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PARTIAL DIFFERENTIAL EQUATIONS AND OPTION PRICING WITH STOCHASTIC VOLATILITY

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Contents

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Introduction

Fixed point theory is a powerful tool for the study of differential equations, integral equations, partial differential equations and their associated problems, as well as for integral and differential inclusions and problems associated them. The approach in this case is based on the following method: under appropriate assumptions on the given problem, one can equivalently transform the problem associated to a differential equation or to an integral equation or to a partial differential equation (respectively to a differential inclusion or to an integral inclusion) into a fixed point equation (respectively inclusion) of the following form $x = t(x)$ (respectively $x \in T(x)$). Using abstract fixed point theorems one obtain existence, uniqueness and other qualitative properties of the solution set (such as data dependence, well-posedness, stability, limit shadowing property, etc.)

The first purpose of this thesis is to present a fixed point theory for the class of singlevalued and multivalued φ-contractions in complete metric space. A φ-contraction is a nontrivial extension of the classical concept of contraction. More precisely, if (X, d) is a metric space and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a mapping, then a multivalued operator $T : X \to P_{b,cl}(X)$ is called a multivalued φ-contraction if:

(i) φ is a strict comparison function (i.e., φ is increasing and $\sum_{n=0}^{\infty} \varphi^n(t) < +\infty$, for each $t > 0$);

(ii) $H(T(x), T(y)) \leq \varphi(d(x, y))$, for all $x, y \in X$ (where H denotes the Pompeiu-Hausdorff metric on $P_{b,cl}(X)$.

Notice that, if $\varphi(t) = kt$ (with $k \in [0, 1[$), then we get the classical notion of multivalued k-contraction in Nadler' sense. Notice also, that the first fixed point theorem for singlevalued φ-contractions in complete metric spaces was given in 1975 by J. Matkowski and I.A. Rus, while the case of multivalued φ-contractions in complete metric spaces was considered for the first time in 1982 by R. Węgrzyk. Since then, some other results involving multivalued φ-contractions were given by I.A. Rus, J.S. Bae, A. Sîntămărian, B.E. Rhoades, A. Petruşel, A. Muntean, X.Y.-Z. Yuan, etc. In the second chapter of the thesis, we will discuss several properties (such as existence, uniqueness, data dependence, approximation, Ulam-Hyers stability, well-posedness, limit shadowing) of the fixed point inclusion $x \in T(x)$ or of the strict fixed point equation $\{x\} = T(x)$, where $T : X \to P_{cl}(X)$ is a multivalued φ-contraction. The case of fixed point theory on a set endowed with two metric is also presented.

The second purpose of the thesis is to apply the abstract results above mentioned to some problems generated by partial differential equations and inclusions and by integral equations and inclusions. This is the aim of the last two sections of Chapter 2. We obtain existence, uniqueness, approximation, data dependence and Ulam-Hyers stability results for the Dirichlet problem associated to a nonlinear partial differential equation involving the Laplace operator, as well as similar results for Fredholm and Volterra integral inclusions and for a Darboux problem associated to a second order differential inclusion of order two. Our results extend and complement some known theorems in the recent literature, such as that given in I.A. Rus, G. Teodoru, A. Cernea, Castro and Ramos, S.-M. Jung, N. Lungu, C. Crăciun and N. Lungu, S. Reich and A.J. Zaslavski, M. Xu.

The third purpose of this thesis is to present some results concerning the problem of option pricing under the stochastic volatility model given in 1993 by S.L. Heston. Modelling the volatility of financial time series via stochastic volatility models has received a great deal of attention in the finance literature. There are two types of volatility models: continuous-time stochastic volatility models such as: Hull and White model (1987), Wiggins(1987), Stein and Stein(1991), Heston(1993), Bates(1996) and discrete-time stochastic generalized autoregressive conditional heteroskedasticity (GARCH) models, see Taylor(1986), Amin and Ng(1993), Heston and Nandi(1993) and so on.

In their famous work [15], in 1973, Black and Scholes transformed the option pricing problem into the task of solving a parabolic partial differential equation with a final condition. Since then, many of the pricing models lead to partial differential equations, which usually are linear and parabolic. In the third chapter of the thesis we will deal with the partial differential equation corresponding to the Heston model. We give an analytical solution for the digital options and we use numerical methods to solve the option pricing equation in the case of European put options in a Foreign Exchange setting.

The structure of the work is as follows.

The first chapter, entitled **Preliminaries** contains the most important notations, notions and results which are used throughout the thesis.

The second chapter, entitled **Fixed points methods for partial differential equations** presents a fixed point theory for multivalued operators satisfying to a nonlinear contraction condition with a comparison type function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$.

In the first section of this chapter, we discussed the theory of the metrical fixed point theorem of R. Węgrzyk [144], by the following perspectives: existence of the (strict) fixed points, uniqueness of the fixed point and of the strict fixed point, data dependence of the fixed point set, sequence of multivalued operators and fixed points, Ulam-Hyers stability of a multivalued fixed point inclusion, well-posedness of the fixed point problem, set-to-set operatorial equations with multivalued φ-contractions, fixed points for the fractal operator generated by a multivalued operator. In the second part, we proved some new fixed point results for non-self multivalued operators defined on a set with two metrics. Then, in the next part of this chapter, we will give some fixed point results for operators satisfying to a compactness condition of Mönch type, condition introduced in another context by D. O'Regan and R. Precup in [93]. Then, in the last two sections of this chapter, we will apply part of the results given below to the study of the existence and Ulam-Hyers stability of some integral, differential and partial differential equations and inclusions.

The following results belong to the author:

- in the paragraph 2.1.1: Theorem 2.1.3 – Theorem 2.1.10. These results generalize some recent theorems given by A. Petruşel and I.A. Rus (see the paper A. Petruşel and I.A. Rus: The theory of a metric fixed point theorem for multivalued operators, Proc. Ninth International Conference on Fixed Point Theory and its Applications, Changhua, Taiwan, July 16–22, 2009, 161–175, 2010) and they have been published in **V.L. Lazăr**, Fixed point theory for multivalued φ-contractions, Fixed Point Theory and Applications, 2011(2011), 12 pages, doi:10.1186/1687-1812-2011-50;

- in the paragraph 2.1.2: Theorem 2.1.13 - Theorem 2.1.16, results which complement and extend some known theorems in the literature given by A. Petruşel, I.A. Rus [101], T.A. Lazăr, A. Petruşel, N. Shahzad [67], A. Chiş-Novac, R. Precup, I.A. Rus [25], M. Frigon, A. Granas [34]. These results are published in T.A. Lazăr, **V.L. Lazăr**, Fixed points for non-self multivalued operators on a set with two metrics, JP Journal of Fixed Point Theory and Applications, 4(2009), No. 3, 183-191;

- in the paragraph 2.1.3: Theorem 2.1.19, Theorem 2.1.20. These theorems extend, to maps satisfying some compactness conditions introduced by D. O'Regan and R. Precup [93], some results given by R.P. Agarwal and D. O'Regan [1] concerning essential maps in the sense of Mönch. They have been published in **V.L. Lazăr**, On the essentiality of the Mönch type maps, Seminar on Fixed Point Theory, 1(2000), 59-62;

- in the section 2.2: Theorem 2.2.2 and Theorem 2.2.3. The theorems concern with existence and Ulam-Hyers stability results for the Dirichlet problem associated to some nonlinear elliptic partial differential equations. Our approach is based on the weakly Picard operator technique. These results have been presented at the 7th International Conference on Applied Mathematics, Baia Mare, September 01-04, 2010 and will be published in **V.L. Lazăr**, Ulam-Hyers stability results for partial differential inclusions, accepted for publication in Creative Math. Inform., 20(2011), No.3. The Ulam stability (Ulam-Hyers, Ulam-Hyers-Rassias, Ulam-Hyers-Bourgin,...) of various functional equations have been investigated by many authors (see [18, 22, 29, 35, 50, 51, 56, 94, 109, 110]). There are also some results for differential equations ([57, 59, 60, 84, 122]), integral equations ([58, 121]), partial differential equations ([79], [80], [129], [130]) and for difference equations [19, 106, 107]).

- in the section 2.3: Theorem 2.3.2-Theorem 2.3.4, results which have been presented at the International Conference on Nonlinear Operators and Differential Equations, Cluj-Napoca, July 5-8, 2011 and submitted for publication to Electronic Journal of Qualitative Theory of Differential Equations, see V.L. Lazăr [75]. Using the multivalued weakly Picard operator technique, the above theorems present existence and Ulam-Hyers stability results for integral inclusions of Fredholm and Volterra type and for some partial differential inclusions.

In the third chapter, entitled **Heston model**, we will present some results concerning the problem of option pricing under a stochastic volatility model, namely the Heston model (see [44], 1993).

The aim of the first section of this chapter is to analyse the problem of digital option pricing under the Heston stochastic volatility model. In this model the variance v, follows the same square-root process as the one used by Cox, Ingersoll and Ross (see [26]) from the short term interest rate. We present an analytical solution for this kind of options, based on S. Heston's original work [44].

In the last section of this chapter we use finite difference and element methods to solve numerically the option pricing equation in the case of European put options under the Heston model, solving partial differential equations by this standard numerical methods being possible for a wide range of option models (see [31, 55, 63, 64, 147, 148]).

Author's contributions are:

- in the section 3.1: Theorem 3.1.1, Theorem 3.1.2, results which have been published in **L.V. Lazăr**, Pricing Digital Call Option in the Heston stochastic volatility model, Studia Univ. Babeş-Bolyai, Mathematica, vol. XLVIII, no.3, 2003, 83-92. These theorems give a closed-form solution for digital options with stochastic volatility in the case of the Heston model;

- in the section 3.2: Theorem 3.2.1. We have considered here the case of European put options under stochastic volatility in a Foreign Exchange setting to illustrate how the finite difference and finite element methods can be used. The option pricing partial differential equation for the Heston model is a convection-diffusion equation with the diffusion term linear in v and so is the convection term up to an additional constant. For more about convection-diffusion equation and numerical methods see [3, 63, 64, 86, 108, 115, 139, 147, 148]. The author's results have been presented at International Conference on Nonlinear Operators and Differential Equations, August 24-27, 2004, Cluj-Napoca and have been published in **V.L. Lazăr**, Finite difference and element methods for pricing options with stochastic volatility, Int. Journal of Pure and Applied Mathematics, vol. 28(2006), No.3, 339-354.

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Cluj-Napoca, September 2011 Ph.D. Student Vasile Lucian Lazăr

Chapter 1

Preliminaries

The purpose of this chapter is to recall some basic notions and results needed in the main part of this Ph.D. thesis. Thus, in the first part we review some concepts and theorems from Nonlinear Analysis, while in the last two paragraphs we will give notions and results from the Options Theory.

1.1 Some Concepts and Results in Nonlinear Analysis

Throught this thesis we will use the classical notations and notions in Nonlinear Analysis, see [8, 9, 13, 38, 39, 40, 47, 62, 97, 100, 125, 126, 140, 153].

