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Kronecker Modules and Matrix Pencils

Summary of PhD Thesis

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Cluj-Napoca

2011

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Introduction

Let $K: 1 \underset{\beta}{\overset{\alpha}{\leftleftarrows}} 2$ be the Kronecker quiver and κ an arbitrary field. The path algebra κK over the Kronecker quiver is the Kronecker algebra. We will consider the category of finite dimensional right modules over this algebra, the category of Kronecker modules, denoted by mod-κK. The category mod-κK can be identified with the category rep-κK of the finite dimensional κ-representations of the Kronecker quiver. Partial results were given by Weierstrass[1] for the classification problem of indecomposable modules in mod-κK. The classification was completed by Kronecker[2] in 1890, hence the name of the quiver, the algebra and the corresponding module category.

The Kronecker algebra is a special case of a tame hereditary algebra. Its importance stems from the fact that it models the behavior of all tame hereditary algebras (see [2, 18]). Moreover it is closely related to other branches of mathematics, such as geometry, linear algebra and applied mathematics (control theory, some problems in engineering, etc.). On one hand the category mod-κK has a geometric interpretation since it is derived equivalent with the category Coh($\mathbb{P}^1(\kappa)$) of coherent sheaves on the projective line – as the Kronecker quiver K is just the Beilinson quiver for $\mathbb{P}^1$ (see [4]). On the other hand, Kronecker modules correspond to matrix pencils in linear algebra (as detailed in Section 3.5). In turn, this correspondence allows one to deal with problems in linear algebra and control theory (related to matrix pencils) in terms of Kronecker modules, by placing these problems in a new setting, which allows the usage of results and techniques of higher abstract algebra. Because of its usefulness, the category mod-κK has been extensively studied and the Auslander-Reiten quiver revealing the indecomposable objects and the so-called irreducible morphisms is well-known. Information about it can be found in many classic textbooks on representation theory (see for example [1, 2, 18]).

The aim of the current thesis is twofold. On one hand it aims to enrich the information available on the category mod-κK by answering basic and important questions, such as which are the conditions for the existence of embeddings, projections or short exact sequences involving various types of Kronecker modules. We mention here that due to the work of Cs. Szántó, we have readily available answers for these questions in the case of indecomposable Kronecker modules over a finite field (see [21, 22]). We generalize some of his results in two

[1] *Karl Theodor Wilhelm Weierstrass* (1815 – 1897), German mathematician, the "father of modern analysis".
[2] *Leopold Kronecker* (1823 – 1891), German mathematician who worked on number theory and algebra.

ways: by outgrowing the case of finite fields, while passing to an arbitrary field κ and by giving similar numerical criteria for the existence of embeddings, projections and short exact sequences involving decomposable Kronecker modules over the arbitrary field. Therefore, the new results presented here are mainly based on [22] and [21], and the current thesis can be regarded in some sense as a continuation of his work. We mention that in turn, our results contribute back by giving information on the Ringel-Hall product of various decomposable Kronecker modules. On the other hand, by exploring the Kronecker module – matrix pencil correspondence, we want to show how our results can be applied immediately in the (partial) solution of the matrix subpencil problem, an important open problem originating from control theory and having applications in engineering and physics.

The current thesis is organized in three chapters, each of them having several sections.

Chapter 1 is dedicated to the presentation of the terminology and the well-known facts about the category of Kronecker modules. Definitions as well as information about the structure of the category and about the indecomposable objects are to be found in Section 1.1. As already mentioned, the main sources are [1, 2, 18]. In Section 1.2 we extract the relevant results from [22] and [21] and present them in a concise way. The theorems in this section represent the starting point of our current work.

Chapter 2 contains the majority of our contribution to the theory of Kronecker modules. In Section 2.1 we give a numerical criteria in terms of the so-called Kronecker invariants (integer parameters which determine the Kronecker modules up to isomorphism) for the existence of an embedding between (decomposable) preinjective Kronecker modules (and dually, for the existence of a projection between preprojective ones). After characterizing the middle terms in certain short exact sequences (Section 2.2), we proceed in Sections 2.3 and 2.4 to show that the conditions governing the existence of the mentioned morphisms and short exact sequences are independent of the underlying field κ – with a detailed proof for the preinjective and preprojective case. The extension monoid product introduced by Reineke (see [17]) proves to be a very convenient tool in dealing with short exact sequences of Kronecker modules in this field independent setting, hence the monoid product is the subject of all the subsequent sections of Chapter 2. In Sections 2.5 and 2.6 we completely describe this product in the case when the modules involved are either preinjective or preprojective, exploring its combinatorial properties. As a result we get the conditions for the existence of the corresponding short exact sequences and the characterization of the modules which may appear as middle terms. In Section 2.7 we make some of the conditions fully explicit and describe the middle terms in $\mathrm{Ext}^1(I, I')$ – where I and I' are preinjectives – by easy to check numerical conditions, resulting in an algorithm (linear in the number of indecomposable components) for the decidability problem. We also propose a method to generate all extensions of I' by I and we give a different (combinatorial) proof for the theorem providing numerical criteria in terms of Kronecker invariants for the existence of a monomorphism $I' \hookrightarrow I$. All these results apply dually to preprojective modules as well.

Chapter 3 exemplifies the usefulness of our new results through an application to an open problem originating from control theory (see for example [14]). Following [10] we make a short introduction to the theory of matrix pencils (Sections 3.1, 3.2 and 3.3), briefly explaining the notions and enlisting the relevant results needed in understanding the matrix subpencil problem presented in Section 3.4. Matrix pencils are determined up to strict equivalence relation by some integer parameters, called the classical Kronecker invariants, in the same way as Kronecker modules are determined up to isomorphism by the Kronecker invariants. This correspondence between Kronecker modules and matrix pencils is detailed in Section 3.5 and finally, our solution to the matrix subpencil problem in an important special case is presented in Section 3.6.

We emphasize that all of our numerical criteria are valid over arbitrary fields and are explicit and easy to check. Basically they are systems of inequalities involving only integers (the Kronecker and the classical Kronecker invariants, respectively). If we wanted to write a motto for our thesis, the following quotation attributed to Leopold Kronecker himself would have certainly been a good candidate: "*God created the integers, all else is the work of man.*"[5]

We also mention that this work is the description of a research in progress. The aim of our ongoing efforts is to find explicit numerical criteria for the existence of embeddings, projections and short exact sequences involving arbitrary Kronecker modules (preprojectives, regulars and preinjectives). As a result we expect to be able to solve the matrix subpencil problem in the general case, by giving an explicit numerical condition as in the special case treated in the last chapter.

The current thesis is based on the following four articles by the author: [25, 26, 27, 28] – the first two written jointly with Cs. Szántó. The results from Section 2.6 and 3.6 are not published yet, these will be included in future papers. The main results included here were also presented at the following international scientific conferences:

1. I. Szöllősi, *Kronecker modules and matrix pencils*, 13^{th} Postgraduate Group Theory Conference, University of Aberdeen, Aberdeen, Scotland, June 23-25, 2011.

2. Cs. Szántó, I. Szöllősi, *Short exact sequences of Kronecker modules (poster)*, New developments in noncommutative algebra and its applications (Workshop), Sabhal Mòr Ostaig, Isle of Skye, Scotland, Jun 26 - Jul 2, 2011.

3. I. Szöllősi, *Computing the extensions of preinjective and preprojective Kronecker modules*, Groups and Semigroups: Interactions and Computations (Conference), University of Lisbon, Lisbon, Portugal, July 25-29, 2011.

4. I. Szöllősi, *Computational methods in the theory of Kronecker modules*, A^3 Abstract Algebra and Algorithms Conference, Eszterházy Károly College, Eger, Hungary, August 14-17, 2011.

5. I. Szöllősi, *On short exact sequences of Kronecker modules*, Algebraic Representation Theory, Uppsala University, Uppsala, Sweden, September 1-3, 2011.

Acknowledgments

I would like to thank to my supervisor, Dr. Andrei Mărcuş for backing me through my studies, for his help and encouragement. I'm extremely grateful to Dr. Szántó Csaba – his mathematical insight, constructive advices and generous help made this thesis possible, despite the short time frame and demanding conditions. Special thanks for Dr. Septimiu Crivei and Dr. Gabriela Olteanu for including me as a member into their research teams. Dr. Septimiu Crivei's guidance and friendship since my undergraduate years have been among the decisive factors in choosing my vocation.

Naturally, a lot of thanks go to my family and friends.

Last but not least I would like to express my gratitude for the countless free and open source programmers and community members out there, who donate their time, programming and/or mathematical skills to create and make available for free a whole range of indispensable tools in the academic work, from operating systems (like GNU/Linux), typesetting systems, document processors and graphic editors (LaTeX, LyX, Inkscape, etc.) to computer algebra software, (like GAP, Maxima, etc.) Their contribution to the progress of science and social welfare is enormous and much appreciated.

I wish to thank for the financial support provided from programs co-financed by the **Sectoral Operational Programme Human Resources Development**, Contract POSDRU 6/1.5/S/3 - "Doctoral studies: through science towards society".

Key words and phrases: Kronecker algebra, (preinjective and/or preprojective) Kronecker modules, short exact sequence, extension monoid product, matrix pencil, the matrix subpencil problem.

Chapter 1

Preliminaries

1.1 The category of Kronecker modules

Let K be the **Kronecker quiver**, i.e. the quiver having two vertices and two parallel arrows:

$$K:\ 1 \underset{\beta}{\overset{\alpha}{\leftleftarrows}} 2$$

and κ an arbitrary field. The path algebra of the Kronecker quiver is the **Kronecker algebra** and we will denote it by κK. The κ-base of the path algebra κK is determined by the set of all paths in K. A finite dimensional right module over the Kronecker algebra is called a **Kronecker module**. We denote by mod-κK the category of finite dimensional right modules over the Kronecker algebra.

A (finite dimensional) κ-linear representation of the quiver K is a quadruple $M = (V_1, V_2; \varphi_\alpha, \varphi_\beta)$ where V_1, V_2 are finite dimensional κ-vector spaces (corresponding to the vertices) and $\varphi_\alpha, \varphi_\beta : V_2 \to V_1$ are κ-linear maps (corresponding to the arrows). Thus a κ-linear representation of K associates vector spaces to the vertices and compatible κ-linear functions (or equivalently, matrices) to the arrows. Let us denote by rep-κK the category of finite dimensional κ-representations of the Kronecker quiver. Given two such representations $M = (V_1, V_2; \varphi_\alpha, \varphi_\beta)$ and $M' = (V_1', V_2'; \varphi_\alpha', \varphi_\beta')$, a morphism in rep-$\kappa K$ between M and M' is a pair of κ-linear functions $f = (f_1, f_2)$, where $f_1 : V_1 \to V_1'$ and $f_2 : V_2 \to V_2'$ so that the following diagram commutes (i.e. $f_1\varphi_\alpha = \varphi_\alpha' f_2$ and $f_1\varphi_\beta = \varphi_\beta' f_2$):

$$\begin{array}{ccc} V_1 & \underset{\varphi_\beta}{\overset{\varphi_\alpha}{\leftleftarrows}} & V_2 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_2} \\ V_1' & \underset{\varphi_\beta'}{\overset{\varphi_\alpha'}{\leftleftarrows}} & V_2' \end{array} .$$

The category of Kronecker modules (mod-κK) is equivalent with the category of κ-linear representations of the Kronecker quiver (rep-κK) via the functors F and G, defined as follows:

- F : mod-$\kappa K \to$ rep-κK with $F(M) = (F(M)_1, F(M)_2; F(M)_\alpha, F(M)_\beta) = (V_1, V_2; \varphi_\alpha, \varphi_\beta)$, where $V_1 = M\varepsilon_1$, $V_2 = M\varepsilon_2$ and $\varphi_\alpha, \varphi_\beta : V_2 \to V_1$, such that $\varphi_\alpha(x) = x\alpha$ and $\varphi_\beta(x) = x\beta$ for any $x \in V_2$. If $f : M \to M'$ is a morphism of Kronecker modules, then $F(f) : F(M) \to F(M')$, $F(f) = (f_1, f_2)$, where $f_1 = f|_{F(M)_1}$ and $f_2 = f|_{F(M)_2}$.

- G : rep-$\kappa K \to$ mod-κK with $G(M) = M_1 \oplus M_2$ being a κ-space, where the the right κK-module structure is given by $x\varepsilon_1 = (x_1 + x_2)\varepsilon_1 = x_1$, $x\varepsilon_2 = (x_1 + x_2)\varepsilon_2 = x_2$, $x\alpha = \varphi_\alpha(x_2)$, $x\beta = \varphi_\beta(x_2)$ and $G(f) = f_1 \oplus f_2$.

The equivalence of the categories mod-κK and rep-κK made clear, from now on we will identify a module $M \in$ mod-κK with its corresponding κ-linear representation of the quiver K and we will use sometimes the notation $M : V_1 \underset{\varphi_\beta}{\overset{\varphi_\alpha}{\leftleftarrows}} V_2$ or $M : V_1 \underset{[\varphi_\beta]}{\overset{[\varphi_\alpha]}{\leftleftarrows}} V_2$, where $[\varphi_\alpha]$ and $[\varphi_\beta]$ are the matrices of the linear maps φ_α and φ_β in some basis.

