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On the Dini-Hadamard subdifferential calculus in Banach spaces

PhD thesis Summary

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Contents

Chapter 1

Introduction

It is well known that nonsmooth functions, sets with nonsmooth boundaries and set-valued mappings appear naturally and frequently in various areas of mathematics and applications, especially in those related to optimization, stability, variational systems and control systems. Actually, the study of the local behavior of nondifferentiable objects is accomplished in the framework of *nonsmooth analysis* whose origin goes back in the early 1960's, when control theorists and nonlinear programmers attempted to deal with necessary optimality conditions for problems with nonsmooth data or with nonsmooth functions (such as the pointwise maximum of several smooth functions) that arise even in many problems with smooth data. Since then, nonsmooth analysis has come to play an important role in functional analysis, optimization, mechanics and plasticity, differential equations (as in the theory of viscosity solutions), control theory etc, becoming an active and fruitful area of mathematics. As Penot said very nice, while there are some reasons of being afraid of nonsmooth analysis, namely the abundance of concepts, some uncertainty of terminology, the lack of coherence in notations and the feeling of unsecurity, there are also other reasons of being seduced by this famous area, since any sort of function or set can be treated, new operations such as taking infima or suprema are no more out of reach and finally the passages from functions to sets and to set-valued mappings bring a unification of mathematics.

There has been a good amount of interest in generalizations of the pointwise derivative for the purposes of optimization. This has lead to many definitions of *generalized gradients*, *subgradients* and other kind of objects under various names. They were introduced first in the classical theory of real functions and in the theory of distributions (see for instance Bruckner [29], Saks [134], Schwartz [135] and Sobolev [137]). And all this work in order to solve optimization problems where classical differentiability assumptions are no longer appropriate. One of the most widely used *subdifferential* (set of subgradients), appropriate for applications to optimization, is the one who first appeared for convex functions in the context of convex analysis (see for more details [100, 127, 130] and the references therein). It has found many significant theoretical and practical uses in optimization, economics (see for instance [10]), mechanics and has proven to be a very interesting mathematical construct.

But the attempt to extend this success to functions which are no more convex has proven to be more difficult.

We mention here two main approaches. The first one uses a *generalized directional derivative* f^∂ of $f : X \to \mathbb{R} \cup \{+\infty\}$ of some type and then *tangentially* defines a convex subdifferential via the formula $\partial f(x) := \{x^* \in X^* : x^* \leq f^\partial(x, \cdot)\}$. As an example, the *Clarke subdifferential*, who in fact uses a positively homogeneous directional derivative, was the first concept of a subdifferential defined for a general nonconvex function and has been introduced in 1973 by Clarke (see for instance [34, 35]), who actually coined the term *nonsmooth analysis* in the 1970's and who performed a real pioneering work in the field of nonsmooth analysis, spread far beyond the scope of convexity. But unfortunately, as stated in [16], at some *abnormal points* of certain even Lipschitz nonsmooth functions, the Clarke subdifferential may include some *extraneous subgradients*. Moreover, the Clarke normal cone happens to be a linear subspace or even the whole space (for example, $N^C(\operatorname{graph} f; (0,0)) = \mathbb{R}^2$ in case $f(x) = |x|$, $x \in \mathbb{R}$). And this because, in general, a convex set often provides a subdifferential that is too large for a lot of optimization problems.

The second approach to define general subdifferentials satisfying useful calculus rules is to take limits of primitive subdifferential constructions which do not possess such calculus. It is important that limiting constructions depend not only on the choice of primitive objects but also on the character of the limit: *topological* or *sequential*.

The topological way allows one to develop useful subdifferentials in general infinite dimensional settings, but the biggest drawback is the fact that it may lead to broad constructions and in general they have an intrinsically complicated structure usually following a three-step procedure. Namely, the definition of ∂f for a Lipschitz function which requires considering restrictions to finite-dimensional (or separable) subspaces with intersections over the collection of all such subspaces, then the definition of a normal cone of a set C at a given point x as the cone generated by the subdifferential of the distance function to C and finally the definition of ∂f for an arbitrary lower semicontinuous function by means of the normal cone to the epigraph of f. In this line of development, many infinite dimensional extensions of the nonconvex constructions in [89, 90] were introduced and strongly developed by Ioffe in a series of publications starting from 1981 (see [65, 66, 67] for the bibliographies and commentaries therein) on the basis of topological limits of Dini-Hadamard ε-subdifferentials. Such constructions, also called *approximate subdifferentials*, are well defined in more general spaces, but all of them (including also their nuclei) may be broader than the Kruger-Mordukhovich extension even for Lipschitz functions on Banach spaces with Fréchet differentiable renorms.

The sequential way usually leads to more convenient objects, but it requires some special geometric properties of spaces in question (see for instance [18]). Thus, because the convexity is no longer inherent in the procedure, we are able to define smaller subdifferentials and also to exclude some points from the set of stationary points. The sequential nonconvex subdifferential constructions in Banach spaces were first introduced in Kruger-Mordukhovich [77, 78] on the basis of sequential limits of Fréchet ε-normals and subdifferentials. Such limiting normal cone and subdifferential appeared as infinite dimensional extensions of the corresponding finite dimensional constructions in Mordukhovich [89, 90] motivated by applications to optimization and control. Useful properties of those and related constructions were revealed mainly for Banach spaces with Fréchet differentiable renorms.

But, of course, many authors work with an abstract concept of a subdifferential (see for instance [67, 70, 114] and the references therein), satisfying a list of comprehensive axioms.

Let us finally emphasize that while the subdifferential theory in finite dimensions has been well developed, there still exist many open questions in infinite dimensional spaces.

In the following we give a description of how is this thesis organized, underlining its most important results.

In **Chapter 2** we begin our exposure with some preliminary notions and results, referring to the almost all the important subdifferentials that have been widely used during the last decades, namely the Fréchet subdifferential, the Dini-Hadamard one, the proximal subdifferential, some subdifferentials associated with bornologies, the limiting ones, the Mordukhovich and the approximate subdifferentials of Ioffe, the Clarke generalized gradient, the Michel-Penot subdifferential and finally we detail a method of introducing an abstract subdifferential.

Chapter 3 deals with a detailed study of the Dini-Hadamard subdifferential and related constructions. We show that these constructions have a variety of nice properties in the general Banach space setting. Most of these properties (including some efficient representations, variational descriptions, some dissipative and differentiability properties etc.) are collected in this chapter. After presenting some concepts of smallness for sets, we further revisit the notion of the so-called spongious sets introduced by Treiman [140] in the 1986's and we illustrate the relationship between them and the classical neighborhoods, answering the question how further can we go in Treiman's definition with the replacement of a neighborhood by a sponge. A fruitful relationship between directionally convergent sequences and sponges is further provided, in fact, one of the key ingredients that plays a prominent role in various descriptions of the Dini-Hadamard subdifferential and other related constructions. We even introduce a directionally limes inferior which naturally induces a corresponding directionally lower semicontinuity. Some spongious continuity notions are studied next in Subsection 3.1.4, where the intention is to generalize the classical Lipschitz continuity by introducing a kind of spongiously continuity notion, for the purpose of the fourth chapter. In the final part of this subsection we complete the picture by illustrating with some examples the relationships between the new and the old constructions. Some favorable classes of nonsmooth functions are further described, among which we mention the class of approximately convex functions, approximately starshaped and the one of directionally approximately starshaped introduced by Penot [108]. The great novelty here comes from an important characterization of directionally approximately starshaped functions by means of sponges. In Subsection 3.1.6 we propose some various kinds of topological notions for operators very useful in deriving exact difference formulae for the Dini-Hadamard subdifferential. An entire section is dedicated to the study of derivatives of the Dini-Hadamard type. After recalling the definition and the main properties of the

Dini-Hadamard directional derivative, we describe a natural way of introducing a new directional derivative via directionally convergent sequences. In fact, the latter construction enable us to perfect the Dini-Hadamard subdifferential by introducing the Dini-Hadamard-like one, while the secret motivation was the simple fact that it enjoys similar properties like the Fréchet subdifferential in terms of sponges. In Section 3.3.2 we emphasize the key role of the calmness property in describing Dini-Hadamard subgradients and we correct a simple assertion given by Treiman [140] without proof. In the final part of this subsection we furnish some examples illustrating the error and we show how it looks like a sponge which is not a neighborhood, having in mind the crucial importance of such an example in providing various links between the key notions. A cornerstone of the theory developed in this thesis turns out to be a variational description of the Dini-Hadamard subdifferential presented in Subsection 3.3.3. Also valid for the Dini-Hadamard-like subdifferential, this kind of property facilitate almost all the proofs that involves such constructions. Although it is broad enough, the sponge notion has a great minus. Namely, the cartesian product of two sponges is not anymore a sponge, like we were familiar in the case of neighborhoods. This is the reason why we need to introduce also a decoupled Dini-Hadamard construction on product spaces. A smooth variational description is obtained in Subsection 3.3.5, while the full usage will be done in Subsection 4.2.5. Next we make some connections between the subgradients studied until now in this chapter and the corresponding normal cones. We even provide an alternative description for the contingent normal cone and we distinguish between various analytic and geometrical constructions. Our primary goal in Subsection 3.3.7. is to underline some links between the Dini-Hadamard subdifferential and other well known subdifferentials in the literature, while in Subsection 3.3.8 we give a key description of directionally approximately starshaped functions via Dini-Hadamard subgradients. At the end of this chapter we introduce a new type of derivative object for multifunctions by means of the decoupled Dini-Hadamard subdifferential of the indicator function to the graphical set. We also define some differentiability notions, illustrating a relationship between the decoupled Dini-Hadamard-like coderivative of spongiously Lipschitz mappings and the Dini-Hadamard-like subdifferential of their scalarization.

Chapter 4 is devoted to a thorough study on the Dini-Hadamard subdifferential calculus in Banach spaces. After discussing some ways of developing a set of basic tools for the subdifferential analysis, we present the most important type of calculus rules that a subdifferential may obey. A weak fuzzy sum rule for the Dini-Hadamard subdifferential (see Ioffe [59]) and also some weak fuzzy calculus for the Dini-Hadamard coderivative (see [74]) are next furnished, reminding us the poor calculus available for this kind of constructions. The main attention in the second part of the chapter is paid in finding some exact calculus rules for the Dini-Hadamard subdifferential and the related constructions in arbitrary Banach spaces. Thus, after presenting a few sums and differences involving smooth functions we provide in Subsection 4.2.3 an exact subdifferential formula for the difference of two directionally approximately starshaped functions. It is worth emphasizing here that this kind of result for the Dini-Hadamard ε-subdifferential was the first one who appeared in literature. Later, Penot [119] has also provided similar formulae but by using some dissipativity assumptions. In fact, the main idea was to extend some assertions given by Penot [1] for approximately starshaped functions, by means of the Fréchet subdifferential, motivated by the fact that there exists directionally approximately starshaped functions for which the statements in there do not apply. Although the Dini-Hadamard subdifferential and its ε-enlargement are well known in variational analysis and generalized differentiation they are not widely used due to the lack of calculus. However, as already mentioned, the essential difference rule holds for such subdifferentials under natural assumptions. On the other hand, it happens that the exact difference rule for the Dini-Hadamard coderivative obtained in Subsection 4.2.4 alongside the relationship between the coderivative of the given function and the subdifferential of its scalarization to be essential in developing exact formulae for various compositions. Thus, the results in Subsection 4.2.5 follow somehow the ones obtained by Mordukhovich [96] in terms of Fréchet subgradients. Actually, the key method here is to furnish such calculus first for the Dini-Hadamard-like subdifferential and then for the Dini-Hadamard subdifferential and the Fréchet one. We observe also here that our estimates obtained for the Fréchet subdifferential, involving also the Dini-Hadamard-like subdifferential of the scalarization function remains true also for those single-valued mappings which are only spongiously Lipschitz and strongly spongiously continuous. Finally we present some particular cases, among which one can mention some product and quotient rules, emphasizing the essential role of the upper counterparts of Dini-Hadamard-like subgradients. Thus we extend some assertions given by Mordukhovich for Lipschitz functions, by means of the Fréchet subdifferential, to the class of spongiously Lipschitz and approximately starshaped functions.

In **Chapter 5** we turn our attention to the formulation of optimality conditions via the Dini-Hadamard subdifferential for the following general cone-constrained optimization problem

$$(\mathcal{P}) \qquad \inf_{x\in\mathcal{A}} f(x).$$
$$\mathcal{A} = \{x \in C : k(x) \ \in -K\}$$

having the difference of two functions as objective. The general framework we work under is the following one: X, Z two Banach spaces, $C \subseteq X$ is a convex and closed set, $K \subseteq Z$ is a nonempty convex and closed con, $k : X \to Z$ is a K-convex and K-epi closed function and $g, h : X \to \mathbb{R} \cup \{+\infty\}$ are such that $\operatorname{dom} g \subseteq \operatorname{dom} h$ and $f := g - h$. In fact we extend a result given by Amahroq, Penot and Syam [1] in the particular instance when $K = \{0\}$ and $k(x) = 0$ for all $x \in X$ and by means of the Fréchet subdifferential, underlining the fact that although they make use of some exact subdifferential formulae for the limiting subdifferential, they provide an incorrect argumentation, since these are valid in Asplund spaces. However, the statement in [1, Proposition 6] is true in Banach spaces, too, and it can be proven in the lines of the proof of our Theorem.

The author's original contributions are:

In Chapter 3: Remark 3.1.7, Lemma 3.1.8, Remark 3.1.9, Example 3.1.10, Example 3.1.11, Example 3.1.12, Example 3.1.13, Remark 3.1.14, Remark 3.1.16, Remark 3.1.17, Remark 3.1.18, Proposition 3.1.19, Proposition 3.1.20, Example 3.1.21, Proposition 3.1.28, Lemma 3.2.1, Lemma 3.3.1, Remark 3.3.2, Remark 3.3.3, Lemma 3.3.8, Lemma 3.3.9, Remark 3.3.10, Example 3.3.11, Example 3.3.12, Example 3.3.14, Theorem 3.3.15, Remark 3.3.16, Theorem 3.3.17, Remark 3.3.18, Proposition 3.3.19, Corollary 3.3.20, Proposition 3.3.21, Remark 3.3.24, Theorem 3.3.25, Theorem 3.3.26, Proposition 3.3.27, Proposition 3.3.28, Corollary 3.3.29, Corollary 3.3.30, Corollary 3.3.31, Remark 3.3.32, Proposition 3.3.33, Proposition 3.3.34, Proposition 3.3.35, Proposition 3.3.36, Proposition 3.4.1, Proposition 3.4.2, Remark 3.4.3, Proposition 3.4.4.

We also mention here that the following notions were introduced: *directional lower limit*, *directional upper limit*, *spongiously Lipschitz function*, *strongly spongiously continuous function*, *spongiously pseudo-dissipative operator* (Definition 3.1.23), *spongiously gap-continuous operator*, *Dini-Hadamard-like directional derivative*, *Dini-Hadamard-like ε-subdifferential*, *decoupled Dini-Hadamard-like subdifferential*, *Dini-Hadamard-like differentiability* (Definition 3.3.22), *Dini-Hadamard-like decoupled differentiability* (Definition 3.3.23), *spongiously local minimizer*, *spongiously decoupled local minimizer*, *decoupled Dini-Hadamard-like coderivative* and *decoupled Dini-Hadamard normal cone.*

In Chapter 4: Proposition 4.2.1, Proposition 4.2.2, Proposition 4.2.3, Remark 4.2.4, Theorem 4.2.5, Remark 4.2.6, Theorem 4.2.7, Remark 4.2.8, Theorem 4.2.9, Corollary 4.2.10, Corollary 4.2.11, Corollary 4.2.12, Corollary 4.2.13, Theorem 4.2.14, Corollary 4.2.15, Remark 4.2.16, Corollary 4.2.17, Corollary 4.2.18, Corollary 4.2.19, Theorem 4.2.20, Corollary 4.2.21, Corollary 4.2.22, Theorem 4.2.23, Corollary 4.2.24, Corollary 4.2.25, Theorem 4.2.26, Corollary 4.2.27.

In Chapter 5: Remark 5.1.2, Proposition 5.1.3, Theorem 5.1.4, Remark 5.1.5, Remark 5.1.6. We also mention that we have introduced the notion of a *spongiously local ε-blunt minimizer* (Definition 5.1.1) in this chapter.

There are also other original discussion and details presented in this thesis, too, but they are unnumbered.

At the end of this short Introduction we specify that the results in this paper are partially included in the following papers: A. Baias and D.-M. Nechita [10], A. Baias and D.-M. Nechita [9], R.I. Boţ and D.-M. Nechita [28], D.-M. Nechita [101], D.-M. Nechita [102], D.-M. Nechita [103] and D.-M. Nechita [104].

Keywords: Dini-Hadamard ε-subdifferential, Dini-Hadamard-like constructions, Fréchet ε-subdifferential, porous set, sponge, approximately convex functions, approximately starshaped functions, directionally approximately starshaped functions, directionally convergent sequences, variational description, dissipative operator, spongiously gap-continuity, spongiously Lipschitz function, strongly spongiously continuous function, coderivatives, subdifferential calculus, optimality conditions, constrained optimization problems, spongiously local ε-blunt minimizer

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Chapter 2

Subdifferentials: an overview

In general the calculus rules for the three main classes of objects of nonsmooth analysis, namely subdifferentials, normal cones and coderivatives, are strongly related and every result for each of them can, in principle, be obtained from the calculus rules for any other basic operation with any other object. Thus, the choice of a sequence in which the results are proved or presented is often a matter of taste or personal preferences. It is our aim in this section to present an axiomatic approach, but also the most important subdifferential constructions in Banach spaces, alongside their key properties, while their importance will be seen throughout the entire work.

2.1 Examples of subdifferentials

Firstly, we present in Subsection 2.1.1 the most important subdifferentials that can be obtained by means of various directional derivatives. Namely, the classical subdifferential of convex analysis, the Dini-Hadamard subdifferential, the Clarke and the Clarke-Rockafellar subdifferentials and the Michel-Penot one. Then we revisit in Subsection 2.1.2 the Fréchet (the analytical and the geometrical version) and the Mordukhovich subdifferentials, emphasizing their strong relationship. The proximal subdifferential is presented in Subsection 2.1.3, while the topological constructions by Ioffe are defined in Subsection 2.1.4. Some bornological subdifferentials are further detailed in Subsection 2.1.5. In Subsection 2.1.6 we present a simple way to derive limiting subdifferentials, while in Subsection 2.1.7 we finally refer to some second order constructions.

2.2 An axiomatic approach

To conclude this chapter, we present an abstract way to introduce a subdifferential, alongside the normal cone and the coderivative that can be naturally associated to.

Chapter 3

The Dini-Hadamard subdifferential and related constructions

Nonsmooth phenomena have been well known for a long time in mathematics and applied sciences, too. To deal with nonsmooth functions, sets with nonsmooth boundaries and set-valued mappings, various kinds of generalized derivatives were introduced in the classical theory of real functions and in the theory of distributions. It is our intention in this chapter to illustrate the main tools of working with the Dini-Hadamard subdifferential, generated with the help of an interesting directional derivative who was fascinating the mathematicians since even the 1970's. The key ingredient in almost all the proofs involving such subgradients will proved to be a variational description via sponges, similar to the one that exists for the Fréchet subdifferential. A decoupled version is also introduced on product spaces, while the great importance will be seen in Chapter 4. Some special classes of nonsmooth functions are also revisit in order to provide then some exact calculus rules. Finally, it is worthwhile to emphasize that by providing examples of sponges which are not neighborhoods one can easily construct further the beautiful theory that is moving around this nice subdifferential construction.