1.2 Basic Option Theory

1.3 Stochastic Volatility Models

In this two paragraphs we recall classic notions and results from the option pricing theory, see [2, 4, 12, 15, 26, 44, 45, 49, 63, 64, 72, 77, 137, 146, 147, 145].

Chapter 2

Fixed Points Methods for Partial Differential Equations

The purpose of this chapter is to present a fixed point theory for multivalued operators satisfying a nonlinear contraction condition with a comparison type function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$. In the first section of this chapter, we will discuss the theory of the metrical fixed point theorem of R. Węgrzyk [144], by the following perspectives: existence of the fixed points, uniqueness of the fixed point and the strict fixed point, data dependence of the fixed point set, sequence of multivalued operators and fixed points, Ulam–Hyers stability of a multivalued fixed point inclusion, well-posedness of the fixed point problem, set-to-set operatorial equations, fixed points for the fractal operator generated by a multivalued operator. In the second part of this section, we proved some new results of the fixed point theory for a set with two metrics for non-self multivalued generalized contractions. As consequences, new open operators principles are obtained. In the last part of this section, we will give some fixed point results for operators satisfying to a compactness condition of Mönch type, condition introduced in another context by D. O'Regan and R. Precup in [93]. Then, in the last two sections of this chapter, we will apply part of the results given below to the study of the existence and Ulam-Hyers stability of some integral, diferential and partial differential equations and inclusions.

2.1 Fixed Point Theorems

2.1.1 Fixed Point Theory for Multivalued φ-Contractions

Our results generalize some recent theorems given in A. Petruşel and I.A. Rus (The theory of a metric fixed point theorem for multivalued operators, Proc. Ninth International Conference on Fixed Point Theory and its Applications, Changhua, Taiwan, July 16–22, 2009, 161–175, 2010). For the single-valued case see I.A. Rus [120] and [116].

Let $T : X \to P(X)$ be a multivalued operator. Then, the operator $\hat{T} : P(X) \to P(X)$ defined by

$$\hat{T}(Y) := \bigcup_{x \in Y} T(x), \quad \text{for } Y \in P(X)$$

is called the fractal operator generated by T.

It is known that if (X, d) is a metric spaces and $T : X \to P_{cp}(X)$, then the following conclusions hold:

(a) if T is upper semicontinuous, then $T(Y) \in P_{cp}(X)$, for every $Y \in P_{cp}(X)$;

(b) the continuity of T implies the continuity of $\hat{T} : P_{cp}(X) \to P_{cp}(X)$.

If $T : X \to P(X)$, then $T^0 := 1_X,\ T^1 := T, \ldots, T^{n+1} = T \circ T^n,\ n \in \mathbb{N}$ denote the iterate operators of T.

By definition, a periodic point for a multivalued operator $T : X \to P_{cp}(X)$ is an element $p \in X$ such that $p \in F_{T^m}$, for some integer $m \geq 1$, i.e., $p \in \hat{T}^m(\{p\})$ for some integer $m \geq 1$.

We recall now from Rus et al. [131] the notion of multivalued $\psi-$weakly Picard operator.

Definition 2.1.1 *Let (X, d) be a metric space and $T : X \to P(X)$ be a MWP operator. Then, we define the multivalued operator $T^\infty : Graph(T) \to P(F_T)$ by the formula $T^\infty(x, y) = \{\ z \in F_T \mid$ there exists a sequence of successive approximations of T starting from (x, y) that converges to $z\ \}$.*

Definition 2.1.2 *Let (X, d) be a metric space and let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing function which is continuous in 0 and $\psi(0) = 0$. Then $T : X \to P(X)$ is said to be a multivalued ψ-weakly Picard operator if it is a multivalued weakly Picard operator and there exists a selection $t^\infty : Graph(T) \to Fix(T)$ of T^∞ such that*

$$d(x, t^\infty(x, y)) \leq \psi(d(x, y)), \quad \text{for all } (x, y) \in Graph(T).$$

If there exists $c > 0$ such that $\psi(t) = ct$, for each $t \in \mathbb{R}_+$, then T is called a multivalued c-weakly Picard operator.

Example 2.1.1 *Let (X, d) be a complete metric space and $T : X \to P_{cl}(X)$ be a multivalued a-contraction. Then T is a c-MWP operator, where $c = (1 - a)^{-1}$.*

The following result is known in the literature as Matkowski-Rus's theorem.

Thorem 2.1.1 *(J. Matkowski [82], I. A. Rus [125]) Let (X, d) be a complete metric space and $f : X \to X$ be a φ-contraction, i.e., $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a comparison function and*

$$d(f(x), f(y)) \leq \varphi(d(x, y)) \quad \text{for all } x, y \in X.$$

Then f is a Picard operator, i.e., f has a unique fixed point $x^* \in X$ and $\lim_{n\to+\infty} f^n(x) = x^*$, for all $x \in X$.

The multivalued variant of the above result is the following theorem proved by R. Węgrzyk, (see [144]).

Thorem 2.1.2 *Let (X, d) be a complete metric space and $T : X \to P_{cl}(X)$ be a multivalued φ-contraction, where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a strict comparison function. Then F_T is nonempty and for any $x_0 \in X$ there exists a sequence of successive approximations of T starting from x_0 which converges to a fixed point of T.*

Our first result concerns the case of multivalued φ-contractions.

Thorem 2.1.3 *(V.L. Lazăr, [73]) Let (X, d) be a complete metric space and $T : X \to P_{cl}(X)$ be a multivalued φ-contraction. Then, we have:*

(i) (existence and approximation of the fixed point) T is a MWP operator (see R. Węgrzyk [144]);

(ii) If additionally $\varphi(qt) \leq q\varphi(t)$ for every $t \in \mathbb{R}_+$ (where $q > 1$) and $t = 0$ is a point of uniform convergence for the series $\sum_{n=1}^{\infty} \varphi^n(t)$, then T is a ψ-MWP operator, with $\psi(t) := t + s(t)$, for each $t \in \mathbb{R}_+$ (where $s(t) := \sum_{n=1}^{\infty} \varphi^n(t)$);

(iii) (Data dependence of the fixed point set) Let $S : X \to P_{cl}(X)$ be a multivalued φ-contraction and $\eta > 0$ be such that $H(S(x), T(x)) \leq \eta$, for each $x \in X$. Suppose that $\varphi(qt) \leq q\varphi(t)$ for every $t \in \mathbb{R}_+$ (where $q > 1$) and $t = 0$ is a point of uniform convergence for the series $\sum_{n=1}^{\infty} \varphi^n(t)$. Then, $H(F_S, F_T) \leq \psi(\eta)$;

(iv) (sequence of operators) Let $T, T_n : X \to P_{cl}(X), n \in \mathbb{N}$ be multivalued φ-contractions such that $T_n(x) \xrightarrow{H} T(x)$ as $n \to +\infty$, uniformly with respect to each $x \in X$. Then, $F_{T_n} \xrightarrow{H} F_T$ as $n \to +\infty$.

If, moreover $T(x) \in P_{cp}(X)$, for each $x \in X$, then we additionally have:

(v) (generalized Ulam–Hyers stability of the inclusion $x \in T(x)$) Let $\epsilon > 0$ and $x \in X$ be such that $D(x, T(x)) \leq \epsilon$. Then there exists $x^* \in F_T$ such that $d(x, x^*) \leq \psi(\epsilon)$;

(vi) T is upper semicontinuous, $\hat{T} : (P_{cp}(X), H) \to (P_{cp}(X), H), \hat{T}(Y) := \bigcup_{x \in Y} T(x)$ is a set-to-set φ-contraction and (thus) $F_{\hat{T}} = \{A_T^*\}$;

(vii) $T^n(x) \xrightarrow{H} A_T^*$ as $n \to +\infty$, for each $x \in X$;

(viii) $F_T \subset A_T^*$ and F_T is compact;

(ix) $A_T^* = \bigcup_{n \in \mathbb{N}^*} T^n(x)$, for each $x \in F_T$.

Remark 2.1.1 *For related results to (vi) and (vii)-(ix) see also Andres-Górniewicz [7] and Chifu and Petrușel [24].*

A second result for multivalued φ-contractions is as follows.

Thorem 2.1.4 *(V.L.Lazăr, [73]) Let (X, d) be a complete metric space and $T : X \to P_{cl}(X)$ be a multivalued φ-contraction with $SF_T \neq \emptyset$. Then, the following* *assertions hold:*

(x) $F_T = SF_T = \{x^*\}$ *(see A. Sîntămărian [135]);*

(xi) If, additionally $T(x)$ *is compact for each* $x \in X$*, then* $F_{T^n} = SF_{T^n} = \{x^*\}$ *for* $n \in \mathbb{N}^*$*;*

(xii) If, additionally $T(x)$ *is compact for each* $x \in X$*, then* $T^n(x) \xrightarrow{H} \{x^*\}$ *as* $n \to +\infty$*, for each* $x \in X$*;*

(xiii) Let $S : X \to P_{cl}(X)$ *be a multivalued operator and* $\eta > 0$ *such that* $F_S \neq \emptyset$ *and* $H(S(x), T(x)) \leq \eta$*, for each* $x \in X$*. Then,* $H(F_S, F_T) \leq \beta(\eta)$*, where* $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ *is given by* $\beta(\eta) := \sup\{t \in \mathbb{R}_+ |\ t - \varphi(t) \leq \eta\}$*;*

(xiv) Let $T_n : X \to P_{cl}(X), n \in \mathbb{N}$ *be a sequence of multivalued operators such that* $F_{T_n} \neq \emptyset$ *for each* $n \in \mathbb{N}$ *and* $T_n(x) \xrightarrow{H} T(x)$ *as* $n \to +\infty$*, uniformly with respect to* $x \in X$*. Then,* $F_{T_n} \xrightarrow{H} F_T$ *as* $n \to +\infty$*.*

(xv) (Well-posedness of the fixed point problem with respect to D*) If* $(x_n)_{n\in\mathbb{N}}$ *is a sequence in* X *such that* $D(x_n, T(x_n)) \to 0$ *as* $n \to \infty$*, then* $x_n \xrightarrow{d} x^*$ *as* $n \to \infty$*;*

(xvi) (Well-posedness of the fixed point problem with respect to H*) If* $(x_n)_{n\in\mathbb{N}}$ *is a sequence in* X *such that* $H(x_n, T(x_n)) \to 0$ *as* $n \to \infty$*, then* $x_n \xrightarrow{d} x^*$ *as* $n \to \infty$*;*

(xvii) (Limit shadowing property of the multivalued operator) Suppose additionally that φ *is a sub-additive function. If* $(y_n)_{n\in\mathbb{N}}$ *is a sequence in* X *such that* $D(y_{n+1}, T(y_n)) \to 0$ *as* $n \to \infty$*, then there exists a sequence* $(x_n)_{n\in\mathbb{N}} \subset X$ *of successive approximations for* T*, such that* $d(x_n, y_n) \to 0$ *as* $n \to \infty$*.*

A third result for multivalued φ-contraction is the following.