The category mod-κK has been extensively studied because the Kronecker algebra is a very important example of a tame hereditary algebra. Moreover, the category mod-κK has also a geometric interpretation, since it is derived equivalent with the category $\mathrm{Coh}(\mathbb{P}^1(\mathbb{F}_q))$ of coherent sheaves on the projective line (see [4]). The Kronecker quiver K is just the Beilinson quiver for $\mathbb{P}^1$.

In what follows we put together a short compilation of definitions and well-known facts about the category of Kronecker modules. The calculations, justifications and proofs leading to these results can be found in many standard textbooks on representation theory of algebras, see for example [1, 19, 2, 18]. We have no intent to replicate the arguments here, only to revisit the material needed to understand the new results presented in Chapters 2 and 3.

The **simple Kronecker module**s (up to isomorphism) are

$$S_1 : \kappa \leftleftarrows 0 \quad \text{and} \quad S_2 : 0 \leftleftarrows \kappa.$$

For a Kronecker module M we denote by $\underline{\dim} M$ its **dimension**. The dimension of M is a vector $\underline{\dim} M = ((\dim M)_1, (\dim M)_2) = (m_{S_1}(M), m_{S_2}(M))$, where $m_{S_i}(M)$ is the number of factors isomorphic with the simple module S_i in a composition series of M, $i = \overline{1,2}$. Regarded as a representation, $M : V_1 \underset{\varphi_\beta}{\overset{\varphi_\alpha}{\leftleftarrows}} V_2$, we have that $\underline{\dim} M = (\dim_\kappa V_1, \dim_\kappa V_2)$.

The **defect** of $M \in$ mod-κK with $\underline{\dim} M = (a, b)$ is defined in the Kronecker case as $\partial M = b - a$.

An indecomposable module $M \in$ mod-κK is a member in one of the following three families: preprojective indecomposables, regular indecomposables and preinjective indecomposables. Let us take them in order.

The **preprojective indecomposable Kronecker module**s are determined up to isomorphism by their dimension vector. For $n \in \mathbb{N}$ we will denote by P_n the indecomposable preprojective module of dimension $(n+1, n)$. So P_0 and P_1 are the projective indecomposable modules ($P_0 = S_1$ being simple). It is known that (up to isomorphism) $P_n = (\kappa^{n+1}, \kappa^n; f, g)$, where choosing the canonical basis in κ^n and κ^{n+1}, the matrix of $f : \kappa^n \to \kappa^{n+1}$ (respectively of $g : \kappa^n \to \kappa^{n+1}$) is $\begin{pmatrix} E_n \\ 0 \end{pmatrix}$ (respectively $\begin{pmatrix} 0 \\ E_n \end{pmatrix}$). Thus in this case

$$P_n : \ \kappa^{n+1} \underset{\left(\begin{smallmatrix} E_n \\ 0 \end{smallmatrix}\right)}{\overset{\left(\begin{smallmatrix} 0 \\ E_n \end{smallmatrix}\right)}{\leftleftarrows}} \kappa^n \ ,$$

where E_n is the identity matrix. We have for the defect $\partial P_n = -1$.

We define a **preprojective Kronecker module** P as being a direct sum of indecomposable preprojective modules: $P = P_{a_1} \oplus P_{a_2} \oplus \cdots \oplus P_{a_l}$, where we use the convention that $a_1 \leq a_2 \leq \cdots \leq a_l$.

The **preinjective indecomposable Kronecker module**s are also determined up to isomorphism by their dimension vector. For $n \in \mathbb{N}$ we will denote by I_n the indecomposable preinjective module of dimension $(n, n+1)$. So I_0 and I_1 are the injective indecomposable modules ($P_0 = S_2$ being simple). It is known that (up to isomorphism) $I_n = (\kappa^n, \kappa^{n+1}; f, g)$, where choosing the canonical basis in κ^{n+1} and κ^n, the matrix of $f : \kappa^{n+1} \to \kappa^n$ (respectively of $g : \kappa^{n+1} \to \kappa^n$) is $\begin{pmatrix} E_n & 0 \end{pmatrix}$ (respectively $\begin{pmatrix} 0 & E_n \end{pmatrix}$). Thus in this case

$$I_n : \ \kappa^n \underset{(0\, E_n)}{\overset{(E_n\, 0)}{\leftleftarrows}} \kappa^{n+1} \ ,$$

where E_n is the identity matrix. We have for the defect $\partial I_n = 1$.

We define a **preinjective Kronecker module** I as being a direct sum of indecomposable preinjective modules: $I = I_{a_1} \oplus I_{a_2} \oplus \cdots \oplus I_{a_l}$, where we use the convention that $a_1 \geq a_2 \geq \cdots \geq a_l$.

The **regular indecomposable Kronecker module**s ($\mathbf{R}$) are those indecomposable modules $M \in$ mod-κK which are neither preprojective nor preinjective. A regular indecomposable is isomorphic as representation with one of the following (f_X denotes the multiplication by X, id is the identity function and $n \geq 1$):

- $R_\infty(n) : \ \kappa[X]/(X^n) \underset{id}{\overset{f_X}{\leftleftarrows}} \kappa[X]/(X^n)$;

- $R_\phi(n) \ : \ \kappa[X]/(\phi(X)^n) \underset{f_X}{\overset{id}{\leftleftarrows}} \kappa[X]/(\phi(X)^n)$, where ϕ is a monic polynomial with $\deg \phi \geq 2$, irreducible in κ.

- $R_k(n) : \ \kappa[X]/((X-k)^n) \underset{f_X}{\overset{id}{\leftleftarrows}} \kappa[X]/((X-a)^n)$, where $k \in \kappa$ (hence $R_k(n)$ is just a notation for $R_{X-k}(n)$).

This is consistent with everything claimed about Kronecker modules so far, since we have the following isomorphism g of representations, where ϕ is an arbitrary monic irreducible polynomial, with $\deg \phi = d \geq 1$:

$$\begin{array}{ccc} \kappa^{nd} & \overset{id}{\underset{C_{\phi(X)^n}}{\longleftleftarrows}} & \kappa^{nd} \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ \kappa[X]/(\phi(X)^n) & \overset{id}{\underset{X}{\longleftleftarrows}} & \kappa[X]/(\phi(X)^n) \end{array} ,$$

where $g : \kappa^{nd} \to \kappa[X]/(\phi(X)^n)$, $g(k_0, k_1, \ldots, k_{nd-1}) = k_0 + k_1X + \cdots + k_{nd-1}X^{nd-1} + (\phi(X)^n)$ for any $(k_0, k_1, \ldots, k_{nd-1}) \in \kappa^{nd}$ and $C_{\phi(X)^n}$ is the companion matrix of the polynomial $\phi(X)^n$. The **companion matrix** of an arbitrary monic polynomial $A(X) = X^d + k_{d-1}X^{d-1} + \cdots + k_0 \in \kappa[X]$ of degree d is $C_{A(X)} \in \mathcal{M}_d(\kappa)$, where

$$C_{A(X)} = \begin{pmatrix} 0 & 0 & \cdots & 0 & -k_0 \\ 1 & 0 & \cdots & 0 & -k_1 \\ 0 & 1 & & 0 & -k_2 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & 1 & -k_{d-1} \end{pmatrix}.$$

We also have the isomorphism

$$\begin{array}{ccc} \kappa^{n} & \overset{J_0^{(n)}}{\underset{id}{\longleftleftarrows}} & \kappa^{n} \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ \kappa[X]/(X^n) & \overset{f_X}{\underset{id}{\longleftleftarrows}} & \kappa[X]/(X^n) \end{array} ,$$

where $g : \kappa^n \to \kappa[X]/(X^n)$ is defined as before and $J_0^{(n)} = \begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}$ is the nilpotent matrix of degree n.

To simplify notations and terminology, let us introduce the following set:

$$\mathcal{P} = \{\infty\} \cup \kappa \cup \{\phi | \phi \text{ is a monic irreducible polynomial of degree deg}\phi \geq 2 \text{ over } \kappa\}$$

and call an element $p \in \mathcal{P}$ of this set simply a "point". We will denote by d_p the degree of the point p, where $d_p = \begin{cases} 1 & p \in \{\infty\} \cup \kappa \\ \deg \phi & p \in \mathcal{P} \setminus (\{\infty\} \cup \kappa) \end{cases}$. We also use the convention $R_p(0) = 0$, for any $p \in \mathcal{P}$.

Hence the dimension of a regular indecomposable will be $\underline{\dim} R_p(n) = (nd_p, nd_p)$ and we have for the defect $\partial R_p(n) = 0$. As reveled from the Auslander-Reiten quiver the regular modules lay on so-called tubes (see [18, 19]). Every point $p \in \mathcal{P}$ determines a tube (every irreducible monic polynomial $\phi \in \kappa[X]$ determines a tube $\mathcal{T}_\phi$, in addition, there is the tube $\mathcal{T}_\infty$ of the modules $R_\infty(n)$).

If $\kappa = \bar{\kappa}$ is algebraically closed, then all irreducible polynomials are of the form $\phi(X) = X - k$ and the companion matrix $C_{(X-k)^n}$ is similar to $J_k^{(n)}$, where $J_k^{(n)}$ is the $n \times n$ Jordan block $J_k^{(n)} = \begin{pmatrix} k & 1 & & \\ & k & \ddots & \\ & & \ddots & 1 \\ & & & k \end{pmatrix} = kE_n + J_0^{(n)}$. In this case $\mathcal{P} = \{\infty\} \cup \kappa$ and the regular indecomposables are

$$R_k(n) : \ \kappa^n \underset{E_n}{\overset{pE_n + J_0^{(n)}}{\longleftarrow\!\!\longleftarrow}} \kappa^n \ \text{ for } k \in \kappa \text{ and } R_\infty(n) : \ \kappa^n \underset{J_0^{(n)}}{\overset{E_n}{\longleftarrow\!\!\longleftarrow}} \kappa^n \ .$$

A module $R \in$ mod-κK will be called a **regular Kronecker module** if it is a direct sum of regular indecomposables. If $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$ is a partition, then we use the notation $R_p(\lambda) = R_p(\lambda_1) \oplus R_p(\lambda_1) \oplus \dots \oplus R_p(\lambda_m)$.

The category mod-κK of Kronecker modules is a **Krull-Schmidt category**, hence we have:

Theorem 1.1.1 (Krull-Schmidt). *Every module in $M \in$ mod-κK has the decomposition*

$$M = (P_{c_1} \oplus \dots \oplus P_{c_n}) \oplus (\oplus_{p \in \mathcal{P}} R_p(\lambda^{(p)})) \oplus (I_{d_1} \oplus \dots \oplus I_{d_m})$$

up to isomorphism, where:

- *$(c_1, \dots, c_n)$ is a finite increasing sequence of nonnegative integers;*
- *$\lambda^{(p)} = (\lambda_1, \dots, \lambda_t)$ is a nonzero partition for finitely many $p \in \mathcal{P}$;*
- *$(d_1, \dots, d_m)$ is a finite decreasing sequence of nonnegative integers.*

The increasing sequences $(c_1, \dots, c_n)$ and $(d_1, \dots, d_m)$ together with the partitions $\lambda^{(p)}$ corresponding to every $p \in \mathcal{P}$ are called the **Kronecker invariants** of the module M. Hence Kronecker invariants determine a module $M \in$ mod-κK up to isomorphism.

We end this section with the following well-known lemmas, summarizing some facts on morphisms, extensions and short exact sequences in mod-κK:

Lemma 1.1.2. *Denoting by R, P and I a preprojective, a regular, respectively a preinjective Kronecker module, we have (where $m, n, t, t_1, t_2 \in \mathbb{N}$, $p \in \mathcal{P}$ and d_p is the degree of the point p):*

(a) $\mathrm{Hom}(R,P) = \mathrm{Hom}(I,P) = \mathrm{Hom}(I,R) = \mathrm{Ext}^1(P,R) = \mathrm{Ext}^1(P,I) = \mathrm{Ext}^1(R,I) = 0$.

(b) *For* $n \leq m$, *we have* $\dim_\kappa \mathrm{Hom}(P_n, P_m) = m - n + 1$ *and* $\mathrm{Ext}^1(P_n, P_m) = 0$*; otherwise* $\mathrm{Hom}(P_n, P_m) = 0$ *and* $\dim_\kappa \mathrm{Ext}^1(P_n, P_m) = n - m - 1$*. In particular* $\mathrm{End}(P_n) \cong \kappa$ *and* $\mathrm{Ext}^1(P_n, P_n) = 0$.

(c) *For* $n \geq m$, *we have* $\dim_\kappa \mathrm{Hom}(I_n, I_m) = n - m + 1$ *and* $\mathrm{Ext}^1(I_n, I_m) = 0$*; otherwise* $\mathrm{Hom}(I_n, I_m) = 0$ *and* $\dim_\kappa \mathrm{Ext}^1(I_n, I_m) = m - n - 1$*. In particular* $\mathrm{End}(I_n) \cong \kappa$ *and* $\mathrm{Ext}^1(I_n, I_n) = 0$.

