\langle Tx, y - x\rangle > 0$ implies $\langle Ty, y - x\rangle > 0$.

T is called strictly pseudomonotone (see [55, 68, 69]), if for all $x, y \in D$, $x \neq y$, $\langle Tx, y - x\rangle \geq 0$ implies $\langle Ty, y - x\rangle > 0$.

The operator T is called quasimonotone (see [36,53,55,58,68,69]), if for all $x, y \in D$, $\langle Tx, y-x\rangle > 0$ implies $\langle Ty, y - x\rangle \geq 0$.

Let D be convex. T is called strictly quasimonotone (see [36, 55, 56]), if T is quasimonotone, and for all $x, y \in D$, $x \neq y$ there exists $z \in (x, y)$ such that$\langle Tz, y - x\rangle \neq 0$.

Next we give several definitions for local generalized monotonicity of operators on a Banach space.

Definition 1.2.4. *Let X be a real Banach space, X^* its dual, $D \subseteq X$ an open subset of X, and $T : D \longrightarrow X^*$ an operator. One says that T is locally quasimonotone (respectively, locally* *strictly quasimonotone, locally pseudomonotone, locally strictly pseudomonotone), if for all $z \in D$ there exists an open neighborhood $U_z \subseteq D$ of z, such that the restriction $T|_{U_z}$ is quasimonotone, (respectively, strictly quasimonotone, pseudomonotone, strictly pseudomonotone).*

In what follows, X denotes a real Banach space, and let $C \subseteq D \subseteq X$ with D open and convex, and C closed relative to D, with empty interior, such that the intersection $[x,y] \cap C$ is countable, possibly empty, for all $x, y \in D \setminus C$.

The next result provides, in a Banach space context, a sufficient condition for global strict quasimonotonicity (respectively, pseudomonotonicity, strict pseudomonotonicity).

Theorem 1.2.2. *If $T : D \longrightarrow X^*$ is a hemicontinuous operator with the property that $\langle Tz, y - x\rangle \neq 0$ for all $z \in [x,y] \cap C$, $x, y \in D$, $x \neq y$ and whose restriction $T|_{D\setminus C}$ is locally strictly quasimonotone, (respectively, locally pseudomonotone, locally strictly pseudomonotone), then T is strictly quasimonotone, (respectively, pseudomonotone, strictly pseudomonotone), on D.*

1.3 Applications

1.3.1 Some injectivity results

In this section we apply the results from section 1.2.1 to prove some injectivity results under certain classical hypothesis which are imposed on the complements of the same type of sets as before.

Let C and D be the same sets as in Section 1.2.1.

Proposition 1.3.1. If H is a Hilbert space and $T : D \subseteq H \longrightarrow H$ is a continuous operator which is of class C^1 on $D \setminus C$ and has the property $\langle (dT)_x(y), y\rangle > 0$ for all $x \in D \setminus C$ and all $y \in H \setminus \{0\}$, and C does not contain line segments, then T is injective.

Corollary 1.3.1. If $D \setminus C$ is connected and $f : D \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ is a continuous function which is also of class C^1 on $D \setminus C$ and satisfies the inequality

$$Re\frac{\partial f}{\partial z}(z) > \left|\frac{\partial f}{\partial \bar{z}}(z)\right|,$$

for all $z \in D \setminus C$, and C does not contain line segments, then f is injective.

1.3.2 Applications to convex functions

For the first part of this section our goal is to provide sufficient conditions for convexity of real-valued functions. Taking into account that the convexity of real-valued functions of class C^1 defined on open convex subsets of Hilbert spaces is characterized by the monotonicity of the gradient operator (see [34]), we may get, as a consequence of our results presented in the previous section some global convexity theorem of real-valued functions based on their local convexity. In what follows H denotes a real Hilbert space, and let C and D the same sets as in Section 2.2. We shall need the following definitions and results related to generalized convex functions.

Definition 1.3.1. *Let $\mathfrak{D}$ an open subset of a locally convex space. A real-valued function $f : \mathfrak{D} \longrightarrow \mathbb{R}$ is said to be locally convex if every point $x \in \mathfrak{D}$ has a convex and open neighborhood $U_x \subseteq \mathfrak{D}$ such that the restriction $f\big|_{U_x}$ is convex.*

Theorem 1.3.1. *If H is a Hilbert space and $f : D \subseteq H \longrightarrow \mathbb{R}$ is a function of class C^1 and locally convex on $D \setminus C$, then f is globally convex.*

A real valued function f defined on the open convex subset D of H, is called quasiconvex (see [36,53,62,105]), respectively strictly quasiconvex (see [36,105]), if for all $x, y \in D$ and $t \in [0, 1]$, we have

$$f(y) \leq f(x) \Longrightarrow f(tx + (1 - t)y) \leq f(x),$$

respectively for all $x, y \in D$, $x \neq y$ and $t \in (0, 1)$, we have

$$f(y) \leq f(x) \Longrightarrow f(tx + (1 - t)y) < f(x),$$

or equivalently for all $x, y \in D$ and $t \in [0, 1]$, we have

$$f(tx + (1 - t)y) \leq \max\{f(x), f(y)\},$$

respectively for all $x, y \in D$, $x \neq y$ and $t \in (0, 1)$, we have

$$f(tx + (1 - t)y) < \max\{f(x), f(y)\}.$$

Remark 1.3.1. A differentiable quasiconvex function f can be characterized by its differential (see [55]), i.e. f is quasiconvex on the open convex subset D of H, if and only if, for every pair of points $x, y \in D$ we have

$$f(y) \leq f(x) \Longrightarrow \langle \nabla f(x), y - x \rangle \leq 0,$$

where ∇f denotes the gradient operator.

A real valued differentiable function f defined on the open convex subset D of H, is called pseudoconvex (see [4,35,36,52,54]), respectively strictly pseudoconvex (see [35,36,52–54]) on D, if for every pair of distinct points $x, y \in D$ we have

$$\langle \nabla f(x), y - x \rangle \geq 0 \Longrightarrow f(y) \geq f(x),$$

respectively

$$\langle \nabla f(x), y - x \rangle \geq 0,\ x \neq y \Longrightarrow f(y) > f(x).$$

The next result is well-known, see for instance [30,55,68].

Proposition 1.3.2. *Let f be differentiable on the open convex subset D of H. Then f is pseudoconvex, (respectively strictly pseudoconvex) on D, if and only if, ∇f is pseudomonotone, (respectively strictly pseudomonotone) on D.*

Next we give the definitions of some locally generalized convex functions.

Definition 1.3.2. *Let $\mathfrak{D}$ an open subset of a locally convex space. A real-valued function $f : \mathfrak{D} \longrightarrow \mathbb{R}$ is said to be locally quasiconvex, (respectively locally strictly quasiconvex, locally pseudoconvex, locally strictly pseudoconvex) if every point $x \in \mathfrak{D}$ has a convex and open neighborhood $U_x \subseteq \mathfrak{D}$ such that the restriction $f\big|_{U_x}$ is quasiconvex, (respectively strictly quasiconvex, pseudoconvex, strictly pseudoconvex).*

In what follows we provide, in a Hilbert space context, a sufficient condition for strict quasiconvexity, (respectively, pseudoconvexity, strict pseudoconvexity), of a locally strictly quasiconvex, (respectively, locally pseudoconvex, locally strictly pseudoconvex), function.

Theorem 1.3.2. *Let $f : D \longrightarrow \mathbb{R}$ be a continuously differentiable, locally strictly quasiconvex, (respectively, locally pseudoconvex, locally strictly pseudoconvex), function on $D \setminus C$. If ∇f has the property, that $\langle \nabla f(z), x - y \rangle \neq 0$ for all $z \in [x, y] \cap C$, $x, y \in D$, $x \neq y$ then f is globally strictly quasiconvex, (respectively, globally pseudoconvex, globally strictly pseudoconvex), on D.*

However, as we have seen in Example 1.1.1, the local quasimonotonicity does not imply the global quasimonotonicity. Next we will give an example of a continuously differentiable locally quasiconvex function, which is not globally quasiconvex.

Example 1.3.1. Let us consider the function $F : \mathbb{R} \longrightarrow \mathbb{R}$,

$$F(x) := \begin{cases} -\dfrac{x^2}{2} - x, \text{ if } x < -1, \\ \dfrac{1}{2}, \text{ if } x \in [-1, 1], \\ -\dfrac{x^2}{2} + x, \text{ if } x > 1. \end{cases}$$

It can be easily checked that F is an antiderivative of f given in Example 1.1.1, consequently F is continuously differentiable.

We know that any monotone (increasing/decreasing) function from $\mathbb{R}$ to $\mathbb{R}$ is quasiconvex. Since F is locally monotone we obtain that F is locally quasiconvex.

On the other hand, for $x = -2$ and $y = 2$ we have:
$F\left(\frac{1}{2} \cdot (-2) + \left(1 - \frac{1}{2}\right) \cdot (2)\right) = F(0) = \frac{1}{2} > \max\{F(-2), F(2)\} = 0$, which shows that F is not globally quasiconvex.

Chapter 2

$\theta-$monotone operators and $\theta-$convex functions

2.1 $\theta-$monotone operators

2.1.1 On some properties of $\theta-$monotone operators

In this section we present some properties of multivalued $\theta-$monotone operators. As a main result of the section we show, in a Hilbert space context, that under some mild requirements imposed on the function θ, a $\theta-$monotone operator is locally bounded. We also establish a condition on the function θ that ensures that the inverse of a $\theta-$monotone operator is $\theta-$monotone too.

In what follows we introduce the concept of $\theta-$monotonicity for an operator. Let X be a real Banach space with its dual denoted by X^*, and let $T : X \rightrightarrows X^*$ be a multivalued operator. We denote by $D(T) = \{x \in X : Tx \neq \emptyset\}$ its domain and by $R(T) = \bigcup_{x\in D(T)} Tx$ its range. The graph of the operator T is the set $G(T) = \{(x, u) \in X \times X^* : u \in Tx\}$. Let $\theta : X \times X \longrightarrow \mathbb{R}$ be a given function with the property that $\theta(x, y) = \theta(y, x)$ for all $x, y \in D(T)$.

Definition 2.1.1. *We say that T is $\theta-$monotone, if*

$$\langle u - v, x - y\rangle \geq \theta(x, y)\|x - y\| \text{ for all } (x, u),\ (y, v) \in G(T). \tag{2. 1}$$

T is called strictly $\theta-$monotone if in (2.1) *equality holds only for $x = y$.*

To this respect single-valued $\theta-$monotone operators are those $\theta-$monotone operators $T : X \rightrightarrows X^*$, which satisfy the condition $card(Tx) = 1$, for all $x \in D(T)$. It can be easily observed that the concept of $\theta-$monotonicity generalizes several concepts of monotonicity known in literature.