The results presented in this chapter are mainly based on [9, 28, 101, 102, 103].

3.1 Preliminary notions and results

3.1.1 Small sets

Firstly, let us specify that are many concepts of *smallness* (see [14, 150]) which frequently appear in analysis, namely measure theoretic (null sets of different kind), topological (sets of the first Baire category), metric (σ-porous sets or directionally σ-porous sets), analytic (countable unions of sets that can be represented as subsets of graphs of certain classes of Lipschitz functions).

In this subsection we focus our attention in presenting some of the above concepts with a significant role in describing a typical behavior of the Dini-Hadamard directional derivative and the corresponding subdifferential in separable Banach spaces.

3.1.2 Sponges versus neighborhoods

The notion of a sponge was introduced by Treiman [140] and, as we will see bellow it turns out to be very useful for characterizing the Dini-Hadamard subdifferential. Actually, the idea behind this concept was the fact that a neighborhood is in general not broad enough to characterize this kind of subdifferential constructions.

Definition 3.1.5 *(cf. [140, Definition 2.2]) A set $S \subseteq X$ is said to be a* sponge *around $\overline{x} \in X$ if for all $h \in X \setminus \{0\}$ there exist $\lambda > 0$ and $\delta > 0$ such that $\overline{x} + [0, \lambda] \cdot B(h, \delta) \subseteq S$.*

Observe here that, if for $h \in S_X$ the same statement as above holds true, then we obtain an equivalent notion. A nice example is provided in the following.

Example 3.1.6 (cf. [140, Example 2.3]) Let $f : X \to \mathbb{R}$ be a locally Lipschitz and Gâteaux differentiable function at $\overline{x} \in X$ with $x^* \in X^*$ its Gâteaux derivative at this point. Then for all $\varepsilon > 0$ the sets

$$S_1 := \{x \in X : \ f(x) - f(\overline{x}) \geq \langle x^*, x - \overline{x} \rangle - \varepsilon \|x - \overline{x}\|\}$$

and

$$S_2 := \{x \in X : \ f(x) - f(\overline{x}) \leq \langle x^*, x - \overline{x} \rangle + \varepsilon \|x - \overline{x}\|\}$$

are sponges around $\overline{x}$.

As one can easily observe from the definition above, the singular point 0 is ignored. Let us emphasize also here that the sponges enjoy a nice relationship with the cone-porous sets. Indeed, accordingly to Cobzaş [38, Proposition 1], if S is a sponge around $\overline{x}$ then the complementary set $\mathcal{C}(S) \cup \{\overline{x}\}$ is cone porous in any direction $v \in S_X$ and hence it is a porous set. We remind also here that every neighborhood of a point $\overline{x} \in X$ is a sponge around $\overline{x}$ and that the converse is not true (see, for instance, Example 3.3.11). However, in case S is a convex set or X is a finite dimensional space (here one can make use of the fact that the unit sphere is compact), then S is also a neighborhood of $\overline{x}$.

Remark 3.1.7 Trying to answer the question how further can we go with the replacement of a neighborhood by a sponge, let us first present some properties that a sponge can successfully enjoy:

(A) : for all $h \in X \setminus \{0\}$ there exist $\lambda > 0$ and a sponge S' around h such that for all $u \in S'$, $\overline{x} + [0, \lambda] \cdot u \subseteq S$.

(B) : for all $h \in X \setminus \{0\}$ and all $d \in X \setminus \{0\}$ there exists $\delta > 0$ such that for all $u \in B(h, \delta) \cap (h + [0, \delta] \cdot B(d, \delta))$, $\overline{x} + [0, \delta] \cdot u \subseteq S$.

Moreover, every set S which satisfies one of the above properties is a sponge around $\overline{x}$.

In fact, the beautiful equivalence between the above properties shows us that by replacing in the definition of a sponge the neighborhood with a sponge we do not obtain a different notion.

3.1.3 Relationships between directionally convergent sequences and sponges

Inspired by a definition introduced in [115] we say that a sequence (x_n) of X *converges to* $\overline{x}$ *in the direction* $d \in X \setminus \{0\}$ (and we write $(x_n) \underset{d}{\longrightarrow} \overline{x}$) if there exist sequences $(t_n) \to 0$, $t_n \geq 0$ and $(d_n) \to d$ such that $x_n = \overline{x} + t_n d_n$ for each $n \in \mathbb{N}$. Further, a sequence (x_n) *directionally converges* to $\overline{x}$ if there exists $d \in X \setminus \{0\}$ such that $(x_n) \underset{d}{\longrightarrow} \overline{x}$. Our notion, slightly different from the one proposed by Penot [115], allows us to consider also the constants sequences among the ones which are directionally convergent. Motivated by this observation, we call the *directional lower limit* of f at $\overline{x}$ in the direction $d \in X \setminus \{0\}$ the following limit

$$\liminf_{x \underset{d}{\longrightarrow} \overline{x}} f(x) := \sup_{\delta > 0} \inf_{x \in B(\overline{x},\delta) \cap (\overline{x} + [0,\delta] \cdot B(d,\delta))} f(x).$$

Similarly one can define the *directional upper limit* of f at $\overline{x}$ in the direction $d \in X \setminus \{0\}$, since the lower properties symmetrically induce the corresponding upper ones

$$\limsup_{x \underset{d}{\longrightarrow} \overline{x}} f(x) := -\liminf_{x \underset{d}{\longrightarrow} \overline{x}} (-f)(x) = \inf_{\delta > 0} \sup_{x \in B(\overline{x},\delta) \cap (\overline{x} + [0,\delta] \cdot B(d,\delta))} f(x).$$

Moreover, one can surely observe that

$$\liminf_{x \to \overline{x}} f(x) \leq \liminf_{x \underset{d}{\longrightarrow} \overline{x}} f(x) \leq \limsup_{x \underset{d}{\longrightarrow} \overline{x}} f(x) \leq \limsup_{x \to \overline{x}} f(x) \text{ for all } d \in X \setminus \{0\}. \tag{3.1}$$

Actually, as it was first observed by Penot (see [115, Lemma 2.1]), the concept of a directionally convergent sequence is clearly related to the so-called spongious sets introduced above. So, let us finally illustrate this fruitful relationship.

Lemma 3.1.8 *A subset S of X is a sponge around $\overline{x}$ if and only if for any sequence (x_n) directionally convergent to $\overline{x}$ there exists $n_0 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $n \geq n_0$, $x_n \in S$.*

A direct consequence of this result will be the fact that the two notions of limes inferior listed above coincide in finite dimensions.

3.1.4 Spongious continuity notions

For the purpose of Section 4.2, we generalize in this subsection the well-known Lipschitz continuity for single-valued mappings, actually very important in variational analysis and its applications. Namely, we say that $f : X \to Y$ is *spongiously Lipschitz* at $\overline{x} \in X$ if there exist $K > 0$ and a sponge S around $\overline{x}$ such that for all $x \in S$ the following inequality

$$\|f(x) - f(\overline{x})\| \leq K\|x - \overline{x}\|$$

holds true. Clearly if f is Lipschitz at $\overline{x}$ then it is spongiously Lipschitz at $\overline{x}$, too, but not viceversa (see, for instance, Example 3.1.11 bellow). However, the two notions agree if $\dim X < +\infty$. Of course, f is called spongiously Lipschitz around $\overline{x}$ if the inequality $\|f(x) - f(y)\| \leq K\|x - y\|$ is satisfied for every $x, y \in S$.

Further, we say that f is *strongly spongiously continuous* at $\overline{x} \in X$ if for any sponge S_2 around $f(\overline{x})$ there exists a sponge S_1 around $\overline{x}$ such that $f(S_1) \subseteq S_2$. In fact, this special strange continuity is a strongest version of the following one, mentioned by Penot in [115]. Namely, f is said to be *spongiously continuous* at $\overline{x}$ if for each neighborhood V of $f(\overline{x})$ there exists a sponge S of $\overline{x}$ such that $f(S) \subset V$, i.e. if f is *directionally continuous* at $\overline{x}$ in the following sense: for any $d \in X \setminus \{0\}$ and any directionally convergent sequence $(x_n) \underset{d}{\longrightarrow} \overline{x}$ one has $(f(x_n)) \to f(\overline{x})$. However, when Y is finite dimensional, the two notions coincide. Let us also mention that we cannot establish any precise relationship between the classical continuity and the strongly spongiuosly continuity one. Nevertheless, any mapping which is either continuous or strongly spongiuosly continuous at $\overline{x}$ is clearly spongiously continuous at $\overline{x}$, too.

Remark 3.1.9 Observe also that, in contrast to the Lipschitz continuity, which actually hides the classical continuity of the mapping involved, the simple fact that f is spongiously Lipschitz at $\overline{x}$ doesn't necessarily implies that f is strongly spongiously continuous at $\overline{x}$, too, unless Y is finite dimensional.

The following examples intend to complete the picture involving the four continuity conditions studied above.

Example 3.1.10 Consider X an infinite Banach space and S a sponge around 0_X which is not a neighborhood of 0_X (see for instance Example 3.3.11 bellow). Further let us define the function $f : X \to X$ as follows

$$f(x) = \begin{cases} 0_X, & \text{if } x \in S, \\ \frac{x}{\|x\|}, & \text{otherwise.} \end{cases}$$

Then f is spongiously Lipschitz and strongly spongiously continuous at 0_X, but not Lipschitz or even continuous at 0_X.

The following example describes a similar situation for the case when the second space in question is a finite dimensional one.

Example 3.1.11 Consider S a sponge around $\overline{x} \in X$ which is not a neighborhood of $\overline{x}$ and define the function $f : X \to \mathbb{R}$ as follows

$$f(x) = \begin{cases} 0, & \text{if } x \in S, \\ -1, & \text{otherwise.} \end{cases}$$

Then f is spongiously Lipschitz and strongly spongiously continuous at $\overline{x}$, but not Lipschitz or even continuous at $\overline{x}$.

We have seen above that in case we are working in infinite dimensional settings, we are able to find functions which are strongly spongiously continuous, but not continuous. Now, it is the time to construct a function which is continuous, but not spongiously continuous.

Example 3.1.12 Consider X a Banach space and S a sponge around $\overline{x} \in X$ which is not a neighborhood of $\overline{x}$. That means that for any natural number $n \in \mathbb{N}^*$ one can find at least one element x_n (we fix one) such that $x_n \in \overline{B}(\overline{x}, \frac{1}{n}) \setminus S$. Further, define the function $f : X \to X$ as follows

$$f(x) = \begin{cases} x_{n+1}, & \text{if } x \in \overline{B}(\overline{x}, \frac{1}{n}) \setminus B(\overline{x}, \frac{1}{n+1}), n \in \mathbb{N}^* \\ x, & \text{otherwise,} \end{cases}$$

Then f is Lipschitz continuous and spongiously Lipschitz at $\overline{x}$, but it is not strongly spongiously continuous at $\overline{x}$.

Following the idea above one can furnish a similar example in case only the second space is infinite dimensional.

Example 3.1.13 Consider X a Banach space and S a sponge around $0_X \in X$ which is not a neighborhood of 0_X. Then for any natural number $n \in \mathbb{N}^*$ one can find at least one element x_n (we fix one) such that $x_n \in \overline{B}(0_X, \frac{1}{n}) \setminus S$. Further, define the function $f : \mathbb{R} \to X$ as follows

$$f(x) = \begin{cases} x_{n+1}, & \text{if } x \in \left[-\frac{1}{n}, -\frac{1}{n+1}\right) \bigcup \left(\frac{1}{n+1}, \frac{1}{n}\right], n \in \mathbb{N}^* \\ 0_X, & \text{otherwise,} \end{cases}$$

Then f is Lipschitz continuous and spongiously Lipschitz at 0, but it is not strongly spongiously continuous at 0.

Thus, the spongious continuity notion seems to be more general than the continuity one in case X is infinite dimensional and Y is finite dimensional, while the situation changes when X is finite dimensional and Y is infinite dimensional. But, of course, the two notions coincide in finite dimensional settings. Finally, the following remark intends to capture all the aspects presented above.

Remark 3.1.14 For a given function $f : X \to Y$ finite at $\overline{x}$ the following hold:

(a) *If X and Y are both infinite dimensional then*

$$\begin{array}{ccc}
f \text{ continuous at } \overline{x} & \begin{array}{c} \textit{(Ex. 3.1.10)} \\ \not\Longleftarrow \\ \not\Longrightarrow \\ \textit{(Ex. 3.1.12)} \end{array} & f \text{ strongly spongiously continous at } \overline{x} \\
\not\Downarrow \ \Uparrow & & \not\Uparrow \ \not\Downarrow \ \textit{(Ex. 3.1.12)} \\
f \text{ Lipschitz at } \overline{x} & \begin{array}{c} \textit{(Ex. 3.1.10)} \\ \not\Longleftarrow \\ \Longrightarrow \\ (\textit{def.}) \end{array} & f \text{ spongiously Lipschitz at } \overline{x}
\end{array}$$

(b) *If X and Y are both finite dimensional then*

$$\begin{array}{ccc}
f \text{ continuous at } \overline{x} & \Longleftrightarrow & f \text{ strongly spongiously continous at } \overline{x} \\
\not\Downarrow \ \Uparrow & & \Uparrow \ \not\Downarrow \\
f \text{ Lipschitz at } \overline{x} & \Longleftrightarrow & f \text{ spongiously Lipschitz at } \overline{x}
\end{array}$$

(c) *If X is finite dimensional and Y is infinite dimensional then*

$$\begin{array}{ccc}
f \text{ continuous at } \overline{x} & \begin{array}{c} \textit{(def.)} \\ \Longleftarrow \\ \not\Longrightarrow \\ \textit{(Ex. 3.1.13)} \end{array} & f \text{ strongly spongiously continous at } \overline{x} \\
\not\Downarrow \ \Uparrow & & \textit{(Ex. 3.1.13)} \not\Uparrow \ \not\Downarrow \\
f \text{ Lipschitz at } \overline{x} & \begin{array}{c} \Longleftrightarrow \\ \textit{(def.)} \end{array} & f \text{ spongiously Lipschitz at } \overline{x}
\end{array}$$

(d) *If X is infinite dimensional and Y is finite dimensional then*

$$\begin{array}{ccc}
f \text{ continuous at } \overline{x} & \begin{array}{c} \textit{(Ex. 3.1.11)} \\ \not\Longleftarrow \\ \Longrightarrow \\ \textit{(def.)} \end{array} & f \text{ strongly spongiously continous at } \overline{x} \\
\not\Downarrow \ \Uparrow & & \textit{(Rem. 3.1.9)} \ \Uparrow \ \not\Downarrow \\
f \text{ Lipschitz at } \overline{x} & \begin{array}{c} \textit{(Ex. 3.1.11)} \\ \not\Longleftarrow \\ \Longrightarrow \\ \textit{(def.)} \end{array} & f \text{ spongiously Lipschitz at } \overline{x}
\end{array}$$

3.1.5 Favorable classes of nonsmooth functions

We begin this subsection by recalling some generalized convexity notions for functions, while their essential importance and influence will be discovered in the sequel.

Definition 3.1.15 *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a given function and $\overline{x} \in \operatorname{dom} f$. Then f is said to be*

(i) approximately convex *at $\overline{x}$, if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for every $x, y \in B(\overline{x}, \delta)$ and every $t \in [0, 1]$ one has*

$$f((1-t)y + tx) \leq (1-t)f(y) + tf(x) + \varepsilon t(1-t)\|x - y\|. \tag{3.2}$$

(ii) approximately starshaped *at $\overline{x}$, if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for every $x \in B(\overline{x}, \delta)$ and every $t \in [0, 1]$ one has*

$$f((1-t)\overline{x} + tx) \leq (1-t)f(\overline{x}) + tf(x) + \varepsilon t(1-t)\|x - \overline{x}\|. \tag{3.3}$$

(iii) directionally approximately starshaped *at $\overline{x}$, if for any $\varepsilon > 0$ and any $u \in S_X$ there exists $\delta > 0$ such that for every $s \in (0, \delta)$, every $v \in B(u, \delta)$ and every $t \in [0, 1]$, when $x := \overline{x} + sv$, one has*

$$f((1-t)\overline{x} + tx) \leq (1-t)f(\overline{x}) + tf(x) + \varepsilon t(1-t)\|x - \overline{x}\|. \tag{3.4}$$

The approximately convex functions have been introduced in [106] (see also [8, 107]), while the approximately starshaped and the directionally approximately starshaped ones have been object of study in [108].

Remark 3.1.16 The set of approximately convex functions at a given point $\overline{x} \in X$ is a convex cone containing the functions which are strictly differentiable at $\overline{x}$, being stable under finite suprema and moreover the most of the well-known subdifferentials coincide and share several properties of the convex subdifferential (see [106]) on this particular class of functions. An example of an approximately convex function at every $x \in \mathbb{R}$, which is not convex, is $x \mapsto |x| - x^2$.

Remark 3.1.17 One can easily see that if f is approximately convex at $\overline{x}$, then it is approximately starshaped at $\overline{x}$, too. Nevertheless, the reverse implication does not hold. The following example in this sense has been inspired by [108, Example 6.10]. We define $f : \mathbb{R} \to \mathbb{R}$ as follows: $f(0) = 0$, $f(x) = 1/(2n+1)(x - 1/(2n)) + 1/(4n^2)$, for $x \in [1/(2n+1), 1/(2n)]$, $n \geq 1$, $f(x) = 1/(2n)x$, for $x \in [1/(2n), 1/(2n-1))$, $n \geq 1$, $f(x) = +\infty$, for $x \geq 1$, while for $x < 0$ we take $f(x) = f(-x)$. Then f is approximately starshaped at 0, but not approximately convex at 0.

Remark 3.1.18 By a straightforward calculation one can show that if f is approximately starshaped at $\overline{x}$, then it is directionally approximately starshaped at $\overline{x}$, too. In order to give an example for the failure of the reverse implication we first characterize the class of directionally approximately starshaped functions by means of sponges. A direct consequence of Proposition 3.1.19 will be the fact that, in finite dimensional spaces, the two classes of functions coincide.

Proposition 3.1.19 *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a given function and $\overline{x} \in \operatorname{dom} f$. Then f is directionally approximately starshaped at $\overline{x}$ if and only if for any $\varepsilon > 0$ there exists a sponge S around $\overline{x}$ such that for every $x \in S$ and every $t \in [0,1]$ one has*

$$f((1-t)\overline{x} + tx) \leq (1-t)f(\overline{x}) + tf(x) + \varepsilon t(1-t)\|x - \overline{x}\|. \tag{3.5}$$

It is worth emphasizing here that in view of Remark 3.1.7, the above characterization via sponges it is also equivalent with the following one.