(d) *If* $p \neq p'$, *then* $\mathrm{Hom}(R_p(t), R_{p'}(t)) = \mathrm{Ext}^1(R_p(t), R_{p'}(t)) = 0$.

(e) $\dim_\kappa \mathrm{Hom}(P_n, I_m) = n + m$ *and* $\dim_\kappa \mathrm{Ext}^1(I_m, P_n) = m + n + 2$.

(f) $\dim_\kappa \mathrm{Hom}(P_n, R_p(t)) = \dim_\kappa \mathrm{Hom}(R_p(t), I_n) = d_p t$ *and* $\dim_\kappa \mathrm{Ext}^1(R_p(t), P_n) = \dim_\kappa \mathrm{Ext}^1(I_n, R_p(t)) = d_p t$.

(g) $\dim_\kappa \mathrm{Hom}(R_p(t_1), R_p(t_2)) = \dim_\kappa \mathrm{Ext}^1(R_p(t_1), R_p(t_2)) = d_p \min(t_1, t_2)$.

Lemma 1.1.3. *If there is a short exact sequence* $0 \to M' \to M \to M'' \to 0$ *of Kronecker modules, then* $\underline{\dim} M = \underline{\dim} M' + \underline{\dim} M''$ *and* $\partial M = \partial M' + \partial M''$.

1.2 Ringel-Hall algebras in the Kronecker case

Let κ now be a finite field with $|\kappa| = q$ (for the sake of this section, only). For a module $X \in$ mod-κK, $[X]$ will denote its isomorphism class and $tM := M \oplus ... \oplus M$ (t-times).

The **Ringel-Hall algebra** $\mathcal{H}(\kappa K, \mathbb{Q})$ associated to the Kronecker algebra κK is the free $\mathbb{Q}$-vector space having as basis the isomorphism classes in mod-κK together with a multiplication (the **Ringel-Hall product**) defined by:

$$[N_1][N_2] = \sum_{[M]} F^M_{N_1 N_2} [M],$$

where the structure constants $F^M_{N_1 N_2} = |\{M \supseteq U |\ U \cong N_2,\ M/U \cong N_1\}|$ are called **Ringel-Hall numbers**. Notice that $\mathcal{H}(\kappa K, \mathbb{Q})$ is an associative, usually noncommutative algebra with unit element the isomorphism class of the zero module.

More generally for $M, N_1, ..., N_t \in$ mod-κK we can define

$$F^M_{N_1 \ldots N_t} = |\{M = M_0 \supseteq M_1 \supseteq ... \supseteq M_t = 0 |\ M_{i-1}/M_i \cong N_i, \forall 1 \leq i \leq t\}|.$$

We then have (using associativity)

$$[N_1]...[N_t] = \sum_{[M]} F^M_{N_1 \ldots N_t} [M].$$

In what follows we will present some known facts about the Ringel-Hall algebra associated to the Kronecker algebra.

Lemma 1.2.1 ([22]). *For $N_1, N_2 \in$ mod-kK with $\mathrm{Ext}^1(N_1, N_2) = 0$ and $\mathrm{Hom}(N_2, N_1) = 0$ we have $[N_1][N_2] = [N_1 \oplus N_2]$.*

It results from the previous section that the isomorphism class of a module $M \in$ mod-kK is of the form $[M] = [P \oplus R \oplus I]$, where P, R and I is a preprojective, a regular respectively a preinjective module. From Lemma 1.1.2 (a) and Lemma 1.2.1 follows that $[P \oplus R \oplus I] = [P][R][I]$, so for $[M'] = [P' \oplus R' \oplus I']$

$$[M][M'] = [P][R][I][P'][R'][I'].$$

This shows, that in order to obtain all the Ringel-Hall numbers we should be able to describe the Hall products of the form

$$[I][I'], [I][P], [I][R], [R][R'], [R][P], [P][P'].$$

We have a formula list for the Ringel-Hall products of specific modules in the Kronecker case due to Szántó (see for example [22] and [21]).

If $F^M_{N_1 \dots N_t} \neq 0$ then we will call $[M]$ a term in $[N_1]...[N_t]$ and use the notation $[M] \in \{[N_1]...[N_t]\}$ ($\{[N_1]...[N_t]\}$ denoting the set of all terms in $[N_1]...[N_t]$).

Using the definitions above and the associativity in the Ringel-Hall algebra one can easily check the following lemma:

Lemma 1.2.2. Let $N_1, N_2, M, M' \in$ mod-κK. We have the following:

(a) $\{[N_1][N_2]\} = \{[M] | F^M_{N_1 N_2} \neq 0\} = \{[M] | \exists$ *short exact sequence* $0 \to N_2 \to M \to N_1 \to 0\}$.

(b) $M' \hookrightarrow M \Leftrightarrow F^M_{XM'} \neq 0$ *for some* $X \in$ mod-$\kappa K \Leftrightarrow \exists$ *short exact sequence* $0 \to M' \to M \to X \to 0$ *for some* $X \in$ mod-$\kappa K \Leftrightarrow [M] \in \{[X][M']\}$ *for some* $X \in$ mod-κK.

(c) $M \twoheadrightarrow M \Leftrightarrow F^M_{M'X} \neq 0$ *for some* $X \in$ mod-$\kappa K \Leftrightarrow \exists$ *short exact sequence* $0 \to X \to M \to M' \to 0$ *for some* $X \in$ mod-$\kappa K \Leftrightarrow [M] \in \{[M'][X]\}$ *for some* $X \in$ mod-κK.

So, in order to check the existence of an embedding, projection or short exact sequence, all we have to do is to look for nonzero Ringel-Hall numbers in the corresponding Ringel-Hall product, i.e. to look for the terms in the product. As a consequence of the formulas from [22] and [21] we deduce the following:

Corollary 1.2.3. *We have the following terms in the Ringel-Hall product of various Kronecker modules:*

(a) $\{[P][R][I]\} = \{[P \oplus R \oplus I]\}$, *where* $P, R, I \in$ mod-κK *are arbitrary preprojective, regular respectively preinjective modules;*

(b) $\{[R][R']\} = \{[R'][R]\}$, *moreover this set contains only regulars (for* $R, R' \in \text{mod-}\kappa K$ *arbitrary regulars)*

(c) $\{[I_i][I_j]\} = \begin{cases} \{[I_i \oplus I_j]\} & i-j \geq -1 \\ \{[I_j \oplus I_i], [I_{j-1} \oplus I_{i+1}], ..., [I_{j-[\frac{j-i}{2}]} \oplus I_{i+[\frac{j-i}{2}]}]\} & i-j < -1 \end{cases}$

(d) $\{[P_i][P_j]\} = \begin{cases} \{[P_i \oplus P_j]\} & i-j \leq -1 \\ \{[P_j \oplus P_i], [P_{j+1} \oplus P_{i-1}], ..., [P_{j+[\frac{i-j}{2}]} \oplus P_{i-[\frac{i-j}{2}]}]\} & i-j > -1 \end{cases}$

(e) $\{[I_{n-1-i}][P_i]\} = R_n \cup \{[P_i \oplus I_{n-1-i}]\}$, *where*

$$\mathcal{R}_n = \{[R_{p_1}(t_1) \oplus ... \oplus R_{p_s}(t_s)] \,|\, s \in \mathbb{N}^*, p_i \neq p_j \text{ if } i \neq j, t_1 d_{p_1} + ... + t_s d_{p_s} = n\}$$

(f) $\{[I_m]\mathcal{R}_n\} = \{\mathcal{R}_n[I_m]\} \cup \{\mathcal{R}_{n-1}[I_{m+1}]\} \cup ... \cup \{[I_{m+n}]\}$, *with* $[I_{m+n}] \in \{[I_m][R_n]\}$ *for any* $R_n = R_{p_1}(t_1) \oplus ... \oplus R_{p_s}(t_s)$ *such that* $p_1, \dots p_s \in \mathcal{P}$ *are pairwise distinct points and* $t_1 d_{p_1} + ... + t_s d_{p_s} = n$.

(g) $\{\mathcal{R}_n[P_m]\} = \{[P_m]\mathcal{R}_n\} \cup \{[P_{m+1}]\mathcal{R}_{n-1}]\} \cup ... \cup \{[P_{m+n}]\}$, *with* $[P_{m+n}] \in \{[R_n][P_m]\}$ *for any* $R_n = R_{p_1}(t_1) \oplus ... \oplus R_{p_s}(t_s)$ *such that* $p_1, \dots p_s \in \mathcal{P}$ *are pairwise distinct points and* $t_1 d_{p_1} + ... + t_s d_{p_s} = n$.

As we will see, the generalization of Corollary 1.2.3 plays a crucial role in our study of short exact sequences of Kronecker modules (in Chapter 2) and the matrix subpencil problem (in Chapter 3). Although in this section we have put a finiteness condition on the underlying field κ, we will show in Section 2.3 how to get rid of this restriction, at least in the case of preprojective and preinjective Kronecker modules. By replacing the Ringel-Hall product with Reineke's extension monoid product, we will get the analogue versions of the various product rules presented in Corollary 1.2.3, allowing us to deal with short exact sequences of Kronecker modules over arbitrary fields.

Chapter 2

Short exact sequences of Kronecker modules

2.1 Monomorphisms between preinjectives and epimorphisms between preprojectives

Our aim is to give a numerical criteria in terms of Kronecker invariants for $I' \hookrightarrow I$ (where I', I are preinjectives) and for $P \twoheadrightarrow P'$ (where P', P are preprojectives).

We begin with two (dual) lemmas which permits us to split the "smaller" modules I' and P'.

Lemma 2.1.1 ([25]). *Let $N_1, N_2, M_1, M_2 \in$ mod-κK be Kronecker modules (where κ is an arbitrary field) such that $\mathrm{Ext}^1(N_1, N_2) = 0$ and $\mathrm{Hom}(N_2, M_1) = 0$. Then there exists an exact sequence of the form*

$$0 \to N_1 \oplus N_2 \to M_1 \oplus M_2 \to Y \to 0$$

if and only if there is a module X with exact sequences

$$0 \to N_2 \to M_2 \to X \to 0$$

$$0 \to N_1 \to M_1 \oplus X \to Y \to 0.$$

Dually we have:

Lemma 2.1.2. *Let N_1, N_2, M_1, M_2 be finite dimensional right modules over the Kronecker algebra κK (where κ is a field) such that $\mathrm{Ext}^1(N_1, N_2) = 0$ and $\mathrm{Hom}(M_2, N_1) = 0$. Then there exists an exact sequence of the form*

$$0 \to Y \to M_1 \oplus M_2 \to N_1 \oplus N_2 \to 0$$

if and only if there is a module X with exact sequences

$$0 \to X \to M_1 \to N_1 \to 0$$

$$0 \to Y \to X \oplus M_2 \to N_2 \to 0.$$

The following lemma gives the criteria for the existence of a monomorphism $f : I_n \to I_{n_1} \oplus \cdots \oplus I_{n_p}$ with $n \geq n_1 \geq \cdots \geq n_p \geq 0$.

Lemma 2.1.3. *We consider preinjective modules in* mod-κK *where κ is an arbitrary field. Let $n \geq n_1 \geq \cdots \geq n_p \geq 0$ be integers such that $s = \sum_{i=1}^{p} n_i \geq n$. Then there exists a monomorphism $f : I_n \to I_{n_1} \oplus \cdots \oplus I_{n_p}$. Moreover* $\mathrm{coker} f \cong I_{m_1} \oplus ... \oplus I_{m_{p-1}}$ *where $n \geq m_1 \geq \cdots \geq m_{p-1} \geq 0$.*

Based on the previous lemmas, we are ready now to give the numerical criteria for the existence of a monomorphism $f : I' \to I$ where I, I' are preinjectives.

Theorem 2.1.4 ([25])**.** *Suppose $d_1 \geq ... \geq d_n > 0$ and $c_1 \geq ... \geq c_m > 0$ are integers. We have a monomorphism*

$$f : I_{d_1} \oplus ... \oplus I_{d_n} \oplus dI_0 \to I_{c_1} \oplus ... \oplus I_{c_m} \oplus cI_0$$

if and only if $d \leq c$ and $d_i + ... + d_n \leq \sum_{c_j \leq d_i} c_j$ for $i = \overline{1, n}$ (the empty sum being 0).

Remark 2.1.5. Using the notation $I' = (a_0 I_0) \oplus ... \oplus (a_n I_n) \oplus ...$, $I = (b_0 I_0) \oplus ... \oplus (b_n I_n) \oplus ...$ we have a monomorphism $f : I' \to I$ if and only if

$$\begin{aligned} a_0 &\leq b_0 \\ a_1 &\leq b_1 \\ a_1 + 2a_2 &\leq b_1 + 2b_2 \\ &\vdots \\ a_1 + 2a_2 + ... + na_n &\leq b_1 + 2b_2 + ... + nb_n \\ &\vdots \end{aligned}$$

So one can see that in the preinjective case "a kind of" weighted dominance describes the numerical criteria for the embedding.