If $\theta(x, y) = 0$ for all $x, y \in D(T)$ we obtain the concept of *Minty-Browder monotonicity,* respectively the concept of *strict Minty-Browder monotonicity* (see [22, 23, 91, 92]), i.e.

$$\langle u - v, x - y\rangle \geq 0 \text{ for all } (x, u), (y, v) \in G(T),$$

respectively,

$$\langle u - v, x - y\rangle > 0 \text{ for all } (x, u), (y, v) \in G(T),\ x \neq y.$$

If $\theta(x, y) = r\|x - y\|$ for all $x, y \in D(T)$, where $r > 0$, we obtain the concept of *strong monotonicity* (see for instance [125]), i.e.

$$\langle u - v, x - y\rangle \geq r\|x - y\|^2 \text{ for all } (x, u), (y, v) \in G(T).$$

If $\theta(x,y) = f(\|x-y\|)$ for all $x,y \in D(T)$, $x \neq y$ and $\theta(x,x) = 0$ for all $x \in D(T)$, where $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is an increasing function, with $\lim_{t\downarrow 0} f(t) = 0$ and $\lim_{t\to\infty} f(t) = \infty$, then the $\theta-$monotonicity becomes the *uniform monotonicity* (see for instance [70]), i.e.

$$\langle u-v, x-y\rangle \geq f(\|x-y\|)\|x-y\| \text{ for all } (x,u),(y,v) \in G(T),\ x \neq y.$$

If $\theta(x,y) = -\epsilon$ for all $x,y \in D(T)$, where $\epsilon > 0$, we obtain the concept of $\epsilon-$*monotonicity* (see [65, 96]), i.e.

$$\langle u-v, x-y\rangle \geq -\epsilon\|x-y\| \text{ for all } (x,u),(y,v) \in G(T).$$

If $\theta(x,y) = -C\|x-y\|^{\gamma-1}$ for all $x,y \in D(T)$, where $C > 0$ and $\gamma > 1$, we obtain the concept of $\gamma-$*paramonotonicity* (see [66]), i.e.

$$\langle u-v, x-y\rangle \geq -C\|x-y\|^{\gamma} \text{ for all } (x,u),(y,v) \in G(T).$$

For $\gamma = 2$, hence for $\theta(x,y) = -C\|x-y\|$ for all $x,y \in D(T)$, where $C > 0$, the $\gamma-$paramonotonicity becomes the $C-$*relaxed monotonicity* (see for instance [125]), i.e.

$$\langle u-v, x-y\rangle \geq -C\|x-y\|^{2} \text{ for all } (x,u),(y,v) \in G(T).$$

If $\theta(x,y) = -\min\{\sigma(x),\sigma(y)\}$, for all $x,y \in D(T)$ and $\theta(x,y) = 0$ otherwise, where $\sigma : D(T) \longrightarrow (0,\infty)$ is a given function, we obtain the concept of *premonotonicity*, introduced in [62], i.e.

$$\langle u-v, x-y\rangle \geq -\min\{\sigma(x),\sigma(y)\}\|x-y\| \text{ for all } (x,u),(y,v) \in G(T).$$

Recall that the operator $T : X \rightrightarrows X^*$ is *locally bounded* in $x \in X$, if there exists a neighborhood $U \subseteq X$ of x, such that the set $T(U)$ is a bounded subset of X^*.

Let $f : X \longrightarrow \mathbb{R}$ be a function. We say that f is *lower semicontinuous* in $x \in X$, if for every $\epsilon > 0$ there exists a neighborhood $U \subseteq X$ of x, such that $f(x) - \epsilon \leq f(y)$ for all $y \in U$. Equivalently, this can be expressed as $\liminf_{y\to x} f(y) \geq f(x)$. We say that f is lower semicontinuous on $U \subseteq X$ if f is lower semicontinuous in every $x \in U$.

The next result provides, in a finite dimensional Hilbert space context, the local boundedness of a $\theta-$monotone operator in the interior of its domain. This is a major result, since the $\theta-$monotonicity property of the operators is a weaker condition than the Minty-Browder monotonicity, and still one of the fundamental property of the Minty-Browder monotone operators remains true.

Theorem 2.1.1. *Let $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a $\theta-$monotone operator. If the function $\theta(\cdot,y)$ is lower semicontinuous on $int(D(T))$ for all $y \in int(D(T))$, then T is locally bounded in the interior of its domain $D(T)$.*

2.1.2 Maximal $\theta-$monotone operators

In this section the concept of maximal $\theta-$monotone operator is considered. It is shown that a maximal $\theta-$monotone operator has convex and closed images and that, under some circumstances its graph is $\|\cdot\| \times bdw^*$-closed, where by bdw^* we denote weak*-convergence for bounded nets. Finally, for a single-valued operator, we present some conditions that ensure its maximal $\theta-$monotonicity. This result is a generalization of a well-known result established for the classical Minty-Browder monotonicity.

Definition 2.1.2. *Let $T : X \rightrightarrows X^*$ be a $\theta-$monotone operator. One says that T is maximal $\theta-$monotone, if for every operator $T' : X \rightrightarrows X^*$, which is $\theta-$monotone with $G(T) \subseteq G(T')$, one has $T = T'$.*

The next result provides the convexity and closedness of the images of a maximal $\theta-$monotone operator.

Theorem 2.1.2. *Let $T : X \rightrightarrows X^*$ be a maximal $\theta-$monotone operator. Then Tx is convex and closed for all $x \in D(T)$.*

The $\|\cdot\| \times \|\cdot\|$ closedness of the graph of a maximal $\theta-$monotone operator holds under mild assumptions.

Proposition 2.1.1. *Let $T : X \rightrightarrows X^*$ be a maximal $\theta-$monotone operator. If the function $\theta(\cdot, y) : X \longrightarrow \mathbb{R}$ is lower semicontinuous on $D(T)$ for every $y \in D(T)$, then $G(T)$ is $\|\cdot\| \times \|\cdot\|$-closed.*

2.1.3 Locally $\theta-$monotone operators

In this section we introduce the local $\theta-$monotonicity concept of a multivalued operator. Further we give under some conditions involving the function θ, a sufficient condition that ensures the $\theta-$monotonicity of an operator. We present next the concept of local $\theta-$monotonicity, respectively, of local central $\theta-$monotonicity for operators.

Definition 2.1.3. *Let $T : X \rightrightarrows X^*$ be an operator. One says that T is locally $\theta-$monotone, respectively, locally central $\theta-$monotone, if for all $z \in D(T)$ there exists an open neighborhood $U_z \subseteq X$ of z, such that*

$$\langle u - v, x - y\rangle \geq \theta(x, y)\|x - y\|, \text{ for all } x, y \in U_z \cap D(T),\ u \in Tx,\ v \in Ty \tag{2. 2}$$

respectively

$$\langle u - v, x - z\rangle \geq \theta(x, z)\|x - z\|, \text{ for all } x \in U_z \cap D(T),\ u \in Tx,\ v \in Tz. \tag{2. 3}$$

The notion of strict local $\theta-$monotonicity, respectively, the notion of strict local central $\theta-$monotonicity is obtained if in (2.2), respectively in (2.3) we have equality only for $x = y$, respectively, for $x = z$.

Definition 2.1.4. *Let $D \subseteq X$ be convex. One says that the function θ has the (m) property on D, if*

$$\theta(x, z) + \theta(z, y) \geq \theta(x, y)$$

for all $z \in (x, y)$, $x, y \in D$, $x \neq y$.

The next result provides a sufficient condition for the $\theta-$monotonicity of an operator.

Theorem 2.1.3. *Let $T : X \rightrightarrows X^*$ be a locally central $\theta-$monotone operator, having a convex domain $D(T)$. If the function θ has the (m) property on $D(T)$, then T is $\theta-$monotone.*

2.2 $\theta-$convex functions

In this section we introduce the concept of $\theta-$convexity for real valued functions in Hilbert spaces. This concept generalizes some convexity notions known in literature, such as strong convexity and $\epsilon-$convexity. We will show that this notion is strongly connected with the notion of $\theta-$monotonicity, namely that a differentiable $\theta-$convex function has as differential a $2\theta-$monotone operator with the same θ. Further, we will give an analytical condition upon θ that provides the $\theta-$convexity of a differentiable real valued function. Everywhere in the sequel D denotes an open and convex subset of a real Hilbert space H, while the Frèchet differential of a function $f : D \longrightarrow \mathbb{R}$ at $x \in D$ will be identified with $\nabla f(x)$.

Definition 2.2.1. *Let $\theta : D \times D \longrightarrow \mathbb{R}$ be a given function with the property that $\theta(x, y) = \theta(y, x)$ for all $x, y \in D$. One says that the function $f : D \longrightarrow \mathbb{R}$ is $\theta-$convex, if for all $x, y \in D$ and all $z \in (x, y)$ we have*

$$\frac{f(z) - f(x)}{\|z - x\|} + \frac{f(z) - f(y)}{\|z - y\|} + \theta(x, z) + \theta(z, y) \leq 0. \tag{2. 4}$$

It can be easily observed that (2.4) is equivalent to $f((1-t)x+ty) \leq (1-t)f(x)+tf(y)-t(1-t)(\theta(x,(1-t)x+ty)+\theta((1-t)x+ty, y))\|x-y\|$, for all $t \in [0, 1]$ and all $x, y \in D$. Obviously, if $\theta(x, y) = \frac{c}{2}\|x - y\|$ for all $x, y \in D$ where $c \in \mathbb{R}_+ \setminus \{0\}$, we obtain the concept of strong convexity on D, while if $\theta(x, y) = 0$ for all $x, y \in D$, we obtain the concept of "classical" convexity on D.

We say that the function $f : D \longrightarrow \mathbb{R}$ is locally $\theta-$convex, if for every $x_0 \in D$ there exists an open and convex neighborhood $U_{x_0} \subseteq D$ of x_0 such that the restriction of f on U_{x_0}, $f|_{U_{x_0}}$ is $\theta-$convex.

The next result connects the $\theta-$convexity property of a differentiable function with the $2\theta-$monotonicity property of its differential.

Proposition 2.2.1. *If $f : D \longrightarrow \mathbb{R}$ is a differentiable $\theta-$convex function, where $\theta(x, \cdot) : D \longrightarrow \mathbb{R}$ is radially continuous and $\theta(x, x) = 0$ for all $x \in D$, then ∇f is $2\theta-$monotone, with the same θ. If $D = X$ and ∇f is hemicontinuous, then ∇f is maximal $2\theta-$monotone.*

Next we will give a condition involving the function θ, such that the $2\theta-$monotonicity of the differential of a differentiable function provides the $\theta-$convexity property of that function.

Theorem 2.2.1. *If $f : D \longrightarrow \mathbb{R}$ is a continuously differentiable function, the function $s : [0, 1] \longrightarrow \mathbb{R}$, $s(t) = \theta(x, x + t(y - x))$ is integrable with*

$$\int_0^1 s(t)dt \geq \frac{\theta(x, y)}{2}$$

for all $x, y \in D$, $x \neq y$, and ∇f is $2\theta-$monotone, then f is $\theta-$convex.

Theorem 2.2.2. *If $f : D \longrightarrow \mathbb{R}$ is a differentiable function, and the function θ has the property that $2\theta(u, v) \geq \theta(x, z) + \theta(z, y)$ for all $x, y \in D$, $x \neq y$, $z \in (x, y)$, $u \in (x, z)$, $v \in (z, y)$, and ∇f is $2\theta-$monotone, then f is $\theta-$convex.*

2.3 Applications to surjectivity results

In what follows we will provide some surjectivity results involving $\theta-$monotone operators in the case when $X = \mathbb{R}^n$.

Recall that an operator having $\|\cdot\| \times \|\cdot\|$ closed graph in $X \times X^*$ is called *outer semi-continuous*.

Obviously if the operator T is $\theta-$monotone, then $T + \lambda I$ is $\theta-$monotone (with the same θ), for all $\lambda > 0$, where I denotes the identity operator. Even more, $T + \lambda I$ is θ' monotone, with $\theta'(x, y) = \theta(x, y) + \lambda\|x - y\|$.

Theorem 2.3.1. *If the operator $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is $\theta-$monotone, convex valued, outer semi-continuous and $D(T) = \mathbb{R}^n$, as well as the function $\theta(\cdot, y) : \mathbb{R}^n \longrightarrow \mathbb{R}$ is lower semicontinuous for all $y \in \mathbb{R}^n$ and the function $\theta(\cdot, 0) : \mathbb{R}^n \longrightarrow \mathbb{R}$ is bounded below, then $T + \lambda I$ is surjective for all $\lambda > 0$.*

The next Minty's type theorem ensures the surjectivity of $T+\lambda I$, when T is maximal $\theta-$monotone.