Proposition 3.1.20 *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a given function and $\overline{x} \in \operatorname{dom} f$. Then f is directionally approximately starshaped at $\overline{x}$ if and only if for any $\varepsilon > 0$, $h \in X \setminus \{0\}$ and any $d \in X \setminus \{0\}$ there exists $\delta > 0$ such that for every $s \in (0, \delta)$, $v \in B(h, \delta) \cap (h + [0, \delta] \cdot B(d, \delta))$ and every $t \in [0,1]$, with $x := \overline{x} + sv$, the relation (3.5) above holds true.*

We come now to the announced example of a function which is directionally approximately starshaped at a point, but fails to be approximately starshaped at that point.

Example 3.1.21 Let $\overline{x} \in X$, $S \subseteq X$ be a sponge around $\overline{x}$, which is not a neighborhood of $\bar{x}$ (see, for instance, Example 3.3.11), and the function real-valued $f : X \to \mathbb{R}$,

$$f(x) = \begin{cases} 0, & \text{if } x \in S, \\ -\|x - \overline{x}\|, & \text{otherwise.} \end{cases}$$

Then the function f is directionally approximately starshaped at $\overline{x}$ but not approximately starshaped at $\overline{x}$.

3.1.6 On some various kinds of dissipative properties for operators

The notions presented in this subsection play a special role in deriving exact difference formulae for the Dini-Hadamard and Dini-Hadamard-like subdifferentials. We actually propose various kinds of topological notions for operators, showing the relation between them and those introduced by Penot in [119].

The results in this subsection are mainly based on [9] and [28].

Definition 3.1.22 (cf. [119, Definition 1]) *A set-valued mapping $F : X \rightrightarrows X^*$ is said to be* approximately pseudo-dissipative *at $\overline{x} \in X$ if for every $\varepsilon > 0$ one can find some $\delta > 0$ such that*

$$\forall x \in B(\overline{x}, \delta),\ \exists x^* \in F(x),\ \exists \overline{x}^* \in F(\overline{x}) \quad \langle x^* - \overline{x}^*, x - \overline{x} \rangle \leq \varepsilon \|x - \overline{x}\|.$$

Let us describe in the following two ways of extending the approximately pseudo-dissipativity property. Observe also here that the first one bellow is obtained by replacing neighborhoods with sponges in Definition 3.1.22.

Definition 3.1.23 *A set-valued mapping $F : X \rightrightarrows X^*$ is said to be* spongiously pseudo-dissipative *at $\overline{x} \in X$ if for any $\varepsilon > 0$ there exists S a sponge around $\overline{x}$ such that for any $x \in S$ there exist $x^* \in F(x)$ and $\overline{x}^* \in F(\overline{x})$ so that*

$$\langle x^* - \overline{x}^*, x - \overline{x} \rangle \leq \varepsilon \|x - \overline{x}\|$$

or, equivalently, if for any $\varepsilon > 0$ and any $u \in S_X$ there exists $\delta > 0$ such that for any $t \in (0, \delta)$ and $v \in B(u, \delta)$ there exist $x^* \in F(x)$ and $\overline{x}^* \in F(\overline{x})$ so that

$$\langle x^* - \overline{x}^*, v \rangle \leq \varepsilon \|v\|.$$

Definition 3.1.24 (cf. [119, Definition 1]) *A set-valued mapping $F : X \rightrightarrows X^*$ is said to be* directionally approximately pseudo-dissipative *at $\overline{x} \in X$ if for every $u \in S_X$ and $\varepsilon > 0$ one can find some $\delta > 0$ such that*

$$\forall v \in B(u, \delta),\ \forall t \in (0, \delta)\ \exists x^* \in F(\overline{x} + tv),\ \exists \overline{x}^* \in F(\overline{x}) \quad \langle x^* - \overline{x}^*, v \rangle \leq \varepsilon.$$

Since the last inequality can be changed into $\langle x^* - \overline{x}^*, x - \overline{x}\rangle \leq \varepsilon\|x - \overline{x}\|$ with $x := \overline{x} + tv$, ε being arbitrary, one sees that F is directionally approximately pseudo-dissipative at $\overline{x}$ whenever it is approximately pseudo-dissipative at $\overline{x}$. Moreover, an easy covering argument shows that both properties coincide when X is finite dimensional.

In fact this latter conditions are not very restrictive ones, since the following coarse continuity (introduced in [1]) ensures the approximately pseudo-dissipativity and the *spongiously gap-continuity* studied in [28], as well. Let us formulate now this concept.

Definition 3.1.25 (cf. [1, Definition 2]) *A set-valued mapping $F : X \rightrightarrows Y$ between a topological space X and a metric space Y is said to be* gap-continuous *at $\overline{x} \in X$ if for any $\varepsilon > 0$ one can find some $\delta > 0$ such that for every $x \in B(\overline{x}, \delta)$*

$$\operatorname{gap}(F(\overline{x}), F(x)) < \varepsilon,$$

where for two subsets A and B of Y

$$\operatorname{gap}(A, B) := \inf\{d(a, b) : a \in A, b \in B\},$$

with the convention that if one of the sets is empty, then $\operatorname{gap}(A, B) := +\infty$.

When defining a spongiously gap-continuous mapping one only has to replace in the above definition the neighborhood $B(\overline{x}, \delta)$ of $\overline{x}$ with a sponge S around $\overline{x}$. Therefore, every gap-continuous mapping at a point is spongiously gap-continuous and moreover it is also spongiously pseudo-dissipative and directionally approximately pseudo-dissipative at that point, too. Furthermore, every set-valued mapping which is either Hausdorff upper semicontinuous or lower semicontinuous at a given point is gap-continuous at that point (see [118]). Thus, the gap-continuity is a sort of semicontinuity notion which is satisfied in many situations when no other semicontinuity notion holds. Moreover, in case the mapping is single-valued, it coincides with the classical continuity. Clearly, when X is a finite dimensional space then the gap-continuity coincides with the spongiously gap-continuity as well as the approximately pseudo-dissipativity property agrees with the spongiously pseudo-dissipativity and with the directionally approximately pseudo-dissipativity one.

It is worth emphasizing also here that a set-valued mapping $F : X \rightrightarrows Y$ is spongiously gap-continuous at $\overline{x}$ if and only if for all $u \in X \setminus \{0\}$, $\operatorname{gap}(F(\overline{x} + tv), F(\overline{x})) \to 0$, as $(t, v) \to (0_+, u)$, i.e. the notion of spongiously gap-continuity defined in [28] is equivalent to that of *directionally-gap continuity* introduced later by Penot [119]. In fact, although the notion of a sponge is not explicitly mention in [119], there are some directionally convergent sequences that are successfully used in there. We refer the reader to the papers by Penot [118, 119] for more discussions and some criteria ensuring the gap-continuity and also the approximately pseudo-dissipativity.

On the other hand, it can be shown that $F : X \rightrightarrows Y$ is spongiously gap-continuous at $\overline{x}$ if and only if for any $\varepsilon > 0$ there exists a sponge S around $\overline{x}$ such that for every $x \in S$ one has

$$F(x) \cap (F(\overline{x}) + \varepsilon B_Y) \neq \emptyset. \tag{3.6}$$

When $\overline{x} \in X$ and $S \subseteq X$ is a sponge around $\overline{x}$, which is not a neighborhood of $\overline{x}$, then $F : X \rightrightarrows \mathbb{R}$ defined by $F(x) = \{0\}$ for $x \in S$ and $F(x) = \emptyset$, otherwise, is not gap-continuous, but spongiously gap-continuous at $\overline{x}$.

Proposition 3.1.28 *Let $F, G : X \rightrightarrows Y$ be two set-valued mappings. If F is spongiously gap-continuous at $\overline{x} \in X$ and there exists a sponge S around $\overline{x}$ such that $F(x) \subseteq G(x)$ for all $x \in S$, then G is spongiously gap-continuous at $\overline{x}$.*

Note also that the above property holds true also for spongiously pseudo-dissipative set-valued mappings.

3.2 Derivatives of the Dini-Hadamard type

Our aim now is to show that starting with the Dini-Hadamard directional derivative prezented bellow one can easily define an interesting derivative-like object by means of directionally convergent sequences.

But first we recall that

$$d^{DH} f(\overline{x}; h) := \liminf_{\substack{u \to h \\ t \downarrow 0}} \frac{f(\overline{x} + tu) - f(\overline{x})}{t} = \sup_{\delta > 0} \inf_{\substack{u \in B(h,\delta) \\ t \in (0,\delta)}} \frac{f(\overline{x} + tu) - f(\overline{x})}{t} \tag{3.7}$$

denotes the Dini-Hadamard directional derivative of f at $\overline{x}$, while

$$d^{DH,+} f(\overline{x}; h) := \limsup_{\substack{u \to h \\ t \downarrow 0}} \frac{f(\overline{x} + tu) - f(\overline{x})}{t} = \inf_{\delta > 0} \sup_{\substack{u \in B(h,\delta) \\ t \in (0,\delta)}} \frac{f(\overline{x} + tu) - f(\overline{x})}{t},$$

stands for its upper counterpart, also labeled the *upper Dini-Hadamard directional derivative.*

Of course that

$$d^{DH,+} f(\overline{x}; h) = -d^{DH}(-f)(\overline{x}; h),$$

so that only lower derivatives may be considered.

After presenting a survey of certain elementary facts (see, for instance, [3, 15, 63, 73, 111, 122]), we introduce the following construction (see [101]),

$$\widetilde{D}_d f(\overline{x}; h) := \sup_{\delta > 0} \inf_{\substack{u \in B(h,\delta) \cap (h + [0,\delta] \cdot B(d,\delta)) \\ t \in (0,\delta)}} \frac{f(\overline{x} + tu) - f(\overline{x})}{t}, \tag{3.8}$$

simply labeled as the *Dini-Hadamard-like directional derivative* of f at $\overline{x}$ in the direction $h \in X$ through $d \in X \setminus \{0\}$. It extends somehow the Dini-Hadamard directional derivative, while the essential idea was inspired by the relationship between sponges and directionally convergent sequences. In fact, as we can easily see, directionally convergent sequences are used in place of the usual ones.

One always have the inequality $d^{DH} f(\overline{x}; h) \leq \widetilde{D}_d f(\overline{x}; h)$, which holds as equality in finite dimensional spaces if we take into account Lemma 3.1.8 and the fact that the unit sphere is compact.

Furthermore, like in the case of the Dini-Hadamard directional derivative, for $\overline{x} \in \operatorname{dom} f$ and $d \in X \setminus \{0\}$, $\widetilde{D}_d f(\overline{x}; \cdot)$ is in general not convex, but positive homogeneous and hence $\widetilde{D}_d f(\overline{x}; 0)$ is either 0 or $-\infty$. Moreover, we may (formally) write

$$\widetilde{D}_d f(\overline{x}; h) = \liminf_{\substack{u \xrightarrow[d]{} h \\ t \xrightarrow[1]{} 0}} \frac{f(\overline{x} + tu) - f(\overline{x})}{t} = \liminf_{\substack{u \xrightarrow[d]{} h \\ t \downarrow 0}} \frac{f(\overline{x} + tu) - f(\overline{x})}{t}.$$

Finally, the next lemma is especially important for various results involving Dini-Hadamard-like constructions.

Lemma 3.2.1 *Let $f : X \to \overline{\mathbb{R}}$ be a given function and $\overline{x}, h \in X$. Then the following statements are true:*

(i) $\widetilde{D}_d f(\overline{x}; h) \leq \liminf\limits_{n \to +\infty} \dfrac{f(\overline{x} + t_n u_n) - f(\overline{x})}{t_n}$,
whenever $(u_n) \xrightarrow[d]{} h$ *and* $(t_n \downarrow 0)$, *with* $d \in X \setminus \{0\}$.

(ii) If for some $d \in X \setminus \{0\}$, $\widetilde{D}_d f(\overline{x}; h) = l \in \mathbb{R} \cup \{-\infty\}$, *then there exist sequences* $(u_n) \xrightarrow[d]{} h$ *and* $(t_n \downarrow 0)$ *such that* $\lim\limits_{n \to +\infty} \dfrac{f(\overline{x} + t_n u_n) - f(\overline{x})}{t_n} = l$.

3.3 Subgradients of the Dini-Hadamard type

3.3.1 Basic definitions and some properties

In this section we introduce a new subdifferential construction and study its relationship with the Dini-Hadamard one. As usual, our standard framework is that of Banach spaces unless otherwise stated.

We start with the definition of the Dini-Hadamard subdifferential. Namely, the following set

$$\partial_\varepsilon^{DH} f(\overline{x}) := \{x^* \in X^* : \langle x^*, h\rangle \leq d^{DH} f(\overline{x}; h) + \varepsilon \|h\| \text{ for all } h \in X\}, \tag{3.9}$$

where $\varepsilon \geq 0$, is called the *Dini-Hadamard ε-subdifferential* of f at $\overline{x}$, while $\partial^{DH} f(\overline{x}) := \partial_0^{DH} f(\overline{x})$ stands for the simply called *Dini-Hadamard subdifferential* of f at $\overline{x}$. When $\overline{x} \notin \operatorname{dom} f$, we set $\partial_\varepsilon^{DH} f(\overline{x}) := \emptyset$ for all $\varepsilon \geq 0$.

Similarly, following the two steps procedure of constructing the Dini-Hadamard ε-subdifferential, but employing a directional convergence in place of the usual one, we can define (see [101]) the *Dini-Hadamard-like ε-subdifferential* of f at $\overline{x}$, i.e.

$$\widetilde{\partial}_\varepsilon f(\overline{x}) := \{x^* \in X^* : \langle x^*, h\rangle \leq \widetilde{D}_d f(\overline{x}; h) + \varepsilon \|h\| \ \ \forall h \in X \ \forall d \in X \setminus \{0\}\}.$$

We put $\widetilde{\partial}_\varepsilon f(\overline{x}) := \emptyset$ if $\overline{x} \notin \operatorname{dom} f$. In case $\varepsilon = 0$, $\widetilde{\partial} f(\overline{x}) := \widetilde{\partial}_0 f(\overline{x})$ simply denotes the *Dini-Hadamard-like subdifferential* of f at $\overline{x}$.

Notice that although the two directional derivatives $d^{DH} f(\overline{x}; \cdot)$ and $\widetilde{D}_d f(\overline{x}; \cdot)$ are in general not convex, the Dini-Hadamard ε-subdifferential of f at $\overline{x} \in \operatorname{dom} f$ and the Dini-Hadamard-like ε-one are always convex sets.

For $f : X \to \mathbb{R} \cup \{+\infty\}$, $\overline{x} \in \operatorname{dom} f$ and $\varepsilon \geq 0$, we define $f_\varepsilon : X \to \mathbb{R} \cup \{+\infty\}$ as being

$$f_\varepsilon(x) := f(x) + \varepsilon\|x - \overline{x}\|. \tag{3.10}$$

Lemma 3.3.1 *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a given function and $\overline{x} \in \operatorname{dom} f$. Then for all $\varepsilon \geq 0$ it holds*

$$\partial_\varepsilon^{DH} f(\overline{x}) = \partial^{DH} f_\varepsilon(\overline{x}). \tag{3.11}$$

Remark 3.3.2 Let us notice that one can replace in (3.11) the Dini-Hadamard subdifferential by the Fréchet one (see, for instance, [1]) and also by the Dini-Hadamard-like one. On the other hand, in case f is convex, by a classical subdifferential sum formula provided by the convex analysis, one has $\partial^{DH} f_\varepsilon(x) = \widetilde{\partial} f_\varepsilon(x) = \partial f(x) + \varepsilon \overline{B}_{X^*}$ for all $x \in X$ and all $\varepsilon \geq 0$. Let us emphasize also here that for the Dini-Hadamard ε-subdifferential and for its natural extension the following monotonicity property holds, namely:

$$\partial_{\varepsilon_1}^{DH} f(x) \subseteq \partial_{\varepsilon_2}^{DH} f(x), \tag{3.12}$$

$$\widetilde{\partial}_{\varepsilon_1} f(x) \subseteq \widetilde{\partial}_{\varepsilon_2} f(x), \tag{3.13}$$

when $\varepsilon_2 \geq \varepsilon_1 \geq 0$ and $x \in X$.

Remark 3.3.3 Finally, using (3.12) and (3.13) one can show that for a given function $f : X \to \mathbb{R} \cup \{+\infty\}$ and a given point $\overline{x} \in \operatorname{dom} f$, $\partial_\eta^{DH} f$ is spongiously gap-continuous at $\overline{x}$ for all $\eta > 0$, whenever $\partial^{DH} f$ is spongiously gap-continuous at $\overline{x}$, while $\widetilde{\partial}_\eta f$ is spongiously pseudo-dissipative at $\overline{x}$ for all $\eta > 0$, whenever $\widetilde{\partial} f$ is spongiously pseudo-dissipative at $\overline{x}$.

3.3.2 The key role of the calmness property in describing Dini-Hadamard subgradients

In this section, with the notion of calmness (first developed by Clarke [33] in order to express a kind of constraint qualification, in the context of optimal value functions; see also [34]), we light up some properties about the Dini-Hadamard subdifferential of an arbitrary function on a Banach space. Although for the study of this kind of subgradients the notion is fundamental, this is no longer true when speaking about Dini-Hadamard-like subgradients.

Definition 3.3.4 *A function $f : X \to \overline{\mathbb{R}}$ is said to be* calm at $\overline{x} \in \operatorname{dom} f$ *if there exists $c \geq 0$ and $\delta > 0$ such that $f(x) - f(\overline{x}) \geq -c\|x - \overline{x}\|$ for all $x \in B(\overline{x}, \delta)$.*

When $-f$ is calm at $\overline{x}$, then f is said to be *quiet* at $\overline{x}$, and if f is both calm and quiet at $\overline{x}$, f is said to be *stable* at $\overline{x}$.

The following proposition is the first ingredient who emphasizes the strongly relationship between calmness and the Dini-Hadamard or contingent subdifferentiability.

Proposition 3.3.5 (cf. [54, Proposition 2.2]) *Let $f : X \to \overline{\mathbb{R}}$ be a given function and $\overline{x} \in \operatorname{dom} f$. Then the following assertions are equivalent:*

(i) *f is calm at $\overline{x}$.*

(ii) *For every $x \in X$ and every sequence $((x_n, r_n))_n \in X \times (0, 1]$ converging to $(x, 0_+)$ for m large enough, the sequence $(r^{-1}(f(\overline{x} + r_n x_n) - f(\overline{x})))_{n \geq m}$ is bounded bellow.*

(iii) *There exists some constant $c \in \mathbb{R}_+$ such that for every $h \in X$:*

$$d^{DH} f(\overline{x}; h) \geq -c\|h\|.$$

(iv) *The lower derivative $d^{DH} f(\overline{x}; \cdot)$ does not take the value $-\infty$ on X.*

(v) $d^{DH} f(\overline{x}; 0) = 0$.

Moreover, if f is tangentially convex at $\overline{x}$ (i.e. $d^{DH} f(\overline{x}; \cdot)$ is a convex function), then we have the additional equivalence:

(vi) *$\partial^{DH} f(\overline{x})$ is nonempty.*

The following interesting lemma, due to Penot [119, Lemma 19], provides a sufficient condition for a function to be tangentially convex.

Lemma 3.3.6 *If the function $f : X \to \mathbb{R} \cup \{+\infty\}$ is approximately convex at $\overline{x} \in$ core(dom f), then f has a directional derivative at $\overline{x}$ which is sublinear and finite, so that f is tangentially convex at $\overline{x}$.*

A similar result as above is also true for quiet functions, but we refer the reader to the paper by Giner [54] for more details and discussions in this direction.