Theorem 2.1.4 can be easily dualized for preprojectives:

Theorem 2.1.6 ([25])**.** *Suppose $d_1 \geq ... \geq d_n > 0$ and $c_1 \geq ... \geq c_m > 0$ are integers. We have an epimorphism*

$$f : cP_0 \oplus P_{c_m} \oplus ... \oplus P_{c_1} \to dP_0 \oplus P_{d_n} \oplus ... \oplus P_{d_1}$$

if and only if $d \le c$ *and* $d_i + ... + d_n \le \sum_{c_j \le d_i} c_j$ *for* $i = \overline{1,n}$ *(the empty sum being 0).*

2.2 Some particular preinjective and preprojective short exact sequences

Applying Theorems 2.1.4 and 2.1.6 we describe first the possible middle terms in certain preinjective and preprojective short exact sequences.

The proposition below is well-known:

Proposition 2.2.1. *If* I', I'' *are preinjectives and* X *is a middle term in* $\mathrm{Ext}^1(I', I'')$ *then* X *is also preinjective. Dually if* P', P'' *are preprojectives and* Y *is a middle term in* $\mathrm{Ext}^1(P', P'')$ *then* Y *is also preprojective.*

Corollary 2.2.2 ([25])**.** *Suppose* $d_1 \ge ... \ge d_n > 0$, $c_1 \ge ... \ge c_m > 0$ *and* $a > 0$ *are integers. Then we have:*

(a) $I_{c_1} \oplus ... \oplus I_{c_m} \oplus cI_0$ *is a middle term in* $\mathrm{Ext}^1(I_a, I_{d_1} \oplus ... \oplus I_{d_n} \oplus dI_0)$ *if and only if* $m + c = n + d + 1$, $\sum_{i=1}^m c_i = a + \sum_{j=1}^n d_j$, $d \le c$ *and* $d_i + ... + d_n \le \sum_{c_j \le d_i} c_j$ *for* $i = \overline{1,n}$.

(b) $I_{c_1} \oplus ... \oplus I_{c_m} \oplus cI_0$ *is a middle term in* $\mathrm{Ext}^1(aI_0, I_{d_1} \oplus ... \oplus I_{d_n} \oplus dI_0)$ *if and only if* $m + c = n + d + a$, $\sum_{i=1}^m c_i = \sum_{j=1}^n d_j$, $d \le c$ *and* $d_i + ... + d_n \le \sum_{c_j \le d_i} c_j$ *for* $i = \overline{1,n}$.

Dually we have for the preprojectives:

Corollary 2.2.3 ([25])**.** *Suppose* $d_1 \ge ... \ge d_n > 0$, $c_1 \ge ... \ge c_m > 0$ *and* $a > 0$ *are integers. Then we have:*

(a) $cP_0 \oplus P_{c_m} \oplus ... \oplus P_{c_1}$ *is a middle term in* $\mathrm{Ext}^1(dP_0 \oplus P_{d_n} \oplus ... \oplus P_{d_1}, P_a)$ *if and only if* $m + c = n + d + 1$, $\sum_{i=1}^m c_i = a + \sum_{j=1}^n d_j$, $d \le c$ *and* $d_i + ... + d_n \le \sum_{c_j \le d_i} c_j$ *for* $i = \overline{1,n}$.

(b) $cP_0 \oplus P_{c_m} \oplus ... \oplus P_{c_1}$ *is a middle term in* $\mathrm{Ext}^1(dP_0 \oplus P_{d_n} \oplus ... \oplus P_{d_1}, aP_0)$ *if and only if* $m + c = n + d + a$, $\sum_{i=1}^m c_i = \sum_{j=1}^n d_j$, $d \le c$ *and* $d_i + ... + d_n \le \sum_{c_j \le d_i} c_j$ *for* $i = \overline{1,n}$.

2.3 The field independence of preinjective and preprojective short exact sequences

We prove that the possible middle terms in preinjective and preprojective short exact sequences do not depend on the base field κ. More precisely denote now by $I_n^{(\kappa)}$ the preinjective indecomposable in mod-κK of dimension $(n, n+1)$. Then we have that

Theorem 2.3.1 ([25])**.** *Suppose* $d_1 \geq ... \geq d_n \geq 0$, $c_1 \geq ... \geq c_m \geq 0$ *and* $e_1 \geq ... \geq e_p \geq 0$ *are integers,* κ *and* κ' *are arbitrary fields and we have the short exact sequence in* mod-κK

$$0 \to I_{d_1}^{(\kappa)} \oplus ... \oplus I_{d_n}^{(\kappa)} \to I_{c_1}^{(\kappa)} \oplus ... \oplus I_{c_m}^{(\kappa)} \to I_{e_1}^{(\kappa)} \oplus ... \oplus I_{e_p}^{(\kappa)} \to 0.$$

Then we have a similar short exact sequence in mod-$\kappa' K$*:*

$$0 \to I_{d_1}^{(\kappa')} \oplus ... \oplus I_{d_n}^{(\kappa')} \to I_{c_1}^{(\kappa')} \oplus ... \oplus I_{c_m}^{(\kappa')} \to I_{e_1}^{(\kappa')} \oplus ... \oplus I_{e_p}^{(\kappa')} \to 0.$$

Dually we have:

Theorem 2.3.2 ([25])**.** *Suppose* $d_1 \geq ... \geq d_n \geq 0$, $c_1 \geq ... \geq c_m \geq 0$ *and* $e_1 \geq ... \geq e_p \geq 0$ *are integers,* κ *and* κ' *are arbitrary fields and we have the short exact sequence in* mod-κK

$$0 \to P_{d_n}^{(\kappa)} \oplus ... \oplus P_{d_1}^{(\kappa)} \to P_{c_m}^{(\kappa)} \oplus ... \oplus P_{c_1}^{(\kappa)} \to P_{e_p}^{(\kappa)} \oplus ... \oplus P_{e_1}^{(\kappa)} \to 0.$$

Then we have a similar short exact sequence in mod-$\kappa' K$*:*

$$0 \to P_{d_n}^{(\kappa')} \oplus ... \oplus P_{d_1}^{(\kappa')} \to P_{c_m}^{(\kappa')} \oplus ... \oplus P_{c_1}^{(\kappa')} \to P_{e_p}^{(\kappa')} \oplus ... \oplus P_{e_1}^{(\kappa')} \to 0.$$

2.4 Extensions of Kronecker modules over arbitrary fields

For $d \in \mathbb{N}^2$ let $M_d = \{[M] | M \in \text{mod-}\kappa K, \underline{\dim} M = d\}$ be the set of isomorphism classes of Kronecker modules of dimension d. Following Reineke in [17] for subsets $\mathcal{A} \subset M_d$, $\mathcal{B} \subset M_e$ we define

$$\mathcal{A} * \mathcal{B} = \{[X] \in M_{d+e} \,|\, \exists\, 0 \to N \to X \to M \to 0 \text{ exact for some } [M] \in \mathcal{A}, [N] \in \mathcal{B}\}.$$

So the product $\mathcal{A} * \mathcal{B}$ is the set of isoclasses of all extensions of modules M with $[M] \in \mathcal{A}$ by modules N with $[N] \in \mathcal{B}$. This is in fact Reineke's extension monoid product using isomorphism classes of modules instead of modules. It is important to know (see [17]) that the product above is associative, i.e. for $\mathcal{A} \subset M_d, \mathcal{B} \subset M_e, \mathcal{C} \subset M_f$, we have $(\mathcal{A}*\mathcal{B})*\mathcal{C} = \mathcal{A}*(\mathcal{B}*\mathcal{C})$. Also $\{[0]\} * \mathcal{A} = \mathcal{A} * \{[0]\} = \mathcal{A}$. We will call the operation "$*$" simply the **extension monoid product**.

Remark 2.4.1. For $M, N \in \text{mod-}\kappa K$ and κ finite, the product $\{[M]\} * \{[N]\} = \{[M][N]\}$ (see Section 1.2).

The aim of this section is to describe the products of the form $\{[M]\} * \{[N]\}$, i.e to find all extensions of N by M. It is important to note that by saying "an extension of N by M" we mean a module X, which is a middle term in $\text{Ext}^1(M, N)$. We emphasize that all the results are valid over an arbitrary field κ.

Theorem 2.4.2. *We have the following rules for the monoid product of various Kronecker modules:*

(a) $\{[P]\}*\{[R]\}*\{[I]\}=\{[P\oplus R\oplus I]\}$, *where* $P,R,I\in$ mod-κK *are arbitrary preprojective, regular respectively preinjective modules;*

(b) $\{[R]\}*\{[R']\}=\{[R']\}*\{[R]\}$, *moreover this set contains only regulars (for* $R,R'\in$ mod-κK *arbitrary regulars)*

(c) $\{[I_i]\}*\{[I_j]\}=\begin{cases}\{[I_i\oplus I_j]\} & i-j\geq -1\\ \{[I_j\oplus I_i],[I_{j-1}\oplus I_{i+1}],...,[I_{j-[\frac{j-i}{2}]}\oplus I_{i+[\frac{j-i}{2}]}]\} & i-j<-1\end{cases}$

(d) $\{[P_i]\}*\{[P_j]\}=\begin{cases}\{[P_i\oplus P_j]\} & i-j\leq -1\\ \{[P_j\oplus P_i],[P_{j+1}\oplus P_{i-1}],...,[P_{j+[\frac{i-j}{2}]}\oplus P_{i-[\frac{i-j}{2}]}]\} & i-j>-1\end{cases}$

(e) $\{[I_{n-1-i}]\}*\{[P_i]\}=\mathcal{R}_n\cup\{[P_i\oplus I_{n-1-i}]\}$, *where*

$$\mathcal{R}_n=\{[R_{p_1}(t_1)\oplus...\oplus R_{p_s}(t_s)]\,|\,s\in\mathbb{N}^*,p_i\neq p_j \text{ if } i\neq j,t_1d_{p_1}+...+t_sd_{p_s}=n\}.$$

(f) $\{[I_m]\}*\mathcal{R}_n=\mathcal{R}_n*\{[I_m]\}\cup\mathcal{R}_{n-1}*\{[I_{m+1}]\}\cup...\cup\{[I_{m+n}]\}$.

(g) $\mathcal{R}_n*\{[P_m]\}=\{[P_m]\}*\mathcal{R}_n\cup\{[P_{m+1}]\}*\mathcal{R}_{n-1}\cup...\cup\{[P_{m+n}]\}$.

2.5 The extension monoid product of preinjective (preprojective) Kronecker modules

Let $I'=I_{a_1}\oplus\cdots\oplus I_{a_p}$ and $I=I_{b_1}\oplus\cdots\oplus I_{b_n}$ be preinjective Kronecker modules, where $a_1\geq\cdots\geq a_p$ and $b_1\geq\cdots\geq b_n$. Our aim is to describe the set $\{[I']\}*\{[I]\}$, i.e. the isoclasses appearing in the extension monoid product of $[I']$ and $[I]$. All results of this section can be dualized in a natural way for preprojective modules.

Lemma 2.5.1 ([26]). *For* $a_1\geq\cdots\geq a_n\geq 0$, $c_1\geq\cdots\geq c_p\geq 0$ *and* $b\geq 0$ *nonnegative integers, we have that*

$$[I_{c_1}\oplus\cdots\oplus I_{c_p}]\in\{[I_{a_1}\oplus\cdots\oplus I_{a_n}]\}*\{[I_b]\}$$

if and only if $p=n+1$, $c_1=a_1$, $\dots$, $c_{k-1}=a_{k-1}$, $c_{k+1}\geq a_k$, $\dots$, $c_{n+1}\geq a_n$ *for some* $k\in\{1,\dots,n+1\}$ *and* $\sum_{i=1}^n a_i+b=\sum_{i=1}^{n+1}c_i$.

Lemma 2.5.1 may also be stated in the following equivalent way:

Lemma 2.5.2 ([26]). *For* $a_1\geq\cdots\geq a_n\geq 0$, $c_1\geq\cdots\geq c_p\geq 0$ *and* $b\geq 0$ *nonnegative integers, we have that*

$$[I_{c_1}\oplus\cdots\oplus I_{c_p}]\in\{[I_{a_1}\oplus\cdots\oplus I_{a_n}]\}*\{[I_b]\}$$

if and only if $p=n+1$, $c_1=a_1$, $\dots$, $c_{k-1}=a_{k-1}$, $c_k=b-\sum_{i=k}^n m_i$, $c_{k+1}=a_k+m_k$, $\dots$, $c_{n+1}=a_n+m_n$ *for some* $k\in\{1,\dots,n+1\}$ *and* $m_i\geq 0$, $i=\overline{k,n}$.