Theorem 2.3.2. *Let $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a maximal $\theta-$monotone operator with $D(T) = \mathbb{R}^n$. If $\theta(\cdot, y) : \mathbb{R}^n \longrightarrow \mathbb{R}$ is lower semicontinuous for all $y \in \mathbb{R}^n$ and the function $\theta(\cdot, 0) : \mathbb{R}^n \longrightarrow \mathbb{R}$ is bounded below, then the operator $T + \lambda I$ is surjective for all $\lambda > 0$.*

2.4 Final remarks and comments

Since the concepts of $\theta-$monotonicity and $\theta-$convexity contain many monotonicity, respectively, convexity concepts as particular cases, the possibilities of further investigations are considerable.

For instance, it is natural to introduce a new subdifferential concept, the so-called θ-subdifferential.

Let X be a real Banach space and $f : X \longrightarrow \mathbb{R} \cup \{\infty\}$ a proper function. One says that $x^* \in X^*$ is a $\theta-$subgradient of f in $x \in dom(f) = \{x \in X : f(x) < \infty\}$, if $\langle x^*, y - x\rangle \leq f(y) - f(x) - \theta(x, y)\|x - y\|$, $(\forall)y \in X$. The set

$$\partial_\theta f(x) = \{x^* \in X^* : \langle x^*, y - x\rangle \leq f(y) - f(x) - \theta(x, y)\|x - y\|, (\forall)y \in X\}$$

is called the $\theta-$*subdifferential* of f at $x \in dom(f)$.

The investigation of generical differentiability of $\theta-$convex functions in Asplund spaces is also a good starting point for further researches, since this result was already established for approximative convex and $\gamma-$paraconvex functions (see [97, 116]).

It is worthwhile to investigate the applicability of these concepts in the field of optimization or variational inequalities.

Chapter 3

Variational inequalities

3.1 Generalized variational inequalities

In what follows, unless is otherwise specified, we assume that X be a real Banach space and X^* be the topological dual of X. We denote by $\langle x^*, x\rangle$ the value of the linear and continuous functional $x^* \in X^*$ in $x \in X$. Consider the set $K \subseteq X$ and let $A : K \longrightarrow X^*$ and $a : K \longrightarrow X$ be given operators. The general variational inequalities considered, are some generalizations of the classic variational inequalities of Stampacchia type and of Minty type.

Recall that Stampacchia variational inequality, $VI_S(A, K)$, consists in finding an element $x \in K$, such that $\langle A(x), y - x\rangle \geq 0$ for all $y \in K$, where the set K is convex and closed (see, for instance, [43, 74, 87]).

The problem that we shall study in what follows is the so called general variational inequality of Stampacchia type, $VI_S(A, a, K)$, which consists in finding an element $x \in K$, such that

$$\langle A(x), a(y) - a(x)\rangle \geq 0, \quad \text{for all } y \in K, \tag{3. 1}$$

Obviously, when $a \equiv id_K$, then (3.1) reduces to Stampacchia variational inequality $VI_S(A, K)$.

Interchanging the role of A and a in $VI_S(A, a, K)$, we obtain the inverted general variational inequality problem of Stampacchia type, $VI_{iS}(A, a, K)$, which consist of finding an element $x \in K$ such that

$$\langle A(y) - A(x), a(x)\rangle \geq 0, \quad \text{for all } y \in K. \tag{3. 2}$$

Recall that Minty variational inequality, $VI_M(A, K)$, consists in finding an element $x \in K$, such that $\langle A(y), y - x\rangle \geq 0$ for all $y \in K$, where the set K is convex and closed (see, for instance, [43, 63, 87]).

The general variational inequality of Minty type, $VI_M(A, a, K)$, consists in finding an element $x \in K$, such that

$$\langle A(y), a(y) - a(x)\rangle \geq 0, \quad \text{for all } y \in K. \tag{3. 3}$$

Obviously, when $a \equiv id_K$, then (3.3) reduces to Minty variational inequality $VI_M(A, K)$.

Interchanging the role of A and a in $VI_M(A, a, K)$, we obtain the inverted general variational inequality problem of Minty type, $VI_{iM}(A, a, K)$, which consist of finding an element $x \in K$ such that

$$\langle A(y) - A(x), a(y)\rangle \geq 0, \quad \text{for all } y \in K. \tag{3. 4}$$

Let $K \subseteq X$ be nonempty and convex, and let $T : K \rightrightarrows X^*$ and $f : K \longrightarrow X$ be given operators. Consider the following problem. Find an element $x \in K$, such that

$$(\forall)y \in K\ (\exists)u \in T(x) : \langle u, f(y) - f(x)\rangle \geq 0. \tag{3. 5}$$

Obviously, when T is single valued, then (3.5) reduces to the general variational inequality of Stampacchia type, $VI_S(T, f, K)$. Let us denote by $S_w(T, f, K)$ the set of solutions of (3.5).

Moreover, consider the following problem. Find an element $x \in K$, such that

$$(\exists)u \in T(x) : (\forall)y \in K\ \langle u, f(y) - f(x)\rangle \geq 0. \tag{3. 6}$$

It is easy to observe, that also in this case, if T is single valued, then (3.6) reduces to the general variational inequality of Stampacchia type, $VI_S(T, f, K)$. Let us denote by $S(T, f, K)$ the set of solutions of (3.6).

Further, consider the following problem. Find an element $x \in K$, such that

$$(\forall)y \in K\ (\forall)v \in T(y) : \langle v, f(y) - f(x)\rangle \geq 0. \tag{3. 7}$$

It can be easily observed, that if T is single valued, then (3.7) reduces to the general variational inequality of Minty type, $VI_M(T, f, K)$. Let us denote by $M(T, f, K)$ the set of solutions of (3.7).

3.2 Operators of Type ql

3.2.1 Some characterizations of monotonicity of real valued functions of one real variable

In this section we present some characterizations of monotonicity of real valued functions of one real variable, and generalizing these characteristics we introduce several notions of monotonicity of operators already known in literature. Relied on the one of above mentioned characteristic, in the next section, we introduce the notion of operator of type ql.

Proposition 3.2.1. *Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a function. The function f is monotone increasing (decreasing), if and only if, for every $a, b \in I$, $a \leq b$, and every $z \in [a, b] \cap I$ one has $f(z) \in [f(a), f(b)]$, (respectively $f(z) \in [f(b), f(a)]$).*

3.2.2 Some properties of operators of type ql

In this section we introduce the notion of operator of type ql. We show that this concept is a generalization of the notion of monotonicity of real valued functions of one real variable. We show that this notion may be viewed as well, as a generalization of the concept of linear operator, even more, when an operator takes its values in $\mathbb{R}$ then is of type ql if and only if is quasilinear. We obtain some results that will be used in the next section for establishing the existence of the solutions of some general variational inequalities.

Relying on Proposition 3.2.1 we introduce the concept of the operator of type ql.

Definition 3.2.1. *Let X and Y be two real linear spaces. One says that the operator $A : D \subseteq X \longrightarrow Y$ is of type ql, if for every $x, y \in D$ and every $z \in [x, y] \cap D$ one has $A(z) \in [A(x), A(y)]$. One says that the operator $A : D \subseteq X \longrightarrow Y$ is of type strict ql, if for every $x, y \in D$, $x \neq y$ and every $z \in (x, y) \cap D$ one has $A(z) \in (A(x), A(y))$.*

We have the following result, which proof is straightforward.

Proposition 3.2.2. *Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a function. Then f is of type ql, if and only if f is monotone (increasing or decreasing).*

The next result is obvious.

Proposition 3.2.3. *Let X and Y be two real linear spaces and let $A : X \longrightarrow Y$ be a linear operator. Then A is of type ql.*

Definition 3.2.2. *Let X be a real vector space, Y a topological space and let $A : D \subseteq X \longrightarrow Y$ be an operator. One says that A is continuous on line segments at $x \in D$, if for every sequence $\{t_n\} \subseteq \mathbb{R}$ of real numbers convergent to 0 and every $y \in D$ with $x+t_n y \in D$ we have $A(x+t_n y) \longrightarrow A(x)$, $n \longrightarrow \infty$. A is said continuous on line segments in D if it has this continuity property in every $x \in D$.*

Lemma 3.2.1. *Let X be a real linear space and let Y be a real linear space that is also a metric space, let $D \subseteq X$ convex, and $A : D \longrightarrow Y$ an operator continuous on line segments and of type ql. Then for every $x, y \in D$ one has $A([x, y]) = [A(x), A(y)]$.*

The next theorem ensures that the image of an operator of type ql is convex.

Theorem 3.2.1. *Let X be a real linear space and let Y be a real linear space that is also a metric space and let $A : D \subseteq X \longrightarrow Y$ be an operator, continuous on line segments and of type ql, with its domain D convex. Then $A(D)$ is convex.*

In what follows we show that the class of real valued operators of type ql coincides with the class of quasilinear functions.

Definition 3.2.3. *Let X be a real linear space and let $D \subseteq X$ convex. A function $f : D \longrightarrow \mathbb{R}$ is called quasiconcave (see* [4]*), if $-f$ is quasiconvex, i.e.*

$$f((1-t)x + ty) \geq \min\big\{f(x), f(y)\big\}, \text{ for every } \ x, y \in D, \text{ and } \ t \in [0, 1].$$

A function that is quasiconvex and quasiconcave at the same time is called quasilinear.

Proposition 3.2.4. *Let X be a real linear space, let $D \subseteq X$ convex and let $f : D \longrightarrow \mathbb{R}$ be a function. Then f is of type ql if and only if f is quasilinear.*

Next we give a method to obtain new operators of type ql from existing ones.

Proposition 3.2.5. *Let X, Y, Z be real linear spaces, $D \subseteq X$, and let $A : D \longrightarrow Y$, $B : A(D) \longrightarrow Z$ be two operator of type ql. Then $B \circ A : D \longrightarrow Z$ is of type ql.*

At this point is time to give some examples of operators of type ql, that are nontrivial in the sense that are neither linear operators nor quasilinear functions. The first one provides an operator of type ql in finite dimension.

Example 3.2.1. Let us consider the operator $A : [-1, 1] \times [-1, 1] \longrightarrow \mathbb{R}^3$,

$$A(x, y) = \left(\frac{2x + 2y}{(x + y - 1)^2 + 3}, \frac{(x + y)^2 + 4}{(x + y - 1)^2 + 3}, \frac{(x + y - 2)^2}{(x + y - 1)^2 + 3} \right).$$

Then A is a continuous operator of type ql.

The next example provides an operator of type ql in a general infinite dimensional setting.

Example 3.2.2. Let $D = \{f \in C_{[a,b]} | f(a) \geq 0\} \subseteq C_{[a,b]}$ and consider the operator $S : D \longrightarrow \mathbb{R}^{\mathbb{R}}$, $S(f)(x) = (f(a))^2 x$. Then S is a nonlinear operator of type ql.

Definition 3.2.4. *Let X be a real linear space and let $D \subseteq X$. The convex hull of the set D is defined as the set*

$$co(D) = \left\{ \sum_{i=1}^{n} \lambda_i x_i : x_i \in D, \sum_{i=1}^{n} \lambda_i = 1, \lambda_i \geq 0, \text{ for all } i \in \{1, 2, \ldots, n\}, n \in \mathbb{N} \right\}.$$

We have the following result:

Theorem 3.2.2. *Let X and Y be two real linear spaces, let $D \subseteq X$ be convex and let $A : D \longrightarrow Y$ be an operator of type ql. Then for every finite number of elements $x_1, x_2, \ldots, x_n \in D$ and for every $x \in co\{x_1, x_2, \ldots, x_n\}$ we have $A(x) \in co\{A(x_1), A(x_2), \ldots, A(x_n)\}$.*

3.3 Existence of the Solutions of Some Generalized Variational Inequalities

3.3.1 Stampacchia type variational inequalities

In this section will be presented some existence results for the general variational inequality of Stampacchia type, that was introduced in Section 4.2. Our results are relying on the notion of KKM application and a celebrated result due to Ky Fan. By examples is shown, that the condition that one of the operators involved in these variational inequalities is of type ql is essential. As consequences of the presented results, some well known classical results are obtained.