Thorem 2.1.5 *(V.L. Lazăr, [73]) Let* (X, d) *be a complete metric space and* $T : X \to P_{cp}(X)$ *be a multivalued* φ*-contraction such that* $T(F_T) = F_T$*. Then, we have:*

(xviii) $T^n(x) \xrightarrow{H} F_T$ *as* $n \to +\infty$*, for each* $x \in X$*;*

(xix) $T(x) = F_T$*, for each* $x \in F_T$*;*

(xx) If $(x_n)_{n\in\mathbb{N}} \subset X$ *is a sequence such that* $x_n \xrightarrow{d} x^* \in F_T$ *as* $n \to \infty$*, then*

$T(x_n) \stackrel{H}{\to} F_T$ *as* $n \to +\infty$.

For compact metric spaces, we have:

Thorem 2.1.6 *(V.L. Lazăr, [73]) Let* (X,d) *be a compact metric space and* $T : X \to P_{cl}(X)$ *be a multivalued* φ*-contraction. Then, we have:*

(xxi) (Generalized well-posedness of the fixed point problem with respect to D) if $(x_n)_{n\in\mathbb{N}}$ *is a sequence in* X *such that* $D(x_n, T(x_n)) \to 0$ *as* $n \to \infty$*, then there exists a subsequence* $(x_{n_i})_{i\in\mathbb{N}}$ *of* $(x_n)_{n\in\mathbb{N}}$ $x_{n_i} \stackrel{d}{\to} x^* \in F_T$ *as* $i \to \infty$.

Remark 2.1.2 *For the particular case* $\varphi(t) = at$ *(with* $a \in [0,1[$*), for each* $t \in \mathbb{R}_+$ *see Petruşel and Rus [102].*

We need now another important concept in the theory of strict fixed points for multivalued operators.

Definition 2.1.3 *(Petruşel-Rus [103], [102]) Let* (X,d) *be a metric space. Then, by definition,* $T : X \to P(X)$ *is called a multivalued Picard operator if and only if:*

(i) $(SF)_T = F_T = \{x^*\}$;

(ii) $T^n(x) \stackrel{H_d}{\to} \{x^*\}$ *as* $n \to \infty$*, for each* $x \in X$.

The problem is to give sufficient conditions such that T is a multivalued Picard operator.

Thorem 2.1.7 *(V.L. Lazăr) Let* (X,d) *be a complete metric space and let* $T : X \to P_{cp}(X)$ *be a multivalued* φ*-contraction with* $(SF)_T \neq \emptyset$*. Then, we have:*

(i) $F_T = (SF)_T = \{x^*\}$;

ii) $T^n(x) \stackrel{H_d}{\to} \{x^*\}$ *as* $n \to +\infty$*, for each* $x \in X$,

i.e., T *is a multivalued Picard operator.*

Another result of this type involves the so-called multivalued (δ,φ)-contractions.

Definition 2.1.4 *Let (X, d) be a metric space. Then, by definition, $T : X \to P_b(X)$ is called a multivalued strong (δ, φ)-contraction if $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a comparison function and*

$$\delta(T(Y)) \leq \varphi(\delta(Y)), \text{ for each } Y \in P_b(X),$$

with Y a non-singleton set.

Thorem 2.1.8 *(V.L. Lazăr) Let (X, d) be a complete metric space and $T : X \to P(X)$ be a multivalued strong (δ, φ)-contraction such that $T(X) \in P_b(X)$. Then T is a multivalued Picard operator.*

Recall now that a self-multivalued operator $T : X \to P_{cl}(X)$ on a metric space (X, d) is called (ϵ, φ)-contraction if $\epsilon > 0, \varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a strict comparison function and

$$x, y \in X \text{ with } x \neq y \text{ and } d(x, y) < \epsilon \text{ implies } H(T(x), T(y)) \leq \varphi(d(x, y)).$$

For the case of periodic points we have the following results.

Thorem 2.1.9 *(V.L. Lazăr, [73]) Let (X, d) be a metric space and $T : X \to P_{cp}(X)$ be a continuous (ϵ, φ)-contraction. Then, the following conclusions hold:*

(i) $\hat{T}^m : P_{cp}(X) \to P_{cp}(X)$ is a continuous (ϵ, φ)-contraction, for each $m \in \mathbb{N}^*$;

(ii) if, additionally, there exists some $A \in P_{cp}(X)$ such that a subsequence $(\hat{T}^{m_i}(A))_{i \in \mathbb{N}^*}$ of $(\hat{T}^m(A))_{m \in \mathbb{N}^*}$ converges in $(P_{cp}(X), H)$ to some $X^* \in P_{cp}(X)$, then there exists $x^* \in X^*$ a periodic point for T.

As a consequence, another existence result for periodic points of a multivalued operator is the following.

Thorem 2.1.10 *(V.L. Lazăr, [73]) Let (X, d) be a compact metric space and $T : X \to P_{cp}(X)$ be a continuous (ϵ, φ)-contraction. Then, there exists $x^* \in X$ a periodic point for T.*

Remark 2.1.3 *We also refer to [111, 112] for some results of this type for multivalued operators of Reich's type.*

2.1.2 Fixed Points for Non-self Multivalued Operators on a Set with Two Metrics

Let (X, d) be a metric space, $x_0 \in X$ and $r > 0$. Denote by $B(x_0; r) := \{x \in X | d(x_0, x) < r\}$ the open ball centred in x_0 with radius r and by $\widetilde{B}(x_0; r) := \{x \in X | d(x_0, x) \leq r\}$ the closed ball centred in x_0 with radius r. We will denote by $\bar{B}^d(x_0, r)$ the closure of $B(x_0, r)$ in (X, d).

Also, we denote by $I(f) := \{Y \subseteq X | f(Y) \subset Y\}$ the set of all invariant subsets for f, by $I_b(f) := \{Y \in I(f) | Y \text{ is bounded}\}$ the set of all bounded invariant subsets for f and by $I_{b,cl}(f) := \{Y \in I_b(f) | \text{ Y is closed }\}$.

An operator $f : Y \subseteq X \to X$ is said to be an a-contraction if $a \in [0, 1[$ and $d(f(x), f(y)) \leq ad(x, y)$, for all $x, y \in Y$.

The following result is well-known consequence of the Banach-Caccioppoli fixed point principle:

Thorem 2.1.11 *(Granas-Dugundji, [40]) Let (X, d) a complete metric space, $x_0 \in X$ and $r > 0$. If $f : B(x_0; r) \to X$ is an a-contraction and $d(x_0, f(x_0)) < (1 - a)r$, then f has a unique fixed point.*

Let us remark that if $f : \widetilde{B}(x_0; r) \to X$ is an a-contraction such that $d(x_0, f(x_0)) \leq (1 - a)r$, then $\widetilde{B}(x_0; r) \in I_{b,cl}(f)$ and again f has a unique fixed point in $\widetilde{B}(x_0; r)$.

Let E be a Banach space and $Y \subset E$. Given an operator $f : Y \to E$, the operator $g : Y \to E$ defined by $g(x) := x - f(x)$ is called the field associated with f. An operator $f : Y \to E$ is said to be open if for any open subset U of Y the set $f(U)$ is open in E too.

As a consequence of the above result, one can obtain the following domain invariance theorem for contraction type fields.

Thorem 2.1.12 *(Granas-Dugundji, [40]) Let E be a Banach space and Y be an open subset of E. Consider $f : U \to E$ be an a-contraction. Let $g : U \to E$*

$g(x) := x - f(x)$, the associated field. Then:

(a) $g : U \to E$ is an open operator;

(b) $g : U \to g(U)$ is a homeomorphism. In particular, if $f : E \to E$, then the associated field g is a homeomorphism of E into itself.

Recall two important facts.

Lemma 2.1.1 *Let (X, d) be a metric space. Then $D_d(\cdot, Y) : (X, d) \to \mathbb{R}_+$, $x \mapsto D_d(x, Y)$, (where $Y \in P(X)$) is nonexpansive and hence continuous.*

Lemma 2.1.2 *Let X be a normed space. Then for all $x, y \in X$ and for all $A \in P_{cl}(X)$ we have: $D(x, A + y) = D(y, x - A)$.*

Let us also mention that, if Y is a subset of a metric space X and $\varepsilon > 0$, then $V^0(Y; \varepsilon)$ denotes the open ε-neighborhood of Y, i.e., $V^0(Y; \varepsilon) = \{x \in X | \ D(x, Y) < \varepsilon\}$.

The following result is a local Węgrzyk type theorem on a set with two metrics.

Thorem 2.1.13 *(T.A. Lazăr, V.L. Lazăr, [68]) Let X be a nonempty set and d, d' two metrics on X, $x_0 \in X$, $r > 0$. Suppose that*

(i) (X, d) is a complete metric space;

(ii) there exists $c > 0$ such that $d(x, y) \leq cd'(x, y)$ for each $x, y \in X$;

(iii) $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a strict comparison function such that the function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, $\psi(t) := t - \varphi(t)$ strict increasing and continuous with

$$\sum_{n=0}^{\infty} \varphi^n(\psi(r)) \leq \varphi(r).$$

Let $T : \bar{B}^d_{d'}(x_0; r) \to P_{cl}(X)$ be a multivalued φ-contraction with respect to d' such that

$$D_{d'}(x_0, T(x_0)) < r - \varphi(r).$$

Suppose that $T : \bar{B}^d_{d'}(x_0; r) \to P((X, d))$ *is closed.*
Then $F_T \neq \emptyset$.

As a consequence we can obtain a result on the open ball.

Thorem 2.1.14 *(T.A. Lazăr, V.L. Lazăr, [68]) Let* X *be a nonempty set and* d, d' *two metrics on* X, $x_0 \in X$, $r > 0$. *Suppose that*

(i) (X, d) *is a complete metric space;*

(ii) there exists $c > 0$ *such that* $d(x, y) \leq cd'(x, y)$ *for each* $x, y \in X$*;*

(iii) $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *be a strict comparison function such that the function* $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, $\psi(t) := t - \varphi(t)$ *is strictly increasing and continuous in* r, *with*

$$\sum_{n=0}^{\infty} \varphi^n(\psi(s)) \leq \varphi(s) \text{ for all } s \in]0, r[.$$

Let $T : B_{d'}(x_0; r) \to P_{cl}(X)$ *be a multivalued* φ*-contraction respect to metric* d' *such that* $D_{d'}(x_0, T(x_0)) < r - \varphi(r)$.

Then $F_T \neq \emptyset$.

Using the above theorem we can obtain an open operator principle in a Banach space.