The notion that we recall below was introduced by Treiman in [140] and, as we will prove in Theorem 3.3.15, it turns out to be very useful for characterizing both the Dini-Hadamard subdifferential and the Dini-Hadamard-like one.

Definition 3.3.7 *Let $f : X \to \overline{\mathbb{R}}$ be a given function finite at $\overline{x}$ and $\varepsilon \geq 0$. We say that $x^* \in X^*$ is an H_ε-*subgradient *of f at $\overline{x}$ if there exists a sponge S around $\overline{x}$ such that for all $x \in S$*

$$f(x) - f(\overline{x}) \geq \langle x^*, x - \overline{x}\rangle - \varepsilon\|x - \overline{x}\|.$$

The following important lemma was inspired by some statements one can find in Treiman's paper [140].

Lemma 3.3.8 *Let $f : X \to \overline{\mathbb{R}}$ be a given function finite at $\overline{x}$ and $\varepsilon \geq 0$. The following statements are true:*
(i) If $x^* \in \partial^{DH}_\varepsilon f(\overline{x})$, then x^* is an H_γ-subgradient of f at $\overline{x}$ for all $\gamma > \varepsilon$.
(ii) If f is calm at $\overline{x}$ and x^* is an H_ε-subgradient of f at $\overline{x}$, then $x^* \in \partial^{DH}_\varepsilon f(\overline{x})$.

Let us emphasize here that one can obtain a similar result without any calmness assumption on the state function, by making use of the Dini-Hadamard-like subdifferential.

Lemma 3.3.9 *Let $f : X \to \overline{\mathbb{R}}$ be a given function finite at $\overline{x}$ and $\varepsilon \geq 0$. The following statements are true:*

(i) If $x^* \in \widetilde{\partial}_\varepsilon f(\overline{x})$, then x^* is an H_γ-subgradient of f at $\overline{x}$ for all $\gamma > \varepsilon$.

(ii) If x^* is an H_ε-subgradient of f at $\overline{x}$, then $x^* \in \widetilde{\partial}_\varepsilon f(\overline{x})$.

Remark 3.3.10 Moreover, one can even conclude that whenever $\overline{x} \in \operatorname{dom} f$, $\varepsilon \geq 0$ and $\gamma > \varepsilon$ the following set

$$S := \{x \in X : \ f(x) - f(\overline{x}) \geq \langle x^*, x - \overline{x}\rangle - \gamma\|x - \overline{x}\|\} \tag{3.14}$$

is a sponge around $\overline{x}$ not only for those elements $x^* \in \partial_\varepsilon^{DH} f(\overline{x})$, but also for $x^* \in \widetilde{\partial}_\varepsilon f(\overline{x})$.

In the following we provide an example of a sponge around a point which is not a neighborhood of that point.

Example 3.3.11 We consider again the space $C[0,1]$ endowed with the supremum norm. Let $\overline{x} \in S_{C[0,1]}$ be an element in this space with the property that $|\overline{x}|$ attains its maximum at exactly one point of the interval $[0,1]$. Let further $x^* \in X^*$ be an element in $\partial^{DH}(-\|\cdot\|_\infty)(\overline{x})$, which is a nonempty set. As the Fréchet subdifferential of $-\|\cdot\|_\infty$ at $\overline{x}$ is empty, there exists an $\alpha > 0$ such that for all $\delta > 0$ there is some $x \in B(\overline{x}, \delta)$ satisfying

$$\|\overline{x}\|_\infty - \|x\|_\infty + \alpha\|x - \overline{x}\|_\infty < \langle x^*, x - \overline{x}\rangle. \tag{3.15}$$

As seen above, the set

$$S := \{x \in C[0,1] : \ \|\overline{x}\|_\infty - \|x\|_\infty + \alpha\|x - \overline{x}\|_\infty \geq \langle x^*, x - \overline{x}\rangle\} \tag{3.16}$$

is a sponge around $\overline{x}$ (take $\varepsilon := 0$ and $\gamma := \alpha$ in Remark 3.3.10). It remains to show that S is not a neighborhood of $\overline{x}$. Supposing the contrary, there must exist a $\bar{\delta} > 0$ such that $B(\overline{x}, \bar{\delta}) \subseteq S$. But this is a contradiction to (3.15) and, consequently, S fails to be a neighborhood of $\overline{x}$.

Example 3.3.12 The following example shows that, in the case of the Dini-Hadamard subdifferential, more precisely in the second assertion of Lemma 3.3.8 one cannot renounce at the hypotheses that f is calm at $\overline{x}$. Indeed, take S a sponge around $\overline{x} \in X$, which is not a neighborhood of $\bar{x}$ and define $f : X \to \mathbb{R}$ as being

$$f(x) = \begin{cases} 0, & \text{if } x \in S, \\ -1, & \text{otherwise.} \end{cases}$$

Then, for all $\varepsilon \geq 0$, 0 is an H_ε-subgradient of f at $\bar{x}$, but f is not calm at $\bar{x}$ and, consequently, $0 \notin \partial_\varepsilon^{DH} f(\overline{x})$.

Example 3.3.13 Both assertions of Lemma 3.3.8 have been given by Treiman in [140] without proof for f a lower semicontinuous function on X and without assuming for (ii) that f is calm at $\bar{x}$. The following example, which has been kindly provided to us by Jean-Paul Penot, shows that even for lower semicontinuous functions one cannot renounce at the calmness hypotheses in order to get the desired conclusion. Let X be an infinite dimensional Banach space and a sequence of elements $(e_n)_{n\geq 1}$ on the unit sphere of X such that $\|e_n - e_m\| > 1/2$ for all $n, m \geq 1$, $n \neq m$. Define $f : X \to \mathbb{R}$ as being $f(x) = -1/2^n$ when $n \geq 1$ is such that $x = 1/4^n e_n$ and $f(x) = 0$, otherwise. The function f is lower semicontinuous and it fulfills $f(x) \geq f(0)$ for all $x \in X \setminus \bigcup_{n\geq 1} \{1/4^n e_n\}$. Since $X \setminus \bigcup_{n\geq 1} \{1/4^n e_n\}$ is a sponge around 0, for all $\varepsilon \geq 0$, 0 is an H_ε-subgradient of f at 0. On the other hand, as f fails to be calm at 0, 0 cannot be a Dini-Hadamard ε-subgradient of f at 0.

Example 3.3.14 Although in finite dimensions the Dini-Hadamard ε-subdifferential coincide with the corresponding Dini-Hadamard-like one (see for instance Remark 3.3.32) this is in general not the case. Indeed, let us consider the function $f : X \to \mathbb{R}$ as being

$$f(x) = \begin{cases} 0, & \text{if } x \in S, \\ a, & \text{otherwise,} \end{cases}$$

where $a < 0$ and S is a sponge around $\overline{x} \in X$ which is not a neighborhood of $\overline{x}$. Then taking into account the second assertion of Lemma 3.3.9, one can easily conclude that for all $\varepsilon \geq 0$, $0 \in \widetilde{\partial}_\varepsilon f(\overline{x}) \setminus \partial_\varepsilon^{DH} f(\overline{x})$, since 0 is an H_ε-subgradient of f at $\overline{x}$, but f is not calm at $\overline{x}$.

3.3.3 Some useful characterizations of subgradients

Next we provide a variational description of the Dini-Hadamard ε-subdifferential similar to the one that exists for the Fréchet ε-subdifferential, but by replacing neighborhoods with sponges.

Theorem 3.3.15 *Let $f : X \to \overline{\mathbb{R}}$ be an arbitrary function finite at $\overline{x}$. Then for all $\varepsilon \geq 0$ one has*

$$\begin{array}{l} x^* \in \partial_\varepsilon^{DH} f(\overline{x}) \Leftrightarrow f \textit{ is calm at } \overline{x} \textit{ and } \forall \alpha > 0 \textit{ there exists } S \textit{ a sponge} \\ \textit{around } \overline{x} \textit{ such that } \forall x \in S,\ f(x) - f(\overline{x}) \geq \langle x^*, x - \overline{x}\rangle - (\alpha + \varepsilon)\|x - \overline{x}\|. \end{array} \tag{3.17}$$

Remark 3.3.16 By making use of Theorem 3.3.15 one can easily prove that for every $\varepsilon \geq 0$ the Dini-Hadamard ε-subdifferential of $f : X \to \overline{\mathbb{R}}$ at $\overline{x}$ with $|f(\overline{x})| < +\infty$ can be also characterized at follows

$$\begin{array}{ll} x^* \in \partial_\varepsilon^{DH} f(\overline{x}) \Leftrightarrow & f \text{ is calm at } \overline{x} \text{ and } \forall \alpha > 0\ \forall u \in S_X\ \exists \delta > 0 \text{ such that} \\ & \forall s \in (0, \delta)\ \forall v \in B(u, \delta) \text{ for } x := \overline{x} + sv \text{ one has} \\ & f(x) - f(\overline{x}) \geq \langle x^*, x - \overline{x}\rangle - (\alpha + \varepsilon)\|x - \overline{x}\|. \end{array}$$

It is worth emphasizing here that also the Dini-Hadamard-like subdifferential enjoys a similar variational description in the absence of any calmness assumption.

Theorem 3.3.17 *Let $f : X \to \overline{\mathbb{R}}$ be an arbitrary function finite at $\overline{x}$. Then for all $\varepsilon \geq 0$ one has*

$$\begin{array}{ll} x^* \in \widetilde{\partial}_\varepsilon f(\overline{x}) \Leftrightarrow & \forall \alpha > 0 \textit{ there exists } S \textit{ a sponge around } \overline{x} \textit{ such that} \\ & \forall x \in S,\ f(x) - f(\overline{x}) \geq \langle x^*, x - \overline{x}\rangle - (\alpha + \varepsilon)\|x - \overline{x}\|. \end{array} \tag{3.18}$$

Remark 3.3.18 Of course that a similar statement as in Remark 3.3.16, but without any additional calmness assumption holds also true for the ε-extension of the Dini-Hadamard-like subdifferential. Namely, given a function $f : X \to \overline{\mathbb{R}}$ finite at $\overline{x}$ one has

$$x^* \in \widetilde{\partial}_\varepsilon f(\overline{x}) \Leftrightarrow \begin{array}{l} \forall \alpha > 0\ \forall u \in S_X\ \exists \delta > 0 \text{ such that} \\ \forall s \in (0, \delta)\ \forall v \in B(u, \delta) \text{ for } x := \overline{x} + sv \text{ one has} \\ f(x) - f(\overline{x}) \geq \langle x^*, x - \overline{x} \rangle - (\alpha + \varepsilon)\|x - \overline{x}\|. \end{array}$$

To a more careful look we can see that also in the case of the Dini-Hadamard-like subdifferential it is a sort of calmness condition that is hiding behind. So, we say that a function $f : X \to \overline{\mathbb{R}}$ is *weakly calm* at $\overline{x} \in \operatorname{dom} f$ if $\widetilde{D}_d f(\overline{x}; 0) \geq 0$ for all $d \in X \setminus \{0\}$. Actually, unlike the case of the Dini-Hadmamard subdifferential, this last assumption is automatically fulfilled. It is worth mentioning also here that although the weakly calmness assumption is a more general one, it does coincide with the classical calmness condition in finite dimensions.

Proposition 3.3.19 *Let $f : X \to \overline{\mathbb{R}}$ be a given function finite at $\overline{x}$. If X is finite dimensional then f is calm at $\overline{x}$ if and only if f is weakly calm at $\overline{x}$.*

Let us also present a direct consequence of Theorem 3.3.17 and [28, Theorem 2.3], interesting in itself.

Corollary 3.3.20 *Let $f : X \to \overline{\mathbb{R}}$ be a given function finite at $\overline{x}$ and $\varepsilon \geq 0$. Then the following equality $\partial_\varepsilon^{DH} f(\overline{x}) = \widetilde{\partial}_\varepsilon f(\overline{x})$ holds true provided that f is calm at $\overline{x}$. Conversely, if additionally $\widetilde{\partial}_\varepsilon f(\overline{x}) \neq \emptyset$ then f is calm at $\overline{x}$.*

The following estimate for Dini-Hadamard-like subgradients of the *minimum function*

$$(\wedge f_i)(x) := \min\{f_i(x) : i = 1, ..., n\},$$

where $f_i : X \to \overline{\mathbb{R}}$ and $n \geq 2$, uses the above variational descriptions.

Proposition 3.3.21 *Given f_i as above, the following inclusion*

$$\widetilde{\partial}_\varepsilon(\wedge f_i)(\overline{x}) \subseteq \bigcap_{j \in I(\overline{x})} \widetilde{\partial}_\varepsilon(f_j)(\overline{x})$$

where $I(\overline{x}) := \{j \in \{1, ..., n\} : f_j(\overline{x}) = (\wedge f_i)(\overline{x})\}$, holds true.

Finally, we notice that a similar result holds also true for Dini-Hadamard subgradients if the additional calmness assumption is imposed on $f_1, ..., f_n$.

3.3.4 A convenient modification of Dini-Hadamard-like subgradients on product spaces

We propose in this subsection another Dini-Hadamard-like subdifferential construction on the product of two Banach spaces X and Y, which clearly can be extended to the product of any finite number of such spaces.

First of all, as we have already seen, the Dini-Hadamard-like subdifferential of a given function $f : X \to \overline{\mathbb{R}}$ at a point $\overline{x}$ with $|f(\overline{x})| < +\infty$ can be described via the following variational description

$$\widetilde{\partial} f(\overline{x}) := \{x^* \in X^* : \forall \varepsilon > 0 \ \exists S \text{ a sponge around } \overline{x} \text{ such that } \forall x \in S \\ f(x) - f(\overline{x}) \geq \langle x^*, x - \overline{x}\rangle - \varepsilon\|x - \overline{x}\|\}, \tag{3.19}$$

which in fact ensures us that the Dini-Hadamard subdifferential coincides with the Dini-Hadamard-like one on calm functions.

Further, because of the very special structure of the spongious sets (for instance the cartesian product of two sponges is, in general, not a sponge), it seems that the following decoupled constructions are the most suitable tools to derive exact subdifferential formulae for Dini-Hadamard and Dini-Hadamard-like subgradients, which is in fact the main goal of the next chapter.

Thus, given a function $f : X \times Y \to \overline{\mathbb{R}}$ defined on a product of two Banach spaces X and Y, the following subdifferential construction

$$\widetilde{\partial}_{\!\scriptscriptstyle\rm{r}} f(\overline{x}, \overline{y}) := \{(x^*, y^*) \in X^* \times Y^* : \forall \varepsilon > 0 \ \exists S_1 \text{ a sponge around } \overline{x}, \\ \exists S_2 \text{ a sponge around } \overline{y} \text{ such that } \forall (x, y) \in S_1 \times S_2 \\ f(x, y) - f(\overline{x}, \overline{y}) \geq \langle (x^*, y^*), (x - \overline{x}, y - \overline{y})\rangle - \varepsilon\|(x - \overline{x}, y - \overline{y})\|\}, \tag{3.20}$$

denotes the *decoupled Dini-Hadamard-like (lower) subdifferential* of f at $(\overline{x}, \overline{y})$, where $X \times Y$ is a Banach space with respect to the *sum norm*

$$\|(x, y)\| := \|x\| + \|y\|$$

imposed on $X \times Y$ unless otherwise stated. It is interesting to observe that the last notion is actually quite different than the Dini-Hadamard-like one, since, at first sight, neither $\widetilde{\partial} f(\overline{x}, \overline{y}) \not\subseteq \widetilde{\partial}_{\!\scriptscriptstyle\rm{r}} f(\overline{x}, \overline{y})$ nor the opposite inclusion $\widetilde{\partial}_{\!\scriptscriptstyle\rm{r}} f(\overline{x}, \overline{y}) \not\subseteq \widetilde{\partial} f(\overline{x}, \overline{y})$ is valid.

3.3.5 On a smooth variational description of Dini-Hadamard-like subgradients

In order to furnish a smooth variational description for Dini-Hadamard-like subgradients we need to introduce also some special kinds of differentiability for single-valued mappings, weaker than the classical Fréchet one in infinite dimensional spaces. In fact, as one can easily observe, for the same reason mentioned above, there are at least two different ways of defining such a construction on a product of two given spaces. Let us describe in the following the procedure.

Definition 3.3.22 *A single-valued mapping $f : X \to Z$ is said to be* Dini-Hadamard-like differentiable *at $\overline{x} \in X$ if there is a linear continuous operator $\widetilde{\nabla} f(\overline{x}) : X \to Z$ such that for any $\varepsilon > 0$ there exists S a sponge around $\overline{x}$ with the property that for any $x \in S$*

$$\|f(x) - f(\overline{x}) - \widetilde{\nabla} f(\overline{x})(x - \overline{x})\| \leq \varepsilon\|x - \overline{x}\|.$$

Definition 3.3.23 *A single-valued mapping $f : X \times Y \to Z$ is said to be* Dini-Hadamard-like decoupled differentiable *at $(\overline{x}, \overline{y}) \in X \times Y$ if there is a linear continuous operator $\widetilde{\nabla}_{\wr} f(\overline{x}, \overline{y}) : X \times Y \to Z$ such that for any $\varepsilon > 0$ there exists S_1 a sponge around $\overline{x}$ and S_2 a sponge around $\overline{y}$ with the property that for any $(x, y) \in S_1 \times S_2$*

$$\|f(x, y) - f(\overline{x}, \overline{y}) - \widetilde{\nabla}_{\wr} f(\overline{x}, \overline{y})(x - \overline{x}, y - \overline{y})\| \leq \varepsilon \|(x - \overline{x}, y - \overline{y})\|. \tag{3.21}$$

Remark 3.3.24 It is important to emphasize here that, in the particular setting $Z = \overline{\mathbb{R}}$, the function f is Dini-Hadamard-like differentiable at $\overline{x}$ with

$$\widetilde{\partial} f(\overline{x}) = \widetilde{\partial}^{+} f(\overline{x}) = \{\widetilde{\nabla} f(\overline{x})\},$$

whenever the sets $\widetilde{\partial} f(\overline{x})$ and $\widetilde{\partial}^{+} f(\overline{x}) := -\widetilde{\partial}(-f)(\overline{x})$ are nonempty simultaneously. On the other hand, if f Dini-Hadamard-like differentiable at $\overline{x}$ then $\operatorname{card}(\widetilde{\nabla} f(\overline{x})) = 1$.