Recall that an operator $T : X \longrightarrow X^*$ is called weak to $\|\cdot\|$-sequentially continuous at $x \in X$, if for every sequence x_n that converge weakly to x we have that $T(x_n) \longrightarrow T(x)$ in the topology of the norm of X^*. If the range of T is a subset of X, we say that T is weak to weak-sequentially continuous at $x \in X$, if for every sequence x_n that converge weakly to x we have that $T(x_n)$ converge weakly to $T(x)$. One of the main results of this section is the following theorem.

Theorem 3.3.1. *If A is weak to $\|\cdot\|$-sequentially continuous, a is of type ql and weak to weak-sequentially continuous and K is weakly compact and convex, then $VI_S(A, a, K)$ admits solutions.*

As an immediate consequence we obtain the following result.

Corollary 3.3.1. If A is weak to $\|\cdot\|$-sequentially continuous and K is weakly compact and convex, then Stampacchia variational inequality, $VI_S(A, K)$, admits solutions.

In finite dimension we obtain the following result, which is a generalization of Lemma 3.1 from [131].

Corollary 3.3.2. If $X = \mathbb{R}^n$, A is continuous, a is of type ql and continuous, and K is compact and convex, then $VI_S(A, a, K)$ admits solutions.

The following classic result, (see [57, 74]), is well known.

Corollary 3.3.3. If $X = \mathbb{R}^n$, A is continuous, and K is compact and convex, then Stampacchia variational inequality, $VI_S(A, K)$, admits solutions.

The next result extends Theorem 3.2 from [131].

Theorem 3.3.2. *Let X be a Banach space and let $K \subseteq X$ be weakly compact and convex. Suppose that the following conditions are satisfied:*

(a) a of type ql,

(b) if $(x_n) \subseteq K$ converge weakly to $x \in K$ then $\liminf_{n \longrightarrow \infty} \langle Ax_n, a(y) \rangle \leq \langle A(x), a(y) \rangle$, for all $y \in K$,

(c) the function $K \longrightarrow \mathbb{R}$, $x \longrightarrow \langle A(x), a(x) \rangle$ is sequentially weakly lower semicontinuous.

Then, $VI_S(A, a, K)$ admits solutions.

The next example shows, that without the assumption that the operator a being of type ql, the conclusion of Theorem 3.3.1 fails even in finite dimension.

Example 3.3.1. Let us consider the operator $A : K \longrightarrow \mathbb{R}^2$, $A(x, y) = (1, -x)$, where $K = [-1, 1] \times [-1, 1] \subseteq \mathbb{R}^2$ and let $a : K \longrightarrow K$, $a(x, y) = \left(x^2 y, xy\right)$. Then obviously A and a are continuous, K is compact and convex but the general variational inequality problem of Stampacchia type has no solutions.

3.3.2 Minty type variational inequalities

In this section, some generalizations of Minty's classical theorem concerning on the coincidence of the solutions of Stampacchia variational inequality, respectively of Minty variational inequality are obtained.

Recall that Minty's theorem claims that if the operator $A : K \longrightarrow X^*$ is hemicontinuous and monotone in Minty-Browder sense then the solutions of the classical Stampacchia variational inequality $VI_S(A, K)$ and the solutions of the classical Minty variational inequality $VI_M(A, K)$ coincide (see [43, 87]). Actually we need to assume less.

Theorem 3.3.3. *(Minty) Let $A : K \longrightarrow X^*$ be an operator.*

i) If A is hemicontinuous on K, and K is convex, then every $x \in K$ which solves $VI_M(A, K)$ is also a solution of $VI_S(A, K)$.

ii) If, instead, A is monotone on the convex set K, then every $x \in K$ which solves $VI_S(A, K)$ is also a solution of $VI_M(A, K)$

Recall the following definitions (see [101]):

Definition 3.3.1. *Let X be a real Banach space, let X^* be its dual, and let $A : D \subseteq X \longrightarrow X^*$ and $a : D \longrightarrow X$ be given operators. One says that A is monotone relative to a, if for all $x, y \in D$, we have $\langle A(x) - A(y), a(x) - a(y) \rangle \geq 0$.*

In what follows we obtain some results for the problems $VI_S(A, a, K)$ and $VI_M(A, a, K)$, that may be viewed as generalizations of Minty's theorem.

Theorem 3.3.4. *Let $K \subseteq X$ be a convex set, and let $A : K \longrightarrow X^*$ and $a : K \longrightarrow X$ be given operators. Then, the following statements hold.*

i) If A is monotone relative to a on K, then every $x \in K$ which solves $VI_S(A,a,K)$ is also a solution of $VI_M(A,a,K)$.

ii) If A is hemicontinuous and a is of type strict ql, then every $x \in K$ which solves $VI_M(A,a,K)$ is also a solution of $VI_S(A,a,K)$.

The condition that the operator a is of type strict ql in the hypothesis of Theorem 3.3.4 "*ii)*" is essential. The author wishes to thank professor C. Zălinescu, for pointing out this fact.

Example 3.3.2. Let $K = [-1,1]$, $A, a : K \longrightarrow \mathbb{R}$, $A(x) = x$, $a(x) = \begin{cases} -1, \text{ if } x \in [-1,0), \\ 1, \text{ if } x \in [0,1]. \end{cases}$ Then K is convex, A is continuous, a is of type ql, but is not of type strict ql. We show that $x_0 = \dfrac{1}{2}$ is a solution of $VI_M(A,a,K)$ but is not solution of $VI_S(A,a,K)$.

Proof. Indeed $A(y) \cdot (a(y) - a(x_0)) = y(a(y) - 1) = \begin{cases} -2y, \text{ if } y \in [-1,0), \\ 0, \text{ if } y \in [0,1]. \end{cases}$ Hence, $A(y) \cdot (a(y) - a(x_0)) \geq 0$ for all $y \in [-1,1]$, consequently x_0 is a solution of $VI_M(A,a,K)$.

On the other hand, for $y = -\dfrac{1}{2}$, we have $A(x_0)(a(y) - a(x_0)) = -1 < 0$, consequently x_0 is not a solution of $VI_S(A,a,K)$. □

3.3.3 The inverted problems

In what follows we conclude similar result for the inverted problems $VI_{iS}(A,a,K)$ and $VI_{iM}(A,a,K)$.

The following existence result for the problem $VI_{iS}(A,a,K)$ can be proved in a similar way to the proof of Theorem 3.3.1.

Theorem 3.3.5. *If A is weak to norm and a is weak to weak continuous, A is of type ql and K is weakly compact, then the inverted general variational inequality problem of Stampacchia type $VI_{iS}(A,a,K)$ admits solutions.*

We have the following Minty type theorem.

Theorem 3.3.6. *i) Let $A : K \longrightarrow X^*$ be monotone relative to a. If $x \in K$ is a solution of the inverted general variational inequality problem of Stampacchia type $VI_{iS}(A,a,K)$, then x is a solution of the inverted general variational inequality problem of Minty type $VI_{iM}(A,a,K)$.*

ii) Let $A : K \longrightarrow X^*$ be of type strict ql and let a be continuous on line segments. If $x \in K$ is a solution of the inverted general variational inequality problem of Minty type $VI_{iM}(A,a,K)$, then x is a solution of the inverted general variational inequality problem of Stampacchia type $VI_{iS}(A,a,K)$.

3.3.4 Multivalued variational inequalities

In this section, unless is otherwise specified, we assume that X be a real Banach space and X^* be the topological dual of X. Let $K \subseteq X$ be convex, and let $T : K \rightrightarrows X^*$ and $f : K \longrightarrow X$ be given operators. First, we shall study the problem of finding an element $x \in K$, such that

$$\sup_{u \in T(x)} \langle u, f(y) - f(x) \rangle \geq 0, \text{ for all } y \in K. \tag{3. 8}$$

In order to continue our analysis we need the following notion. Let X_1, X_2 be Hausdorff topological spaces and let $T : X_1 \rightrightarrows X_2$ be a multivalued operator with nonempty values. T is said to be upper semicontinuous if, for every $x_0 \in X_1$ and for every open set N containing $T(x_0)$, there exists a neighborhood M of x_0 such that $T(M) \subseteq N$.

Theorem 3.3.7. *Let $K \subseteq X$ be nonempty, weakly compact and convex, and let $f : K \longrightarrow K$ be of type ql and weak to norm-sequentially continuous. Let $T : K \rightrightarrows X^*$ weak to weak* upper semicontinuous on K, such that $T(x)$ is nonempty, weak* compact for every $x \in K$. Then, $S_w(T, f, K) \neq \emptyset$. If in addition T is f-pseudomonotone, then $M(T, f, K) \neq \emptyset$.*

In what follows, we present the main result of this section.

Theorem 3.3.8. *Let $K \subseteq X$ be nonempty, weakly compact and convex, and let $f : K \longrightarrow K$ be of type ql and weak to norm continuous. Let $T : K \rightrightarrows X^*$ weak to weak* upper semicontinuous on K, such that $T(x)$ is nonempty, weak* compact and convex for every $x \in K$. Then, $S(T, f, K) \neq \emptyset$.*

3.4 Applications to fixed point theorems

In this section we prove Brouwer's fixed point theorem, respectively Kakutani fixed point theorem. Recall, that Brouwer's fixed point theorem states, that if $F : K \longrightarrow K$ is a continuous function, where $K \subseteq \mathbb{R}^n$ is a compact and convex, then F admits a fixed point, that is there exists $x \in K$ such that $F(x) = x$, (see, for instance, [61]).

Now, according to Theorem 3.3.5, if A is weak to norm sequentially continuous and is of type ql, a is weak to weak sequentially continuous, and K is weakly compact and convex, then the inverted general variational inequality of Stampacchia type $VI_{iS}(A, a, K)$ admits solutions. Let $K \subseteq \mathbb{R}^n$ compact and convex, $A : K \longrightarrow K$, $A \equiv id_K$ and $a : K \longrightarrow \mathbb{R}^n$, $a(x) = x - F(x)$. Obviously A and a are continuous. Hence, the assumptions of Theorem 3.3.5 are satisfied, consequently, there exists $x_0 \in K$ such that

$$\langle y - x_0, x_0 - F(x_0)\rangle \geq 0, (\forall) y \in K.$$

Since $Im(F) \subseteq K$, for $y = F(x_0) \in K$ we obtain $\langle F(x_0) - x_0, x_0 - F(x_0)\rangle \geq 0$. Hence, $-\|F(x_0) - x_0\|^2 \geq 0$, so we have $F(x_0) = x_0$.

In what follows we give a similar proof for Kakutani's fixed point theorem. According to Theorem 3.3.8, in a Banach space context X, if $K \subseteq X$ be nonempty, weakly compact and convex, and $f : K \longrightarrow K$ is of type ql and weak to norm continuous, respectively $T : K \rightrightarrows X^*$ is weak to weak* upper semicontinuous on K, such that $T(x)$ is nonempty, weak* compact and convex for every $x \in K$, then, $S(T, f, K) \neq \emptyset$, i.e. there exists $x \in K$, $u \in T(x)$ such that

$$\langle u, f(y) - f(x)\rangle \geq 0, \ (\forall) y \in K.$$

Recall, that Kakutani's fixed point theorem states, that if $F : K \rightrightarrows K$ is a a set valued map with closed graph and the property that $F(x)$ is non-empty and convex for all $x \in K$, where $K \subseteq \mathbb{R}^n$ is a compact and convex, then F has a fixed point, i.e. there exists $x \in K$ such that $x \in F(x)$. However this statement is equivalent to the following. Let K be a non-empty, compact and convex subset of some Euclidean space $\mathbb{R}^n$. Let $F : K \rightrightarrows K$ be an upper semicontinuous set-valued map on K with the property that $F(x)$ is non-empty, closed, and convex for all $x \in K$. Then F has a fixed point.