We are ready for the main theorem of this section:

Theorem 2.5.3 ([26]). *If* $a_1 \geq \dots a_p \geq 0$*,* $b_1 \geq \dots \geq b_n \geq 0$ *and* $c_1 \geq \dots \geq c_r \geq 0$ *are nonnegative integers, then* $[I_{c_1} \oplus \dots \oplus I_{c_r}] \in \{[I_{a_1} \oplus \dots \oplus I_{a_p}]\} * \{[I_{b_1} \oplus \dots \oplus I_{b_n}]\}$ *if and only if* $r = n + p$*,* $\exists \beta : \{1, \dots, n\} \to \{1, \dots, n+p\}$*,* $\exists \alpha : \{1, \dots, p\} \to \{1, \dots, n+p\}$ *both functions strictly increasing with* $\mathrm{Im}\alpha \cap \mathrm{Im}\beta = \emptyset$ *and* $\exists m_j^i \geq 0$*,* $1 \leq i \leq n$*,* $1 \leq j \leq p$*, such that* $\forall \ell \in \{1, \dots, n+p\}$

$$c_\ell = \begin{cases} b_i - \sum_{\substack{\beta(i)<\alpha(j)\\ 1\leq j\leq p}} m_j^i, \ \textit{where } i = \beta^{-1}(\ell) & \ell \in \mathrm{Im}\beta \\ a_j + \sum_{\substack{\beta(i)<\alpha(j)\\ 1\leq i\leq n}} m_j^i, \ \textit{where } j = \alpha^{-1}(\ell) & \ell \in \mathrm{Im}\alpha \end{cases}. \tag{2.5.1}$$

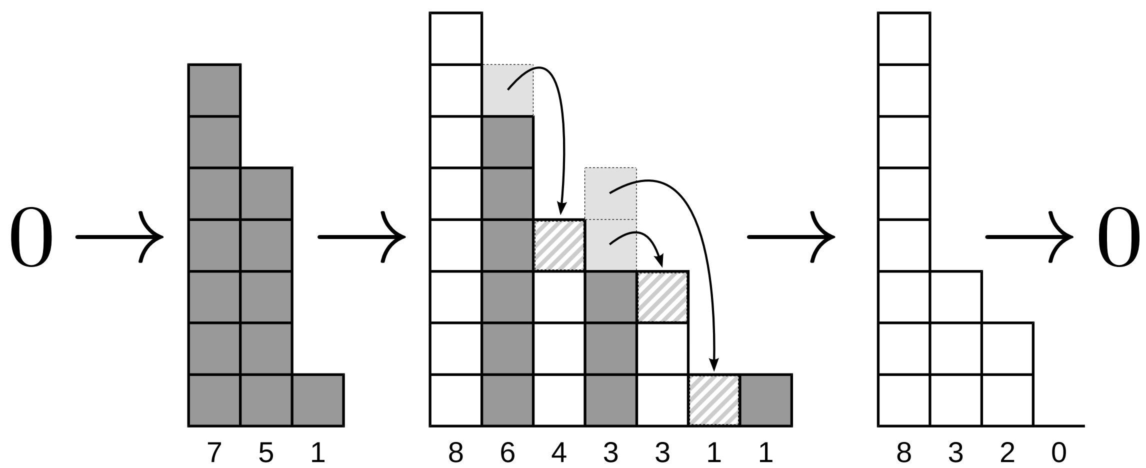
Example 2.5.4. In mod-κK we have that

$$[I_8 \oplus I_6 \oplus I_4 \oplus I_3 \oplus I_3 \oplus I_1 \oplus I_1] \in \{[I_8 \oplus I_3 \oplus I_2 \oplus I_0]\} * \{[I_7 \oplus I_5 \oplus I_1]\}.$$

Using the notations from Theorem 2.5.3, we have $p = 4$, $n = 3$, $r = 7$ and the to strictly increasing functions are $\beta : \{1, 2, 3\} \to \{1, \dots, 7\}$ with $\beta(1) = 2$, $\beta(2) = 4$, $\beta(3) = 7$ and $\alpha : \{1, \dots, 4\} \to \{1, \dots, 7\}$ with $\alpha(1) = 1$, $\alpha(2) = 3$, $\alpha(3) = 5$, $\alpha(4) = 6$. For the values m_j^i, $1 \leq i \leq n$, $1 \leq j \leq p$, we have $m_2^1 = m_3^2 = m_4^2 = 1$ and $m_j^i = 0$ in all other cases. Hence there exists a short exact sequence

$$0 \to I_7 \oplus I_5 \oplus I_1 \to I_8 \oplus I_6 \oplus I_4 \oplus I_3 \oplus I_3 \oplus I_1 \oplus I_1 \to I_8 \oplus I_3 \oplus I_2 \oplus I_0 \to 0,$$

illustrated as follows:



So, less formally, Theorem 2.5.3 claims that $[I_{c_1} \oplus \dots \oplus I_{c_r}] \in \{[I_{a_1} \oplus \dots \oplus I_{a_p}]\} * \{[I_{b_1} \oplus \dots \oplus I_{b_n}]\}$ if and only if the sequence $c_1 \geq \dots \geq c_r \geq 0$ is obtained by merging the sequences $a_1 \geq \dots \geq a_p \geq 0$ and $b_1 \geq \dots \geq b_n \geq 0$ and by applying the "box dropping rule" illustrated in the picture above. This rule says that in the middle term boxes can be dropped only to the right and only from columns corresponding to elements of the sequence $b_1 \geq \dots \geq b_n \geq 0$

on top of columns corresponding to elements of the sequence $a_1 \geq \cdots \geq a_p \geq 0$. The values m_j^i from the theorem denote the number of boxes dropped from the column corresponding to the element b_i on top of the column corresponding to the element a_j.

As immediate consequences, we get the following two corollaries:

Corollary 2.5.5. *Let $I, I', I'' \in$ mod-κK be preinjective Kronecker modules, where $I = I_{c_1} \oplus \cdots \oplus I_{c_r}$, $I' = I_{a_1} \oplus \cdots \oplus I_{a_p}$ and $I'' = I_{b_1} \oplus \cdots \oplus I_{b_n}$. Then there is a short exact sequence*

$$0 \to I_{b_1} \oplus \cdots \oplus I_{b_n} \to I_{c_1} \oplus \cdots \oplus I_{c_r} \to I_{a_1} \oplus \cdots \oplus I_{a_p} \to 0$$

if and only if there is a short exact sequence

$$0 \to I_{b_1+m} \oplus \cdots \oplus I_{b_n+m} \to I_{c_1+m} \oplus \cdots \oplus I_{c_r+m} \to I_{a_1+m} \oplus \cdots \oplus I_{a_p+m} \to 0$$

for some $m \in \mathbb{N}$.

Corollary 2.5.6. *For $a_1 \geq \cdots \geq a_n \geq 0$, $c_1 \geq \cdots \geq c_p \geq 0$ and $b \geq 0$ nonnegative integers, we have that*

$$[I_{c_1} \oplus \cdots \oplus I_{c_p}] \in \{[I_b]\} * \{[I_{a_1} \oplus \cdots \oplus I_{a_n}]\}$$

if and only if $p = n+1$, $c_1 = a_1 - m_1$, ..., $c_{k-1} = a_{k-1} - m_{k-1}$, $c_k = b + \sum_{i=1}^{k-1} m_i$, $c_{k+1} = a_k$, ..., $c_{n+1} = a_n$ for some $k \in \{1, \ldots, n+1\}$ and $m_i \geq 0$, $i = \overline{k, n}$.

Theorem 2.5.3 can be easily dualized for preprojectives:

Theorem 2.5.7 ([26])**.** *If $a_1 \geq \ldots a_p \geq 0$, $b_1 \geq \cdots \geq b_n \geq 0$ and $c_1 \geq \cdots \geq c_r \geq 0$ are nonnegative integers, then $[P_{c_r} \oplus \cdots \oplus P_{c_1}] \in \{[P_{b_n} \oplus \cdots \oplus P_{b_1}]\} * \{[P_{a_p} \oplus \cdots \oplus P_{a_1}]\}$ if and only if $r = n + p$, $\exists \beta : \{1, \ldots, n\} \to \{1, \ldots, n+p\}$, $\exists \alpha : \{1, \ldots, p\} \to \{1, \ldots, n+p\}$ both functions strictly increasing with $\mathrm{Im}\alpha \cap \mathrm{Im}\beta = \emptyset$ and $\exists m_j^i \geq 0$, $1 \leq i \leq n$, $1 \leq j \leq p$, such that $\forall \ell \in \{1, \ldots, n+p\}$*

$$c_\ell = \begin{cases} b_i - \sum_{\substack{\beta(i)<\alpha(j) \\ 1 \leq j \leq p}} m_j^i, \ \text{where } i = \beta^{-1}(\ell) & \ell \in \mathrm{Im}\beta \\ a_j + \sum_{\substack{\beta(i)<\alpha(j) \\ 1 \leq i \leq n}} m_j^i, \ \text{where } j = \alpha^{-1}(\ell) & \ell \in \mathrm{Im}\alpha \end{cases}.$$

Corollary 2.5.5 also has an obvious analogue version for preprojective Kronecker modules. We have seen that Theorem 2.5.3 describes the combinatorics of the monoid product of two arbitrary preinjective Kronecker modules. An even more general result has been proved in [28]:

Theorem 2.5.8 ([28])**.** *For $a_1, \ldots, a_n, c_1, \ldots, c_r \in \mathbb{N}$, $c_1 \geq \cdots \geq c_r \geq 0$, $r, n \geq 2$ we have that*

$$[I_{c_1} \oplus \cdots \oplus I_{c_r}] \in \{[I_{a_1}]\} * \cdots * \{[I_{a_n}]\}$$

if and only if $r = n$, $\exists \sigma \in \mathcal{S}_n$ *a permutation and* $\exists m_j^i \geq 0$ *nonnegative integers,* $1 \leq j < i \leq n$, *such that* $\forall \ell \in \{1, \ldots, n\}$

$$c_\ell = a_{\sigma(\ell)} + \sum_{i=\sigma(\ell)+1}^{n} m^i_{\sigma(\ell)} - \sum_{j=1}^{\sigma(\ell)-1} m_j^{\sigma(\ell)},$$

and the following conditions hold:

(i) $m_j^i > 0 \Longrightarrow \sigma^{-1}(i) < \sigma^{-1}(j)$,

(ii) $a_j > a_i \Longrightarrow \sigma^{-1}(i) > \sigma^{-1}(j)$.

The proof is somewhat similar to that of Theorem 2.5.3. Using this theorem one can give an easy proof for an interesting corollary, which was observed in [23]:

Corollary 2.5.9. *Suppose that* $0 \leq a_1 \leq \cdots \leq a_n$. *Then* $[I_{c_1} \oplus \cdots \oplus I_{c_n}] \in \{[I_{a_1}]\} * \cdots * \{[I_{a_n}]\}$ *if and only if* $c_1 \geq \cdots \geq c_n \geq 0$, $|\mu| = |\lambda|$ *and* $\mu \trianglelefteq \lambda$, *where* $\mu = (c_1, c_2, \ldots, c_n)$ *and* $\lambda = (a_n, a_{n-1}, \ldots, a_1)$[1].

Remark 2.5.10. Theorem 2.5.8 and the previous corollary apply dually to preprojective modules as well.

2.6 The extension monoid product of a preinjective and a preprojective Kronecker module

In what follows, we are going to give some results needed in the solution of the matrix subpencil problem in Chapter 3.

Lemma 2.6.1. *Let* $d_1 \geq \cdots \geq d_q \geq 0$ *and* $c_1 \geq \cdots \geq c_r \geq 0$ *be nonnegative integers. Then* $[I_{c_1} \oplus \cdots \oplus I_{c_r}] \in \{[I_{d_1} \oplus \cdots \oplus I_{d_q}]\} * \{[P_n]\}$ *if and only if* $r = q - 1$ *and we have* $c_1 = d_1 + m_1, \ldots, c_l = d_l + m_l$, $c_{l+1} = d_{l+2}, \ldots, c_{q-1} = d_q$ *for some* $l \in \{1, \ldots, q-1\}$ *with* $m_i \geq 0$, $i = \overline{1, l}$ *and* $\sum_{i=1}^{l} m_i = d_{l+1} + n + 1$.

Lemma 2.6.2. *Let* $d_1 \geq \cdots \geq d_q \geq 0$ *and* $c_1 \geq \cdots \geq c_{q-1} \geq 0$ *be nonnegative integers. Then* $[I_{c_1} \oplus \cdots \oplus I_{c_{q-1}}] \in \{[I_{d_1} \oplus \cdots \oplus I_{d_q}]\} * \{[P_n]\}$ *if and only if* $[I_{d_1+n+1} \oplus \cdots \oplus I_{d_q+n+1}] \in \{[I_0]\} * \{[I_{c_1+n+1} \oplus \cdots \oplus I_{c_{q-1}+n+1}]\}$.