Thorem 2.1.15 *(T.A. Lazăr, V.L. Lazăr, [68]) Let* X *be a linear space and* $\|\cdot\|, \|\cdot\|'$ *be two norms on* X. *We denote by* d, d' *the metrics induced by the norms* $\|\cdot\|, \|\cdot\|'$. *Suppose that* (X, d) *is complete. Let* U *an open set with respect to the norm* $\|\cdot\|'$ *and let* $T : U \to P_{cl}(X)$ *be a multivalued operator.*

Suppose that:

(i) there exists $c > 0$ *such that* $d(x, y) \leq cd'(x, y)$ *for all* $x, y \in X$*;*

(ii) $T : U \to P_{cl}(X)$ *is a* φ*-contraction with respect to the norm* $\|\cdot\|'$, *i.e.*

$$H_{d'}(T(x_1), T(x_2)) \leq \varphi(d'(x_1, x_2)) \text{ for all } x_1, x_2 \in U;$$

(iii) $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *be a strict comparison function, such that the function* $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, $\psi(t) := t - \varphi(t)$ *is strictly increasing and continuous on* $\mathbb{R}_+$ *and there exists* $r_0 > 0$ *such that* $\sum_{n=1}^{\infty} \varphi^n(\psi(s)) \leq \varphi(s)$, *for each* $s \in]0, r_0[$.

Then, the multivalued field $G : U \to P(X), G(x) = x - T(x)$ *is an open operator in the topology generated by the norm* $\|\cdot\|'$.

Let us establish now a fixed point result for multivalued Caristi type operators defined on a ball from a space X endowed with two metrics.

Thorem 2.1.16 *(T.A. Lazăr, V.L. Lazăr, [68]) Let* X *be a nonempty set, and* d, d' *two metrics on* X, $x_0 \in X$ *and* $r > 0$. *Suppose that:*

(i) (X, d) *is a complete metric space;*

(ii) there exists $c > 0$ *such that* $d(x, y) \leq cd'(x, y)$ *for each* $x, y \in X$*;*

Let $\varphi : X \to \mathbb{R}_+$ *be a function with* $\varphi(x_0) < r$.

Consider $T : \bar{B}^d_{d'}(x_0; r) \to P_{cl}(X)$ *such that for each* $x \in \bar{B}^d_{d'}(x_0; r)$ *there exists* $y \in T(x)$ *such that* $d'(x, y) \leq \varphi(x) - \varphi(y)$.

If T *is a closed operator with respect to the metric* d *(i.e., Graph*(T) *is a closed set with respect to the product topology on* $X \times X$ *generated by* d*), then* $F_T \neq \emptyset$.

2.1.3 On the Essentiality of the Mönch Type Maps

In this paragraph, some results by R.P. Agarwal and D. O'Regan [1] concerning essential maps in the sense of Mönch are extended to maps satisfying some compactness conditions introduced by D. O'Regan and R. Precup [93].

We begin by presenting the fixed point theorem of Mönch [87], which has been particularly useful in establishing existence of solutions to nonlinear boundary value problems in Banach spaces [42].

Thorem 2.1.17 *Let X be a Banach space, D be a closed, convex subset of X and $x_0 \in D$. Let $f : D \to D$ be a continuous map with the property:*

$$\left(\left. \begin{array}{l} C \subset D, C - countable \\ \overline{C} = \overline{co}(\{x_0\} \cup f(C)) \end{array} \right\} \Rightarrow \overline{C} \text{ is compact} \right) \tag{2.1.1}$$

Then f has a fixed point.

The following result is a Leray-Schauder type theorem:

Thorem 2.1.18 *Let X be a Banach space, U an open subset of X and $x_0 \in U$. Let $f : \overline{U} \to X$ be a continuous map with the property:*

$$\left(\left. \begin{array}{l} C \subset \overline{U}, C - countable \\ \overline{C} = \overline{co}(\{x_0\} \cup f(C)) \end{array} \right\} \Rightarrow \overline{C} \text{ is compact} \right) \tag{2.1.2}$$

If $x \neq (1-\lambda)x_0 + \lambda f(x)$, for any $x \in \partial U$ and $\lambda \in (0,1)$, then f has a fixed point in $\overline{U}$.

Remark 2.1.4 *Note that 2.1.1 and 2.1.2 can be replaced by the more general assumptions*

$$\left(\left. \begin{array}{l} M \subset D \\ M = co(\{x_0\} \cup f(M)) \\ \overline{M} = \overline{C}, C \subset M, C - countable \end{array} \right\} \Rightarrow \overline{M} \text{ is compact} \right) \tag{2.1.3}$$

and

$$\left(\left. \begin{array}{l} M \subset \overline{U} \\ M = co(\{x_0\} \cup f(M)) \\ \overline{M} = \overline{C}, C \subset M, C - countable \end{array} \right\} \Rightarrow \overline{M} \text{ is compact} \right) \tag{2.1.4}$$

respectively.

The conditions (2.1.3) and (2.1.4) are useful to establish generalizations of Mönch's fixed point theorems for multivalued maps (see[93]).

Definition 2.1.5 *We let $M(\overline{U}, X)$ denote the set of all continuous maps $f : \overline{U} \to X$, which satisfy 2.1.4 with $x_0 = 0$.*

Definition 2.1.6 *We let $f \in M_{\partial U}(\overline{U}, X)$ if $f \in M(\overline{U}, X)$ and $x \neq f(x)$ for $x \in \partial U$.*

Definition 2.1.7 *A map $f \in M_{\partial U}(\overline{U}, X)$ is essential if for every $g \in M_{\partial U}(\overline{U}, X)$ with $g/_{\partial U} = f/_{\partial U}$ there exists $x \in U$ with $x = g(x)$.*

These definitions, corresponding to condition 2.1.2, have been given by R.P. Agarwal and D.O'Regan [1].

The first result is an example of the zero map being essential in $M_{\partial U}(\overline{U}, X)$.

Thorem 2.1.19 *(V.L. Lazăr, [69]) Let X be a Banach space and let U an open subset of X with $0 \in U$. Then the zero map is essential in $M_{\partial U}(\overline{U}, X)$.*

The following result is a nonlinear alternative of Leray-Schauder type for Mönch type maps.

Thorem 2.1.20 *(V.L. Lazăr, [69]) Let X be a Banach space and let U be an open subset of X with $0 \in U$. Suppose $f \in M(\overline{U}, X)$ satisfies $x \neq \lambda f(x)$ for any $x \in \partial U$ and $\lambda \in (0, 1]$. Then f is essential in $M_{\partial U}(\overline{U}, X)$. In particular f has a fixed point in U.*

2.2 Ulam-Hyers Stability Results for Partial Differential Equations

We will present first some notions and results from the weakly Picard operator theory (see [118]; see also [126], pp. 119-126).

Let (X, d) be a metric space and $f : X \to X$ an operator. We denote by $F_f := \{x \in X \mid f(x) = x\}$, the fixed point set of the operator f. By definition f is weakly Picard operator if the sequence of successive approximations, $f^n(x)$, converges for all $x \in X$ and the limit (which may depend on x) is a fixed point of f.

If f is a weakly Picard operator then we consider the operator $f^\infty : X \to X$ defined by $f^\infty(x) := \lim\limits_{n\to\infty} f^n(x)$. It is clear that $f^\infty(X) = F_f$. Moreover, f^∞ is a set retraction of X to F_f.

If f is a weakly Picard operator and $F_f = \{x^*\}$, then by definition f is a Picard operator. In this case f^∞ is the constant operator, $f^\infty(x) = x^*$, $\forall\ x \in X$. The following class of weakly Picard operators is very important in our considerations.

Definition 2.2.1 *Let (X, d) be a metric space and let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing function which is continuous in 0 and $\psi(0) = 0$. An operator $f : X \to X$ is said to be a ψ-weakly Picard operator if it is weakly Picard operator and*

$$d(x, f^\infty(x)) \leq \psi(d(x, f(x))), \quad \textit{for all } x \in X.$$

In the case that $\psi(t) = ct$ with $c > 0$, we say that f is c-weakly Picard operator.

Example 2.2.1 Let (X, d) be a complete metric space and $f : X \to X$ an operator with closed graph. We suppose that f is graph α-contraction, i.e.,

$$d(f^2(x), f(x)) \leq \alpha d(x, f(x)),\ \forall\ x \in X.$$

Then f is a c-weakly Picard operator, with $c = (1 - \alpha)^{-1}$.

Example 2.2.2 Let (X, d) be a complete metric space, $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ a function and $f : X \to X$ an operator with closed graphic. We suppose that:

(i) f is a φ-Caristi operator, i.e., $d(x, f(x)) \leq \varphi(x) - \varphi(f(x)),\ \forall\ x \in X$;

(ii) there exists $c > 0$ such that $\varphi(x) \leq cd(x, f(x)),\ \forall\ x \in X$.

Then f is a c-weakly Picard operator.

On the other hand, by the analogy with the notion of the Ulam-Hyers stability in the theory of functional equation (see [51, 56, 22, 18, 29, 35, 41, 46, 50, 94, 109, 110]), the following concept was introduced by I.A. Rus in [124].

Definition 2.2.2 *Let (X, d) be a metric space and $f : X \to X$ an operator. By definition, the fixed point equation*

$$x = f(x) \tag{2.2.1}$$

is generalized Ulam-Hyers stable if there exists $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ *increasing, continuous in* 0 *and* $\psi(0) = 0$ *such that for each* $\varepsilon > 0$ *and each solution* y^* *of the inequation*

$$d(y, f(y)) \leq \varepsilon \tag{2.2.2}$$

there exists a solution x^* *of the equation (2.2.1) such that*

$$d(y^*, x^*) \leq \psi(\varepsilon).$$

If, in particular, there exists $c > 0$ *such that* $\psi(t) := ct$, *for each* $t \in \mathbb{R}_+$, *the equation (2.2.1) is said to be Ulam-Hyers stable.*

The following results are important for our considerations.

Lemma 2.2.1 *(I.A. Rus [124]) If* f *is a* ψ*-weakly Picard operator, then the fixed point equation (2.2.1) is generalized Ulam-Hyers stable.*

Lemma 2.2.2 *(I.A. Rus [124]) Let* (X, d) *be a metric space,* $f : X \to X$ *an operator and* $X = \bigcup\limits_{i \in I} X_i$ *a partition of* X *such that* $f(X_i) \subset X_i$, $\forall\ i \in I$. *If the equation (2.2.1) is Ulam-Hyers stable in each* (X_i, d), $i \in I$, *then it is Ulam-Hyers stable in* (X, d).

I will consider first the Dirichlet problem associated to a nonlinear elliptic equation. Let Ω be a bounded domain in $\mathbb{R}^n$ with sufficiently smooth border $\partial\Omega$. Consider the following problem.

$$\begin{cases} \Delta u = f(x, u(x)) \\ u_{|\partial\Omega} = 0 \end{cases} \tag{2.2.3}$$

where f is a continuous function on $\overline{\Omega} \times \mathbb{R}$.

Throughout this paragraph I will denote by $||\cdot||_C$ the supremum norm in $C(\overline{\Omega}, \mathbb{R})$.

Lemma 2.2.3 *In the above conditions, the Dirichlet problem 2.2.3 is equivalent to the following integral equation*

$$u(x) = -\int\limits_{\Omega} G(x, s) f(s, u(s)) ds, \tag{2.2.4}$$

where G *denotes the usual Green function corresponding to the Laplace operator.*

We recall now Matkowski-Rus fixed point theorem.