Let $T : K \rightrightarrows K$, $T(x) = id_K(x) - F(x)$, which is obviously nonempty, convex and closed valued, and since $T(x) \subseteq K - K$, we have $T(x)$ bounded for every $x \in K$. Hence $T(x)$ is compact for all $x \in K$. We show that T is also upper semicontinuous. Indeed, according to Remark **??**, it is enough to show, that for every convergent sequence $(x_n) \subseteq K$, $x_n \longrightarrow x^0 \in K$ and every sequence $z_n \in T(x_n)$, $n = 1, 2, ...$, there exists $z^0 \in T(x^0)$, and a subsequence $(z_{n_k}) \subseteq (z_n)$ such that $z_{n_k} \longrightarrow z^0$, $k \longrightarrow \infty$.

Let $(x_n) \subseteq K$, $x_n \longrightarrow x^0 \in K$ and let $z_n \in T(x_n)$, $n \geq 1$. Then $z_n = x_n - y_n$, where $y_n \in F(x_n)$ for all $n = 1, 2, ...$. Since F is compact valued and upper semicontinuous, according to Remark **??** there exists $y^0 \in F(x^0)$, and a subsequence $(y_{n_k}) \subseteq (y_n)$ such that $y_{n_k} \longrightarrow y^0$, $k \longrightarrow \infty$. But then $z_{n_k} \longrightarrow x^0 - y^0 \in T(x^0)$, $k \longrightarrow \infty$.

Consider further $f : K \longrightarrow K$, $f \equiv id_K$, which is obviously is of type ql and continuous, therefore K, T and f satisfy the assumptions of Theorem 3.3.8. Hence, according to Theorem 3.3.8 there exists $x_0 \in K$, $u \in T(x_0)$ such that

$$\langle u, y - x_0 \rangle \geq 0,\ (\forall) y \in K.$$

But then $u = x_0 - v_0$, for some $v_0 \in F(x_0) \subseteq K$, hence for $y = v_0$ we obtain $\langle x_0 - v_0, v_0 - x_0 \rangle \geq 0$. Thus $x_0 = v_0 \in F(x_0)$.

Chapter 4

The sum problems in Banach spaces

4.1 Preliminaries

4.1.1 Interiority notions and Fenchel conjugate

Consider X a separated locally convex space and X^* its topological dual space. We denote by $w(X, X^*)$ ($w(X^*, X)$) the weak topology on X induced by X^* (the weak* topology on X^* induced by X). When there is no danger of confusion, the notation w (w^*) is used. For a non-empty set $D \subseteq X$, we denote by $\mathrm{co}(D), \mathrm{cone}(D), \mathrm{coneco}(D), \mathrm{aff}(D), \mathrm{lin}(D), \mathrm{int}(D), \mathrm{cl}(D)$, its *convex hull, conic hull, convex conic hull, affine hull, linear hull, interior, relative interior*, and *closure*, respectively. We have $\mathrm{cone}(D) = \cup_{t \geq 0} tD$ and if $0 \in D$ then obviously $\mathrm{cone}(D) = \cup_{t>0} tD$.

The *algebraic interior* (the *core*) of D is the set (see [60, 113, 133])

$$\mathrm{core}(D) = \{u \in X |\ \forall x \in X,\ \exists \delta > 0 \text{ such that } \forall \lambda \in [0, \delta] : u + \lambda x \in D\},$$

while its *relative algebraic interior* (sometimes called also *intrinsic core*) is the set (see [60, 133])

$$\mathrm{icr}(D) = \{u \in X |\ \forall x \in \mathrm{aff}(D - D),\ \exists \delta > 0 \text{ such that } \forall \lambda \in [0, \delta] : u + \lambda x \in D\}.$$

We consider also the *strong quasi-relative interior* (sometimes called *intrinsic relative algebraic interior*) of D (see [13, 64, 133, 134]), denoted by $\mathrm{sqri}(D)$,

$$\mathrm{sqri}(D) = \begin{cases} \mathrm{icr}(D), & \text{if } \mathrm{aff}(D) \text{ is a closed set,} \\ \emptyset, & \text{otherwise.} \end{cases}$$

Let us consider non-empty sets $V \subseteq X$. By $U + V$ we denote the usual *Minkowski sum* of the sets $U, V \subseteq X$, that is $U + V = \{u + v : u \in U, v \in V\}$, while for $\alpha \in \mathbb{R}$, $\alpha U = \{\alpha x : x \in U\}$. By convention we take $U + \emptyset = \emptyset + U = \emptyset + \emptyset = \alpha\emptyset = \emptyset$.

We say that the function $f : X \longrightarrow \overline{\mathbb{R}}$ is convex if

$$\forall x, y \in X,\ \forall t \in [0, 1]\ : f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y),$$

with the conventions $(+\infty) + (-\infty) = +\infty$, $0 \cdot (+\infty) = +\infty$ and $0 \cdot (-\infty) = 0$ (see [133]). We consider $\mathrm{dom}\, f = \{x \in X : f(x) < +\infty\}$ the *domain* of f and $\mathrm{epi}\, f = \{(x, r) \in X \times \mathbb{R} : f(x) \leq r\}$ its *epigraph*. We call f *proper* if $\mathrm{dom}\, f \neq \emptyset$ and $f(x) > -\infty$ for all $x \in X$. By $\mathrm{cl}\, f$ we denote the *lower semicontinuous hull* of f, namely the function whose epigraph is the closure of $\mathrm{epi}\, f$ in

$X \times \mathbb{R}$, that is $\operatorname{epi}(\operatorname{cl} f) = \operatorname{cl}(\operatorname{epi} f)$. We consider also $\operatorname{co} f$, the *convex hull* of f, which is the greatest convex function majorized by f.

The *Fenchel-Moreau conjugate* of f is the function $f^* : X^* \longrightarrow \overline{\mathbb{R}}$ defined by

$$f^*(x^*) = \sup_{x \in X} \{\langle x^*, x\rangle - f(x)\} \ \forall x^* \in X^*.$$

We mention here some important properties of conjugate functions. We have the so-called *Young-Fenchel inequality*

$$f^*(x^*) + f(x) \geq \langle x^*, x\rangle \ \forall x \in X \ \forall x^* \in X^*.$$

The *Fenchel-Moreau Theorem* states that if f is proper, then f is convex and lower semicontinuous if and only if $f^{**} = f$ (see [133]). Moreover, if f is convex and $(\operatorname{cl} f)(x) > -\infty$ for all $x \in X$, then $f^{**} = \operatorname{cl} f$ (see [133, Theorem 2.3.4]).

Having $f, g : X \longrightarrow \overline{\mathbb{R}}$ two functions we consider their *infimal convolution*, namely the function denoted by $f \Box g : X \longrightarrow \overline{\mathbb{R}}$, $f \Box g(x) = \inf_{u \in X}\{f(u) + g(x-u)\}$ for all $x \in X$. We say that the infimal convolution is *exact at* $x \in X$ if the infimum in its definition is attained. Moreover, $f \Box g$ is said to be *exact* if it is exact at every $x \in X$. We refer to [95] for more properties of the infimal convolution operation.

Let us also note that everywhere within this chapter we write min (max) instead of inf (sup) when the infimum (supremum) is attained.

Consider Y another separated locally convex space. For a function $h : X \longrightarrow Y$ we denote by $Im(h) = \{h(u) : u \in U\}$ the *image* of the set $U \subseteq X$ through h, while for $D \subseteq Y$ we use the notation $h^{-1}(D) = \{x \in X : h(x) \in D\}$. Given a continuous linear mapping $A : X \longrightarrow Y$, its *adjoint operator*, $A^* : Y^* \longrightarrow X^*$ is defined by $\langle A^*y^*, x\rangle = \langle y^*, Ax\rangle$ for all $y^* \in Y^*$ and $x \in X$.

4.1.2 Maximal monotone operators and representative functions

Consider further X a nontrivial Banach space, X^* its topological dual space and X^{**} its bidual space. A set-valued operator $S : X \rightrightarrows X^*$ is said to be *monotone* if

$$\langle y^* - x^*, y - x\rangle \geq 0, \text{ whenever } y^* \in S(y) \text{ and } x^* \in S(x).$$

The monotone operator S is called *maximal monotone* if its graph

$$G(S) = \{(x, x^*) : x^* \in S(x)\} \subseteq X \times X^*$$

is not properly contained in the graph of any other monotone operator $S' : X \rightrightarrows X^*$. For S we consider also its *domain* $D(S) = \{x \in X : S(x) \neq \emptyset\} = \operatorname{pr}_X(G(S))$ and its *range* $R(S) = \cup_{x \in X} S(x) = \operatorname{pr}_{X^*}(G(S))$.

Definition 4.1.1. *For $S : X \rightrightarrows X^*$ a monotone operator, we call* representative function *of S a convex and lower semicontinuous function $h_S : X \times X^* \longrightarrow \overline{\mathbb{R}}$ (in the strong topology of $X \times X^*$) fulfilling*

$$h_S \geq c \text{ and } G(S) \subseteq \{(x, x^*) \in X \times X^* : h_S(x, x^*) = \langle x^*, x\rangle\}.$$

Theorem 4.1.1. *Let X be a nonzero Banach space and $f : X \times X^* \longrightarrow \overline{\mathbb{R}}$ a proper, convex and lower semicontinuous function such that $f \geq c$ and $f^*(x^*, x^{**}) \geq \langle x^{**}, x^*\rangle$ for all $(x^*, x^{**}) \in X^* \times X^{**}$. Then the operator whose graph is the set $\{(x, x^*) \in X \times X^* : f(x, x^*) = \langle x^*, x\rangle\}$ is maximal monotone and it holds $\{(x, x^*) \in X \times X^* : f(x, x^*) = \langle x^*, x\rangle\} = \{(x, x^*) \in X \times X^* : f^*(x^*, x) = \langle x^*, x\rangle\}$.*

The following particular class of maximal monotone operators has been recently introduced in [88], being also studied in [129].

Definition 4.1.2. *An operator $S : X \rightrightarrows X^*$ is said to be* strongly-representable *whenever there exists a proper, convex and strong lower semicontinuous function $h : X \times X^* \longrightarrow \overline{\mathbb{R}}$ such that*

$$h \geq c, h^*(x^*, x^{**}) \geq \langle x^{**}, x^* \rangle \forall (x^*, x^{**}) \in X^* \times X^{**}$$

and

$$G(S) = \{(x, x^*) \in X \times X^* : h(x, x^*) = \langle x^*, x \rangle\}.$$

In this case h is called a strong-representative *of S.*

Definition 4.1.3. *(*see [51]*) Gossez's monotone closure of a maximal monotone operator $S : X \rightrightarrows X^*$ is $\overline{S} : X^{**} \rightrightarrows X^*$,*

$$G(\overline{S}) = \{(x^{**}, x^*) \in X^{**} \times X^* : \langle x^* - y^*, x^{**} - y \rangle \geq 0, (\forall)(y, y^*) \in G(S)\}.$$

A maximal monotone operator $S : X \rightrightarrows X^*$ is of Gossez type (D) if for any $(x^{**}, x^*) \in G(\overline{S})$, there exists a bounded net $\{(x_\alpha, x^*_\alpha)\}_{\alpha \in \mathfrak{I}} \subseteq G(S)$ which converges to (x^{**}, x^*) in the $w^* \times \|\cdot\|$ topology of $X^{**} \times X^*$.

In [121] Simons introduced a new class of maximal monotone operators, called operators of negative infimum type (NI).

Definition 4.1.4. *(*see [121]*) A maximal monotone operator $S : X \rightrightarrows X^*$ is of Simons type (NI) if*

$$\inf_{(y,y^*) \in G(S)} \langle y^* - x^*, y - x^{**} \rangle \geq 0, (\forall)(x^*, x^{**}) \in X^* \times X^{**}.$$

Remark 4.1.1. Marques Alves and Svaiter recently proved that the class of strongly-representable operators, the class of maximal monotone operators of type (NI) and the class of maximal monotone operators of Gossez type (D) coincide (cf. [89, Theorem 1.2] and [90, Theorem 4.4]).

