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THE METHOD OF MONOTONE ITERATIONS FOR MIXED MONOTONE OPERATORS

Ph.D. Thesis Summary

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Contents

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Mixed monotone operator, fixed point, coupled fixed point, the method of monotone iterations, Thompson's metric, ordered linear space, ordered Banach space, normal cone, ordered metric space, φ-contraction, nonlinear Fredholm integral equations, second-order two-point boundary value problems, systems of partially monotone operators

Introduction

It is well known that the method of monotone iterations (*MMI* for short), coupled with the method of lower and upper solutions, offers an effective and flexible mechanism for proving theoretical as well as constructive existence (and uniqueness) results for a variety of nonlinear equations and systems.

These two methods go back at least to E. Picard in the 1890s, in the study of the Dirichlet problem for nonlinear second order (ordinary and partial) differential equations (see, for example, [77], [78], [79]). Later on, these methods were formulated and further developed in an abstract framework, while being applied to a variety of non-linear problems.

In essence, *MMI* describes the following abstract phenomena. Consider an equation or a system of equations, modeled as a fixed point problem

$$x = T(x) \quad (x \in X) \tag{1}$$

where $X = (X, \leq)$ is an ordered set and $T : X \to X$ is an increasing operator with respect to the order on X (i.e., $T(x) \leq T(y)$ whenever $x \leq y$). If $x_0, y_0 \in X$ are such that

$$\begin{cases} x_0 \leq y_0 \\ x_0 \leq T(x_0) \\ y_0 \geq T(y_0) \end{cases}, \tag{2}$$

then the sequences $(x_n), (y_n)$ defined recursively by

$$\begin{cases} x_{n+1} = T(x_n) \\ y_{n+1} = T(y_n) \end{cases} \quad (n \in \mathbb{N}) \tag{3}$$

or, equivalently (using the functional powers of T), by

$$\begin{cases} x_n = T^n(x_0) \\ y_n = T^n(y_0) \end{cases} \quad (n \in \mathbb{N})$$

satisfy

$$x_0 \leq x_1 \leq \ldots \leq x_n \leq \ldots \leq y_m \leq \ldots \leq y_1 \leq y_0. \tag{4}$$

Moreover, if x is a solution of (1) such that $x_0 \leq x \leq y_0$, then

$$x_n \leq x \leq y_m \quad \text{for any } n, m \in \mathbb{N}.$$

In the presence of some kind of *convergence* on X (e.g., the convergence induced by a metric or by a topology) and under suitable conditions (see, for example, Krasnosel′skiĭ [46]), (x_n) is convergent to a (minimal) solution x^* of (1) and (y_n) is convergent to a (maximal) solution y^* of (1), meaning that any solution x of (1) such that $x_0 \leq x \leq y_0$ will lie *between* x^* and y^*. Additional assumptions may also ensure that $x^* = y^*$, hence the uniqueness of the solution in the order interval $[x_0, y_0]$ follows.

The chief advantage of the *MMI* over other methods used in the study of (1) is that it provides a constructive two-sided approximation of the solution, while simultaneously establishing the existence and, in some cases, the uniqueness of the solution, under certain requirements.

This led to an increasing interest in extended the *MMI* to a other classes of operators. It has been observed (e.g., Babkin [6]) that the monotonicity can be easily weakened, by considering a one sided Lipschitz condition on the operator T. If X is a real linear space and there exists some scalar $M > 0$ such that

$$T(y) - T(x) \geq -M(y - x) \quad \text{for any } x \leq y,$$

then T may be replaced with the increasing operator

$$\widetilde{T}(x) = \frac{T(x) + Mx}{M + 1} \quad (x \in X),$$

since T and $\widetilde{T}$ have identical fixed points.

Also, the case when T is decreasing ($T(x) \geq T(y)$ whenever $x \leq y$) was considered, either by modifying the iterative technique (see Picard [77]), or by studying the fixed points of the increasing operator $T^2 = T \circ T$ with additional assumptions that ensure that solutions of (1) can be obtained from solutions of $x = T^2(x)$.

An important step in extending the *MMI* to a larger class of operators was done by considering the class of, so called, *heterotonic* operators (in the terminology of Opoĭtsev [74]), i.e., the operators T that can be expressed as

$$T(x) = A(x, x) \quad (x \in X)$$

with $A : X^2 \to X$ being *mixed monotone*, i.e., increasing with respect to the first variable and decreasing with respect to the second one. This class clearly contains both the increasing and the decreasing operators.