Of course, a similar statement holds also true for Dini-Hadamard-like decoupled differentiable functions with values in $\overline{\mathbb{R}}$ and the proof it can be done in the lines o the proof of the result above. More precisely, if the sets $\widetilde{\partial}_{\wr} f(\overline{x}, \overline{y})$ and $\widetilde{\partial}_{\wr}^{+} f(\overline{x}, \overline{y})$ are nonempty simultaneously, where of course the latter construction denotes the decoupled Dini-Hadamard-like upper subdifferential of f at $\overline{x}$, then f is Dini-Hadamard-like decoupled differentiable at $(\overline{x}, \overline{y})$ with

$$\widetilde{\partial}_{\wr} f(\overline{x}, \overline{y}) = \widetilde{\partial}_{\wr}^{+} f(\overline{x}, \overline{y}) = \{\widetilde{\nabla}_{\wr} f(\overline{x}, \overline{y})\}.$$

The next theorem provides an important variational description of Dini-Hadamard-like subgradients of nonsmooth functions in terms of smooth supports similar to the one that exists for Fréchet subgradients (we refer the reader to [94] where results of this type were developed in a thorough study by Mordukhovich). Here one also make use of the following property which is weaker than mere local minimization as it involves a sponge in place of a neighborhood. Namely, we say that a point $\overline{x} \in X$ is a *spongiously local minimizer* of a function $f : X \to \overline{\mathbb{R}}$ if f is finite at $\overline{x}$ and if there exists S a sponge around $\overline{x}$ such that $f(x) \geq f(\overline{x})$ for every $x \in S$. Let us finally emphasize here that when speaking about the product of two Banach spaces $X \times Y$ one can use also the next property. Intentionally, a point $(\overline{x}, \overline{y}) \in X \times Y$ is said to be a *spongiously decoupled local minimizer* of a function $f : X \times Y \to \overline{\mathbb{R}}$ if f is finite at $(\overline{x}, \overline{y})$ and if there exist S_1 a sponge around $\overline{x}$ and S_2 a sponge around $f(\overline{x})$ so that $f(x, y) \geq f(\overline{x}, \overline{y})$, whenever $(x, y) \in S_1 \times S_2$.

Theorem 3.3.25 *Let $f : X \to \overline{\mathbb{R}}$ be finite at $\overline{x}$. Then*

(i) Given $x^* \in X^*$, we assume that there is a function $s : S \to \mathbb{R}$ defined on a sponge S around $\overline{x}$ and Dini-Hadamard-like differentiable at $\overline{x}$ such that $\widetilde{\nabla} s(\overline{x}) = x^*$ and $f(x) - s(x)$ achieves a spongiously local minimum at $\overline{x}$. Then $x^* \in \widetilde{\partial} f(\overline{x})$.

(ii) Conversely, for every $x^* \in \widetilde{\partial} f(\overline{x})$ there is a function $s : X \to \mathbb{R}$ Dini-Hadamard-like differentiable at $\overline{x}$ with $\widetilde{\nabla} s(\overline{x}) = x^*$ and such that $s(\overline{x}) = f(\overline{x})$ and $s(x) \leq f(x)$ whenever $x \in X$.

The next similar result expressed in terms of Dini-Hadamard-like decoupled subgradients follows the lines of the proof of the above result.

Theorem 3.3.26 *Let $f : X \times Y \to \overline{\mathbb{R}}$ be finite at $(\overline{x}, \overline{y})$. Then*

(i) Given $(x^*, y^*) \in X^* \times Y^*$, we assume that there is a function $s : S_1 \times S_2 \to \mathbb{R}$ defined on a product of two sponges S_1 around $\overline{x}$ and S_2 around $f(\overline{x})$, respectively, and Dini-Hadamard-like decoupled differentiable at $(\overline{x}, \overline{y})$ such that $\widetilde{\nabla}_t s(\overline{x}, \overline{y}) = (x^*, y^*)$ and $f(x, y) - s(x, y)$ achieves a spongiously decoupled local minimum at $(\overline{x}, \overline{y})$. Then $(x^*, y^*) \in \widetilde{\partial}_t f(\overline{x}, \overline{y})$.

(ii) Conversely, for every $(x^*, y^*) \in \widetilde{\partial}_t f(\overline{x}, \overline{y})$ there is a function $s : X \times Y \to \mathbb{R}$ Dini-Hadamard-like decoupled differentiable at $(\overline{x}, \overline{y})$ with $\widetilde{\partial}_t f(\overline{x}, \overline{y}) = (x^*, y^*)$ and such that $s(\overline{x}, \overline{y}) = f(\overline{x}, \overline{y})$ and $s(x, y) \leq f(x, y)$ whenever $(x, y) \in X \times Y$.

3.3.6 Relationships between subgradients and normal cones

We begin our exposure in this subsection with a few remarks (see [3, 111]). First of all, recall that the Dini-Hadamard normal cone to a set $C \subseteq X$ at $\overline{x} \in C$, naturally introduced via the Dini-Hadamard subdifferential to the indicator function, can also be expressed by means of the polar cone to the contingent one, which is in fact the contingent normal cone, i.e.

$$N^{DH}(\overline{x}; C) := \partial^{DH} \delta_C(\overline{x}) = T^\circ(\overline{x}; C) := N(\overline{x}; C), \tag{3.22}$$

where the contingent cone can be viewed (see [3]) in the following way

$$T(\overline{x}; C) = \bigcap_{\substack{\varepsilon > 0 \\ \delta > 0}} \bigcup_{t \in (0, \delta)} (t^{-1}(C - \overline{x}) + \varepsilon B),$$

i.e. the set of all vectors v so that one can find sequences $t_n \downarrow 0, u_n \to v$ with the property that $\overline{x} + t_n u_n \in C$ for all $n \in \mathbb{N}$ and where given a subcone $K \subseteq X$, its polar cone K° is defined by

$$K^\circ := \{x^* \in X^* : \sup_{x \in K} \langle x^*, x \rangle \leq 0\}.$$

On the other hand, the Dini-Hadamard subdifferential of a given function f enjoys also the following geometrical description

$$\partial^{DH} f(\overline{x}) = \{x^* \in X^* : (x^*, -1) \in N((\overline{x}, f(\overline{x})); \operatorname{epi} f))\}, \tag{3.23}$$

which clearly implies, in view of (3.22), the following one

$$\partial^{DH} f(\overline{x}) = \{x^* \in X^* : (x^*, -1) \in N^{DH}((\overline{x}, f(\overline{x})); \operatorname{epi} f))\}. \tag{3.24}$$

In fact, the latter actually says that the analytic Dini-Hadamard subdifferential ∂_a^{DH}, as introduced in (3.9), always agrees with the geometrical one, ∂_g^{DH}, as defined in (3.24). However, this is no longer the case for the Dini-Hadamard-like subdifferential. The reason is that, for this particular construction, one can state a similar result like in (3.24) only by making use of the corresponding decoupled one.

Proposition 3.3.27 *Let $f : X \to \overline{\mathbb{R}}$ be a given function finite at $\overline{x}$. Then*

$$\widetilde{\partial} f(\overline{x}) = \{x^* \in X^* : (x^*, -1) \in \widetilde{N}_{\mathfrak{r}}((\overline{x}, f(\overline{x})); \operatorname{epi} f)\}, \tag{3.25}$$

where $\widetilde{N}_{\mathfrak{r}}((\overline{x}, \overline{y}); C) := \widetilde{\partial}_{\mathfrak{r}}\delta((\overline{x}, \overline{y}); C)$ stands for the decoupled Dini-Hadamard-like normal cone *to $C \subset X \times Y$ at $(\overline{x}, \overline{y})$.*

A valuable characterization of the Dini-Hadamard-like normal cone, similar to the one that exist for the Fréchet normal cone (see for instance [94, Definition 1.1]), but by replacing the usual convergence with a directional one, will be provided in the sequel.

Proposition 3.3.28 *Let C be a nonempty subset of X and $\overline{x} \in C$. Then*

$$\widetilde{N}(\overline{x}; C) = \{x^* \in X^* : \inf_{\delta \in (0,1)} \sup_{x \in (\overline{x} + (0,\delta) \cdot B(u,\delta)) \cap C} \frac{\langle x^*, x - \overline{x} \rangle}{\|x - \overline{x}\|} \leq 0 \ \forall u \in S_X\}, \tag{3.26}$$

where $\widetilde{N}(\overline{x}; C) := \widetilde{\partial} \delta_C(\overline{x})$.

The following result is a direct consequence of Corollary 3.3.20, since the indicator function $\delta_C(\overline{x})$, where $\overline{x} \in C \subseteq X$, is obviously calm at $\overline{x}$. See also relation (3.22) above.

Corollary 3.3.29 *Let C be a nonempty subset of X and $\overline{x} \in C$. Then the following equalities*

$$N^{DH}(\overline{x}; C) = \widetilde{N}(\overline{x}; C) = N(\overline{x}; C) \tag{3.27}$$

hold true.

Now, we show the links between the analytic Dini-Hadamard-like subdifferential and the geometrical one, concluding that this kind of construction doesn't follows at all the behavior of the Fréchet subdifferential (see, for instance, the results in [94, Section 1.3]).

Corollary 3.3.30 *Let $f : X \to \overline{\mathbb{R}}$ be an arbitrary function and $\overline{x} \in X$. Then*

$$\widetilde{\partial}_g f(\overline{x}) \subsetneq \widetilde{\partial}_a f(\overline{x}), \tag{3.28}$$

where $\widetilde{\partial}_g f(\overline{x}) := \{x^* \in X^* : (x^*, -1) \in \widetilde{N}((\overline{x}, f(\overline{x})); \operatorname{epi} f)\}$ stands for the geometric Dini-Hadamard-like subdifferential of f at $\overline{x}$, while $\widetilde{\partial}_a f(\overline{x}) := \widetilde{\partial} f(\overline{x})$ denotes the analytical one.

Finally, let us illustrate the relationships between various subgradients studied above, which are in fact direct consequences of the discussions made in this subsection.

Corollary 3.3.31 *Let $f : X \to \overline{\mathbb{R}}$ be an arbitrary function and $\overline{x} \in X$. Then*

$$\partial_a^{DH} f(\overline{x}) = \partial_g^{DH} f(\overline{x}) = \widetilde{\partial}_g f(\overline{x}) \subsetneq \widetilde{\partial}_a f(\overline{x}), \tag{3.29}$$

while the equalities hold true in case f is calm at $\overline{x}$.

3.3.7 Connections with other subgradients

In the following, let us first present the relationship between the Fréchet, the Dini-Hadamard and the Dini-Hadamard-like subdifferential. Namely, the following inclusions hold true,

$$\widehat{\partial}_\varepsilon f(\overline{x}) \subseteq \partial_\varepsilon^{DH} f(\overline{x}) \subseteq \widetilde{\partial}_\varepsilon f(\overline{x}), \tag{3.30}$$

whenever $\varepsilon \geq 0$. However, in the infinite dimensional framework one can often obtain strict inclusions. To see this for the first inclusion, in case $\varepsilon = 0$, consider, for instance, the function $f : C[0,1] \to \mathbb{R},\ f(x) = -\|x\|_\infty$. Then $\widehat{\partial} f(x) = \emptyset$ for all $x \in C[0,1]$, while $\partial^{DH} f(\overline{x}) \neq \emptyset$ when $\overline{x} \in S_{C[0,1]}$ is chosen such that $|\overline{x}| : [0,1] \to \mathbb{R}, |\overline{x}|(t) = |\overline{x}(t)|$, attains its maximum at exactly one point of the interval $[0,1]$ (see [49, Exercise 8.28]). For a similar example in ℓ_1 we refer to [49, Exercise 8.26]. As regards the second inclusion, we refer the reader to Example 3.3.14 above.

Remark 3.3.32 When X is finite dimensional, we always have equality in (3.30).

To make us an idea about how further can we go with the choice of f such that the inclusions in (3.30) hold true as equalities in arbitrary Banach spaces, we give the following proposition along with some observations.

Proposition 3.3.33 *Let the function $f : X \to \mathbb{R} \cup \{+\infty\}$ be approximately starshaped at $\overline{x} \in \operatorname{dom} f$. Then for all $\varepsilon \geq 0$ it holds*

$$\widehat{\partial}_\varepsilon f(\overline{x}) = \partial_\varepsilon^{DH} f(\overline{x}) = \widetilde{\partial}_\varepsilon f(\overline{x}).$$

On the other hand, Example 3.3.14 above ensures us that the following equality $\partial^{DH} f(\overline{x}) = \widetilde{\partial} f(\overline{x})$ does not hold in case f is only directionally approximately starshaped at $\overline{x} \in \operatorname{dom} f$, since f is directionally approximately starshaped at $\overline{x}$, but $0 \in \widetilde{\partial} f(\overline{x}) \backslash \partial^{DH} f(\overline{x})$. The same is, of course true with the equality $\widehat{\partial} f(\overline{x}) = \partial^{DH} f(\overline{x})$. Moreover, the function in Example 3.3.14 shows also that in general the class of approximately starshaped functions does not coincide with the one of directionally approximately starshaped functions.

Thanks to the following result (actually based on [106, Theorem 3.6], but take also into account Proposition 3.3.34 above and the property 2.1. from Birge [16] which holds true for an arbitrary function f), the Dini-Hadamard-like subdifferential as well as the Dini-Hadamard one agrees with a great number of well-known subdifferentials such as the Clarke-Rockafellar, the Clarke's generalized gradient, the Michel-Penot subdifferential, the Mordukhovich one, the Fréchet subdifferential and the geometric subdifferential of Ioffe, whenever the given function is lower semicontinuous and approximately convex at a given point of the domain.

Proposition 3.3.34 *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function on X and approximately convex at $\overline{x} \in \operatorname{dom} f$. Then we have*

$$\begin{aligned} \partial^{C-R} f(\overline{x}) &= \partial^C f(\overline{x}) = \partial^\diamond f(\overline{x}) = \partial^M f(\overline{x}) = \widehat{\partial} f(\overline{x}) \\ &= \partial^{DH} f(\overline{x}) = \widetilde{\partial} f(\overline{x}) = \partial' f(\overline{x}) = \partial^G f(\overline{x}), \end{aligned} \tag{3.31}$$

where $\partial' f(\overline{x}) := \left\{ x^* \in X^* : \langle x^*, v\rangle \leq \lim_{t\downarrow 0} \frac{f(\overline{x}+tv)-f(\overline{x})}{t} \right\}$.

If additionally f *is Lipschitz at* $\overline{x}$ *then one can complete the above list of equalities with the approximate subdifferential* $\partial^A f(\overline{x})$.

In fact, the following inclusions hold true whenever $f : X \to \overline{\mathbb{R}}$ is an arbitrary function and $\overline{x} \in \operatorname{dom} f$. Namely,

$$\partial^{DH} f(\overline{x}) \subseteq \widetilde{\partial} f(\overline{x}) \subseteq \partial^{\diamond} f(\overline{x}) \subseteq \partial^{C} f(\overline{x}) \tag{3.32}$$

and

$$\widetilde{\partial} f(\overline{x}) \subseteq \partial^{G} f(\overline{x}) \subseteq \partial^{C-R} f(\overline{x}) \tag{3.33}$$

where the latter two follows from [67, Proposition 4.2]. Note also here that in case f is lower semicontinuous at $\overline{x}$ then $\partial^{C} f(\overline{x}) \subseteq \partial^{C-R} f(\overline{x})$, while the equalities $\partial^{DH} f(\overline{x}) = \widetilde{\partial} f(\overline{x})$ and $\partial^{C} f(\overline{x}) = \partial^{C-R} f(\overline{x})$ are available for locally Lipschitz functions.

We finally observe that the generalized convexity notion used in Proposition 3.3.34 is essentially, since even in the case of a locally Lipschitz function on a separable Banach space one cannot obtain (in general) all the above equalities.

Proposition 3.3.35 *Let* $U \subseteq X$ *be an open subset of a separable Banach space and let* f *be a locally Lipschitz function on* U. *Then*

$$\partial^{C-R} f(\overline{x}) = \partial^{C} f(\overline{x}) = \partial^{\diamond} f(\overline{x}) = \partial^{DH} f(\overline{x}) = \widetilde{\partial} f(\overline{x}) = \partial^{G} f(\overline{x}) \tag{3.34}$$

for all $x \in U$ *of a residual subset of* U. *On the other hand,*

$$\partial^{DH} f(\overline{x}) = \widetilde{\partial} f(\overline{x}) = \partial^{\diamond} f(\overline{x}) \tag{3.35}$$

outside of a directionally σ*-porous subset of* U.

To conclude this subsection let us mention that in the first statement above the passage from residual to a complement of a directionally σ-porous set is impossible in principle (see the discussion after [73, Theorem 3.6]).

3.3.8 A key description of directionally approximately starshaped functions via the Dini-Hadamard-like subdifferential

Along the aforementioned nice stability properties for directionally approximately starshaped functions, we also provide another one, by making use of the Dini-Hadamard-like subdifferential.

Proposition 3.3.36 *Let the function* $f : X \to \mathbb{R} \cup \{+\infty\}$ *be directionally approximately starshaped at* $\overline{x} \in \operatorname{dom} f$. *Then for every* $\alpha > 0$ *and every* $\varepsilon \geq 0$ *there exists a sponge* S *around* $\overline{x}$ *such that for every* $x \in S$ *one has*

$$f(x) - f(\overline{x}) \geq \langle \overline{x}^*, x - \overline{x}\rangle - (\alpha + \varepsilon)\|x - \overline{x}\| \ \forall \overline{x}^* \in \widetilde{\partial}_{\varepsilon} f(\overline{x}), \tag{3.36}$$

$$f(\overline{x}) - f(x) \geq \langle x^*, \overline{x} - x\rangle - (\alpha + \varepsilon)\|x - \overline{x}\| \ \forall x^* \in \widetilde{\partial}_{\varepsilon} f(x). \tag{3.37}$$

It is to be further noted that a similar statement holds also true if we replace the Dini-Hadamard-like subdifferential with the Dini-Hadamard one (see [28, Lemma 3.2]). Moreover the assertions stated in Proposition 3.3.36 are more general even then those presented in [1, Lemma 1], where approximately starshaped functions are characterize by means of the Fréchet subdifferential.

3.4 About a Dini-Hadamard-like coderivative of single-valued mappings; differentiability properties

Now let us describe the main derivative-like object for single-valued mappings that we are going to use mainly in Section 4.2.

It is worth emphasizing here that this kind of objects are called *coderivatives* because they provide a pointwise approximation of set-valued (in particular, single-valued) mappings between given spaces using elements of dual spaces. We will see bellow that in the case of smooth single-valued mappings the coderivative reduce to the classical adjoint derivative operator at the point in question. But for general nonsmooth and set-valued mappings the coderivative is constructed via normal vectors to graphs and it is not dual to any derivative objects related to tangential approximations in initial spaces.

Given a single-valued mapping $f : X \to Y$ between two Banach spaces, the following construction

$$\widetilde{D}^*_{\dagger} f(\overline{x})(y^*) := \{x^* \in X^* : (x^*, -y^*) \in \widetilde{N}_{\dagger}((\overline{x}, f(\overline{x})); \operatorname{graph} f)\} \tag{3.38}$$

defines the *decoupled Dini-Hadamard-like coderivative* of f at $\overline{x}$, where

$$\widetilde{N}_{\dagger}((\overline{x}, \overline{y}); C) := \widetilde{\partial}_{\dagger}\delta((\overline{x}, \overline{y}); C) \tag{3.39}$$

stands for the *decoupled Dini-Hadamard-like normal cone* to $C \subset X \times Y$ at $(\overline{x}, \overline{y})$ and where

$$\operatorname{graph} f := \{(x, y) \in X \times Y : f(x) = y\}$$

denotes the graph of f.