4.2 About some stable strong duality problems

4.2.1 Conjugate duality

Let V be a separated locally convex space and $F : V \longrightarrow \overline{\mathbb{R}}$ a proper function. Recall that the optimization problem

$$(PG) : \inf_{v \in V} F(v)$$

is called the primal problem (see [15]).

Consider W an other separated locally convex space, and the function $\Phi : V \times W \longrightarrow \overline{\mathbb{R}}$ satisfying $\Phi(v, 0) = F(v)$ for all $v \in V$. The function Φ is called perturbation function. Then, (PG) can be written as

$$(PG) : \inf_{v \in V} \Phi(v, 0).$$

Let V^* and W^* be the topological dual spaces of V and W and assume that V^* and W^*, respectively, are endowed with the weak* ($w(V^*, V)$, respectively, $w(W^*, W)$) topology. The conjugate dual problem of (PG) can be formulated as

$$(DG) : \sup_{w^* \in W^*} -\Phi^*(0, w^*).$$

Let us denote by $v(PG)$ and $v(DG)$ the optimal values of the problem (PG) and (DG), respectively. We say that for the optimization problems (PG) and (DG) strong duality holds, if $v(PG) = v(DG)$ and the dual (DG) has an optimal solution.

For every $v^* \in V^*$ consider the extension of the primal problem (PG)

$$(PG^{v^*}): \inf_{v \in V} \{\Phi(v,0) - \langle v^*, v\rangle\}.$$

Its conjugate dual is given by

$$(DG^{v^*}): \sup_{w^* \in W^*} -\Phi^*(v^*, w^*).$$

We say that for the optimization problems (PG) and (DG) stable strong duality holds, if for all $v^* \in V^*$ for (PG^{v^*}) and (DG^{v^*}) strong duality holds, i.e. $v(PG^{v^*}) = v(DG^{v^*})$ and the dual (DG^{v^*}) has an optimal solution.

In what follows assume that Φ is proper and convex and $0 \in \mathrm{pr}_W(\mathrm{dom}\,\Phi)$. In [15] is considered the following regularity condition:

(RC_2^Φ) : V and W are Fréchet spaces, Φ is lower semicontinuous and $0 \in \mathrm{sqri}(\mathrm{pr}_W(\mathrm{dom}\,\Phi))$.

In [15] was established the following result.

Theorem 4.2.1. *(Theorem 5.5, [15]) Let $\Phi : V \times W \longrightarrow \overline{\mathbb{R}}$ be a proper and convex function such that $0 \in \mathrm{pr}_W(\mathrm{dom}\,\Phi)$. If the regularity conditions (RC_2^Φ) is fulfilled then for (PG) and (DG) stable strong duality holds, i.e.*

$$\sup_{v \in V} \{\langle v^*, v\rangle - \Phi(v,0)\} = \min_{w^* \in W^*} \Phi^*(v^*, w^*) \,\forall v^* \in V^*.$$

In [16] it has been given a closedness type condition, which is equivalent to stable strong duality in case Φ is proper, convex and lower semicontinuous.

Theorem 4.2.2. *(Theorem 2, [16]) Let $\Phi : V \times W \longrightarrow \overline{\mathbb{R}}$ be proper, convex, lower semicontinuous such that $0 \in \mathrm{pr}_W(\mathrm{dom}(\Phi))$ and let $U \subseteq V^*$. The following conditions are equivalent:*

(i) $(\Phi(\cdot,0))^*(v^*) = \sup_{v \in V}\{\langle v^*, v\rangle - \Phi(v,0)\} = \min_{w^* \in W^*} \Phi^*(v^*, w^*)$, *for all* $v^* \in U$.

(ii) $(CQ^\Phi)(U) : \mathrm{pr}_{V^* \times \mathbb{R}}(epi(\Phi^*))$ *is closed regarding* $U \times \mathbb{R}$ *in* $(V^*, w^*) \times \mathbb{R}$ *topology.*

4.2.2 Fenchel duality

Let X be a real, separated, locally convex space, with its dual X^* endowed with the w^* topology, and consider the proper, convex and lower semicontinuous functions $f, g : X \longrightarrow \overline{\mathbb{R}}$. Moreover, assume that $\mathrm{dom}(f) \cap \mathrm{dom}(g) \neq \emptyset$. Consider the following primal problem.

$$(P): \inf_{x \in X} \{(f+g)(x)\}.$$

The dual of (P) is

$$(D): \sup_{y^* \in Y^*} \{-f^*(y^*) - g^*(-y^*)\}.$$

For every $x^* \in X^*$ consider the extension of the primal optimization problem (P^G)

$$(P^{x^*}): -\sup_{x \in X} \{\langle x^*, x\rangle - (f+g)(x)\},$$

and its dual

$$(D^{x^*}): \ -\inf_{y^*\in X^*}\{f^*(y^*)+g^*(x^*-y^*)\}.$$

Theorem 4.2.3. *Let U be a nonempty subset of X^*. The following statements are equivalent:*

(i) $\sup_{x\in X}\{\langle x^*,x\rangle-(f+g)(x)\}=\min_{y^*\in X^*}\{f^*(y^*)+g^*(x^*-y^*)\}$ *for all* $x^*\in U$.

(ii) $(CQ)(U)$: $\{(x^*+y^*,r): f^*(x^*)+g^*(y^*)\le r\}$ *is closed regarding to* $U\times\mathbb{R}$ *in* $(X^*,w^*)\times\mathbb{R}$ *topology.*

We have the following interior point type regularity condition:

(RC_2) : X is a Fréchet space and $0\in \operatorname{sqri}(\operatorname{dom}(f)-\operatorname{dom}(g))$. We have the following result.

Theorem 4.2.4. *If f the regularity condition (RC_2) is fulfilled then*

$$\sup_{x\in X}\{\langle x^*,x\rangle-(f+g)(x)\}=\min_{y^*\in Y^*}\{f^*(y^*)+g^*(x^*-y^*)\},\quad \textit{for all } x^*\in X^*.$$

4.2.3 Stable strong duality for the problem having the composition with a linear continuous operator in the objective function

Let X,Y be real separated locally convex spaces, with their dual X^* and Y^*, respectively, endowed with the w^* topology, and consider the proper, convex and lower semicontinuous functions $f: X\longrightarrow\overline{\mathbb{R}}$ and $g: Y\longrightarrow\overline{\mathbb{R}}$. Moreover, let $A: Y\longrightarrow X$, respectively, $B: X\longrightarrow Y$ be two linear and continuous operators such that $A^{-1}(\operatorname{dom}(f))\cap\operatorname{dom}(g)\neq\emptyset$, respectively, $\operatorname{dom}(f)\cap B^{-1}(\operatorname{dom}(g))\neq\emptyset$. Consider the following primal problems.

$$(P^A): \ \inf_{y\in Y}\{(f\circ A+g)(y)\},$$

respectively,

$$(P^B): \ \inf_{x\in X}\{(f+g\circ B)(x)\}.$$

It can be easily observed that these two formulations are actually equivalent, (by changing X with Y and f with g), but we prefer to simultaneously consider both formulations, since we will make use later of both forms.

The dual problems of (P^A), and of (P^B) are

$$(D^A): \ \sup_{x^*\in X^*}\{-f^*(x^*)-g^*(-A^*y^*)\},$$

and

$$(D^B): \ \sup_{y^*\in Y^*}\{-f^*(-B^*y^*)-g^*(y^*)\},$$

respectively. For every $x^*\in X^*$, respectively, $y^*\in Y^*$ consider the extensions of the primal optimization problems

$$(P^{A^{y^*}}): \ -\sup_{y\in Y}\{\langle y^*,y\rangle-(f\circ A+g)(y)\},$$

respectively,

$$(P^{B^{x^*}}): \ -\sup_{x\in X}\{\langle x^*,x\rangle-(f+g\circ B)(x)\}.$$

Their duals are

$$(D^{A^{y^*}}): \ -\inf_{x^*\in X^*}\{f^*(x^*)+g^*(y^*-A^*x^*)\},$$

respectively,

$$(D^{B^{x^*}}): \ -\inf_{y^*\in Y^*}\{f^*(x^*-B^*y^*)+g^*(y^*)\}.$$

Theorem 4.2.5. *Let U be a nonempty subset of Y^*. The following statements are equivalent:*

(i) $\sup_{y\in Y}\{\langle y^*,y\rangle-(f\circ A+g)(y)\}=\min_{x^*\in X^*}\{f^*(x^*)+g^*(y^*-A^*x^*)\}$ *for all* $y^*\in U$.

(ii) $(CQ^{\Phi_A})(U):\{(A^*x^*+y^*,r):f^*(x^*)+g^*(y^*)\leq r\}$ *is closed regarding to* $U\times\mathbb{R}$ *in* $(Y^*,w^*)\times\mathbb{R}$ *topology.*

Theorem 4.2.6. *Let U be a nonempty subset of X^*. The following statements are equivalent:*

(i) $\sup_{x\in X}\{\langle x^*,x\rangle-(f+g\circ B)(x)\}=\min_{y^*\in Y^*}\{f^*(x^*-B^*y^*)+g^*(y^*)\}$ *for all* $x^*\in U$.

(ii) $(CQ^{\Phi_B})(U):\{(x^*+B^*y^*,r):f^*(x^*)+g^*(y^*)\leq r\}$ *is closed regarding to* $U\times\mathbb{R}$ *in* $(X^*,w^*)\times\mathbb{R}$ *topology.*

We have the following regularity conditions:

$(RC_2^{\Phi_A})$: X and Y are Fréchet spaces and $0\in\operatorname{sqri}(\operatorname{dom}(f)-A(\operatorname{dom}(g)))$, respectively,

$(RC_2^{\Phi_B})$: X and Y are Fréchet spaces and $0\in\operatorname{sqri}(\operatorname{dom}(g)-B(\operatorname{dom}(f)))$. We have the following result.

Theorem 4.2.7. *If one of the regularity conditions* $(RC_2^{\Phi_A}),\}$, *respectively,* $(RC_2^{\Phi_B})$, *is fulfilled then*

$$\sup_{y\in Y}\{\langle y^*,y\rangle-(f\circ A+g)(y)\}=\min_{x^*\in X^*}\{f^*(x^*)+g^*(y^*-A^*x^*)\}\ \textit{for all}\ y^*\in Y^*,$$

respectively,

$$\sup_{x\in X}\{\langle x^*,x\rangle-(f+g\circ B)(x)\}=\min_{y^*\in Y^*}\{f^*(x^*-B^*y^*)+g^*(y^*)\}\ \ \textit{for all}\ x^*\in X^*.$$

4.2.4 Stable strong duality for the problem having the sum of two functions each composed with a linear continuous operator in the objective function

Let X,Y,Z be real separated locally convex spaces, with their duals X^*, Y^* and Z^*, respectively, endowed with the w^* topology, and consider the proper, convex and lower semicontinuous functions $f:X\longrightarrow\overline{\mathbb{R}}$ and $g:Y\longrightarrow\overline{\mathbb{R}}$. Moreover, let $A:Z\longrightarrow X$ and $B:Z\longrightarrow Y$ be two linear and continuous operators such that $A^{-1}(\operatorname{dom}(f))\cap B^{-1}(\operatorname{dom}(g))\neq\emptyset$. Consider the following primal problem.

$$(P^{AB}):\ \inf_{z\in Z}\{(f\circ A+g\circ B)(z)\}.$$

We obtain, the dual of (P^{AB}) being

$$(D^{AB}):\ \sup_{(x^*,y^*)\in X^*\times Y^*}\{-f^*(x^*)-g^*(y^*):A^*x^*+B^*y^*=0\}.$$

For every $z^*\in Z^*$ consider the extension of the primal optimization problem

$$(P^{AB^{z^*}}):\ -\sup_{z\in Z}\{\langle z^*,z\rangle-(f\circ A+g\circ B)(z)\},$$

and its dual

$$(D^{AB^{z^*}}):\ -\inf_{(x^*,y^*)\in X^*\times Y^*}\{f^*(x^*)+g^*(y^*):A^*x^*+B^*y^*=z^*\}$$

Consider the regularity condition

$(CQ^{\Phi_{AB}})(U):\{(A^*x^*+B^*y^*,r):f^*(x^*)+g^*(y^*)\leq r\}$ is closed regarding $U\times\mathbb{R}$ in $(Z^*,w^*)\times\mathbb{R}$ topology.