Thorem 2.2.1 *(J. Matkowski [82], I. A. Rus [125]) Let* (X, d) *be a complete metric space and* $f : X \to X$ *a* φ*-contraction. Then* $F_f = \{x^*\}$ *and* $f^n(x_0) \to x^*$ *as* $n \to \infty$*, for all* $x_0 \in X$*, i.e.,* f *is a Picard operator.*

Remark 2.2.1 *If in the above result, we additionally suppose that the function* $\psi : \mathbb{R}_+ \to \mathbb{R}_+$*,* $\psi(t) := t - \varphi(t)$ *is strictly increasing and onto, then the operator* f *is* ψ*-weakly Picard. Indeed, for each* $x \in X$*, we have*

$$d(x, x^*) \leq d(x, f(x)) + d(f(x), f(x^*)) \leq d(x, f(x)) + \varphi(d(x, x^*)).$$

Hence

$$d(x, x^*) \leq \psi^{-1}(d(x, f(x)), \text{ for each } x \in X.$$

Our first main result is the following existence, uniqueness and stability result for the Dirichlet problem 2.2.3.

Thorem 2.2.2 *(V.L.Lazăr, [74]) Let* Ω *be a bounded domain in* $\mathbb{R}^n$ *such that its border* $\partial\Omega$ *is sufficiently smooth. Denote by* G *denotes the usual Green function corresponding to the Laplace operator. Suppose that:*

(i) $f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$*;*

(ii) there exist $p \in C(\overline{\Omega}, \mathbb{R}_+)$ *with* $\sup\limits_{x\in\overline{\Omega}} \int\limits_{\Omega} G(x, s)p(s)ds \leq 1$ *and a comparison function* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$*, such that for each* $s \in \overline{\Omega}$ *and each* $u, v \in \mathbb{R}$ *we have that*

$$|f(s, u) - f(s, v)| \leq p(s)\varphi(|u - v|);$$

Then, the Dirichlet problem 2.2.3 has a unique solution $u^* \in C(\overline{\Omega}, \mathbb{R})$*. Moreover, if the function* $\psi : \mathbb{R}_+ \to \mathbb{R}_+$*,* $\psi(t) := t - \varphi(t)$ *is strictly increasing and onto, then the Dirichlet problem 2.2.3 is generalized Ulam-Hyers stable with function* ψ^{-1}*, i.e., for each* $\varepsilon > 0$ *and for each* ε*-solution* y^* *of the Dirichlet problem 2.2.3 we have that*

$$|u^*(x) - y^*(x)| \leq \psi^{-1}(\varepsilon), \text{ for each } x \in \overline{\Omega}.$$

Remark 2.2.2 *Theorem 2.2.2 generalizes some known results in the literature, such as Theorem 16.2.1 in I.A. Rus [117].*

A second result concerns with the case of a Dirichlet problem for a partial differential equation with modified argument.

Consider the following problem

$$\begin{cases} \Delta u = f(x, u(g(x))) \\ u_{|\partial\Omega} = 0, \end{cases} \tag{2.2.5}$$

where f is a continuous function on $\overline{\Omega} \times \mathbb{R}$ and $g \in C(\overline{\Omega}, \overline{\Omega})$.

Thorem 2.2.3 *(V.L.Lazăr, [74]) Let Ω be a bounded domain in $\mathbb{R}^n$ such that its border $\partial\Omega$ is sufficiently smooth. Denote by G denotes the usual Green function corresponding to the Laplace operator. Suppose that:*

(i) $f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ and $g \in C(\overline{\Omega}, \overline{\Omega})$;

(ii) there exist $p \in C(\overline{\Omega}, \mathbb{R}_+)$ with $\sup\limits_{x \in \overline{\Omega}} \int\limits_{\Omega} G(x,s)p(s)ds \leq 1$ and a comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, such that for each $s \in \overline{\Omega}$ and each $u, v \in \mathbb{R}$ we have that

$$|f(s,u) - f(s,v)| \leq p(s)\varphi(|u - v|);$$

Then, the Dirichlet problem 2.2.5 has a unique solution $u^* \in C(\overline{\Omega}, \mathbb{R})$. Moreover, if the function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, $\psi(t) := t - \varphi(t)$ is strictly increasing and onto, then the Dirichlet problem 2.2.5 is generalized Ulam-Hyers stable with function ψ^{-1}, i.e., for each $\varepsilon > 0$ and for each ε-solution y^* of the Dirichlet problem 2.2.5 we have that

$$|u^*(x) - y^*(x)| \leq \psi^{-1}(\varepsilon), \text{ for each } x \in \overline{\Omega}.$$

2.3 Ulam-Hyers Stability Results for Partial Differential Inclusions

Using the weakly Picard operator technique, we will present existence and Ulam-Hyers stability results for integral inclusions of Fredholm and Volterra type and for some problems associated to partial differential inclusions.

We present now some Ulam-Hyers stability concepts for the fixed point problem associated to a multivalued operator.

Definition 2.3.1 *(I.A. Rus, [124]) Let (X, d) be a metric space and $T : X \to P(X)$ be a multivalued operator. The fixed point inclusion*

$$x \in T(x), \;\; x \in X \tag{2.3.1}$$

is called generalized Ulam-Hyers stable if and only if there exists $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ increasing, continuous in 0 and $\psi(0) = 0$ such that for each $\varepsilon > 0$ and for each solution $y^* \in X$ of the inequation

$$D_d(y, T(y)) \leq \varepsilon \tag{2.3.2}$$

there exists a solution x^* of the fixed point inclusion (2.3.1) such that

$$d(y^*, x^*) \leq \psi(\varepsilon).$$

If there exists $c > 0$ such that $\psi(t) := ct$, for each $t \in \mathbb{R}_+$, then the fixed point inclusion (2.3.1) is said to be Ulam-Hyers stable.

The following theorem is an abstract result concerning the Ulam-Hyers stability of the fixed point inclusion (2.3.1) for multivalued operators with compact values.

Thorem 2.3.1 *(I.A. Rus, [124]) Let (X, d) be a metric space and $T : X \to P_{cp}(X)$ be a multivalued ψ-weakly Picard operator. Then, the fixed point inclusion (2.3.1) is generalized Ulam-Hyers stable.*

We consider here some integral inclusion of Fredholm and Volterra type. Throughout this section we will denote by $\|\cdot\|$ the supremum norm in $C([a, b], \mathbb{R}^n)$ and by $|\cdot|$ a norm in $\mathbb{R}^n$.

Consider first the following Fredholm type integral inclusion.

$$x(t) \in \int_a^b K(t, s, x(s))ds + g(t), \;\; t \in [a, b]. \tag{2.3.3}$$

The main result concerning the stability of the Fredholm integral inclusion (2.3.3) is the following.

Thorem 2.3.2 *(V.L. Lazăr, [75]) Let* $K : [a,b] \times [a,b] \times \mathbb{R}^n \to P_{cl,cv}(\mathbb{R}^n)$ *and* $g : [a,b] \to \mathbb{R}^n$ *such that:*

(i) there exists an integrable function $M : [a,b] \to \mathbb{R}_+$ *such that for each* $t \in [a,b]$ *and* $u \in \mathbb{R}^n$ *we have* $K(t,s,u) \subset M(s)B(0;1)$*, a.e.* $s \in [a,b]$*;*

(ii) for each $u \in \mathbb{R}^n$ $K(\cdot,\cdot,u) : [a,b] \times [a,b] \to P_{cl,cv}(\mathbb{R}^n)$ *is jointly measurable;*

(iii) for each $(s,u) \in [a,b] \times \mathbb{R}^n$ $K(\cdot,s,u) : [a,b] \to P_{cl,cv}(\mathbb{R}^n)$ *is lower semi-continuous;*

(iv) there exists a continuous function $p : [a,b] \times [a,b] \to \mathbb{R}_+$ *with* $\sup\limits_{t\in[a,b]} \int\limits_a^b p(t,s)ds \leq 1$ *and a strict comparison function* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *such that for each* $(t,s) \in [a,b] \times [a,b]$ *and each* $u, v \in \mathbb{R}^n$ *we have that*

$$H(K(t,s,u), K(t,s,v)) \leq p(t,s) \cdot \varphi(|u-v|); \tag{2.3.4}$$

(v) g *is continuous.*

Then the following conclusions hold:

(a) the integral inclusion (2.3.3) has least one solution, i.e., there exists $x^* \in C([a,b],\mathbb{R}^n)$ *which satisfies (2.3.3), for each* $t \in [a,b]$.

(b) If additionally $\varphi(qt) \leq q\varphi(t)$ *for every* $t \in \mathbb{R}_+$ *(where* $q > 1$*) and* $t = 0$ *is a point of uniform convergence for the series* $\sum\limits_{n=1}^{\infty} \varphi^n(t)$*, then the integral inclusion (2.3.3) is generalized Ulam-Hyers stable with function* ψ *(where* $\psi(t) := t + s(t)$*, for each* $t \in \mathbb{R}_+$ *and* $s(t) := \sum\limits_{n=1}^{\infty} \varphi^n(t)$*), i.e., for each* $\varepsilon > 0$ *and for any* ε*-solution* y *of (2.3.3), that is any* $y \in C([a,b],\mathbb{R}^n)$ *for which there exists* $u \in C([a,b],\mathbb{R}^n)$ *such that*

$$u(t) \in \int\limits_a^b K(t,s,y(s))ds + g(t),\ t \in [a,b]$$

and

$$|u(t) - y(t)| \leq \varepsilon,\ \textit{for each } t \in [a,b]),$$

there exists a solution x^* *of the integral inclusion (2.3.3) such that*

$$|y(t) - x^*(t)| \leq \psi(\varepsilon), \text{ for each } t \in [a, b].$$

Moreover, in this case the continuous data dependence of the solution set of the integral inclusion (2.3.4) holds.

A second application concerns an integral inclusion of Volterra type.

$$x(t) \in \int_a^t K(t, s, x(s))ds + g(t), \; t \in [a, b]. \tag{2.3.5}$$

By a similar method, we can prove the following.