Next let us provide a detail regarding the decoupled Dini-Hadamard-like coderivative of a differentiable singled-valued mapping, which is of special interest for subsequent discussions. But first, recall that given a linear continuous mapping $A \in \mathcal{L}(X, Y)$ from X to Y its adjoint operator $A^* \in \mathcal{L}(Y^*, X^*)$ is defined by $\langle A^*y^*, x\rangle := \langle y^*, Ax\rangle$ for all $x \in X$ and $y^* \in Y^*$. When $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$, A can be identified with an $m \times n$ matrix and A^* coincides with A^T.

Proposition 3.4.1 *Let $f : X \to Y$ be Dini-Hadamard-like differentiable at $\overline{x}$. Then*

$$\widetilde{\nabla} f(\overline{x})^* y^* \in \widetilde{D}^*_{\dagger} f(\overline{x})(y^*), \quad \textit{for all } y^* \in Y^*.$$

If, additionally, Y is finite dimensional then

$$\widetilde{D}^*_{\dagger} f(\overline{x})(y^*) = \{\widetilde{\nabla} f(\overline{x})^* y^*\},$$

whenever $y^* \in Y^*$.

Given a singled valued-mapping $f : X \to Y$ between two Banach spaces, we consider also the following *scalarization* defined by

$$\langle y^*, f\rangle(x) := \langle y^*, f(x)\rangle, \; x \in X, \tag{3.40}$$

for any $y^* \in Y^*$. Let us study, in the following particular setting, the Dini-Hadamard-like differentiability property of the above scalarization function involving adjoint operators.

Proposition 3.4.2 *If the single-valued mapping $f : X \to Y$ is Dini-Hadamard-like differentiable at $\overline{x}$, then so it is its scalarization $\langle y^*, f\rangle$ and*

$$\widetilde{\nabla}\langle y^*, f\rangle(\overline{x}) = \{\widetilde{\nabla} f(\overline{x})^* y^*\}.$$

Remark 3.4.3 Observe that the second two results listed above leads in fact to the following relation between the decoupled Dini-Hadamard-like coderivative of a differentiable single-valued mapping and the corresponding derivative of its scalarization. Namely, if $f : X \to Y$, where Y is finite dimensional, is Dini-Hadamard-like differentiable at $\overline{x}$, then the following equalities

$$\widetilde{D}_{\dagger}^* f(\overline{x})(y^*) = \widetilde{\nabla}\langle y^*, f\rangle(\overline{x}) = \{\widetilde{\nabla} f(\overline{x})^* y^*\}$$

hold true for any $y^* \in Y^*$.

Finally, let us illustrate a relationship between the decoupled Dini-Hadamard-like coderivative (3.38) of a spongiously Lipschitz mapping and the Dini-Hadamard-like subdifferential of their scalarization. In fact, this final result turns out to be one of the main tools in deriving exact calculus rules for Dini-Hadamard-like subgradients (see Section 4.2 bellow).

Proposition 3.4.4 *Let $f : X \to Y$ be spongiously Lipschitz and strongly spongiously continuous at $\overline{x}$. Then*

$$\widetilde{D}_{\dagger}^* f(\overline{x})(y^*) = \widetilde{\partial}\langle y^*, f\rangle(\overline{x}) \quad \textit{for all } y^* \in Y^*. \tag{3.41}$$

Observe here that the second inclusion in Proposition 3.4.4 does not need any additional assumptions on f.

Chapter 4

On the Dini-Hadamard subdifferential calculus in Banach spaces

Although the Dini-Hadamard subdifferential and its upper counterpart are well-known (we refer the reader to [12], [63] and [111] for further references and other details that will be not mentioned here), the use of them has been rather limited so far, apparently for the following reasons. First of all, apart from the obvious smooth and convex case, it is rather typical that $\partial^{DH} f(x)$ is empty at some points. Although for a lower semicontinuous function on a space having a Gâteaux differentiable renorm (more generally on a space on which there exists a Gâteaux differentiable locally Lipschitz bump function), in particular on any separable Banach space, the Dini-Hadamard subdifferential is nonempty on a dense subset of its domain, in general it is very easy to find an example of a Lipschitz, even concave continuous function on a Banach space with $\partial^{DH} f(x)$ being identical to the empty set. The same is of course true for the Dini-Hadamard superdifferential. Another reason is that the existing calculus for the Dini-Hadamard subdifferential or superdifferential is very poor compared, say, to what we know about convex subdifferentials, the generalized gradients of Clarke, or even the Mordukhovich subdifferential or the approximate subdifferentials of Ioffe. Since the Dini-Hadamard directional derivative is neither convex nor concave, it is natural to believe that it cannot admit further developments using the beautiful duality theory.

After a brief overview of the main calculus rules already obtained in literature (most of them weak fuzzy rules), that are available for Dini-Hadamard constructions, it is our aim to provide an exact formula for the Dini-Hadamard ε-subdifferential of the difference of two functions by making use of the star-difference of the Dini-Hadamard ε-subdifferentials of the functions involved. In this investigation an important role will be played by the variational description of the Dini-Hadamard ε-subgradients obtained in Subsection 3.3.3, which in fact represents the counterpart of a well-known variational description for Fréchet ε-subgradients. While in the announced subdifferential formula for the difference of two functions one inclusion follows automatically, in order to guarantee the other one we need some supplementary assumptions on the functions involved. More precisely, we show that in case the two functions are directionally approximately starshaped at a given point and a weak topological assumption is fulfilled, then the opposite inclusion is fulfilled, too. The great novelty of this subdifferential formula involving ε subgradients Dini-Hadamard is the fact that such a result wasn't known to be also valid for this kind of constructions, until now. However, similar characterizations were later given by Penot [119] in terms of dissipative operators. Some exact calculus rules for the Dini-Hadamard-like subdifferential and then for the Dini-Hadamard one are further developed. Among them we mention some formulas for generalized and usual compositions, products and quotients etc. It happens that the calculus rules obtained involve spongiously Lipschitz and strongly spongiously continuous mappings that in finite dimensions coincide with the classical Lipschitz continuity and with the continuity, respectively. We obtain also some assertions given by Mordukhovich [96] in terms of Fréchet subgradients, but by using different assumptions.

Finally, we notice that the results presented in this chapter are mainly based on [9, 28, 104].

4.1 Fuzzy calculus rules

4.1.1 Preliminaries

There are several ways of developing a set of basic tools for subdifferential analysis that can be applied to a wide range of various problems. The difference lies on the starting point. Borwein, Treiman and Zhu [22] use the *nonlocal fuzzy sum rule* [151] as a basic device; the *multidirectional mean value inequality* [36] is a corner stone result in Clarke, Ledyaev, Stern and Wolenski [37]; Ioffe [68] begins with the *local fuzzy sum rule* and Mordukhovich and Shao [93] choose the *Kruger-Mordukhovich extremal principle* [11, 89, 92] as the main tool. Nevertheless, all these basic results are equivalent (the equivalence between the extremal principle and the local fuzzy sum rule was established in [98] and their equivalence to the multidirectional mean value inequality and to the nonlocal fuzzy sum rule was proved in [152]). In fact, they operate in different ways two basic principles, namely, a *smooth variational principle* [21] and a *decoupling lemma* used by Crandall and Lions to study the uniqueness of viscosity solutions [39]. Thus, motivated by the theorem of Zhu [152], Ioffe [69] actually proved that the basic principles of subdifferential calculus (various local and global fuzzy principles, the multidirectional mean value theorem, the extremal principle) are in fact equivalent for any subdifferential, not only for the so-called viscosity [23], or variational [132] β-subdifferentials. The theorem actually says that there is only one fundamental principle behind the calculus of subdifferentials (we mean here an abstract one, which includes all elementary subdifferentials associated with bornologies (not only the viscosity β-subdifferentials), all kind of limiting subdifferentials and all modifications of approximate subdifferentials as well as for ε-versions of all mentioned subdifferentials). Later, Lassonde [80] added four more properties to the list of seven equivalent properties of Ioffe, showing that all of them are satisfied if the space has a ∂-differentiable norm in the sense of Aussel, Corvellec and Lassonde [7].

4.1.2 Trustworthiness and fuzzy calculus rules for the Dini-Hadamard subdifferential

When we refer to calculus of derivatives and subdifferentials, we usually mean rules that enable us to estimate or calculate derivatives or subdifferentials of combinations of functions such as sums and composites, mean value theorems etc. As far as sums are concerned, the following inclusion

$$\partial(f_1 + ... + f_n)(x) \supset \partial f_1(x) + ... + \partial f_n(x)$$

holds true for any β-subdifferential, while the reverse one is obviously not valid (exception to this rule make some limiting subdifferentials in suitable spaces, the approximate subdifferential of Ioffe and the Clarke's generalized gradient in arbitrary Banach spaces). Moreover, the experience of convex analysis suggests us that it is the opposite inclusion that is often needed in applications. As regards the subdifferentials associated with derivatives, as it is the case of the Dini-Hadamard one and certain others, we have a new type of calculus in which the desired inclusion is *almost* satisfied, in a sense that will be made precise bellow.

So, we recall the following important result which incorporates the three types of sum rules (1. the *weak fuzzy sum rule*, 2. the *strong fuzzy sum rule* and 3. the *exact sum rule*) that a subdifferential may obey.

Theorem 4.1.3 *(cf. [71, Theorem 2]) Let X be a Banach space and let $f_1, ...f_k$ satisfying a Lipschitz condition in a neighborhood of a point $x \in X$. Suppose further that $x^* \in \partial(f_1 + ... + f_k)(\overline{x})$ for some subdifferential ∂.*

1. If ∂ is a β-subdifferential and there is a β-differentiable Lipschitz bump function on X, then for any $\varepsilon > 0$ and any weak neighborhood V of the origin in X^* there are $x_1, ..., x_k \in X$ and $x_1^*, ...x_k^* \in X^*$ such that*

$$\|x_i - \overline{x}\| < \varepsilon, \quad x_i^* \in \partial f_i(x_i),\ i = 1, ..., k \quad \text{and} \quad x^* \in x_1^* + ... + x_k^* + V. \tag{4.1}$$

2. If X is an Asplund space and $\partial = \widehat{\partial}$, then for any $\varepsilon > 0$ there are $x_1, ..., x_k \in X$ and $x_1^*, ...x_k^* \in X^*$ such that

$$\|x_i - \overline{x}\| < \varepsilon, \quad x_i^* \in \partial f_i(x_i),\ i = 1, ..., k \quad \text{and} \ \|x_1^* + ... + x_k^* - x^*\| < \varepsilon. \tag{4.2}$$

3. The following inclusion

$$\partial(f_1 + ... + f_k)(\overline{x}) \subseteq \partial f_1(\overline{x}) + ... + \partial f_k(\overline{x}) \tag{4.3}$$

holds true in the following cases: there is a Lipschitz β-differentiable bump function on X, the unit ball in X^* is sequentially weak compact and ∂ is a limiting β-subdifferential, X is an Asplund space and ∂ is the limiting Fréchet (canonical) subdifferential or X is an arbitrary Banach space and ∂ is the approximate subdifferential or the Clarke's generalized gradient.*

But unfortunately, as one can easily observe, the Dini-Hadamard subdifferential enjoy only a weak fuzzy sum rule on appropriate spaces.

4.1.3 Weak fuzzy calculus for the Dini-Hadamard coderivative

Firstly, we mention that it is often sufficient to know only rules for sums of functions and/or marginal functions, in order to get corresponding rules for other operations (see the approach in [74] and the references therein).

We present next some various estimates for the Dini-Hadamard coderivative of set-valued mappings, obtained by Ioffe and Penot [74] in 1996. The general framework is the following one. Let X, Y, Z as well as their products to be weakly trustworthy spaces. In fact, what we need is that the Dini-Hadamard subdifferential satisfies the weak fuzzy calculus rule.

Proposition 4.1.6 (cf. [74, Proposition 5.1]) *Let $F : X \to Y$ and $G : Y \to Z$ be set-valued mappings with closed graphs. Let $\overline{z} \in (G \circ F)(\overline{x})$ and $\overline{x}^* \in D^*(G \circ F)(\overline{x}, \overline{z})(\overline{z}^*)$. Then for any $\overline{y} \in F(\overline{x}) \bigcap G^{-1}(\overline{z})$, any $\varepsilon > 0$ and any weak* neighborhoods U^*, V^* and W^* of zeros in X^*, Y^* and Z^* respectively there are $x \in X, y_1, y_2 \in Y, z \in Z$, and $x^* \in X^*, y_1^*, y_2^* \in Y^*, z^* \in Z$ such that*

$$\begin{cases} \|x - \overline{x}\| < \varepsilon, \|y_i - \overline{y}\| < \varepsilon, \|z - \overline{z}\| < \varepsilon; \\ x^* \in D^*F(x, y_1)(y_1^*), y_2^* \in D^*G(y_2, z)(z^*); \\ x^* \in \overline{x}^* + U^*, y_1^* - y_2^* \in V^*, z^* \in \overline{z}^* + W^*. \end{cases} \tag{4.4}$$

Proposition 4.1.7 (cf. [74, Proposition 5.2]) *Let $F_i : X \to Y, (i = 1, ..., k)$ be set-valued mappings with closed graphs. Set $F(x) = F_1(x) + ... + F_k(x)$ and assume that $\overline{y} \in F(\overline{x})$ and $\overline{x}^* \in D^*F(\overline{x}, \overline{y})(\overline{y}^*)$. Assume further that $\overline{y}_i \in F_i(\overline{x}), \overline{y}_1 + ... + \overline{y}_k = \overline{y}$. Then for any $\varepsilon > 0$ and any weak* neighborhoods U^*, V^* of zeros in X^* and Y^* respectively there are $x_i, y_i, x_i^*, y_i^* (i = 1, ..., k)$ such that*

$$\begin{cases} \|x_i - \overline{x}\| < \varepsilon, \|y_i - \overline{y}_i\| < \varepsilon, \\ x^* \in D^*F_i(x_i, y_i)(y_i^*), \\ x_1^* + ... + x_k^* \in \overline{x}^* + U^*, y_i^* \in \overline{y}^* + W^* (i = 1, ..., k). \end{cases} \tag{4.5}$$

Proposition 4.1.8 (cf. [74, Proposition 5.3]) *Let F_i be as in Proposition 4.1.7 and $F(x) = \bigcap F_i(x)$. Assume that $\overline{y} \in F(\overline{x})$ and $\overline{x}^* \in D^*F(\overline{x}, \overline{y})(\overline{y}^*)$. Then for any $\varepsilon > 0$ and any weak* neighborhoods U^*, V^* of zeros in X^* and Y^* respectively there are $x_i, y_i, x_i^*, y_i^* (i = 1, ..., k)$ such that*

$$\begin{cases} \|x_i - \overline{x}\| < \varepsilon, \|y_i - \overline{y}_i\| < \varepsilon, \\ x^* \in D^*F_i(x_i, y_i)(y_i^*), \\ x_1^* + ... + x_k^* \in \overline{x}^* + U^*, y_1^* + ... + y_k^* \in \overline{y}^* + W^*. \end{cases} \tag{4.6}$$

Of course, if we are suppose to work in a finite dimensional framework, since there the Dini-Hadamard subdifferential coincide with the Fréchet one, one can get more accurate formulas (we refer the reader to [74], mainly to Section 6, but also to [94, 95, 96]).

4.2 Looking for exact calculus rules

Let $f : X \to \overline{\mathbb{R}} := [-\infty, +\infty]$ be an extended real valued function defined on a real Banach space X. It follows directly from the definition of the Dini-Hadamard subdifferential that the following generalized Fermat rule holds, namely

$$0 \in \partial^{DH} f(\overline{x}) \tag{4.7}$$

whenever $\overline{x}$ is a local minimizer for $f : X \to \overline{\mathbb{R}}$. If we consider further the following general constrained minimization problem

$$\min_{x \in C \subseteq X} f(x), \tag{4.8}$$

we can easily observe that it can be successfully (equivalently) reduce to the unconstrained problem

$$\min_{x \in X} f(x) + \delta(x; C), \tag{4.9}$$

which involves the indicator function $\delta(\cdot; C)$ of the set C. Applying now Fermat's rule (4.7), we get

$$0 \in \partial^{DH}[f + \delta(\cdot; C)](\overline{x}),$$

whenever $\overline{x}$ is a local solution to the constrained optimization problem (4.8). To proceed further in constrained optimization and obtain valuable optimality conditions in terms of the initial data, we need to have satisfactory calculus rules for Dini-Hadamard subgradients, which is generally not the case. In particular, the desirable sum rule

$$\partial^{DH}(f_1 + f_2)(\overline{x}) \subset \partial^{DH} f_1(\overline{x}) + \partial^{DH} f_2(\overline{x}) \tag{4.10}$$

does not hold even in the simplest nonsmooth setting, for instance, consider the functions $f_1(x) = |x|$ and $f_2(x) = -|x|$ on the real line. On the other hand, Dini-Hadamard subgradients satisfy, as we have seen above, the so-called weak fuzzy sum rule under natural conditions involving a broad class of Banach spaces (namely, those which admits an equivalent Gâteaux differentiable norm). However, such a fuzzy calculus rule is not very useful for a number of applications which includes necessary optimality conditions in constrained optimization, since they bear a flavor of uncertainty and allow us only to approximately represent Dini-Hadamard subgradients of sums at points of interest via Dini-Hadamard subgradients of separate functions at points nearby. It is to be further noted that an exact calculus rule dealing only with points of interest is certainly more desirable for the majority of applications. In fact, the absence of such calculus rules for the Dini-Hadamard subdifferential and its ε-enlargement significantly has restricted the scope of their applications.

The purpose of this section is to complete the picture made by Ioffe [63] and later by Penot [74]. More precisely, we want to show that the Dini-Hadamard ε-subdifferential enjoys also some nontrivial exact calculus rules. In contrast with the results derived by Ioffe in the particular setting of a finite dimensional space and also in that of a Banach space admitting a Gâteaux differentiable norm, the exact calculus rules obtained in the final part of this chapter makes from the Dini-Hadamard subdifferential a strong and competitive opponent for the Fréchet one.

4.2.1 Sums and differences rules involving smooth functions

Although a desirable sum rule of the inclusion type (4.10) does not hold for Dini-Hadamard-like subgradients involving both nonsmooth functions, this is no longer the case when at least one of them is Dini-Hadamard-like differentiable.

Proposition 4.2.1 *Let $f_1 : X \to \overline{\mathbb{R}}$ be finite at $\overline{x}$ and let $f_2 : X \to \overline{\mathbb{R}}$ be Dini-Hadamard-like differentiable at $\overline{x}$. Then one has the equality*

$$\widetilde{\partial}(f_1 + f_2)(\overline{x}) = \widetilde{\partial} f_1(\overline{x}) + \widetilde{\nabla} f_2(\overline{x}). \tag{4.11}$$

A direct consequence of the latter proposition is the following useful (while elementary) difference rule.

Proposition 4.2.2 *Let $f_1 : X \to \overline{\mathbb{R}}$ be finite at $\overline{x}$ and let $f_2 : X \to \overline{\mathbb{R}}$ be Dini-Hadamard-like differentiable at $\overline{x}$. Then one has the equality*

$$\widetilde{\partial}(f_1 - f_2)(\overline{x}) = \widetilde{\partial} f_1(\overline{x}) - \widetilde{\nabla} f_2(\overline{x}). \tag{4.12}$$

Let us also remark here that one can obtain similar results for Dini-Hadamard subgradients if some additional calmness assumptions are imposed.