We have the following theorem.

Theorem 4.2.8. *The following conditions are equivalent.*

(i) $(CQ^{\Phi_{AB}})(U)$ *is fulfilled.*

(ii) For all $z^* \in U$ *we have* $(f\circ A+g\circ B)^*(z^*) = \inf_{(x^*,y^*)\in X^*\times Y^*}\{f^*(x^*)+g^*(y^*) : A^*x^*+B^*y^* = z^*\}$ *and the infimum is attained.*

We have the following interior point type regularity condition:
$(RC_2^{\Phi_{AB}})$: Z, X and Y are Fréchet spaces and
$(0,0) \in \operatorname{sqri}(\operatorname{dom}(f) \times \operatorname{dom}(g) - (A \times B)(\Delta_Z))$. We have the following result.

Theorem 4.2.9. *If the regularity condition* $(RC_2^{\Phi_{AB}})$ *is fulfilled then*

$$\sup_{z\in Z}\{\langle z^*, z\rangle - (f\circ A + g\circ B)(z)\} = \min_{(x^*,y^*)\in X^*\times Y^*}\{f^*(x^*) + g^*(y^*) : A^*x^* + B^*y^* = z^*\}\,\forall z^* \in Z^*.$$

4.3 The conjugate of some generalized inf-convolution formulas

4.3.1 The inf-convolution formulas $\Box_1$ and $\Box_2$

Let X, Y be two real, separated, locally convex spaces, with their dual X^* and Y^* endowed with the w^*-topology, and consider the proper, convex and lower semicontinuous functions $f, g : X \times Y \longrightarrow \overline{\mathbb{R}}$.

The inf-convolution formulas $\Box_1$ and $\Box_2$ are introduced by $f\Box_1 g : X \times Y \longrightarrow \overline{\mathbb{R}}$

$$(f\Box_1 g)(x, y) = \inf\{f(u, y) + g(v, y) : u, v \in X,\ u + v = x\},$$

respectively, $f\Box_2 g : X \times Y \longrightarrow \overline{\mathbb{R}}$

$$(f\Box_2 g)(x, y) = \inf\{f(x, u) + g(x, v) : u, v \in Y,\ u + v = y\}.$$

We have the following result.

Theorem 4.3.1. *Let* X, Y *be two real separated locally convex spaces, let their duals* X^* *and* Y^* *be endowed with the* w^**-topology, and consider the proper, convex and lower semicontinuous functions* $f, g : X \times Y \longrightarrow \overline{\mathbb{R}}$, *such that* $\operatorname{pr}_Y(\operatorname{dom}(f)) \cap \operatorname{pr}_Y(\operatorname{dom}(g)) \neq \emptyset$ *and let* $V \subseteq X^*$, $V \neq \emptyset$. *The following conditions are equivalent:*

(i) $(f\Box_1 g)^*(x^*, y^*) = (f^*\Box_2 g^*)(x^*, y^*)$ *and* $f^*\Box_2 g^*$ *is exact for every* $(x^*, y^*) \in V \times Y^*$.

(ii) $(CQ^{\Box_1})$: $\{(u^*, v^*, a^* + b^*, r) \in X^* \times X^* \times Y^* \times \mathbb{R} : f^*(u^*, a^*) + g^*(v^*, b^*) \leq r\}$ *is closed regarding the set* $\Delta_V \times Y^* \times \mathbb{R}$ *in the* $(X^*, w^*) \times (X^*, w^*) \times (Y^*, w^*) \times \mathbb{R}$ *topology, where* $\Delta_V = \{(x^*, x^*) : x^* \in V\}$.

Consider the following regularity condition: $(RC_2^{\Box_1})$: X and Y are Fréchet spaces and
$0 \in \operatorname{sqri}(\operatorname{pr}_Y \operatorname{dom}(f) - \operatorname{pr}_Y \operatorname{dom}(g))$. We have the following result.

Theorem 4.3.2. *If the regularity condition* $(RC_2^{\Box_1})$ *is fulfilled then*

$$(f\Box_1 g)^*(x^*, y^*) = (f^*\Box_2 g^*)(x^*, y^*) \ \text{ and } f^*\Box_2 g^* \text{ is exact for every } (x^*, y^*) \in X^* \times Y^*.$$

4.3.2 The inf-convolution formulas $\Box_1^A$ and $\Box_2^A$

Let X and Y be two normed spaces, with their dual X^* and Y^*, and consider the proper, convex and lower semicontinuous functions $f : X \times X^* \longrightarrow \overline{\mathbb{R}}$ and $g : Y \times Y^* \longrightarrow \overline{\mathbb{R}}$. Assume, that the dual spaces of $X \times X^*$ and $Y \times Y^*$ respectively, $X^* \times X^{**}$ and $Y^* \times Y^{**}$ respectively, are endowed with the w^* topology. Moreover, let $A : X \longrightarrow Y$ be a linear and continuous operator and $A^* : Y^* \longrightarrow X^*$, respectively $A^{**} : X^{**} \longrightarrow Y^{**}$ be its adjoint, respectively its biadjoint operator.

Consider the following generalized inf-convolution formulas, $f\Box_1^A g : Y \times Y^* \longrightarrow \overline{\mathbb{R}}$

$$(f\Box_1^A g)(y, y^*) = \inf\{f(x, A^*y^*) + g(y - Ax, y^*) : x \in X\},$$

respectively, $f^*\Box_2^A g^* : Y^* \times Y^{**} \longrightarrow \overline{\mathbb{R}}$,

$$(f^*\Box_2^A g^*)(y^*, y^{**}) = \inf\{f^*(A^*y^*, x^{**}) + g^*(y^*, y^{**} - A^{**}x^{**}) : x^{**} \in X^{**}\}.$$

Theorem 4.3.3. *If* $\mathrm{pr}_{X^*}(\mathrm{dom}(f)) \cap A^*(\mathrm{pr}_{Y^*}(\mathrm{dom}(g))) \neq \emptyset$ *then the following conditions are equivalent.*

(i) $(CQ^{\Box_1^A}) : \{(x^*, y^*, A^{**}x^{**} + y^{**}, r) : f^*(x^*, x^{**}) + g^*(y^*, y^{**}) \leq r\}$ *is closed regarding the set* $\Delta_{A^*}^{Y^*} \times Y^{**} \times \mathbb{R}$ *in the* $(X^*, w^*) \times (Y^*, w^*) \times (Y^{**}, w^*) \times \mathbb{R}$ *topology, where* $\Delta_{A^*}^{Y^*} = \{(A^*y^*, y^*) : y^* \in Y^*\}$.

(ii) $(f\Box_1^A g)^*(y^*, y^{**}) = (f^*\Box_2^A g^*)(y^*, y^{**})$ *and* $f^*\Box_2^A g^*$ *is exact for every* $(y^*, y^{**}) \in Y^* \times Y^{**}$.

Consider the following regularity condition: $(RC_2^{\Box_1^A}) : 0 \in \mathrm{sqri}(\mathrm{pr}_{X^*}(\mathrm{dom}(f)) - A^* \mathrm{pr}_{Y^*}(\mathrm{dom}(g)))$. We have the following result.

Theorem 4.3.4. *If the regularity conditions* $(RC_2^{\Box_1^A})$ *is fulfilled then*
$(f\Box_1^A g)^*(y^*, y^{**}) = (f^*\Box_2^A g^*)(y^*, y^{**})$ *and* $f^*\Box_2^A g^*$ *is exact for every* $(y^*, y^{**}) \in Y^* \times Y^{**}$.

4.3.3 The inf-convolution formulas $\triangle_1^A$ and $\triangle_2^A$

Let X and Y be two normed spaces, with their dual X^* and Y^*, and consider the proper, convex and lower semicontinuous functions $f : X \times X^* \longrightarrow \overline{\mathbb{R}}$ and $g : Y \times Y^* \longrightarrow \overline{\mathbb{R}}$. Assume, that the dual spaces of $X \times X^*$ and $Y \times Y^*$ respectively, $X^* \times X^{**}$ and $Y^* \times Y^{**}$ respectively, are endowed with the w^* topology. Moreover, let $A : X \longrightarrow Y$ be a linear and continuous operator and $A^* : Y^* \longrightarrow X^*$, respectively $A^{**} : X^{**} \longrightarrow Y^{**}$ be its adjoint, respectively its biadjoint operator.

Consider the following generalized inf-convolution formulas, $f\triangle_2^A g : X \times X^* \longrightarrow \overline{\mathbb{R}}$

$$(f\triangle_2^A g)(x, x^*) = \inf\{f(x, x^* - A^*y^*) + g(Ax, y^*) : y^* \in Y^*\},$$

respectively, $f^*\triangle_1^A g^* : X^* \times X^{**} \longrightarrow \overline{\mathbb{R}}$,

$$(f^*\triangle_1^A g^*)(x^*, x^{**}) = \inf\{f^*(x^* - A^*y^*, x^{**}) + g^*(y^*, A^{**}x^{**}) : y^* \in Y^*\}.$$

Theorem 4.3.5. *If* $A(\mathrm{pr}_X(\mathrm{dom}(f))) \cap (\mathrm{pr}_Y(\mathrm{dom}(g))) \neq \emptyset$ *then the following conditions are equivalent.*

(i) $(CQ^{\triangle_2^A}) : \{(x^* + A^*y^*, x^{**}, y^{**}, r) : f^*(x^*, x^{**}) + g^*(y^*, y^{**}) \leq r\}$ *is closed regarding the set* $X^* \times \Delta_{X^{**}}^{A^{**}} \times \mathbb{R}$ *in the* $(X^*, w^*) \times (X^{**}, w^*) \times (Y^{**}, w^*) \times \mathbb{R}$ *topology, where* $\Delta_{X^{**}}^{A^{**}} = \{(x^{**}, A^{**}x^{**}) : x^{**} \in X^{**}\}$.

(ii) $(f\triangle_2^A g)^*(x^*, x^{**}) = (f^*\triangle_1^A g^*)(x^*, x^{**})$ *and* $f^*\triangle_1^A g^*$ *is exact for every* $(x^*, x^{**}) \in X^* \times X^{**}$.

Consider the following regularity condition: $(RC_2^{\triangle_2^A}) : 0 \in \operatorname{sqri}(\operatorname{pr}_Y(\operatorname{dom}(g)) - A(\operatorname{pr}_X(\operatorname{dom}(f))))$. We have the following result.

Theorem 4.3.6. *If the regularity condition* $(RC_2^{\triangle_2^A})$ *is fulfilled then* $(f \triangle_2^A g)^*(x^*, x) = (f^* \triangle_1^A g^*)(x^*, x)$ *and* $f^* \triangle_1^A g^*$ *is exact for every* $(x^*, x) \in X^* \times X$.

4.3.4 The inf-convolution formulas $\bigcirc_1^A$ and $\bigcirc_2^A$

Let X and Y be two normed spaces, with their dual X^* and Y^*, and consider the proper, convex and lower semicontinuous functions $f : X \times X^* \longrightarrow \overline{\mathbb{R}}$ and $g : Y \times Y^* \longrightarrow \overline{\mathbb{R}}$. Assume, that the dual spaces of $X \times X^*$ and $Y \times Y^*$ respectively, $X^* \times X^{**}$ and $Y^* \times Y^{**}$ respectively, are endowed with the w^* topology. Moreover, let $A : X \longrightarrow Y$ be a linear and continuous operator and $A^* : Y^* \longrightarrow X^*$, respectively $A^{**} : X^{**} \longrightarrow Y^{**}$ be its adjoint, respectively its biadjoint operator.