Proof. Follows easily from Corollary 2.5.6 and Lemma 2.6.1. □

Theorem 2.6.3. *Let* $q > n > 0$, $d_1 \geq \cdots \geq d_q \geq 0$, $c_1 \geq \cdots \geq c_{q-n} \geq 0$ *and* $0 \leq a_1 \leq \cdots \leq a_n$ *be nonnegative integers. Then* $[I_{c_1} \oplus \cdots \oplus I_{c_{q-n}}] \in \{[I_{d_1} \oplus \cdots \oplus I_{d_q}]\} * \{[P_{a_1} \oplus \cdots \oplus P_{a_n}]\}$ *if and only if* $[I_{d_1+a_n+1} \oplus \cdots \oplus I_{d_q+a_n+1}] \in \{[I_{a_n-a_1} \oplus \cdots \oplus I_{a_n-a_{n-1}} \oplus I_0]\} * \{[I_{c_1+a_n+1} \oplus \cdots \oplus I_{c_{q-n}+a_n+1}]\}$, *or equivalently there is a short exact sequence*

$$0 \to P_{a_1} \oplus \cdots \oplus P_{a_n} \to I_{c_1} \oplus \cdots \oplus I_{c_{q-n}} \to I_{d_1} \oplus \cdots \oplus I_{d_q} \to 0$$

[1]Here μ and λ are partitions, see Section 1.1 for notations and other details.

if and only if there is a short exact sequence

$$0 \to I_{c_1+a_n+1}\oplus\cdots\oplus I_{c_{q-n}+a_n+1} \to I_{d_1+a_n+1}\oplus\cdots\oplus I_{d_q+a_n+1} \to I_{a_n-a_1}\oplus\cdots\oplus I_{a_n-a_{n-1}}\oplus I_0 \to 0.$$

Corollary 2.6.4. *Let $q > \alpha > 0$, $d_1 \geq \cdots \geq d_q \geq 0$ and $c_1 \geq \cdots \geq c_{q-n} \geq 0$ be nonnegative integers. Then $[I_{c_1}\oplus\cdots\oplus I_{c_{q-\alpha}}] \in \{[I_{d_1}\oplus\cdots\oplus I_{d_q}]\}*\{[\alpha P_0]\}$ if and only if $[I_{d_1+1}\oplus\cdots\oplus I_{d_q+1}] \in \{[\alpha I_0]\} * \{[I_{c_1+1} \oplus \cdots \oplus I_{c_{q-\alpha}+1}]\}$, or equivalently there is a short exact sequence*

$$0 \to \alpha P_0 \to I_{c_1} \oplus \cdots \oplus I_{c_{q-\alpha}} \to I_{d_1} \oplus \cdots \oplus I_{d_q} \to 0$$

if and only if there is a short exact sequence

$$0 \to I_{c_1+1} \oplus \cdots \oplus I_{c_{q-\alpha}+1} \to I_{d_1+1} \oplus \cdots \oplus I_{d_q+1} \to \alpha I_0 \to 0.$$

2.7 Computing the extension monoid product of preinjective and preprojective Kronecker modules

In order to be able to handle computationally in the most efficient way possible the characterization given in Theorem 2.5.3, we must get rid of the condition requiring the existence of the nonnegative integers m_j^i from the theorem. In the following two lemmas, we replace the condition involving the existence by some inequalities depending only on the sequences $(a_1, \ldots, a_p)$, $(b_1, \ldots, b_n)$ and $(c_1, \ldots, c_r)$ and on the functions α and β.

The following theorem characterizes the extension of preinjective Kronecker modules by explicit, easy to check numerical conditions, involving only the decreasing sequences of integers obtained from the dimension vectors of the respective modules.

Theorem 2.7.1 ([27]). *Let $a_1 \geq \ldots a_p \geq 0$, $b_1 \geq \cdots \geq b_n \geq 0$, $c_1 \geq \cdots \geq c_r \geq 0$ be decreasing sequences of nonnegative integers and let $B_j = \{l \in \{0, \ldots, n\} | \sum_{k=1}^{l} b_k + \sum_{k=1}^{j} a_k \geq \sum_{k=1}^{l+j} c_k\}$ for $j = \overline{1,p}$. Then*

$$[I_{c_1} \oplus \cdots \oplus I_{c_r}] \in \{[I_{a_1} \oplus \cdots \oplus I_{a_p}]\} * \{[I_{b_1} \oplus \cdots \oplus I_{b_n}]\}$$

if and only if $r = p + n$, $\sum_{i=1}^{r} c_i = \sum_{i=1}^{p} a_i + \sum_{i=1}^{n} b_i$, $B_j \neq \emptyset$, $a_j \leq c_{\alpha_j}$ and $b_i \geq c_{\beta_i}$ for $j = \overline{1,p}$ and $i = \overline{1,n}$, where

$$\alpha_j = \begin{cases} \min B_1 + 1 & j = 1 \\ \max\{\alpha_{j-1} + 1, \min B_j + j\} & 1 < j \leq p \end{cases}$$

and

$$\beta_i = \begin{cases} \min\{l \in \{1, \ldots, r\} | l \neq \alpha_j, j = \overline{1,p}\} & i = 1 \\ \min\{l \in \{\beta_{i-1} + 1, \ldots, r\} | l \neq \alpha_j, j = \overline{1,p}\} & 1 < i \leq n \end{cases}.$$

Theorem 2.7.1 may seem thorny at first sight, so we are going to show how to use it in order to obtain an algorithm which decides in linear time whether a certain preinjective Kronecker module is an extension of two other preinjective Kronecker modules.

Suppose we are given three preinjective modules $I_{a_1}\oplus\cdots\oplus I_{a_p}$, $I_{b_1}\oplus\cdots\oplus I_{b_n}$ and $I_{c_1}\oplus\cdots\oplus I_{c_r}$ and we want to decide if $[I_{c_1}\oplus\cdots\oplus I_{c_r}]\in\{[I_{a_1}\oplus\cdots\oplus I_{a_p}]\}*\{[I_{b_1}\oplus\cdots\oplus I_{b_n}]\}$. Obviously, if $r\neq p+n$ or $\sum_{k=1}^{r}c_r\neq\sum_{j=1}^{p}a_j+\sum_{i=1}^{n}b_i$, the answer is a quick and unhesitating no, so in what follows, we suppose that $r=p+n$ and $\sum_{k=1}^{r}c_r=\sum_{j=1}^{p}a_j+\sum_{i=1}^{n}b_i$ both hold, and we work only with the decreasing sequences $(a_1,\dots,a_p)$, $(b_1,\dots,b_n)$ and $(c_1,\dots,c_r)$.

So, let us set the initial values $j=i=k=1$ for the integers used to index elements from the sequences $(a_1,\dots,a_p)$, $(b_1,\dots,b_n)$ respectively $(c_1,\dots,c_r)$. In a practical implementation one can repeat the following steps for all successive values of $k=\overline{1,r}$:

1. If $j\leq p$ and $a_j\leq c_k$ and $(a_1+\cdots+a_{j-1})+(b_1+\cdots+b_{i-1})+a_j\geq c_1+\cdots+c_k$, then increase j by one.

2. Else, if $i\leq n$ and $b_i\geq c_k$ and $(a_1+\cdots+a_{j-1})+(b_1+\cdots+b_{i-1})+b_i\geq c_1+\cdots+c_k$, then increase i by one.

3. If none of the steps above can be carried out than stop with a negative answer, i.e. $[I_{c_1}\oplus\cdots\oplus I_{c_r}]\notin\{[I_{a_1}\oplus\cdots\oplus I_{a_p}]\}*\{[I_{b_1}\oplus\cdots\oplus I_{b_n}]\}$.

Finally, if one of the first two steps can be made for $k=r$ too, then return a positive answer, i.e. $[I_{c_1}\oplus\cdots\oplus I_{c_r}]\in\{[I_{a_1}\oplus\cdots\oplus I_{a_p}]\}*\{[I_{b_1}\oplus\cdots\oplus I_{b_n}]\}$.

Remark 2.7.2. If we know that $I'\hookrightarrow I$, then then the possible factors I/I' are explicitly described by the result in Theorem 2.7.1. So, if we are given three preinjective Kronecker modules I, I' and I'' such that $I'\hookrightarrow I$, then we can decide in linear time if $[I'']\in\{[I/\mathrm{Im}f]|f: I'\to I \text{ is a monomorphism}\}$.

It is trivial to see that the algorithm is linear in the number of indecomposables (i.e. in $r=n+p$), since the only cycle in the algorithm runs at most r times and the partial sums $a_1+\cdots+a_j$, $b_1+\cdots+b_i$ and $c_1+\cdots+c_k$ can be computed one term at a time at every iteration.

To develop an algorithm for generating all the middle terms X in the short exact sequence $0\to I'\to X\to I\to 0$, we could use of course "brute force" and generate all possible modules while checking every one in part with the previous method. But we can do a little better than that for example by using the method of non-recursive backtracking (also known as "iterative backtracking") to generate all the middle terms in $\mathrm{Ext}^1(I_{a_1}\oplus\cdots\oplus I_{a_p}, I_{b_1}\oplus\cdots\oplus I_{b_n})$. In general, using the backtracking method, one can find all solutions to some computational problem, by incrementally building solution candidates, and abandoning each partial candidate as soon as it is determined that the candidate cannot possibly be completed to a valid solution (see [13]). In our case the space of possible solutions (or candidates) is a subset of the set all decreasing sequences of nonnegative integers $(c_1,\dots,c_r)$ with a fixed length and a fixed sum.

Remark 2.7.3. The number of middle terms can be huge. For example, in worst case, when we want to compute the middle terms in $\mathrm{Ext}^1(I_n, mI_0)$, where $m \geq n$, the number of middle terms is $\mathcal{P}(n)$, with $\mathcal{P}(n)$ being the number of partitions of the integer n. In practice however, we have found that using the method described, one can generate almost instantly extensions when they are up to around 100000 in number, so it fits its purpose quiet well. Based on Theorem 2.7.1 a similar method can be developed to generate all factors I/I', when $I' \hookrightarrow I$ is given.

Example 2.7.4. Using a non-recursive (iterative) backtracking implementation in GAP [31] it turns out that:

1. There are 18 possible middle terms in $\mathrm{Ext}^1(I_4 \oplus I_3 \oplus I_1 \oplus I_0, I_5 \oplus I_3 \oplus I_3 \oplus I_2 \oplus I_1)$, namely the following: $[I_5 \oplus I_4 \oplus I_3 \oplus I_3 \oplus I_3 \oplus I_2 \oplus I_1 \oplus I_1 \oplus I_0]$, $[I_5 \oplus I_4 \oplus I_3 \oplus I_3 \oplus I_3 \oplus I_1 \oplus I_1 \oplus I_1 \oplus I_1]$, $[I_5 \oplus I_4 \oplus I_3 \oplus I_3 \oplus I_2 \oplus I_2 \oplus I_2 \oplus I_1 \oplus I_0]$, $[I_5 \oplus I_4 \oplus I_3 \oplus I_3 \oplus I_2 \oplus I_2 \oplus I_1 \oplus I_1 \oplus I_1]$, $[I_5 \oplus I_4 \oplus I_3 \oplus I_2 \oplus I_2 \oplus I_2 \oplus I_2 \oplus I_1 \oplus I_1]$, $[I_4 \oplus I_4 \oplus I_4 \oplus I_3 \oplus I_3 \oplus I_2 \oplus I_1 \oplus I_1 \oplus I_0]$, $[I_4 \oplus I_4 \oplus I_4 \oplus I_3 \oplus I_3 \oplus I_1 \oplus I_1 \oplus I_1 \oplus I_1]$, $[I_4 \oplus I_4 \oplus I_4 \oplus I_3 \oplus I_2 \oplus I_2 \oplus I_2 \oplus I_1 \oplus I_0]$, $[I_4 \oplus I_4 \oplus I_4 \oplus I_3 \oplus I_2 \oplus I_2 \oplus I_1 \oplus I_1 \oplus I_1]$, $[I_4 \oplus I_4 \oplus I_4 \oplus I_2 \oplus I_2 \oplus I_2 \oplus I_2 \oplus I_1 \oplus I_1]$, $[I_4 \oplus I_4 \oplus I_3 \oplus I_3 \oplus I_3 \oplus I_2 \oplus I_2 \oplus I_1 \oplus I_0]$, $[I_4 \oplus I_4 \oplus I_3 \oplus I_3 \oplus I_3 \oplus I_2 \oplus I_1 \oplus I_1 \oplus I_1]$, $[I_4 \oplus I_4 \oplus I_3 \oplus I_3 \oplus I_2 \oplus I_2 \oplus I_2 \oplus I_1 \oplus I_1]$, $[I_4 \oplus I_4 \oplus I_3 \oplus I_2 \oplus I_2 \oplus I_2 \oplus I_2 \oplus I_2 \oplus I_1]$, $[I_4 \oplus I_3 \oplus I_3 \oplus I_3 \oplus I_3 \oplus I_3 \oplus I_2 \oplus I_1 \oplus I_0]$, $[I_4 \oplus I_3 \oplus I_3 \oplus I_3 \oplus I_3 \oplus I_3 \oplus I_1 \oplus I_1 \oplus I_1]$, $[I_4 \oplus I_3 \oplus I_3 \oplus I_3 \oplus I_3 \oplus I_2 \oplus I_2 \oplus I_1 \oplus I_1]$ and $[I_4 \oplus I_3 \oplus I_3 \oplus I_3 \oplus I_2 \oplus I_2 \oplus I_2 \oplus I_2 \oplus I_1]$.