Thorem 2.3.3 *(V.L.Lazăr, [75]) Let* $K : [a, b] \times [a, b] \times \mathbb{R}^n \to P_{cl,cv}(\mathbb{R}^n)$ *and* $g : [a, b] \to \mathbb{R}^n$ *such that:*

(i) there exists an integrable function $M : [a, b] \to \mathbb{R}_+$ *such that for each* $t \in [a, b]$ *and* $u \in \mathbb{R}^n$ *we have* $K(t, s, u) \subset M(s)B(0; 1)$, *a.e.* $s \in [a, b]$*;*

(ii) for each $u \in \mathbb{R}^n$ $K(\cdot, \cdot, u) : [a, b] \times [a, b] \to P_{cl,cv}(\mathbb{R}^n)$ *is jointly measurable;*

(iii) for each $(s, u) \in [a, b] \times \mathbb{R}^n$ $K(\cdot, s, u) : [a, b] \to P_{cl,cv}(\mathbb{R}^n)$ *is lower semi-continuous;*

(iv) there exists a continuous function $p : [a, b] \to \mathbb{R}_+^*$ *and a strict comparison function* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *with* $\varphi(\lambda t) \leq \lambda\varphi(t)$, *for each* $t \in \mathbb{R}_+$ *and each* $\lambda \geq 1$, *such that for each* $(t, s) \in [a, b] \times [a, b]$ *and each* $u, v \in \mathbb{R}^n$ *we have that*

$$H(K(t, s, u), K(t, s, v)) \leq p(s) \cdot \varphi(|u - v|); \tag{2.3.6}$$

(v) g *is continuous.*

Then the following conclusions hold:

(a) the integral inclusion (2.3.5) has at least one solution, i.e., there exists $x^* \in C([a, b], \mathbb{R}^n)$ *which satisfies (2.3.5) for each* $t \in [a, b]$*;*

(b) If additionally $\varphi(qt) \leq q\varphi(t)$ *for every* $t \in \mathbb{R}_+$ *(where* $q > 1$*) and* $t = 0$ *is a point of uniform convergence for the series* $\sum_{n=1}^{\infty} \varphi^n(t)$, *then the integral inclusion*

(2.3.3) is generalized Ulam-Hyers stable with function ψ *(where* $\psi(t) := t + s(t)$*, for each* $t \in \mathbb{R}_+$ *and* $s(t) := \sum_{n=1}^{\infty} \varphi^n(t)$*), i.e., for each* $\varepsilon > 0$ *and for any* ε*-solution* y *of (2.3.5), that is, any* $y \in C([a,b],\mathbb{R}^n)$ *for which there exists* $u \in C([a,b],\mathbb{R}^n)$ *such that*

$$u(t) \in \int_a^t K(t,s,y(s))ds + g(t), \ t \in [a,b]$$

and

$$|u(t) - y(t)| \leq \varepsilon, \ \textit{for each } t \in [a,b]),$$

there exists a solution x^* *of the integral inclusion (2.3.5) such that*

$$|y(t) - x^*(t)| \leq \psi(c\varepsilon), \ \textit{for each } t \in [a,b] \ \textit{and some } c > 0.$$

Moreover, in this case the continuous data dependence of the solution set of the integral inclusion (2.3.6) holds.

Let us consider the following Darboux problem for a second order differential inclusion

$$\begin{cases} \dfrac{\partial^2 u}{\partial x \partial y} \in F(x,y,u(x,y)) \\ u(x,0) = \lambda(x,0), \ u(0,y) = \lambda(0,y), \end{cases} \tag{2.3.7}$$

where $F : I_1 \times I_2 \times \mathbb{R}^m \to P_{cl}(\mathbb{R}^m)$ (with $I_i = [0,T_i]$, $i \in \{1,2\}$) and $\lambda(x,y) = \alpha(x) + \beta(y) - \alpha(0)$ (with α, β continuous functions on I_1 respectively I_2 and $\alpha(0) = \beta(0)$).

Denote by $\Pi = I_1 \times I_2$ and let $a > 0$. By L^1 we will denote the Banach space of all measurable Lebesgue functions $\eta : \Pi \to \mathbb{R}^m$, endowed with the norm

$$\|\eta\|_1 = \int\int_{\Pi} e^{-a(x+y)}|\eta(x,y)|dxdy.$$

Let C be the Banach space of continuous functions $u : \Pi \to \mathbb{R}^m$, with the norm $\|u\|_C = \sup_{(x,y)\in\Pi} |u(x,y)|$ and let $\tilde{C}$ be the linear subspace of C consisting of all $\lambda \in C$ such that there exist continuous functions $\alpha \in C(I_1,\mathbb{R}^m)$ and $\beta \in C(I_2,\mathbb{R}^m)$ with $\alpha(0) = \beta(0)$ satisfying $\lambda(x,y) = \alpha(x) + \beta(y) - \alpha(0)$, for all $x,y \in I_1 \times I_2$. Obviously, $\tilde{C}$ with the norm of C is a separable Banach space.

By definition, the Darboux problem (2.3.7) is called Ulam-Hyers stable if for each $\varepsilon > 0$ and for any ε-solution w of (2.3.7), there exists a solution u^* of (2.3.7) such that

$$|w(x,y) - u^*(x,y)| \leq c\varepsilon, \text{ for each } (x,y) \in \Pi \text{ and for some } c > 0.$$

We have the following existence and Ulam-Hyers stability result.

Thorem 2.3.4 *(V.L. Lazăr, [75]) Consider the Darboux Problem (2.3.7) and suppose that the above mentioned conditions hold. Suppose also that the following assumptions hold:*

(i) for each $u \in \mathbb{R}^m$, $F(\cdot,\cdot,u)$ *is measurable;*

(ii) there exists $k > 0$ *such that a.e.* $(x,y) \in I_1 \times I_2$ *the multifunction* $F(x,y,\cdot)$ *is k-Lipschitz;*

(iii) $a > \sqrt{k}$.

Then, the Darboux Problem (2.3.7) has at least one solution and it is Ulam-Hyers stable.

Chapter 3

Heston Model

The purpose of this chapter is to present some results concerning the problem of option pricing under a stochastic volatility model, namely the Heston model (see [44], 1993).

3.1 A Closed-Form Solution for Digital Call Options in Heston Model

The aim of this paragraph is to analyse the problem of digital option pricing under the Heston stochastic volatility model. We present an analytical solution for this kind of options, based on S. Heston's original work [44].

Heston's option pricing formula is derived under the assumption that the stock price and its volatility follow the stochastic processes:

$$dS(t) = S(t)[rdt + \sqrt{v(t)}dW_1(t)] \tag{3.1.1}$$

where S denotes the asset price, t the time, r the risk-neutral drift term, $W_1(t)$ a Wiener process and v the volatility who follows a mean reversion process:

$$dv(t) = k(\theta - v(t))dt + \xi\sqrt{v(t)}dW_2(t), \tag{3.1.2}$$

where $W_2(t)$ is a second Wiener process which is correlated with $W_1(t)$:

$$\mathbf{Cov}[dW_1(t), dW_2(t)] = \rho dt. \tag{3.1.3}$$

Finally, *the market price of volatility risk* is given by:

$$\lambda(S, v, t) = \lambda v \tag{3.1.4}$$

We have the following partial differential equation for the Heston model (3.1.1)-(3.1.3):

$$\begin{aligned} &\frac{\partial V}{\partial t} + \frac{1}{2}\, v\, S^2\, \frac{\partial^2 V}{\partial S^2} + \rho\, \xi\, v\, S\, \frac{\partial^2 V}{\partial S\, \partial v} + \frac{1}{2}\, \xi^2\, v\, \frac{\partial^2 V}{\partial v^2} \\ &\quad + r\, S\, \frac{\partial V}{\partial S} + [\, k\, (\theta - v\,) - \lambda\, v\,]\, \frac{\partial V}{\partial v} - r\, V = 0\,. \end{aligned} \tag{3.1.5}$$

In what follows we solve the partial differential equation (3.1.5) subject to the final condition corresponding to a digital call option, i.e. the payoff is the Heaviside function:

$$DC(S, v, T) = \mathcal{H}(S - K) = \begin{cases} 1 & \text{if } S \geq K \\ 0 & \text{if } S < K \end{cases} \tag{3.1.6}$$

We note that, in the case of digital call option, the solution of the Black-Scholes option price equation in the constant volatility case is:

$$DC(S, t) = e^{-r(T-t)} N(d_2) \tag{3.1.7}$$

where

$$d_2 = \frac{log(S/K) + \left(r - \frac{1}{2}v^2\right)(T - t)}{v\sqrt{T - t}} \tag{3.1.8}$$

with $N(x)$ the cumulative distribution function for the standard normal distribution.

With a new variable $x = \ln[S]$, so $U(x, v, t) = V(S, v, t)$, we find that the equation (3.1.5) is turn into:

$$\begin{aligned} &\frac{\partial U}{\partial t} + \frac{1}{2}\, \xi^2\, v\, \frac{\partial^2 U}{\partial v^2} + \rho\, \xi\, v\, \frac{\partial^2 U}{\partial x \partial v} + \frac{1}{2}\, v\, \frac{\partial^2 U}{\partial x^2} + \left(r - \frac{1}{2}\, v\right) \frac{\partial U}{\partial x} \\ &\qquad + [k\, (\theta - v) - v\, \lambda]\, \frac{\partial U}{\partial v} - r\, U = 0\,. \end{aligned} \tag{3.1.9}$$

Suppose that the solution of the Heston PDE is like the form of Black-Scholes model (3.1.7):

$$DC(S,\ v,\ t) = e^{-r\ \tau}\ P \tag{3.1.10}$$

where the probability P, correspond to $N(d_2)$ in the constant volatility case, are what we are going to find. Actually, P is the conditional probability that the option expires in-the-money (see [44]):

$$P(x, v, T; ln[K]) = Pr[x(T) \geq ln[K] \ / \ x(t) = x, v(t) = v]. \tag{3.1.11}$$

We now substitute the proposed value for $DC(S,\ v,\ t)$ into the pricing equation (3.1.9) and we obtain:

$$\frac{\partial P}{\partial t} + \frac{1}{2}\,\xi^2\,v\,\frac{\partial^2 P}{\partial v^2} + \rho\,\xi\,v\,\frac{\partial^2 P}{\partial x \partial v} + \frac{1}{2}\,v\,\frac{\partial^2 P}{\partial x^2} + \left(r - \frac{1}{2}\,v\right)\frac{\partial P}{\partial x}$$
$$+ \ [k\,(\theta - v) - v\,\lambda]\,\frac{\partial P}{\partial v} = 0 \tag{3.1.12}$$

subject to the terminal condition:

$$P(x,\ v,\ T;\ ln[K]) = 1_{\{x \,\geq\, ln[K]\}}\ . \tag{3.1.13}$$

Remark 3.1.1 *The probabilities are not immediately available in closed-form, but the next proposition shows that their characteristic function satisfy the same partial differential equation (3.1.12).*

Proposition 3.1.1 *(S. Heston, [44]) Suppose that we have given the two processes:*

$$dx(t) = \left(r - \frac{1}{2}\,v(t)\right)\,dt + \sqrt{v(t)}\,dW_1(t) \tag{3.1.14}$$

$$dv(t) = [k\,(\theta - v(t)) - \lambda\,v(t)]\,dt + \xi\,\sqrt{v(t)}\,dW_2(t) \tag{3.1.15}$$

with

$$cov[dW_1(t),\ dW_2(t)] = \rho\,dt \tag{3.1.16}$$

and a twice-differentiable function

$$f(x(t),\ v(t),\ t) = E[g(x(T),\ v(T))\ /\ x(t) = x,\ v(t) = v]. \tag{3.1.17}$$

Then the function f satisfy the partial differential equation:

$$\frac{1}{2}\,\xi^2\,v\,\frac{\partial^2 f}{\partial v^2} + \rho\,\xi\,v\,\frac{\partial^2 f}{\partial x \partial v} + \frac{1}{2}\,v\,\frac{\partial^2 f}{\partial x^2} + \tag{3.1.18}$$
$$\left(r - \frac{1}{2}\,v\right)\frac{\partial f}{\partial x} + [k\,(\theta - v) - v\,\lambda]\,\frac{\partial f}{\partial v} + \frac{\partial f}{\partial t} = 0\ .$$

Remark 3.1.2 *(S. Heston, [44]) Equation (3.1.17) imposes the final condition*

$$f(x,\ v,\ T)\ =\ g(x,\ v) \tag{3.1.19}$$

Depending on the choice of g, the function f represents different objects. Choosing $g(x\ ,\ v)\ =\ e^{i\varphi x}$ the solution is the characteristic function, which is available in closed form.