4.2.2 On the Dini-Hadamard subdifferential formula of the difference of two directionally approximately starshaped functions

Let now $f, g : X \to \overline{\mathbb{R}}$ be two arbitrary functions. By using Theorem 3.3.15 and the fact that the intersection of two sponges around the same point is a sponge around that point, one can prove that for all $\varepsilon, \eta \geq 0$ and all $x \in \operatorname{dom} f \cap \operatorname{dom} g$

$$\partial_{\varepsilon}^{DH} f(x) + \partial_{\eta}^{DH} g(x) \subseteq \partial_{\varepsilon+\eta}^{DH}(f+g)(x). \tag{4.13}$$

From the conventions made for the Dini-Hadamard ε-subdifferential it follows that (4.13) is in fact true for all $x \in X$.

In what follows we give via (4.13) a formula for the *difference* of two functions. To this end we need to introduce the notion of *star-difference* (also called *Pontryagin difference*) of two sets. Namely, given $A, B \subseteq X$ the *star-difference* of A and B is defined as

$$A \overset{*}{-} B := \{x \in X : \ x + B \subseteq A\} = \bigcap_{b \in B} \{A - b\}.$$

We adopt here the convention that $A \overset{*}{-} B := \emptyset$ if $A = \emptyset$, $B \neq \emptyset$ and $A \overset{*}{-} B := X$ in case $B = \emptyset$. Clearly, for B nonempty, one has $A \overset{*}{-} B + B \subseteq A$ and $A \overset{*}{-} B \subseteq A - B$. In general, these inclusions are strict and $A \overset{*}{-} B$ is much smaller than $A - B$. Introduced by Pontrjagin in [121] in the context of linear differential games, this notion has found resonance in different theoretical and practical investigations in the field of nonsmooth analysis (see, for instance, [1, 4, 30, 52, 56, 84, 96, 123]).

When dealing with the difference of two functions $g, h : X \to \mathbb{R} \cup \{+\infty\}$ we assume throughout this paper that $\operatorname{dom} g \subseteq \operatorname{dom} h$. This guarantees that the function $f = g - h : X \to \mathbb{R} \cup \{+\infty\}$ is well-defined and moreover one can easily verify that $g = f + h$ and $\operatorname{dom} f = \operatorname{dom} g$. Notice that such an assumption seems to be necessary also in [1], in order to guarantee that the difference function takes values in $\mathbb{R} \cup \{+\infty\}$, which is the setting considered in the mentioned article, too.

By making use of (4.13), we get the following result.

Proposition 4.2.3 *Let $g, h : X \to \overline{\mathbb{R}}$ be given functions and $f := g - h$. Then for all $\varepsilon, \eta \geq 0$ and all $x \in X$ one has*

$$\partial_{\varepsilon}^{DH} f(x) \subseteq \partial_{\varepsilon+\eta}^{DH} g(x) \overset{*}{-} \partial_{\eta}^{DH} h(x). \tag{4.14}$$

Remark 4.2.4 (a) If for $\eta \geq 0$ and $x \in X$ the set $\partial_{\eta}^{DH} h(x)$ is nonempty, then we have additionally

$$\partial_{\varepsilon}^{DH}(g-h)(x) \subseteq \partial_{\varepsilon+\eta}^{DH} g(x) \overset{*}{-} \partial_{\eta}^{DH} h(x) \subseteq \partial_{\varepsilon+\eta}^{DH} g(x) - \partial_{\eta}^{DH} h(x) \text{ for all } \varepsilon \geq 0.$$

(a)' A similar result holds also true if we use sums in place of differences. Namely, if for some $\eta \geq 0$ and $x \in X$ the set $\partial_{\eta}^{DH,+} h(x)$ is nonempty, then

$$\partial_{\varepsilon}^{DH}(g+h)(x) \subseteq \partial_{\varepsilon+\eta}^{DH} g(x) \overset{*}{+} \partial_{\eta}^{DH} h(x) \subseteq \partial_{\varepsilon+\eta}^{DH} g(x) + \partial_{\eta}^{DH} h(x) \text{ for all } \varepsilon \geq 0,$$

where for two given subsets A, B of X,

$$A \overset{*}{+} B := \{x \in X : x - B \subseteq A\} = \bigcap_{b \in B} \{A + b\} \tag{4.15}$$

denotes the *star-sum* of A and B.

(b) If $\overline{x}$ is a local minimizer of the function $f := g - h$ and f is finite at $\overline{x}$, then

$$0 \in \partial^{DH} g(\overline{x}) \overset{*}{-} \partial^{DH} h(\overline{x})$$

or, equivalently,

$$\partial^{DH} h(\overline{x}) \subseteq \partial^{DH} g(\overline{x}).$$

(c) Apparently very simple, the difference rule above ensures us that whenever $\partial^{DH} f(x) \neq \emptyset$, the following inclusion

$$\partial^{DH,+} f(x) \subseteq \bigcap_{x^* \in \partial^{DH} f(x)} x^*$$

holds true, where $\partial^{DH,+} f(x) := -\partial^{DH}(-f)(x)$ stands for the Dini-Hadamard upper subdifferential of f at $\overline{x}$.

(d) One can also check that whenever $\partial^{DH} f(\overline{x})$ and $\partial^{DH,+} f(\overline{x})$ are nonempty simultaneously, where $\overline{x}$ is such that $|f(\overline{x})| < +\infty$, then $\partial^{DH} f(\overline{x}) = \partial^{DH,+} f(\overline{x})$.

(e) Similar characterizations for the difference of two functions to the one in Proposition 4.2.3 have been given in [1] by means of the Fréchet subdifferential, in [96] by means of the basic subdifferential (see [94, 95]) and in [9] via the Dini-Hadamard-like one.

Actually a whole theory involving star-sums of two functions can be similarly developed, but we skip here the details.

In the following we will show that for some particular classes of functions one gets equality in (4.14).

The following result gives a first refinement of the statement in Proposition 4.2.3, in case $\varepsilon = \eta = 0$.

Theorem 4.2.5 *Let $g, h : X \to \mathbb{R} \cup \{+\infty\}$ be two directionally approximately starshaped functions at $\overline{x} \in \operatorname{dom} g \subseteq \operatorname{dom} h$ such that $\partial^{DH} h$ is spongiously gap-continuous at $\overline{x}$ and $f := g - h$ is calm at $\overline{x}$. Then it holds*

$$\partial^{DH} f(\overline{x}) = \partial^{DH} g(\overline{x}) \overset{*}{-} \partial^{DH} h(\overline{x}). \tag{4.16}$$

Remark 4.2.6 (a) In the hypotheses of Theorem 4.2.5 one has that

$$0 \in \partial^{DH} f(\overline{x}) \Leftrightarrow \partial^{DH} h(\overline{x}) \subseteq \partial^{DH} g(\overline{x}).$$

(b) If $g, h : X \to \mathbb{R} \cup \{+\infty\}$ are convex functions with $\operatorname{dom} g \subseteq \operatorname{dom} h$, $\partial^{DH} h$ is spongiously gap-continuous at $\overline{x} \in \operatorname{dom} g$ and $f := g - h$ is calm at $\overline{x}$, then

$$\partial^{DH} f(\overline{x}) = \partial g(\overline{x}) \overset{*}{-} \partial h(\overline{x}).$$

Theorem 4.2.5 is the main ingredient for the proof of the following result.

Theorem 4.2.7 *Let $g, h : X \to \mathbb{R} \cup \{+\infty\}$ be two directionally approximately starshaped functions at $\overline{x} \in \operatorname{dom} g \subseteq \operatorname{dom} h$ and $f := g - h$ is calm at $\overline{x}$. If for some $\eta \geq 0$ the set-valued mapping $\partial^{DH}_{\eta} h$ is spongiously gap-continuous at $\overline{x}$, then for all $\varepsilon \geq 0$ it holds*

$$\partial^{DH}_{\varepsilon} f(\overline{x}) = \partial^{DH}_{\varepsilon+\eta} g(\overline{x}) \overset{*}{-} \partial^{DH}_{\eta} h(\overline{x}). \tag{4.17}$$

Remark 4.2.8 (a) One should notice that, in the hypotheses of Theorem 4.2.7, for all $\varepsilon \geq 0$ it holds

$$\partial^{DH}_{\varepsilon} f(\overline{x}) = \bigcap_{\mu \geq 0} \left(\partial^{DH}_{\varepsilon+\mu} g(\overline{x}) \overset{*}{-} \partial^{DH}_{\mu} h(\overline{x}) \right). \tag{4.18}$$

(b) As pointed out in Remark 3.3.3, in order to guarantee that $\partial^{DH}_{\eta} h$ is spongiously gap-continuous at $\overline{x}$ for a given $\eta \geq 0$, it is enough to assume that $\partial^{DH} h$ is spongiously gap-continuous at $\overline{x}$.

Remark here also that the results in Theorems 4.2.5 and 4.2.7 remain true also in case the spongiously gap-continuity is replaced by the spongiously pseudo-dissipativity. Moreover, following the lines of the proofs of the above theorems, we can even furnish a formula for the difference of two functions in terms of the Dini-Hadamard-like subdifferential.

Theorem 4.2.9 *Let $g, h : X \to \mathbb{R} \cup \{+\infty\}$ be two directionally approximately starshaped functions at $\overline{x} \in \operatorname{dom} g$ and $f := g - h$. If for some $\eta \geq 0$ the set-valued mapping $\widetilde{\partial}_\eta h$ is spongiously pseudo-dissipative at $\overline{x}$, then for all $\varepsilon \geq 0$ it holds*

$$\widetilde{\partial}_\varepsilon f(\overline{x}) = \widetilde{\partial}_{\varepsilon+\eta} g(\overline{x}) \overset{*}{-} \widetilde{\partial}_\eta h(\overline{x}). \tag{4.19}$$

If the function f is calm at $\overline{x}$ one obtains the result in Theorem 4.2.7. For a similar statement in the particular setting $\varepsilon = \eta = 0$, we refer to [119, Theorem 28]. There the function h is assumed to be directionally approximately starshaped, directionally continuous, directionally stable and tangentially convex at $\overline{x}$, a point from core(dom h). Similar results expressed by means of the Fréchet subdifferential can be found in [1, Theorem 3] and [119, Theorem 26], where the functions are supposed to be approximately starshaped and a very mild assumption on $\widehat{\partial} h$ is required. But, since f may not be calm at $\overline{x}$, or the functions g and h may not be approximately starshaped, or even core(dom h) could be empty (for instance, $\operatorname{core}(\ell^p_+) = \emptyset$ for any $p \in [1, +\infty)$, see [27]), motivates us to formulate results like Theorem 4.2.9.

Two further corollaries of Theorem 4.2.7 follow easily.

Corollary 4.2.10 *Let $g, h : X \to \mathbb{R} \cup \{+\infty\}$ be two directionally approximately starshaped functions at $\overline{x} \in \operatorname{dom} g$ such that $\partial^{DH} h$ is spongiously gap-continuous at $\overline{x}$ and $f := g - h$ is calm at $\overline{x}$. Then the following statements are equivalent:*

(i) there exists $\eta \geq 0$ such that $\partial^{DH}_\eta h(\overline{x}) \subseteq \partial^{DH}_\eta g(\overline{x})$;

(ii) $0 \in \partial^{DH} f(\overline{x})$;

(iii) for all $\eta \geq 0$ $\partial^{DH}_\eta h(\overline{x}) \subseteq \partial^{DH}_\eta g(\overline{x})$.

Corollary 4.2.11 *Let $g, h : X \to \mathbb{R} \cup \{+\infty\}$ be two given functions, $\overline{x} \in \operatorname{dom} g$ and $f := g - h$ be calm at $\overline{x}$. Then the following assertions are true:*

(a) If g is convex, h is directionally approximately starshaped at $\overline{x}$ and $\partial^{DH} h$ is spongiously gap-continuous at $\overline{x}$, then for all $\varepsilon \geq 0$ it holds

$$\partial^{DH}_\varepsilon f(\overline{x}) = (\partial g(\overline{x}) + \varepsilon \overline{B}_{X^*}) \overset{*}{-} \partial^{DH} h(\overline{x}).$$

(b) If g is lower semicontinuous, approximately convex at $\overline{x}$, h is directionally approximately starshaped at $\overline{x}$ and $\partial^{DH} h$ is spongiously gap-continuous at $\overline{x}$, then for all $\varepsilon \geq 0$ it holds

$$\partial^{DH}_\varepsilon f(\overline{x}) = (\partial^{DH} g(\overline{x}) + \varepsilon \overline{B}_{X^*}) \overset{*}{-} \partial^{DH} h(\overline{x}).$$

Note again that the above results remain also true if we replace the Dini-Hadamard subdifferential with the Dini-Hadamard-like one and if we renounced at the calmness assumption (see [9, Corollary 11, Corollary 12]).

The following statement, which significantly improves the result in [28, Corollary 3.6], due to Theorem 4.2.9 and [119, Theorem 26] (see also Proposition 3.3.34), is meant to reveal that the Dini-Hadamard-like subdifferential coincides with the Dini-Hadamard subdifferential and with the Fréchet one not only on approximately starshaped functions but also on some particular differences of approximately starshaped functions.

Corollary 4.2.12 *Let $g, h : X \to \mathbb{R} \cup \{+\infty\}$ be two approximately starshaped functions at $\overline{x} \in \operatorname{dom} g$ with the property that there exists $\eta \geq 0$ such that $\partial^S_\eta h$ is approximately pseudo-dissipative at $\overline{x}$ and $f := g - h$. Then for all $\varepsilon \geq 0$ $\widehat{\partial}_\varepsilon f(\overline{x}) = \partial^{DH}_\varepsilon f(\overline{x}) = \partial^S_\varepsilon f(\overline{x})$.*

Moreover, in case $\overline{x} \in \operatorname{core}(\operatorname{dom} h)$ and $\partial^{DH} h$ is only directionally approximately pseudo-dissipative at $\overline{x}$, then one can guarantee that for any $\varepsilon \geq 0$, $\partial^{DH}_\varepsilon f(\overline{x}) = \partial^S_\varepsilon f(\overline{x})$ (see for this [119, Lemma 22, Lemma 24, Lemma 27] and Remark 4.2.4 (c)).

4.2.3 A difference rule for Dini-Hadamard-like coderivatives

Let us furnish in the following a difference formula involving the coderivative construction studied in Section 3.4.

Corollary 4.2.13 *Let $f_i : X \to Y$, $i = 1, 2$, be two single-valued mappings such that f_2 and the difference $f_1 - f_2$ are spongiously Lipschitz and strongly spongiously continuous at $\overline{x}$. Then*

$$\widetilde{D}^*_{\nmid}(f_1 - f_2)(\overline{x})(y^*) \subseteq \widetilde{D}^*_{\nmid} f_1(\overline{x})(y^*) \stackrel{*}{-} \widetilde{D}^*_{\nmid} f_2(\overline{x})(y^*) \quad \textit{for all } y^* \in Y^*. \tag{4.20}$$

Additionally, if f_1 is strongly spongiously continuous at $\overline{x}$ and f_2 is Dini-Hadamard-like differentiable at $\overline{x}$ then

$$\begin{aligned} \widetilde{D}^*_{\nmid}(f_1 - f_2)(\overline{x})(y^*) &= \widetilde{D}^*_{\nmid} f_1(\overline{x})(y^*) \stackrel{*}{-} \widetilde{D}^*_{\nmid} f_2(\overline{x})(y^*) \\ &= \widetilde{D}^*_{\nmid} f_1(\overline{x})(y^*) - \widetilde{D}^*_{\nmid} f_2(\overline{x})(y^*) \end{aligned} \tag{4.21}$$

for all $y^* \in Y^*$.

4.2.4 Subdifferentials of composition

The primary goal of this subsection is to develop calculus rules for composition first for the Dini-Hadamard-like subdifferential and then for the Dini-Hadamard one, in arbitrary real Banach spaces. The key ingredient in deriving such formulae is played, as we will see bellow, by the smooth variational description of Dini-Hadamard-like decoupled subgradients (see the results in Subsection 3.3.5).

Astonishingly, as direct consequences, one can obtain a variety of calculus rules such as product and quotient rules in a rather surprising generality.

Consider now the following *generalized composition* given by

$$(\varphi \circ f)(x) := \varphi(x, f(x)), \tag{4.22}$$

where $\varphi : X \times Y \to \overline{\mathbb{R}}$ is an extended-real-valued function and $f : X \to Y$ is a single-valued mapping between two Banach spaces.

Our intention is to provide next an exact chain rule involving both Dini-Hadamard-like and Dini-Hadamard-like decoupled subgradients. Let us emphasize also here that although we have paid the main attention to the study of lower subdifferential constructions whose properties symmetrically induce the ones for upper subgradients, there are important issues in variational analysis and optimization that require both lower and upper subgradients. The following result happens to be such an example.

Theorem 4.2.14 *Let $f : X \to Y$ be a spongiously Lipschitz and strongly spongiously continuous single-valued mapping at $\overline{x}$ and $\varphi : X \times Y \to \overline{\mathbb{R}}$ an extended-real-valued function finite at $(\overline{x}, f(\overline{x}))$. Then*

$$\widetilde{\partial}(\varphi \circ f)(\overline{x}) \subseteq \bigcap_{(x^*,y^*)\in\widetilde{\partial}^+_{\nmid}\varphi(\overline{x},f(\overline{x}))} \left[x^* + \widetilde{\partial}\langle y^*, f\rangle(\overline{x})\right]. \tag{4.23}$$

Additionally, if φ is Dini-Hadamard-like decoupled differentiable at $(\overline{x}, f(\overline{x}))$ with $\widetilde{\nabla}_{\nmid}\varphi(\overline{x}, f(\overline{x})) = (x^*, y^*)$, then

$$\widetilde{\partial}(\varphi \circ f)(\overline{x}) = x^* + \widetilde{\partial}\langle y^*, f\rangle(\overline{x}). \tag{4.24}$$

When the function f is additionally calm at $\overline{x}$ (in particular is the case of those Lipshitz functions at $\overline{x}$) then we get the following corollary that provides estimates of Dini-Hadamard subgradients of compositions in terms of Dini-Hadamard subgradients of the scalarization function and Dini-Hadamard-like subgradients.