Consider the following generalized inf-convolution formulas, $f \bigcirc_1^A g : X \times X^* \longrightarrow \overline{\mathbb{R}}$

$$(f \bigcirc_1^A g)(x, x^*) = \inf_{\substack{u, w \in X \\ v^* \in Y^*}} \{f(u, x^*) + g(Aw, v^*) : \ u + w = x, \ A^* v^* = x^*\},$$

respectively, $f^* \bigcirc_2^A g^* : X^* \times X^{**} \longrightarrow \overline{\mathbb{R}}$,

$$(f^* \bigcirc_2^A g^*)(x^*, x^{**}) = \inf_{\substack{u^{**}, w^{**} \in X^{**} \\ v^* \in Y^*}} \{f^*(x^*, u^{**}) + g^*(v^*, A^{**} w^{**}) : \ u^{**} + w^{**} = x^{**}, \ A^* v^* = x^*\}.$$

Due to our best knowledge $\bigcirc_1^A$ and $\bigcirc_2^A$ were not considered until now in the literature. Obviously, when $A \equiv id_X$, $X = Y$ we obtain $f \Box_1 g$ and $f^* \Box_2 g^*$, respectively. The following result provides a closedness type regularity condition that not only ensures that $(f \bigcirc_1^A g)^*(x^*, x^{**}) = (f^* \bigcirc_2^A g^*)(x^*, x^{**})$ and $f^* \bigcirc_2^A g^*$ is exact for every $(x^*, x^{**}) \in X^* \times X^{**}$, but also is equivalent to it.

Theorem 4.3.7. *If* $\operatorname{dom}(g) \times \operatorname{pr}_{X^*}(\operatorname{dom}(f)) \cap ImA \times \Delta_{Y^*}^{A^*} \neq \emptyset$, *where* $\Delta_{Y^*}^{A^*} = \{(y^*, A^* y^*) | y^* \in Y^*\}$, *then the following statements are equivalent.*

(i) $(CQ^{\bigcirc_1^A}) : \{(u^*, A^* v^*, A^{**} u^{**} + v^{**}, r) : f^*(u^*, u^{**}) + g^*(v^*, v^{**}) \leq r\}$ *is closed regarding the set* $\Delta_{X^*} \times Im(A^{**}) \times \mathbb{R}$ *in the* $(X^*, w^*) \times (X^*, w^*) \times (Y^{**}, w^*) \times \mathbb{R}$ *topology, where* $\Delta_{X^*} = \{(x^*, x^*) : x^* \in X^*\}$.

(ii) $(f \bigcirc_1^A g)^*(x^*, x^{**}) = (f^* \bigcirc_2^A g^*)(x^*, x^{**})$ *and* $f^* \bigcirc_2^A g^*$ *is exact for every* $(x^*, x^{**}) \in X^* \times X^{**}$.

Consider the following regularity condition: $(RC_2^{\bigcirc_1^A}) : (0, 0, 0) \in \operatorname{sqri}(\operatorname{dom}(g) \times \operatorname{pr}_{X^*}(\operatorname{dom}(f)) - Im(A) \times \Delta_{Y^*}^{A^*})$, where $\Delta_{Y^*}^{A^*} = \{(y^*, A^* y^*) | y^* \in Y^*\}$. We have the following result.

Theorem 4.3.8. *If the regularity condition* $(RC_2^{\bigcirc_1^A})$ *is fulfilled then* $(f \bigcirc_1^A g)^*(x^*, x^{**}) = (f^* \bigcirc_2^A g^*)(x^*, x^{**})$ *and* $f^* \bigcirc_2^A g^*$ *is exact for every* $(x^*, x^{**}) \in X^* \times X^{**}$.

4.4 The maximal monotonicity of the parallel sums of two maximal monotone operators of Gossez type (D)

4.4.1 The maximal monotonicity of the parallel sum $S||T$

Using stable strong duality we prove the maximal monotonicity of the parallel sum of two maximal monotone operators, (introduced by Pasty in [104]), under the weakest condition knowing so far in literature.

Everywhere in the sequel, unless is other specified, X denotes a Banach space and X^* denotes its dual and X^{**} denotes its bidual.

Definition 4.4.1. *For given monotone operators $S, T : X \rightrightarrows X^*$ their parallel sum, is defined by*

$$S||Tx = (S^{-1} + T^{-1})^{-1}x, \forall x \in X.$$

It can be easily shown that $S||Tx = \bigcup_{y\in X}(S(y) \cap T(x-y))$.

Theorem 4.4.1. *Let $S, T : X \rightrightarrows X^*$ be two maximal monotone operators of Gossez type* (D)*, with strong representative functions h_S and h_T, such that $\mathrm{pr}_{X^*}(\mathrm{dom}(h_S)) \cap \mathrm{pr}_{X^*}(\mathrm{dom}(h_T)) \neq \emptyset$, and consider the function $h : X \times X^* \longrightarrow \overline{\mathbb{R}}$, $h(x, x^*) = \mathrm{cl}_{\|\cdot\|\times\|\cdot\|_*}(h_S \square_1 h_T)(x, x^*)$. Assume that $R(\overline{S}^{-1}) \subseteq X$, (where $\overline{S}$ is the Gossez monotone closure of the operator S), and that one of the following conditions is fulfilled.*

(a) The regularity condition $(RC_2^{\square_1})$ for h_S, respectively h_T holds.

(b) $(CQ^{\square_1})$ for h_S, respectively h_T holds.

Then then h is a strong representative function of $S||T$ and $S||T$ is a maximal monotone operator of Gossez type (D).

4.4.2 The maximal monotonicity of the parallel sum $S||^A T$

In the sequel, unless is otherwise specified, X and Y are Banach spaces, and X^* and Y^*, respectively X^{**} and Y^{**} denote their duals, respectively their biduals.

Definition 4.4.2. *Consider the monotone operators $S : X \rightrightarrows X^*$ and $T : Y \rightrightarrows Y^*$ and let $A : X \longrightarrow Y$ be a linear and continuous operator, and A^* its adjoint operator. The generalized parallel sum $S||^A T : Y \rightrightarrows Y^*$ is defined as follows:*

$$S||^A T := (AS^{-1}A^* + T^{-1})^{-1}.$$

Theorem 4.4.2. *Consider $A : X \longrightarrow Y$ a linear and continuous operator and let us denote by A^* its adjoint operator, and by A^{**} its biadjoint operator. Let $S : X \rightrightarrows X^*$ and $T : Y \rightrightarrows Y^*$ be two maximal monotone operators of Gossez type* (D) *with strong representative functions h_S and h_T respectively, such that $\mathrm{pr}_{X^*}(\mathrm{dom}(h_S)) \cap A^*(\mathrm{pr}_{Y^*}(\mathrm{dom}(h_T))) \neq \emptyset$. Consider the function $h : Y \times Y^* \longrightarrow \overline{\mathbb{R}}$, $h(y, y^*) = \mathrm{cl}_{\|\cdot\|\times\|\cdot\|_*}(h_S \square_1^A h_T)(y, y^*)$. Assume that $R(\overline{S}^{-1}) \subseteq X$, and that one of the following conditions is fulfilled.*

(a) The regularity condition $(RC_2^{\square_1^A})$ for h_S, respectively h_T holds.

(b) $(CQ^{\square_1^A})$ for h_S, respectively h_T holds.

Then h is a strong representative function of $S||^A T$ and $S||^A T$ is a maximal monotone operator of Gossez type (D).

4.4.3 The maximal monotonicity of the operator $S + A^*TA$

In what follows X, respectively Y will be Banach spaces, X^*, respectively Y^* denote their dual spaces, X^{**}, respectively Y^{**} denote their bidual spaces. Consider the monotone operators $S : X \rightrightarrows X^*$ and $T : Y \rightrightarrows Y^*$ and let $A : X \longrightarrow Y$ be a linear and continuous operator, and A^* its adjoint operator. A well known generalized sum involving S and T is defined as follows:

$$M : X \rightrightarrows X^*,\ M := S + A^*TA.$$

In what follows we give some sufficient conditions which ensure the maximal monotonicity of $S + A^*TA$, where S, respectively T are maximal monotone operators of Gossez type (D).

Theorem 4.4.3. *Consider $A : X \longrightarrow Y$ a linear and continuous operator and let us denote by A^* its adjoint operator, and by A^{**} its biadjoint operator. Let $S : X \rightrightarrows X^*$ and $T : Y \rightrightarrows Y^*$ be two maximal monotone operators of Gossez type* (D) *with strong representative functions h_S and h_T respectively, such that $A(\mathrm{pr}_X(\mathrm{dom}(h_S))) \cap (\mathrm{pr}_Y(\mathrm{dom}(h_T))) \neq \emptyset$. Consider the function $h : X \times X^* \longrightarrow \overline{\mathbb{R}}$, $h(x, x^*) = \mathrm{cl}_{\|\cdot\| \times \|\cdot\|_*}(h_S \triangle_2^A h_T)(x, x^*)$. Assume that one of the following conditions is fulfilled.*

(a) The regularity condition $(RC_2^{\triangle_2^A})$ for h_S, respectively h_T holds.

(b) $(CQ^{\triangle_2^A})$ for h_S, respectively h_T holds.

Then h is a strong representative function of $S + A^*TA$ and $S + A^*TA$ is a maximal monotone operator of Gossez type (D).

4.4.4 The maximal monotonicity of $S||_AT$

In what follows X, respectively Y will be Banach spaces, X^*, respectively Y^* denote their dual spaces and X^{**}, respectively Y^{**} denote their bidual spaces. Let $S : X \rightrightarrows X^*$, respectively $T : Y \rightrightarrows Y^*$ be two monotone operators. Moreover, consider the continuous, linear operator $A : X \longrightarrow Y$, and let us denote by A^* its adjoint operator and by A^{**} its biadjoint operator. Recall that the a generalized parallel sum $S||_AT$, (see [109]), of the monotone operators S, respectively T is defined as

$$S||_AT : X \rightrightarrows X^*,\ S||_AT := (S^{-1} + (A^*TA)^{-1})^{-1}.$$

Remark 4.4.1. If $X = Y$, $A \equiv id_X$, we obtain the concept of parallel sum of two operators, introduced by Pasty in [104], that is

$$S||T : X \rightrightarrows X^*,\ S||T := (S^{-1} + T^{-1})^{-1}.$$

Next we will provide some conditions that ensures the maximal monotonicity of the generalized parallel sum $S||_AT$. Due to our best knowledge, in the literature does not exists so far a known condition that provides this result.

Theorem 4.4.4. *Let $A : X \longrightarrow Y$ be a linear and continuous operator, with its adjoint denoted by A^*, and its biadjoint denoted by A^{**}. Let $S : X \rightrightarrows X^*$ and $T : Y \rightrightarrows Y^*$ be two maximal monotone operators of Gossez type* (D)*, with strong representative functions h_S and h_T respectively, such that*

$\operatorname{dom} h_T \times \operatorname{pr}_{X^*}(\operatorname{dom} h_S) \cap \operatorname{Im} A \times \Delta_{Y^*}^{A^*} \neq \emptyset$, *where* $\Delta_{Y^*}^{A^*} = \{(y^*, A^*y^*) : y^* \in Y^*\}$. *Consider the function*

$$h : X \times X^* \longrightarrow \overline{\mathbb{R}},\ h(x, x^*) = \operatorname{cl}_{\|\cdot\| \times \|\cdot\|_*}(h_S \bigcirc_1^A h_T)(x, x^*),$$

and assume that $R(\overline{S}^{-1}) \subseteq X$, *and that one of the following conditions is fulfilled.*

(a) The regularity condition $(RC_2^{\bigcirc_1^A})$ *for* h_S *and* h_T *hold.*

(b) $(CQ^{\bigcirc_1^A})$ *for* h_S *and* h_T *hold.*

Then h *is a strong representative function of* $S||_A T$ *and the generalized parallel sum* $S||_A T$ *is maximal monotone operator of Gossez type* (D).

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