In order to solve the partial differential equation (3.1.18) with the above condition we invert the time direction: $\tau\ =\ T - t$. This mean that we must solve the following equation:

$$\begin{aligned}\frac{1}{2}\ \xi^2\ v\ \frac{\partial^2 f}{\partial v^2}\ +\ \rho\ \xi\ v\ \frac{\partial^2 f}{\partial x \partial v}\ +\ \frac{1}{2}\ v\ \frac{\partial^2 f}{\partial x^2}\ + \\ \left(r\ -\ \frac{1}{2}\ v\right)\ \frac{\partial f}{\partial x}\ +\ [k\ (\theta\ -\ v)\ -\ v\ \lambda]\ \frac{\partial f}{\partial v}\ -\ \frac{\partial f}{\partial t}\ =\ 0\end{aligned} \tag{3.1.20}$$

with the following initial condition:

$$f(x,\ v,\ 0)\ =\ e^{i\varphi x} \tag{3.1.21}$$

Thorem 3.1.1 *(L.V.Lazăr, [70]) The equation (3.1.20) subject to the initial condition (3.1.21) has the following solution:*

$$f(x\ ,\ v\ ,\ \tau)\ =\ e^{C(\tau)\ +\ D(\tau)\ v\ +\ i\varphi x}, \tag{3.1.22}$$

where

$$C(\tau) = ri\varphi\tau + \frac{k\theta}{\xi^2}\left[(k + \lambda + d - \rho\xi\varphi i)\tau - 2\ln\left(\frac{1 - ge^{d\tau}}{1 - e^{d\tau}}\right)\right] \tag{3.1.23}$$

and

$$D(\tau)\ =\ \frac{k\ +\ \lambda\ +\ d\ -\ \rho\ \xi\ \varphi\ i}{\xi^2}\ \left[\frac{1\ -\ e^{d\tau}}{1\ -\ g\ e^{d\tau}}\right] \tag{3.1.24}$$

with

$$d\ =\ \sqrt{(\rho\ \xi\ \varphi\ i\ -\ k\ -\ \lambda)^2\ -\ \xi^2\ (-\ \varphi^2\ -\ i\ \varphi)} \tag{3.1.25}$$

$$g\ =\ \frac{\rho\ \xi\ \varphi\ i\ -\ k\ -\ \lambda\ -\ d}{\rho\ \xi\ \varphi\ i\ -\ k\ -\ \lambda\ +\ d} \tag{3.1.26}$$

Finally we have the following theorem which give us a closed-form solution, for a Digital Call Option in the Heston model:

Thorem 3.1.2 *(V.L. Lazăr, [70]) Consider a Digital call option in the Heston model, with a strike price of K and a time to maturity of τ. Then the current price is given by the following formula:*

$$DC(S\ ,\ v\ ,\ t)\ =\ e^{-r\ \tau}\ P$$

where the probability function, P is given by:

$$P(x,\ v,\ t;\ \ln K)\ =\ \frac{1}{2}\ +\ \frac{1}{\pi}\int_0^\infty \mathcal{R}e\left[\frac{e^{-i\varphi \ln K}\ f(x,\ v,\ \tau,\ \varphi)}{i\ \varphi}\right]\ d\varphi$$

and the characteristic function is:

$$f(x,\ v,\ \tau)\ =\ e^{C(\tau)\ +\ D(\tau)\ v\ +\ i\varphi x}$$

where $C(\tau)$ and $D(\tau)$ are given by (3.1.23) and (3.1.24) respectively.

3.2 Numerical Methods for European Put Options

In this paragraph we took the case of European Put Options under stochastic volatility in a Foreign Exchange setting to illustrate how the finite difference and element methods can be used.

In the Foreign Exchange(FX) setting the risk-neutral drift term r of the underlying price process is equal to the difference between the domestic and foreign interest rates $r_d - r_f$. As shown by J. Hakala and U. Wystup [43] in a Foreign Exchange setting, the value function V, for the Heston model, satisfies the following partial differential equation:

$$\begin{aligned} &\frac{\partial V}{\partial t}\ +\ \frac{1}{2}\ v\ S^2\ \frac{\partial^2 V}{\partial S^2}\ +\ \rho\ \xi\ v\ S\ \frac{\partial^2 V}{\partial S\ \partial v}\ +\ \frac{1}{2}\ \xi^2\ v\ \frac{\partial^2 V}{\partial v^2} \\ &\quad +\ (r_d - r_f)\ S\ \frac{\partial V}{\partial S}\ +\ [\ k\ (\theta\ -\ v\ )\ -\lambda\ v\ ]\ \frac{\partial V}{\partial v}\ -\ r_d\ V\ =\ 0\ . \end{aligned} \tag{3.2.1}$$

An European put option with strike price K and maturing at time T satisfies the PDE (3.2.1) and the following boundary conditions:

$$\begin{aligned}
&V(S,v,T) = max(0, K-S)\\
&V(0,v,t) = Ke^{-r_d\tau}\\
&\tfrac{\partial V}{\partial S}(\infty,v,t) = 0\\
&(r_d - r_f)S\tfrac{\partial V}{\partial S}(S,0,t) + k\theta\tfrac{\partial V}{\partial S}(S,0,t) + \tfrac{\partial V}{\partial t}(S,0,t) - r_dV(S,0,t) = 0\\
&V(S,\infty,t) = Ke^{-r_d\tau}
\end{aligned} \tag{3.2.2}$$

where $\tau = T - t$ denotes the time to maturity.

Similar with the previous paragraph, we set $x = ln\frac{S}{K}$ and so $V(S,v,t) = w(ln\frac{S}{K}, v, t)$.

With this new variable the equation (3.2.1) becomes:

$$\begin{aligned}
\frac{\partial w}{\partial t} + \frac{1}{2}\xi^2 v\frac{\partial^2 w}{\partial v^2} + \rho\xi v\frac{\partial^2 w}{\partial x\partial v} + \frac{1}{2}v\frac{\partial^2 w}{\partial x^2}&\\
+ [k(\theta - v) - \lambda v]\frac{\partial w}{\partial v} + (r_d - r_f - \frac{1}{2}v)\frac{\partial w}{\partial x} - r_dw = 0,&
\end{aligned} \tag{3.2.3}$$

for all $(x,v,t) \in \Omega_\infty \times [0,T)$ with $\Omega_\infty = (-\infty,\infty)\times[0,\infty)$,

with terminal condition

$$w(x,v,T) = g(Ke^x), \tag{3.2.4}$$

where $g(Ke^x)$ is the payoff-function of the option.

We can rearrange the coefficients of the Heston p.d.e. (3.2.3) in a convenient way in order to obtain the following matrix representation:

$$0 \;=\; \frac{\partial w}{\partial t} \;+\; \nabla\cdot A\,\nabla w \;-\; b\cdot\nabla w \;-\; r_d\,w\;, \tag{3.2.5}$$

where

$$A \;:=\; \frac{1}{2}\,v\,\begin{bmatrix} \xi^2 & \rho\,\xi \\ \rho\,\xi & 1 \end{bmatrix}$$

and

$$b \;:=\; \begin{bmatrix} -\,k\,(\theta\;-\;v)\;+\;\lambda\,v\;+\;\frac{1}{2}\,\xi^2 \\ -\,(r_d\;-\;r_f)\;+\;\frac{1}{2}\,v\;+\;\frac{1}{2}\,\xi\,\rho \end{bmatrix}$$

This is a convection-diffusion equation, where the matrix A is called diffusion matrix and b the convection vector. In the Heston p.d.e the diffusion term is linear in v and so is the convection term up to an additional constant. We note that the diffusion matrix A is positive semidefinite for all $v > 0$. For more about convection-diffusion equation and numerical methods see [63, 64, 86, 108, 139, 147, 148].

3.2.1 Finite Difference Method

We consider the Heston partial differential equation (3.2.1) and we describe how the finite difference method (method which is suitable for solving financial problems with two or three random factors) can be used.

As usual, the first thing to do is to discretize the variables.

This means, we solve the problem on a three dimensional grid with:

$$S = i\delta S,\ v = j\delta v \text{ and } \quad t = T - k\delta t \quad i = 0, ..., I,\ j = 0, ..., J,$$

and the contract value can be written as:

$$V(S\ ,\ v\ ,\ t)\ =\ V_{ij}^{k}\ .$$

In order to solve our problem we must impose certain conditions on the solution.

In the finite-difference notation, the final condition for a European put option is the payoff function:

$$V_{ij}^{0} = max(K - i\delta S, 0). \tag{3.2.6}$$

This final condition will get our finite-difference scheme started. Backward time stepping must then be used when calculating the solution at an earlier time step.

Also, we have the boundary conditions around our domain, boundary conditions which are given by (3.2.2).

The Explicit Method

The definition of the first time-derivative of V is

$$\frac{\partial V}{\partial t} = \lim_{\delta t \to 0} \frac{V(S, v, t + \delta t) - V(S, v, t)}{\delta t},$$

so is naturally to approximate this time derivative from our grid of values using:

$$\frac{\partial V}{\partial t}(S, v, t) \approx \frac{V_{ij}^{k} - V_{ij}^{k+1}}{\delta t} + \mathcal{O}(\delta t) \tag{3.2.7}$$

The same idea can be used for approximating the first derivatives with respect to S and v. But in this cases we can choose between: **forward difference, backward difference** and, the most accurate, **central difference**. The central difference has an error of $O(\delta S^2)$ whereas the error in the forward and backward differences is in both much larger, $O(\delta S)$.