Corollary 4.2.15 *Let $f : X \to Y$ be a calm, spongiously Lipschitz and strongly spongiously continuous single-valued mapping at $\overline{x}$ and $\varphi : X \times Y \to \overline{\mathbb{R}}$ an extended-real-valued function finite at $(\overline{x}, f(\overline{x}))$. Then*

$$\partial^{DH}(\varphi \circ f)(\overline{x}) \subseteq \bigcap_{(x^*,y^*)\in\widetilde{\partial}^+_{\nmid}\varphi(\overline{x},f(\overline{x}))} \left[x^* + \partial^{DH}\langle y^*, f\rangle(\overline{x})\right]. \tag{4.25}$$

Additionally, if φ is Dini-Hadamard-like decoupled differentiable at $(\overline{x}, f(\overline{x}))$ with $\widetilde{\nabla}_{\nmid}\varphi(\overline{x}, f(\overline{x})) = (x^*, y^*)$ and $\varphi \circ f$ is calm at $\overline{x}$, then

$$\partial^{DH}(\varphi \circ f)(\overline{x}) \subseteq x^* + \partial^{DH}\langle y^*, f\rangle(\overline{x}). \tag{4.26}$$

Remark 4.2.16 Similar characterizations to the one in Theorem 4.2.14 have been given by Mordukhovich [96] in terms of Fréchet subgradients, involving the Lipshitz continuity of the function f. But, if the function f is only spongiously Lipschitz and strongly spongiously continuous at $\overline{x}$ (see for instance Example 3.1.11), then we cannot apply anymore the results in there. However, by making use of our result above, one gets the following alternative estimate for Fréchet subgradients

$$\widehat{\partial}(\varphi \circ f)(\overline{x}) \subseteq \bigcap_{(x^*,y^*)\in\widehat{\partial}^+\varphi(\overline{x},f(\overline{x}))} \left[x^* + \widetilde{\partial}\langle y^*, f\rangle(\overline{x})\right], \tag{4.27}$$

since $\widehat{\partial}(\varphi \circ f)(\overline{x}) \subseteq \widetilde{\partial}(\varphi \circ f)(\overline{x})$ and $\widehat{\partial}^{+}\varphi(\overline{x}, f(\overline{x})) \subseteq \widetilde{\partial}_{\dagger}^{+}\varphi(\overline{x}, f(\overline{x}))$. Moreover, if φ is Dini-Hadamard-like decoupled differentiable at $(\overline{x}, f(\overline{x}))$ with $\widetilde{\nabla}_{\dagger}\varphi(\overline{x}, f(\overline{x})) = (x^*, y^*)$, then

$$\widehat{\partial}(\varphi \circ f)(\overline{x}) \subseteq x^* + \widetilde{\partial}\langle y^*, f\rangle(\overline{x}). \tag{4.28}$$

As mentioned before, the calmness assumption above can be successfully changed with the classical Lipschitz continuity.

Further the remarkable formula in Theorem 4.2.14 yields the following chain rule for the usual composition $(\varphi \circ f)(x) := \varphi(f(x))$, if we take into account also relation (4.22), with $\varphi = \varphi(y)$.

Corollary 4.2.17 *Let $f : X \to Y$ be a spongiously Lipschitz and strongly spongiously continuous single-valued mapping at $\overline{x}$ and $\varphi : Y \to \overline{\mathbb{R}}$ an extended-real-valued function finite at $f(\overline{x})$. Then*

$$\widetilde{\partial}(\varphi \circ f)(\overline{x}) \subseteq \bigcap_{y^* \in \widetilde{\partial}^{+}\varphi(f(\overline{x}))} \widetilde{\partial}\langle y^*, f\rangle(\overline{x}). \tag{4.29}$$

Additionally, if φ is Dini-Hadamard-like differentiable at $f(\overline{x})$, then

$$\widetilde{\partial}(\varphi \circ f)(\overline{x}) = \widetilde{\partial}\langle \widetilde{\nabla}\varphi(f(\overline{x})), f\rangle(\overline{x}). \tag{4.30}$$

As an immediate consequence of Corollary 4.2.17 we derive the following chain rule formula involving (only!) Dini-Hadamard subgradients.

Corollary 4.2.18 *Let $f : X \to Y$ be a calm, spongiously Lipschitz and strongly spongiously continuous single-valued mapping at $\overline{x}$ and $\varphi : Y \to \overline{\mathbb{R}}$ an extended-real-valued function finite at $f(\overline{x})$. Then*

$$\partial^{DH}(\varphi \circ f)(\overline{x}) \subseteq \bigcap_{y^* \in \partial^{DH,+}\varphi(f(\overline{x}))} \partial^{DH}\langle y^*, f\rangle(\overline{x}) \tag{4.31}$$

$$\subseteq \bigcap_{y^* \in \partial^{DH}\varphi(f(\overline{x}))} \partial^{DH}\langle y^*, f\rangle(\overline{x}), \tag{4.32}$$

where the latter inclusion holds whenever $\partial^{DH,+}\varphi(f(\overline{x})) \neq \emptyset$.

Additionally, if φ is calm and Dini-Hadamard-like differentiable at $f(\overline{x})$, then

$$\partial^{DH}(\varphi \circ f)(\overline{x}) = \partial^{DH}\langle \widetilde{\nabla}\varphi(f(\overline{x})), f\rangle(\overline{x}). \tag{4.33}$$

Finally, an alternative estimate for the Fréchet subdifferential of the usual composition $\varphi \circ f$, similar to the one furnished by Mordukhovich [96, Corollary 3.8], follows easily. It is worth emphasizing also here that our result cannot be obtained from that one, since there are spongiously Lipschitz and strongly spongiously continuous mappings for which the statements in there do not apply.

Corollary 4.2.19 *Let $f : X \to Y$ be a spongiously Lipschitz and strongly spongiously continuous single-valued mapping at $\overline{x}$ and $\varphi : Y \to \overline{\mathbb{R}}$ an extended-real-valued function finite at $f(\overline{x})$. Then*

$$\widehat{\partial}(\varphi \circ f)(\overline{x}) \subseteq \bigcap_{y^* \in \widehat{\partial}^{+}\varphi(f(\overline{x}))} \widetilde{\partial}\langle y^*, f\rangle(\overline{x}). \tag{4.34}$$

4.2.5 Product and quotient rules involving Dini-Hadamard-like subgradients

The striking difference rule obtained in Proposition 4.2.3 together with the results in Theorem 4.2.14 are among the key ingredients in deriving other useful calculus rules for Dini-Hadamard-like subgradients in Banach spaces. The next theorem furnishes a general product rule involving spongiously Lipschitz functions.

Theorem 4.2.20 *Let the functions $\varphi_i : X \to \mathbb{R}$, $i = 1, 2$, be spongiously Lipschitz at $\overline{x}$. Then one has*

$$\widetilde{\partial}(\varphi_1 \cdot \varphi_2)(\overline{x}) \subseteq \bigcap_{x^* \in \widetilde{\partial}(-\varphi_1(\overline{x})\varphi_2)(\overline{x})} \left[\widetilde{\partial}(\varphi_2(\overline{x})\varphi_1)(\overline{x}) - x^*\right]. \tag{4.35}$$

Moreover, the above product rule inclusion becomes equality provided that φ_2 is Dini-Hadamard-like differentiable at $\overline{x}$.

In particular, one obtain a product formula also for the Dini-Hadamard subdifferential.

Corollary 4.2.21 *Let the functions $\varphi_i : X \to \mathbb{R}$, $i = 1, 2$, be stable and spongiously Lipschitz at $\overline{x}$ (in particular Lipschitz at $\overline{x}$). Then one has*

$$\partial^{DH}(\varphi_1 \cdot \varphi_2)(\overline{x}) \subseteq \bigcap_{x^* \in \partial^{DH}(-\varphi_1(\overline{x})\varphi_2)(\overline{x})} \left[\partial^{DH}(\varphi_2(\overline{x})\varphi_1)(\overline{x}) - x^*\right]. \tag{4.36}$$

Moreover, the above product rule inclusion becomes equality provided that $\varphi_1 \cdot \varphi_2$ is calm at $\overline{x}$ and φ_2 is Dini-Hadamard-like differentiable at $\overline{x}$.

As regards the Féchet subdifferential, Mordukovich was the one who obtained a very nice product rule involving Lipschitz functions. In the following we show that the first assertion in there remains also true if we deal with spongiously Lispchitz and approximately starshaped real functions.

Corollary 4.2.22 *Let the functions $\varphi_i : X \to \mathbb{R}$, $i = 1, 2$, be spongiously Lipschitz at $\overline{x}$ and $\varphi_2(\overline{x})\varphi_1$ approximately starshaped at $\overline{x}$. Then one has*

$$\widehat{\partial}(\varphi_1 \cdot \varphi_2)(\overline{x}) \subseteq \bigcap_{x^* \in \widehat{\partial}(-\varphi_1(\overline{x})\varphi_2)(\overline{x})} \left[\widehat{\partial}(\varphi_2(\overline{x})\varphi_1)(\overline{x}) - x^*\right]. \tag{4.37}$$

The following theorem provides a quotient rule for Dini-Hadamard-like subgradients of spongiously Lipschitz functions in Banach spaces.

Theorem 4.2.23 *Let the functions $\varphi_i : X \to \mathbb{R}$, $i = 1, 2$, be spongiously Lipschitz at $\overline{x}$ with $\varphi_2(\overline{x}) \neq 0$. Then one has*

$$\widetilde{\partial}\left(\frac{\varphi_1}{\varphi_2}\right)(\overline{x}) \subseteq \bigcap_{x^* \in \widetilde{\partial}(\varphi_1(\overline{x})\varphi_2)(\overline{x})} \frac{\left[\widetilde{\partial}(\varphi_2(\overline{x})\varphi_1)(\overline{x}) - x^*\right]}{(\varphi_2(\overline{x}))^2}. \tag{4.38}$$

Moreover, the above quotient rule inclusion becomes equality provided that φ_2 is Dini-Hadamard-like differentiable at $\overline{x}$.

A similar quotient rule is also valid for the Dini-Hadamard subdifferential, while for the Fréchet one we obtain again an alternative statement to the corresponding one furnished by Mordukhovich in [96, Theorem 3.11].

Corollary 4.2.24 *Let the functions $\varphi_i : X \to \mathbb{R}$, $i = 1, 2$, be stable and spongiously Lipschitz at $\overline{x}$ with $\varphi_2(\overline{x}) \neq 0$. Then one has*

$$\partial^{DH}\left(\frac{\varphi_1}{\varphi_2}\right)(\overline{x}) \subseteq \bigcap_{x^* \in \partial^{DH}(\varphi_1(\overline{x})\varphi_2)(\overline{x})} \frac{\left[\partial^{DH}(\varphi_2(\overline{x})\varphi_1)(\overline{x}) - x^*\right]}{(\varphi_2(\overline{x}))^2}. \tag{4.39}$$

Moreover, the above quotient rule inclusion becomes equality provided that $\frac{\varphi_1}{\varphi_2}$ is calm at $\overline{x}$ and φ_2 is Dini-Hadamard-like differentiable at $\overline{x}$.

Corollary 4.2.25 *Let the functions $\varphi_i : X \to \mathbb{R}$, $i = 1, 2$, be spongiously Lipschitz at $\overline{x}$ with $\varphi_2(\overline{x}) \neq 0$ and $\varphi_2(\overline{x})\varphi_1$ approximately starshaped at $\overline{x}$. Then one has*

$$\widehat{\partial}\left(\frac{\varphi_1}{\varphi_2}\right)(\overline{x}) \subseteq \bigcap_{x^* \in \widehat{\partial}(\varphi_1(\overline{x})\varphi_2)(\overline{x})} \frac{\left[\widehat{\partial}(\varphi_2(\overline{x})\varphi_1)(\overline{x}) - x^*\right]}{(\varphi_2(\overline{x}))^2}. \tag{4.40}$$

Let us describe now a very simple quotient rule which always holds as equality, being actually independent of that obtained above.

Theorem 4.2.26 *Let the function $f : X \to \mathbb{R}$ be spongiously Lipschitz at $\overline{x}$ with $f(\overline{x}) \neq 0$. Then*

$$\widetilde{\partial}\left(\frac{1}{f}\right)(\overline{x}) = \frac{\widetilde{\partial}(-f)(\overline{x})}{(f(\overline{x}))^2}. \tag{4.41}$$

When speaking about Dini-Hadamard subgradients one can easily observe that this finally result holds true.

Corollary 4.2.27 *Let the function $f : X \to \mathbb{R}$ be quiet and spongiously Lipschitz at $\overline{x}$ such that $\frac{1}{f}$ is calm at $\overline{x}$ with $f(x) \neq 0$. Then*

$$\partial^{DH}\left(\frac{1}{f}\right)(\overline{x}) = \frac{\partial^{DH}(-f)(\overline{x})}{(f(\overline{x}))^2}. \tag{4.42}$$

In particular, if $f : X \to (0, 1]$, its enough to ask for f to be only quiet and spongiously Lipschitz at $\overline{x}$ (or even Lipschitz at $\overline{x}$), in order to obtained the same statement as above.

Chapter 5

Optimality conditions for nonconvex problems

Providing handleable subdifferential formulae is a decisive aspect for the formulation of necessary and sufficient optimality conditions for nonsmooth optimization problems. In the final part of the paper we deal with a cone-constrained optimization problem having the difference of two functions as objective and a convex feasible set. For this problem we investigate the existence of so-called *spongiously local ε-blunt minimizers* for all $\varepsilon > 0$, a notion which represents an extension of the *local ε-blunt minimizer* introduced and investigated in [1]. To this aim we make use of the formula we give in Chapter 4 for the Dini-Hadamard ε-subdifferential of the difference of two functions, but also of some results originating in the convex optimization. In this way we show how nonsmooth and convex techniques can successfully interact when characterizing optimality.

The theory presented in this chapter is based on [28].

5.1 An application

We begin our approach by presenting the following notion.

Definition 5.1.1 *Let $C \subseteq X$ be a nonempty set, $f : X \to \overline{\mathbb{R}}$ be a given function, $\overline{x} \in \operatorname{dom} f \cap C$ and $\varepsilon > 0$. We say that $\overline{x}$ is a* spongiously local ε-blunt minimizer of f on the set C *if there exists a sponge S around $\overline{x}$ such that for all $x \in S \cap C$*

$$f(x) \geq f(\overline{x}) - \varepsilon\|x - \overline{x}\|.$$

In case $C = X$, we simply call $\overline{x}$ a spongiously local ε-blunt minimizer of f.

Remark 5.1.2 It is worth noticing that the above notion generalizes the one of local ε-blunt minimizer introduced by Amahroq, Penot and Syam in [1]. Although in finite dimensional spaces the two notions coincide, this is in general not the case. To see this one only needs to take a look at the Example 3.1.21. There, $\overline{x}$ is a spongiously local ε-blunt minimizer of f for all $\varepsilon > 0$, but not a local ε-blunt minimizer of f for $\varepsilon \in (0, 1)$.

The following characterization of the Dini-Hadamard subdifferential by means of spongiously local ε-blunt minimizers is a direct consequence of Theorem 3.3.15. For a similar statement via the Dini-Hadamard-like subdifferential we refer to [9, Proposition 18], where the function involved may not be necessary calm.

Proposition 5.1.3 *Let $f : X \to \overline{\mathbb{R}}$ be a given function and $\overline{x} \in \operatorname{dom} f$. Then:*

$$0 \in \partial^{DH} f(\overline{x}) \Leftrightarrow \begin{array}{l} f \textit{ is calm at } \overline{x} \textit{ and } \overline{x} \textit{ is a spongiously local } \varepsilon - \textit{blunt minimizer} \\ \textit{of } f \textit{ for all } \varepsilon > 0. \end{array}$$

Consider now another Banach space Z and Z^* its topological dual space. Let $C \subseteq X$ be a convex and closed set and $K \subseteq Z$ be a nonempty convex and closed *cone* with $K^* := \{z^* \in Z^* : \langle z^*, z\rangle \geq 0 \text{ for all } z \in K\}$ its *dual cone*. Consider a function $k : X \to Z$ which is assumed to be K-*convex*, meaning that for all $x, y \in X$ and all $t \in [0, 1]$, $(1-t)k(x) + tk(y) - k((1-t)x + ty) \in K$, and K-*epi closed*, meaning that the K-*epigraph* of k, $\operatorname{epi}_K k := \{(x, z) \in X \times Z : z \in k(x) + K\}$, is a closed set. One can notice that when $Z = \mathbb{R}$ and $K = \mathbb{R}_+$ the notion of K-epi closedness coincide with the classical lower semicontinuity. For $z^* \in K^*$, by $(z^*k) : X \to \mathbb{R}$ we denote the function defined by $(z^*k)(x) = \langle z^*, k(x)\rangle$. Further, let $g, h : X \to \mathbb{R} \cup \{+\infty\}$ be two given functions with $\operatorname{dom} g \subseteq \operatorname{dom} h$ and $f := g - h$.

The next result provides optimality conditions for the cone-constrained optimization problem

$$(\mathcal{P}) \quad \begin{array}{l} \inf\limits_{x \in \mathcal{A}} f(x). \\ \mathcal{A} = \{x \in C : k(x) \in -K\} \end{array}$$

Theorem 5.1.4 *Let be $\overline{x} \in \operatorname{int}(\operatorname{dom} g) \cap \mathcal{A}$. Suppose that g is lower semicontinuous and approximately convex at $\overline{x}$, that f is calm at $\overline{x}$ and that $\bigcup_{\lambda > 0} \lambda(k(C) + K)$ is a closed linear subspace of Z. Then the following assertions are true:*

(a) If $\overline{x}$ is a spongiously local ε-blunt minimizer of f on $\mathcal{A}$ for all $\varepsilon > 0$, then the following relation holds

$$\partial^{DH} h(\overline{x}) \subseteq \partial^{DH} g(\overline{x}) + \bigcup_{\substack{z^* \in K^* \\ (z^*k)(\overline{x}) = 0}} \partial((z^*k) + \delta_C)(\overline{x}). \tag{5.1}$$

(b) Conversely, if h is directionally approximately starshaped at $\overline{x}$, $\partial^{DH} h$ is spongiously gap-continuous at $\overline{x}$ and (5.1) *holds, then $\overline{x}$ is a spongiously local ε-blunt minimizer of f on $\mathcal{A}$ for all $\varepsilon > 0$.*

Remark 5.1.5 For a similar result to Theorem 5.1.4, given in the particular instance when $K = \{0\}$ and $k(x) = 0$ for all $x \in X$ and by means of the Fréchet subdifferential, we refer to [1, Proposition 6]. In the second statement of that result the authors ask for h to be approximately starshaped at $\overline{x}$ with $\widehat{\partial} h$ gap-continuous at $\overline{x}$ and characterize the local ε-blunt minimizers of f for all $\varepsilon > 0$. To this end they make use of some exact subdifferential formulae for the limiting subdifferential, but by providing an incorrect argumentation, since these are valid in Asplund spaces. Nevertheless, the statement in [1, Proposition 6] is true in Banach spaces, too, and it can be proven in the lines of the proof of Theorem 5.1.4.

Remark 5.1.6 Let us also mention that one can obtain also a general result implying the Dini-Hadamard-like subdifferential (see [9, Theorem19]), where f may not be calm at $\overline{x}$. Moreover, accordingly to [119, Lemma 22, Lemma 24 and Lemma 27], this result remains also true in case $\widetilde{\partial} h$ is directionally approximately pseudo-dissipative at $\overline{x}$. Furthermore, in the particular instance when $K = \{0\}$, $k(x) = 0$ for any $x \in X$, g is lower semicontinuous and approximately convex at $\overline{x} \in \operatorname{int}(\operatorname{dom} g) \cap \mathcal{A}$ and h is convex on C and continuous at $\overline{x}$, and hence directionally approximately pseudo-dissipative at $\overline{x}$ (due to the remarkable dissipativity property of the subdifferential in the sense of convex analysis, see [119, Theorem 6]) then $\overline{x}$ is a spongiously local ε-blunt minimizer of f on $\mathcal{A}$ for all $\varepsilon > 0$ if and only if

$$\partial h(\overline{x}) \subseteq \widetilde{\partial} g(\overline{x}) + N(\mathcal{A}, \overline{x}).$$

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