2. The number of possible middle terms in $\mathrm{Ext}^1(I_{19} \oplus I_{15} \oplus I_8 \oplus I_4 \oplus I_1 \oplus I_1, I_{26} \oplus I_{12} \oplus I_{10} \oplus I_8 \oplus I_4)$ is 102501, generated in 2 seconds on a laptop computer.

3. The number of possible middle terms in $\mathrm{Ext}^1(I_{16} \oplus I_{11} \oplus I_7 \oplus I_6 \oplus I_3 \oplus I_1 \oplus I_1 \oplus I_0, I_{20} \oplus I_{19} \oplus I_{18} \oplus I_{10} \oplus I_8 \oplus I_3 \oplus I_2 \oplus I_2)$ is 3322698, and all the 3322698 modules were generated in just under 2 minutes on a laptop computer.

Remark 2.7.5. The methods and results described in this section will work in the case of preprojective modules as well, after switching over the order of arguments and reversing the indexes.

Chapter 3

Matrix pencils

3.1 Polynomial matrices

A **polynomial matrix**, or **λ-matrix** is a matrix $A(\lambda) \in \mathcal{M}_{m,n}(\kappa[\lambda])$ with the entries polynomials in λ:

$$A(\lambda) = \left(a_{ij}(\lambda)\right) = \left(a_{ij}^{(0)}\lambda^l + a_{ij}^{(1)}\lambda^{l-1} + \cdots + a_{ij}^{(l)}\right),$$

where for each element $a_{ij}(\lambda) \in \kappa[\lambda]$, l is the largest of the degrees of the polynomials $a_{ij}(\lambda)$, $i = \overline{1,m}$ and $j = \overline{1,n}$.

In this section we recall some basic notions related to polynomial matrices, such as: the equivalence of polynomial matrices, the invariant polynomials and the elementary divisors. We end with the following well-known result:

Theorem 3.1.1 ([10]). *Two polynomial matrices $A(\lambda)$ and $B(\lambda)$ of the same size are equivalent if and only if they have the same invariant polynomials, or equivalently, if and only if they have the same elementary divisors.*

For details on these notions, see [10].

3.2 The normal forms of a matrix

In this section we do a quick review of the normal (canonical) forms matrices from $\mathcal{M}_n(\kappa)$ can be brought to, namely the first and the second natural normal form (in the case of an arbitrary field κ) and the Jordan normal form (in the case when κ is algebraically closed). These are block-diagonal forms, where the blocs are the companion matrices of the invariant polynomials (in the case of the first normal form), the companion matrices of the elementary divisors (in the case of the second normal form) and the well-known Jordan blocks (in the case when κ is algebraically closed).

3.3 Linear matrix pencils

In this section we introduce the central notion of this chapter, the (linear) matrix pencils – a special case of polynomial matrices. We present the results clarifying under which conditions are two matrix pencils strictly equivalent. We also describe their canonical form, given by the so called classical Kronecker invariants.

A polynomial matrix of the form $A+\lambda B \in \mathcal{M}_{m,n}(\kappa[\lambda])$ is called a **linear matrix pencil**. In the sequel we will usually omit the word "linear" and we refer to linear matrix pencils simply as **matrix pencil**s.

Two matrix pencils $A+\lambda B, A'+\lambda B' \in \mathcal{M}_{m,n}(\kappa[\lambda])$ are told to be **strictly equivalent** if there exist constant square non-singular matrices $P \in \mathcal{M}_m(\kappa)$ and $Q \in \mathcal{M}_n(\kappa)$ such that $A'+\lambda B' = P(A+\lambda B)Q$. We denote by $A'+\lambda B' \sim A+\lambda B$ the strict equivalence of the two pencils.

Note that we have the following equivalence:

$$A'+\lambda B' = P(A+\lambda B)Q \iff A' = PAQ \text{ and } B' = PBQ.$$

Matrix pencils pencils are determined up to strict equivalence by some integer parameters, called the **classical Kronecker invariants**. Classical Kronecker invariants are of four type: minimal indices for columns, minimal indices for rows, finite elementary divisors and infinite elementary divisors. The following theorem based on [10] established the connection between the classical Kronecker invariants and the canonical block-diagonal form of an arbitrary matrix pencil:

Theorem 3.3.1. *An arbitrary matrix pencil $A+\lambda B$ is strictly equivalent to the following block-diagonal form*

$$A+\lambda B \sim \operatorname{diag}(0_{h\times g}, L_{\varepsilon_{g+1}}, \ldots, L_{\varepsilon_p}, L^{\intercal}_{\eta_{h+1}}, \ldots, L^{\intercal}_{\eta_q}, E_{u_1}+\lambda H_{u_1}, \ldots, E_{u_t}+\lambda H_{u_t}, J+\lambda E),$$

where

(a) *the g zero-filled columns and the blocks $L_{\varepsilon_{g+1}}, \ldots, L_{\varepsilon_p}$ correspond to the minimal indices for columns $0=\varepsilon_1=\cdots=\varepsilon_g \le \varepsilon_{g+1} \le \cdots \le \varepsilon_p$,*

(b) *the h zero-filled rows and the blocks $L^{\intercal}_{\eta_{h+1}}, \ldots, L^{\intercal}_{\eta_q}$ correspond to the minimal indices for rows $0=\eta_1=\cdots=\eta_h \le \eta_{h+1} \le \cdots \le \eta_q$,*

(c) *the diagonal blocks $E_{u_1}+\lambda H_{u_1}, \ldots, E_{u_t}+\lambda H_{u_t}$ correspond to infinite elementary divisors $\mu^{u_1}, \ldots, \mu^{u_t}$,*

(d) *the normal form of the last diagonal block $J+\lambda E$ is uniquely determined by the finite elementary divisors of the pencil.*

This matrix is the canonical form of the pencil $A+\lambda B$ in the most general case[1].

[1] Of course, depending on the pencil $A+\lambda B$, some of the "families" of diagonal blocks may be missing.

Without insisting on the details, we just give the form of the diagonal blocks determined by the minimal indices for columns. If the integer $\varepsilon > 0$ is a minimal index for columns of the pencil $A + \lambda B$, then the pencil has a block of the form

$$L_\varepsilon = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda & 1 \end{pmatrix}$$

when written in its canoncial block-diagonal form, where $L_\varepsilon \in \mathcal{M}_{\varepsilon,\varepsilon+1}(\kappa[\lambda])$.

Given an arbitrary matrix pencil $A + \lambda B$, there are several methods (and software implementations) to transform this pencil to its canonical diagonal form and reveal its classical Kronecker invariants. Description of these methods are beyond the scope of this thesis, so we refer the reader to works such as [3, 6, 7, 16, 24].

3.4 The matrix subpencil problem

A pencil $A' + \lambda B'$ is called **subpencil** of $A + \lambda B$ if and only if there are pencils $A_{12} + \lambda B_{12}$, $A_{21} + \lambda B_{21}$, $A_{22} + \lambda B_{22}$ such that

$$A + \lambda B \sim \begin{pmatrix} A' + \lambda B' & A_{12} + \lambda B_{12} \\ A_{21} + \lambda B_{21} & A_{22} + \lambda B_{22} \end{pmatrix}.$$

In this case we also say that the subpencil can be completed to the bigger pencil. We speak about row completion when $A_{12}, B_{12}, A_{22}, B_{22}$ are zero matrices and about column completion when $A_{21}, B_{21}, A_{22}, B_{22}$ are zero.

There is an unsolved challenge in pencil theory with lots of applications in control theory (problems related to pole placement, non-regular feedback, dynamic feedback etc. may be formulated in terms of matrix pencils, for details see [14]):

Challenge: If $A+\lambda B$, $A'+\lambda B'$ are pencils over $\mathbb{C}$, find a necessary and sufficient condition in terms of their classical Kronecker invariants for $A' + \lambda B'$ to be a subpencil of $A + \lambda B$. Also construct the completion pencils $A_{12} + \lambda B_{12}$, $A_{21} + \lambda B_{21}$, $A_{22} + \lambda B_{22}$. A particular case of the challenge above is when we limit ourselves to column or row completions.

3.5 The matrix pencil – Kronecker module correspondence

Next we will translate the notions from Section 3.3 (taken from pencil theory) into the language of Kronecker modules (representations). A matrix pencil $A + \lambda B \in \mathcal{M}_{m,n}(\mathbb{C}[\lambda])$ corresponds to the Kronecker module $M_{A,B} = (\kappa^m, \kappa^n; f_A, f_B)$, where choosing the canonical basis in κ^n and κ^m, the matrix of $f_A : \kappa^n \to \kappa^m$ (respectively of $f_B : \kappa^n \to \kappa^m$) is A (respectively B). The strict equivalence $A + \lambda B \sim A' + \lambda B'$ means the isomorphism of modules $M_{A,B} \cong M_{A',B'}$. It follows easily that a pencil $A' + \lambda B'$ is a subpencil of $A + \lambda B$ if and only if the module $M_{A',B'}$ is a **subfactor** of $M_{A,B}$ i.e. there is a module N such that $M_{A',B'} \twoheadleftarrow N \hookrightarrow M_{A,B}$ or equivalently there is a module L such that $M_{A',B'} \hookrightarrow L \twoheadleftarrow M_{A,B}$ (see [11]). We will call the modules N and L **linking module**s. Based on [11] we prove the following:

Theorem 3.5.1. *$A' + \lambda B' \in \mathcal{M}_{m',n'}(\mathbb{C}[\lambda])$ is a subpencil of $A + \lambda B \in \mathcal{M}_{m,n}(\mathbb{C}[\lambda])$ if and only if $m \geq m'$, $n \geq n'$ and $[M_{A,B}] \in \{[(n-n')I_0]\} * \{[M_{A',B'}]\} * \{[(m-m')P_0]\}$.*

In particular a pencil $A' + \lambda B'$ is a subpencil of $A + \lambda B$ by column completions if and only if $M_{A',B'} \hookrightarrow M_{A,B}$ with factor isomorphic to tI_0 where $t \in \mathbb{N}$ is arbitrary. Respectively, a pencil $A' + \lambda B'$ is a subpencil of $A + \lambda B$ by row completions if and only if $M_{A',B'} \twoheadleftarrow M_{A,B}$ with kernel isomorphic to tP_0 where $t \in \mathbb{N}$ is arbitrary.

A preinjective module $I_{b_1} \oplus ... \oplus I_{b_k}$ corresponds to the matrix pencil with the following classical Kronecker invariants:

- minimal indices for columns: $b_1, ..., b_k$;
- no minimal indices for rows, no finite elementary divisors, no infinite elementary divisors.

A preprojective module $P_{b_1} \oplus ... \oplus P_{b_k}$ corresponds to the matrix pencil with the following classical Kronecker invariants:

- minimal indices for rows: $b_1, ..., b_k$;
- no minimal indices for columns, no finite elementary divisors, no infinite elementary divisors.

A regular module $\bigoplus_{p \in \mathbb{C} \cup \{\infty\}} R_p(\nu^{(p)}) = \bigoplus_{p \in \mathbb{C} \cup \{\infty\}} \left(R_p(\nu_1^{(p)}) \oplus \cdots \oplus R_p(\nu_{s_p}^{(p)}) \right)$ (where $\nu^{(p)}$ is a partition for every point $p \in \mathbb{C} \cup \{\infty\}$) corresponds to the regular matrix pencil with the following classical Kronecker invariants:

- for $p = \infty$ the partition $\nu^{(\infty)}$ describes the dimensions of the diagonal blocks associated to the infinite elementary divisors of the pencil
- for every $p \in \mathbb{C}$, the partition $\nu^{(p)}$ describes the dimensions of the Jordan blocks corresponding to the characteristic value p (determined by the finite elementary divisors of the pencil).

3.6 Solution of the subpencil problem in a particular case

As an application of our results on short exact sequences of Kronecker modules presented in Chapter 2 we show how to solve the matrix subpencil problem in a special case.