So, for the above derivatives we have:

$$\frac{\partial V}{\partial S}(S, v, t) \approx \frac{V_{i+1,j}^k - V_{i-1,j}^k}{2\delta S} + \mathcal{O}(\delta S^2) \tag{3.2.8}$$

and

$$\frac{\partial V}{\partial v}(S, v, t) \approx \frac{V_{i,j+1}^k - V_{i,j-1}^k}{2\delta v} + \mathcal{O}(\delta v^2). \tag{3.2.9}$$

From a Taylor series expansion, we obtain the following approximations for the second derivatives of the option with respect to the underlying and volatility:

$$\frac{\partial^2 V}{\partial S^2}(S, v, t) \approx \frac{V_{i+1,j}^k - 2V_{ij}^k + V_{i-1,j}^k}{\delta S^2} + \mathcal{O}(\delta S^2). \tag{3.2.10}$$

respectively:

$$\frac{\partial^2 V}{\partial v^2}(S, v, t) \approx \frac{V_{i,j+1}^k - 2V_{ij}^k + V_{i,j-1}^k}{\delta v^2} + \mathcal{O}(\delta v^2). \tag{3.2.11}$$

Finally, the second derivative with respect to both S and v can be approximate by

$$\frac{\partial\left(\frac{\partial V}{\partial v}\right)}{\partial S} \approx \frac{\frac{\partial V}{\partial v}(S + \delta S, v, t) - \frac{\partial V}{\partial v}(S - \delta S, v, t)}{2\delta S}.$$

On the other hand, we have

$$\frac{\partial V}{\partial v}(S + \delta S, v, t) \approx \frac{V_{i+1,j+1}^k - V_{i+1,j-1}^k}{2\delta v},$$

so a suitable discretization might be

$$\frac{\frac{V_{i+1,j+1}^k - V_{i+1,j-1}^k}{2\delta v} - \frac{V_{i-1,j+1}^k - V_{i-1,j-1}^k}{2\delta v}}{2\delta S} = \frac{V_{i+1,j+1}^k - V_{i+1,j-1}^k - V_{i-1,j+1}^k + V_{i-1,j-1}^k}{4\delta S \delta v}. \tag{3.2.12}$$

Using the above approximations (3.2.7-3.2.12) in the option pricing p.d.e. 3.2.3

we get the explicit difference scheme:

$$\begin{aligned}
&\frac{V_{ij}^k - V_{ij}^{k+1}}{\delta t} + \frac{1}{2}\ \sigma^2 j\delta v^k \left(\frac{V_{i,j+1}^k - 2\ V_{ij}^k + V_{i,j-1}^k}{\delta v^2}\right) + \\
&\quad + \rho\ \sigma ij\delta v^k \delta S^k \left(\frac{V_{i+1,j+1}^k - V_{i+1,j-1}^k - V_{i-1,j+1}^k + V_{i-1,j-1}^k}{4\delta S\delta v}\right) \\
&\quad + \frac{1}{2} i^2 j\delta v^k\ (\delta S^k)^2 \left(\frac{V_{i+1,j}^k - 2V_{ij}^k + V_{i-1,j}^k}{\delta S^2}\right) \\
&\quad + (r_d - r_f) i\delta S^k \left(\frac{V_{i+1,j}^k - V_{i-1,j}^k}{2\delta S}\right) \\
&\quad + [k(\theta - j\delta v^k) - \lambda j\delta v^k] \left(\frac{V_{i,j+1}^k - V_{i,j-1}^k}{2\delta v}\right) \\
&\quad + r_d V_{ij}^k = \mathcal{O}(\delta t, \delta S^2, \delta v^2). \qquad (3.2.13)
\end{aligned}$$

The explicit method, besides the advantage of being easy to program, has disadvantages concerning the stability and the speed. The method is only stable for sufficiently small timesteps and the upper bound on the timestepsize limits the speed of calculation.

More about the concepts of convergence and stability of the finite difference method for the Heston p.d.e. can be found in T. Kluge [63].

3.2.2 Finite Element Method

In their paper, G. Winkler, T. Apel and U. Wystup [148] have proved the existence of a solution for an European Call Option in the Heston's stochastic volatility model, solution which have been determined numerically using the finite element method. In this paragraph, we focus on how to solve equation (3.2.1) using finite element method subject to the boundary conditions (3.2.2).

First thing to do is to limit the domain Ω_∞ to a bounded one Ω:

$$\begin{aligned}
\Omega\ &=\ (x_{min}\ ,\ x_{max})\ \times\ (v_{min}\ ,\ v_{max}) \\
\partial\Omega\ &=\ \Gamma_1\ \cup\ \Gamma_2\ ,\ \Gamma_1 \cap\ \Gamma_2\ =\ \varnothing\ .
\end{aligned}$$

From the Black-Scholes formulae in the case of a put option, we find that the boundary condition for v_{min} and v_{max} are:

$$\begin{aligned}
\Gamma_a \; &: \; v \; = \; v_{min} \; : \; w(t, v_{min}, x) \; = K \; e^{-r_d(T-t)} \; \Phi(-d_2) - K \; e^{x - r_f(T-t)} \Phi(-d_1) \\
\Gamma_b \; &: \; v \; = \; v_{max} \; : \; w(t, v_{max}, x) \; = \; K \; e^{-r_d(T-t)}
\end{aligned}$$

For x_{min} and x_{max} we have the following conditions:

$$\begin{aligned}
\Gamma_c \; &: \; x \; = \; x_{min} \; : \; \tfrac{\partial}{\partial v} \, w(t, v, x_{min}) \; := \; A \, \nabla w \cdot \vec{n} \; = - \, \tfrac{1}{2} \, v \, K \, e^{x - r_f(T-t)} \\
\Gamma_d \; &: \; x \; = \; x_{max} \; : \; w(t, v, x_{max}) \; = \; \lambda \, w(t, v_{max}, x_{max}) \; + \; (1 - \lambda) \, w(t, v_{min}, x_{max}), \\
&\qquad \lambda \; = \; \tfrac{v \; - \; v_{min}}{v_{max} \; - \; v_{min}}
\end{aligned}$$

The three edges $\Gamma_a \cup \Gamma_b \cup \Gamma_d = \Gamma_1$ form a Dirichlet-type boundary conditions which must give the same limits on the corner of the domain, when moving to a corner from two distinct vertices. To guarantee the existence of a continuous solution on the entire domain Ω we will make the following assumption: on the boundary Γ_d we interpolate linearly between the values of the right and the left boundary.

Further we choose a semidiscretization in time which yields a series of two-dimensional boundary value problems. Let the difference quotient:

$$D_t^+ w(t^k, v, x) = \frac{w(t^{k+1}, v, x) - w(t^k, v, x)}{\tau} \tag{3.2.14}$$

where $\tau \; = \; t^{k+1} \; - \; t^k$ is constant for all k.

The computational working backwards in time, so in each timestep t^k we can assume we know $w(t^{k+1})$. Using a standard finite difference σ-rule (see [148]), where for $\sigma = 1$ we have a purely implicit method, for $\sigma = 0$ an explicit method and for $\sigma = \frac{1}{2}$ the Crank-Nicolson scheme, our problem reduces to solving a partial differential equation of the form:

$$Lw(t^k, v, x) = f(t^{k+1}, v, x) \tag{3.2.15}$$

where

$$Lw(t, v, x) = -\sigma \nabla \cdot A \nabla w + \sigma b \cdot \nabla w + (\sigma r_d + \frac{1}{\tau}) w \tag{3.2.16}$$

and

$$f(t, v, x) = (1 - \sigma) \nabla \cdot A \nabla w - (1 - \sigma) b \cdot \nabla w - \left((1 - \sigma) r_d - \frac{1}{\tau} \right) w \tag{3.2.17}$$

The next step is to choose appropriate spaces. Using the fact that the matrix A is positive definite, G. Winkler, T. Apel and U. Wystup [148], define the following spaces:

Definition 3.2.1 *Let $\Omega \subset \mathbb{R}^d$ be a bounded open domain and u and w be two functions. Let the $d \times d$-matrix $A(x)$ be symmetric and positive definite for all $x \in \Omega$ and let the constant $\hat{c}$ be positive. Then we define:*

$$(u, w)_A = \int_\Omega A\nabla u \cdot \nabla w + \int_\Omega \hat{c} u w, \tag{3.2.18}$$

$$\|u\|_A^2 = (u, u)_A = \int_\Omega A\nabla u \cdot \nabla u + \int_\Omega \hat{c} u^2. \tag{3.2.19}$$

Remark 3.2.1 *1. The constant $\hat{c}$ will be taken to be $c - \frac{1}{2}\nabla \cdot b$*
2. If $\hat{c} = \tilde{c} - \frac{1}{2}\nabla \cdot b$ we use the symbol $(\cdot, \cdot)_{\tilde{A}}$.

Lemma 3.2.1 *(G. Winkler, T. Apel and U. Wystup [148])The bilinear form $(u, w)_A$ defines a scalar product and satisfies the Cauchy-Schwarz inequality. $\|\cdot\|$ defines a norm.*

Definition 3.2.2 *We have the spaces*

$$\begin{aligned} &V := \{\psi / \ \|\psi\|_A < \infty\}, \\ &V_0 := \{\psi \in V / \ \psi = 0 \text{ on } \Gamma_1\}, \\ &V_* := \{\psi \in V / \ \psi \textit{ satisfies the boundary conditions on } \Gamma_1\}. \end{aligned} \tag{3.2.20}$$

In order to solve equation (3.2.15) we multiply it with a test function ψ taken from the function space V_0 and integrate it over the domain Ω :

$$\int_\Omega L w \psi = \int_\Omega f \psi. \tag{3.2.21}$$

From (3.2.21), we obtain the following weak formulation: We must search, for each k, a function $w^k = w(t^k, v, x) \in V_*$ such that for all $\psi \in V_0$ we have:

$$a(w^k \ , \ \psi) \ = \ \langle \ F \ , \ \psi \ \rangle \ . \tag{3.2.22}$$

where

$$a(w^k,\psi) = \int\limits_{\Omega} A\nabla w^k \cdot \nabla\psi + \frac{1}{2}\int\limits_{\Omega}(b\cdot\nabla w^k\psi - w^k b\cdot\nabla\psi)$$
$$+ \int\limits_{\Omega}(c - \frac{1}{2}\nabla\cdot b)w^k\psi$$
$$+ \frac{1}{2}\int\limits_{\Gamma_c}\left[\frac{1}{2}v - (r_d - r_f) + \frac{1}{2}\xi\rho\right]w^k\psi, \tag{3.2.23}$$

$$\langle\ F^k\ ,\ \psi\ \rangle = f(t^{k+1};\psi) + \int\limits_{\Gamma_c}\underbrace{A\nabla w^k\cdot\begin{bmatrix}0\\1\end{bmatrix}}_{g_2}\psi \tag{3.2.24}$$

$$f(t^{k+1};\psi) = \int\limits_{\Omega} A\nabla w^{k+1}\cdot\nabla\psi - \int\limits_{\Omega}(b\cdot\nabla w^{k+1})\psi - \int\limits_{\Omega}(\tilde{c} - \frac{1}{2}\nabla\cdot b)w^{k+1}\psi.$$

Thorem 3.2.1 *(V.L. Lazăr, [71]) The equation*

$$a(w,\psi) = \langle\ F\ ,\ \psi\ \rangle,\quad \forall\ \psi\ \in\ V_0, \tag{3.2.25}$$

where the bilinear functional a *and the linear functional* F *are given by (3.2.23) respectively by (3.2.24) has a unique solution* $w\ \in\ V_*$ *, if*

$$min\left\{\frac{1}{2}v - (r_d - r_f) + \frac{1}{2}\xi\rho\quad,\quad v\in[v_{min}, v_{max}]\right\} > -2 \tag{3.2.26}$$

and

$$\hat{c} = (c - \frac{1}{2}\nabla\cdot b) = r_d + \frac{1}{\sigma\tau} - \frac{1}{2}(k+\lambda) > 0. \tag{3.2.27}$$

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