Babeş-Bolyai University, Cluj-Napoca
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HABILITATION THESIS

The role of generalized convexity in vector optimization and related variational problems

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Abstract

This thesis presents some of the scientific results published by the author, alone or in collaboration, after the defense of the doctoral thesis *Contribution à l'optimisation vectorielle* (supervised by Michel Théra) at University of Limoges (France), in 1995, and the recognition of the doctoral diploma by the National Ministry of Education in Romania, in 1998.

The thesis consists of two parts. The first overviews the published scientific results, while the second presents the current status of the author's research, as well as various directions for future research and career development opportunities.

The results presented in the first part have been selected from 17 articles published in 13 journals indexed by ISI Web of Science: Annals of Operations Research (2015), Carpathian Journal of Mathematics (2016), Journal of Global Optimization (2007), Journal of Mathematical Analysis and Applications (2000), Journal of Optimization Theory and Applications (2001), Journal of Nonlinear and Convex Analysis (2011), Mathematical Methods of Operations Research (2003), Nonlinear Analysis: Theory, Methods and Applications (2010 and 2012), Operations Research Letters (2006, 2008 and 2010), Optimization (2005 and 2013), Optimization Letters (2015), Proceedings of the American Mathematical Society (2003), and Set-Valued and Variational Analysis (2015).

After the defense of his doctoral thesis, the author has worked as visiting professor (maître de conférences invité) at the University of Limoges for three semesters (in 1998, 2001 and 2004). Over the last two decades the author has made numerous research visits, so that some of the results included in this thesis have been obtained jointly with collaborators from abroad: Ovidiu Bagdasar (Derby, United Kingdom), Joël Benoist (Limoges, France), Jonathan Michael Borwein (Newcastle, Australia), Daishi Kuroiwa (Matsue, Japan), Davide La Torre (Milan, Italy) and Matteo Rocca (Varese, Italy).

As this habilitation thesis aims to become a useful reference text for potential doctoral students, the results are presented in a unifying style, being accompanied by definitions and relevant examples. The material is structured in four chapters.

In Chapter 1 we present some useful results concerning convex cones. We introduce a Gerstewitz (Tammer) type scalarization function associated to any solid convex cone, which is vectorially closed in a real linear space (Section 1.1). Within Banach spaces we show that every reproducing closed convex cone possesses a lattice-like property, by means of the polar cone's extreme directions (Section 1.2). Then, in Euclidean spaces we establish quantitative estimates for the Phelps cones in order to be contained in a given convex cone with nonempty interior or to contain a given pointed convex cone (Section 1.3). Also, we present some results which link the lexicographic cone to certain polyhedral canonical cones (Section 1.4).

Chapter 2 is devoted to some special classes of generalized convex functions. First we study scalar functions, by deriving a "local max - global min" property of explicitly quasiconvex functions (Section 2.1). In Section 2.2 we consider set-valued functions and vector-valued functions. Among other results concerning convex set-valued functions it is shown how certain affine set-valued functions can be constructed starting from two given convex sets. Also, we give sufficient conditions for an affine set-valued function to have an affine inverse (Subsection 2.2.1). In what concerns the cone-convex and cone-quasiconvex set-valued functions we present several interesting characterizations. One of them is a set-valued counterpart of the classical dual characterization of convex functions obtained by Crouzeix, asserting that a real-valued function is convex if and only if its linear perturbations are quasiconvex (Subsection 2.2.2). The next two subsections are reserved for vector-valued (i.e., single-valued) functions. We introduce a notion of explicit quasiconvexity with respect to a relatively solid convex cone, which extends the componentwise explicit quasiconvexity (Subsection 2.2.3). The class of these functions is then extended to the class of the so-called lexicographic quasiconvex functions, which preserve several important properties of explicitly quasiconvex functions (Subsection 2.2.4).

The role of generalized convexity in vector optimization is highlighted within Chapter 3, which represents the core of this thesis. After recalling some basic definitions and results (Section 3.1) we study the (weakly) radiant sets wit respect to a cone. In particular, we show that the boundary of any closed radiant set w.r.t. a convex cone with nonempty interior is homeomorphic with a hyperplane (Section 3.2). Section 3.3. is devoted to shaded sets. First we establish some characterizations of simply shaded and completely shaded sets (Subsection 3.3.1). Then, we study the continuity of conical sections and their marginal functions. We deduce an interesting characterization of polyhedral cones (Subsection 3.3.2). By choosing certain nonzero points in each face of the ordering cone, we introduce projective and polyhedral domination selections: (Subsection 3.3.3). These selections play a key role in establishing the contractibility (among other topological properties) of the efficient frontier of a given completely shaded set (Subsection 3.3.4). The simply shaded sets are shown to satisfy the same topological properties but only in the three-dimensional Euclidean space. However, in this case a more detailed description of the efficient points can be done (Subsection 3.3.5). In Section 3.4 we study the class of weakly shaded sets. We give necessary and/or sufficient conditions for a set to be weakly shaded (Subsection 3.4.1), which are then used for proving that several important classes of vector (multicriteria) optimization problems are Pareto reducible (Subsection 3.4.2). The results obtained in this subsection allow us to establish the contractibility of the efficient solutions' set (Subsection 3.4.3).

Chapter 4 concludes the thesis. Here we study several variational type problems, which are related to vector optimization. Under suitable generalized convexity assumptions we obtain sufficient conditions for scalarization and decomposition of classical vector variational inequalities (Section 4.1), generalized vector variational inequalities (Section 4.2), and vector equilibrium problems (Section 4.3).

The second part of the thesis overviews the present state of the author's research and current research directions for the immediate future, as well as the topics the author would wish to explore together with potential doctoral students (once he obtains the necessary qualification). These research directions could be harmonized with some core topics of the doctoral programme *PhD Program in Methods and Models for Economic Decisions* at University of Varese, in which the author is officially involved since 2015.