Let us consider matrix pencils $A + \lambda B$, $A' + \lambda B'$ over $\mathbb{C}$, having only minimal indices for columns among their classical Kronecker invariants. In this case, using the notations from

Section 3.3 $A+\lambda B \sim \operatorname{diag}(L_{\varepsilon_1},\dots,L_{\varepsilon_p})$ and $A'+\lambda B' \sim \operatorname{diag}(L_{\varepsilon'_1},\dots,L_{\varepsilon'_q})$, where $\varepsilon_1 \le \dots \le \varepsilon_p$ and $\varepsilon'_1 \le \dots \le \varepsilon'_q$ are the minimal indices for columns. Hence, as explained in Section 3.5, one may identify the pencil $A+\lambda B$ with the module $M_{A,B} = I = I_{\varepsilon_p} \oplus \dots \oplus I_{\varepsilon_1} \in$ mod-$\mathbb{C}K$ and the pencil $A'+\lambda B'$ with the module $M_{A',B'} = I' = I_{\varepsilon'_q} \oplus \dots \oplus I_{\varepsilon'_1} \in$ mod-$\mathbb{C}K$. Using this identification, we have that $A'+\lambda B'$ is a subpencil of $A+\lambda B$ if and only if I' is a subfactor of I, that is if and only if there exists a Kronecker module $L \in$ mod-$\mathbb{C}K$ such that $I' \hookrightarrow L \twoheadleftarrow I$. We have the following theorem:

Theorem 3.6.1. *If $I' = a_n I_n \oplus \dots \oplus a_0 I_0$ and $I = c_n I_n \oplus \dots \oplus c_0 I_0$ are preinjective Kronecker modules, then I' is a subfactor of I (i.e. $\exists L$ such that $I' \hookrightarrow L \twoheadleftarrow I$) if and only if*

$$b_1 \le \frac{1}{2}\left(\sum_{i=1}^{n}(i+1)c_i - \sum_{i=2}^{n}(i+1)b_i\right) \quad \textit{and} \quad b_0 \ge a_0,$$

where

$$b_k = \begin{cases} \sum_{i=0}^{n}(i+1)c_i - \sum_{i=1}^{n}(i+1)b_i & k=0 \\ \sum_{i=1}^{n} i a_i - \sum_{i=2}^{n} i b_i & k=1 \\ \left\lfloor \min\left(\frac{\sum_{i=k}^{n} i a_i - \sum_{i=k+1}^{n} i b_i}{k}, \frac{\sum_{i=k}^{n}(i+1)c_i - \sum_{i=k+1}^{n}(i+1)b_i}{k+1}\right)\right\rfloor & 2 \le k < n \\ \min(a_n, c_n) & k=n \end{cases}.$$

In this case (the values $b_0,\dots,b_n$ being nonnegative) one of the linking modules is $L = b_n I_n \oplus \dots \oplus b_0 I_0$.

Example 3.6.2. Consider the following matrix pencils written in canonical diagonal form and having only minimal indices for columns among their classical Kronecker invariants:

$$A+\lambda B = \begin{pmatrix} \lambda & 1 & 0 & 0 & & & & & & & & & & \\ 0 & \lambda & 1 & 0 & & & & & & & & & & \\ 0 & 0 & \lambda & 1 & & & & & & & & & & \\ & & & & \lambda & 1 & 0 & 0 & & & & & & \\ & & & & 0 & \lambda & 1 & 0 & & & & & & \\ & & & & 0 & 0 & \lambda & 1 & & & & & & \\ & & & & & & & & \lambda & 1 & 0 & 0 & & \\ & & & & & & & & 0 & \lambda & 1 & 0 & & \\ & & & & & & & & 0 & 0 & \lambda & 1 & & \\ & & & & & & & & & & & & \lambda & 1 \end{pmatrix} \in \mathcal{M}_{10,14}(\mathbb{C}[\lambda])$$

and

$$A'+\lambda B' = \begin{pmatrix} 0 & \lambda & 1 & 0 & 0 & 0 & 0 & & & & & \\ & 0 & \lambda & 1 & 0 & 0 & 0 & & & & & \\ & 0 & 0 & \lambda & 1 & 0 & 0 & & & & & \\ & 0 & 0 & 0 & \lambda & 1 & 0 & & & & & \\ & 0 & 0 & 0 & 0 & \lambda & 1 & & & & & \\ & & & & & & & \lambda & 1 & 0 & & \\ & & & & & & & 0 & \lambda & 1 & & \\ & & & & & & & & & & \lambda & 1 \end{pmatrix} \in \mathcal{M}_{8,12}(\mathbb{C}[\lambda]).$$

The pencil $A+\lambda B$ has $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 3$, and $\varepsilon_4 = 1$ as its minimal indices for columns, while in the case of the pencil $A'+\lambda B'$ these are $\varepsilon'_1 = 0$, $\varepsilon'_2 = 5$, $\varepsilon'_3 = 2$ and $\varepsilon'_4 = 1$. Hence the corresponding modules are $M_{A,B} = I_3 \oplus I_3 \oplus I_3 \oplus I_1$ and $M_{A',B'} = I_5 \oplus I_2 \oplus I_1 \oplus I_0$. Written using the multiplicative notation used in Theorem 3.6.1, $M_{A',B'} = \bigoplus_{i=0}^{5} a_i I_i$ and $M_{A,B} = \bigoplus_{i=0}^{5} c_i I_i$, where $(a_0, a_1, \ldots, a_5) = (1,1,1,0,0,1)$ and $(c_0, c_1, \ldots, c_5) = (0,1,0,3,0,0)$. We use the recursive formula from the theorem to compute the sequence $(b_0, b_1, \ldots, b_5) = (2,1,2,1,0,0)$ and to find out that the inequalities

$$b_1 \leq \frac{1}{2}\left(\sum_{i=1}^{5}(i+1)c_i - \sum_{i=2}^{5}(i+1)b_i\right) \quad \text{and} \quad b_0 \geq a_0$$

are satisfied. So $A' + \lambda B'$ is a subpencil of $A + \lambda B$ or equivalently, $M_{A',B'}$ is a subfactor of $M_{A,B}$, i.e. $\exists L$ such that $M_{A',B'} \hookrightarrow L \twoheadleftarrow M_{A,B}$. Moreover, we can take the linking module L to be $L = \bigoplus_{i=0}^{5} b_i I_i = I_3 \oplus I_2 \oplus I_2 \oplus I_1 \oplus I_0 \oplus I_0$. We could use at this point Theorem 2.1.4 to verify the existence of the embedding $M_{A',B'} \hookrightarrow L$ and Corollary 2.6.4 to verify the existence of the projection $L \twoheadleftarrow M_{A,B}$ with the kernel equal to $2P_0$. The matrix pencil corresponding to the module L is

$$L_1 + \lambda L_2 = \begin{pmatrix} 0 & 0 & \lambda & 1 & 0 & 0 & & & & & & & & \\ & & 0 & \lambda & 1 & 0 & & & & & & & & \\ & & 0 & 0 & \lambda & 1 & & & & & & & & \\ & & & & & & \lambda & 1 & 0 & & & & & \\ & & & & & & 0 & \lambda & 1 & & & & & \\ & & & & & & & & & \lambda & 1 & 0 & & \\ & & & & & & & & & 0 & \lambda & 1 & & \\ & & & & & & & & & & & & \lambda & 1 \end{pmatrix} \in \mathcal{M}_{8,14}(\mathbb{C}[\lambda]).$$

Let us construct now the completion matrices $A_{12} + \lambda B_{12}$, $A_{21} + \lambda B_{21}$, $A_{22} + \lambda B_{22}$, i.e. those matrix blocks for which the following equivalence holds:

$$A + \lambda B \sim \begin{pmatrix} A' + \lambda B' & A_{12} + \lambda B_{12} \\ A_{21} + \lambda B_{21} & A_{22} + \lambda B_{22} \end{pmatrix}.$$

Since we have an embedding $M_{A',B'} \overset{f}{\hookrightarrow} L$, we must have $f = (F_1, F_2)$, where $F_1 \in \mathcal{M}_{14,12}(\mathbb{C})$ and $F_2 \in \mathcal{M}_8(\mathbb{C})$ are full-rank matrices such that $(L_1 + \lambda L_2)F_1 = F_2(A' + \lambda B')$. Also, for the projection $M_{A,B} \overset{g}{\twoheadrightarrow} L$, we have $g = (G_1, G_2)$, where $G_1 \in \mathcal{M}_{14}(\mathbb{C})$ and $G_2 \in \mathcal{M}_{10,8}(\mathbb{C})$ are full-rank matrices such that $(L_1 + \lambda L_2)G_1 = G_2(A + \lambda B)$. Calculations show that these matrices can be taken to be:

$$F_1 = \begin{pmatrix} 0&0&0&0&1&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&0&0&1 \end{pmatrix} \qquad \text{and} \qquad F_2 = E_8,$$

where E_8 is the 8×8 identity matrix and

$$G_1 = \begin{pmatrix} 0&0&0&0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&1 \end{pmatrix} \quad \text{and} \quad G_2 = \begin{pmatrix} 1&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&1 \end{pmatrix}.$$

Since F_2 and G_1 are full-rank matrices square matrices, they are invertible. In our case $F_2^{-1} = F_2$ and $G_1^{-1} = G_1^{\intercal}$. The matrices $G_1^{-1}F_1$ and $F_2^{-1}G_2$ are also full-rank matrices, so there are non-singular square matrices C_1, C_2, D_1 and D_2 such that $G_1^{-1}F_1 = C_1 \begin{pmatrix} E_{12} \\ 0 \end{pmatrix} C_2$ and $F_2^{-1}G_2 = D_1 \begin{pmatrix} E_8 & 0 \end{pmatrix} D_2$, respectively. In our case these matrices are

$$C_1 = \begin{pmatrix} 0&0&-1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&-1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&-1&0&0&0&0&0&0&0&0&0 \\ -1&0&0&0&0&0&0&0&0&0&0&0&1&0 \\ -1&0&0&0&0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0&0&0&0 \\ -1&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&-1&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&-1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&-1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&-1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&-1&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&-1&0&0 \end{pmatrix},$$

$$C_2 = \begin{pmatrix} 0&0&0&0&-1&0&0&0&0&0&0&0 \\ -1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&-1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&-1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&-1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&-1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&-1&0&0&0 \\ 0&0&0&0&0&0&0&0&0&-1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&-1&0 \\ 0&0&0&0&0&0&0&0&0&0&0&-1 \end{pmatrix},$$

$$D_1 = E_8, \quad \text{and} \quad D_2 = \begin{pmatrix} 1&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0 \end{pmatrix}.$$

Using these matrices we can write:

$$\begin{aligned} A' + \lambda B' &= F_2^{-1}F_2(A' + \lambda B') = F_2^{-1}(L_1 + \lambda L_2)F_1 \\ &= F_2^{-1}(L_1 + \lambda L_2)G_1 G_1^{-1}F_1 = F_2^{-1}G_2(A + \lambda B)G_1^{-1}F_1 \\ &= D_1 \begin{pmatrix} E_8 & 0 \end{pmatrix} D_2(A + \lambda B)C_1 \begin{pmatrix} E_{12} \\ 0 \end{pmatrix} C_2. \end{aligned}$$

So $D_1^{-1}(A' + \lambda B')C_2^{-1} = \begin{pmatrix} E_8 & 0 \end{pmatrix} D_2(A + \lambda B)C_1 \begin{pmatrix} E_{12} \\ 0 \end{pmatrix}$, hence

$$\begin{aligned} A' + \lambda B' &= \begin{pmatrix} E_8 & 0 \end{pmatrix} \begin{pmatrix} D_1 & \\ & E_2 \end{pmatrix} D_2(A + \lambda B)C_1 \begin{pmatrix} C_2 & \\ & E_2 \end{pmatrix} \begin{pmatrix} E_{12} \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} E_8 & 0 \end{pmatrix} D_2(A + \lambda B)C' \begin{pmatrix} E_{12} \\ 0 \end{pmatrix}, \end{aligned}$$

where

$$C' = \begin{pmatrix} 0&1&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&1&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&1&0&0 \end{pmatrix}.$$

Obviously, $A + \lambda B \sim D_2(A + \lambda B)C'$, where

$$D_2(A + \lambda B)C' = \begin{pmatrix} 0&\lambda&1&0&0&0&0&&&&&&0&0 \\ &0&\lambda&1&0&0&0&&&&&&0&0 \\ &0&0&\lambda&1&0&0&&&&&&1&0 \\ &0&0&0&\lambda&1&0&&&&&&0&\lambda \\ &0&0&0&0&\lambda&1&&&&&&0&0 \\ &&&&&&&\lambda&1&0&&&0&0 \\ &&&&&&&0&\lambda&1&&&0&0 \\ &&&&&&&&&&\lambda&1&0&0 \\ 0&0&0&0&1&0&\lambda&0&0&0&0&0&0&0 \\ \lambda&0&0&0&0&0&0&1&0&0&0&0&0&0 \end{pmatrix} = \begin{pmatrix} A' + \lambda B' & A_{12} + \lambda B_{12} \\ A_{21} + \lambda B_{21} & A_{22} + \lambda B_{22} \end{pmatrix},$$

with the completion pencils

$$A_{12} + \lambda B_{12} = \begin{pmatrix} 0&0 \\ 0&0 \\ 1&0 \\ 0&\lambda \\ 0&0 \\ 0&0 \\ 0&0 \\ 0&0 \end{pmatrix}, \quad A_{21} + \lambda B_{21} = \begin{pmatrix} 0&0&0&0&1&0&\lambda&0&0&0&0&0 \\ \lambda&0&0&0&0&0&0&1&0&0&0&0 \end{pmatrix}, \quad A_{22} + \lambda B_{22} = \begin{pmatrix} 0&0 \\ 0&0 \end{pmatrix}.$$

Remark 3.6.3. The calculations were verified using the computer algebra system Maxima [32].

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