In this case, the *MMI* is described in terms of the mixed monotone operator A, rather than the heterotonic operator T; the condition on the lower and upper solutions (2) is replaced by

$$\begin{cases} x_0 \leq y_0 \\ x_0 \leq A(x_0, y_0) \\ y_0 \geq A(y_0, x_0) \end{cases}, \tag{5}$$

((x_0, y_0) is called a *coupled lower-upper fixed point* for A), the approximating sequences in (3) are defined by

$$\begin{cases} x_{n+1} = A(x_n, y_n) \\ y_{n+1} = A(y_n, x_n) \end{cases} \quad (n \in \mathbb{N}), \tag{6}$$

while the concept of solution for (1) may be weakened, by considering *quasi-solutions*, i.e., pairs $(x, y) \in X^2$

satisfying

$$\begin{cases} x = A(x, y) \\ y = A(y, x) \end{cases} \tag{7}$$

((x, y) is called a *coupled fixed point* of A). In this conditions, $(x_n), (y_n)$ satisfy property (4) and, if $(x, y) \in X^2$ is a quasi-solution of (1) such that

$$x_0 \leq x \leq y_0, \quad x_0 \leq y \leq y_0, \tag{8}$$

then

$$x_n \leq x \leq y_m, \quad x_n \leq y \leq y_m \quad \text{for any } n, m \in \mathbb{N}.$$

Following the theory of increasing and concave operators developed by Krasnosel′skiĭ and his students during the 1960s ([7], [8], [46], [47], [48], [49], [97], [98]), Opoĭtsev [73], [74], [75] studied, in the 1970s, a particular class of heterotonic operators (the, so called, *pseudoconcave* heterotonic operators) and established the first (to the best of our knowledge) conditions that ensured the convergence of the approximating sequences $(x_n), (y_n)$ to a unique solution of (1), in the framework of ordered Banach spaces.

Opoĭtsev also showed in [74] that, in the general case of heterotonic operators, continuity or compactness properties of A, coupled with additional properties of the cone in the ordered Banach space X, ensure that (x_n) and (y_n) are convergent to x^*, respectively to y^*, where (x^*, y^*) is an extremal quasi-solution of (1) (the extremality of (x^*, y^*) means that the components of any quasi-solution of (1) satisfying (8) will lie *between* x^* and y^*). The additional step of obtaining solutions from quasi-solution can be achieved by further ensuring (through several types of conditions) that if (x, y) is a quasi-solution of (1) such that x and y are comparable, then $x = y$.

In the past three decades, the results of Opoĭtsev have been "*rediscovered*" or extended by several authors (see *Section 3.7* for more details), regrettably none of them making any reference to the original results. However, in our view, the paper of Opoĭtsev [74] from 1978 contains essential abstract results on the topic of fixed points for mixed monotone (or heterotonic) operators in ordered Banach spaces and must be considered as the most significant contribution among all the published papers so far.

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The aim of the present thesis is to develop a detailed and unitary study on the fixed points of mixed monotone operators, based on the *MMI*, from the very general setting (in abstract ordered sets) to the most commonly used framework (that of ordered Banach spaces), while providing applications to several illustrative non-linear problems.

We advance in the construction of the theory in a step-by-step manner, by assuming less and obtaining the most within the chosen assumptions. In this way, we believe that the essential assumptions of the theory will become more clear and perfectly justified, while non-essential conditions can be put aside, when not required. Also, by introducing and studying new concepts that, we believe, are important (if not fundamental), we can express the abstract results of the theory in a more clear manner, while pointing out the essentials. In this direction, most of the fundamental results are obtained by logical association of intermediary independent results, rather than by a direct proof that is too lengthy and whose lines of reasoning are hard to grasp.

The focus is put on the fixed point results that are based on the *MMI* and which provide both the existence and uniqueness (in some predefined set) of the fixed point in a constructive manner. In this way, the *MMI* is used at its full power, by providing an effective way of approximating the solution (both from the left and from the right, with increasing precision, in a constructive way and by an explicit iterative scheme), while proving its existence and establishing a region where no other solutions can be found.

We are less concerned with the fixed points that are not the limit of an iterative scheme, since other methods may behave better in those cases. We also exclude here any nonconstructive results based on the axiom of choice, that employ Zorn's Lemma directly to prove the existence of fixed points (e.g., Guo and Lakshmikantham [34, Theorem 3 and 4], Sun [100, Theorem 1]).

Note also that it is not our intention to make a survey (even a partial one) of the fixed point results for mixed monotone operators, nor to list a large collection of already available results, but merely to follow some particular directions of research in this field and establish their fundamental concepts and results, both within the available theory and by following new ideas and new methods of investigation.

Since the fixed point theory for mixed monotone operators has the advantage that it contains both the theory for increasing and for decreasing operators in one unitary approach, we can obtain the classical results of the two particular cases, while also obtain new results, especially in the case of decreasing operators (which possesses more difficulties than the case of increasing operators). Clearly, it is an elementary task to rewrite the iterative method and the associated fixed point results obtained in the present thesis, from the more general case of mixed monotone operators to the mentioned particular cases, hence we omit any details on this direction.

Some of the characteristic features of this thesis are as follows:

- presents a systematic investigation of the *MMI* for mixed monotone operators that is suitable for the study of a variety of nonlinear equations;
- provides new constructive criteria for the existence, uniqueness and approximation of the fixed points of mixed monotone operators, both in the presence and in the absence of a lower and upper solution;
- introduces and studies new concepts and notions, in order to obtain a unitary and simplified theory;
- formulates for the first time the *MMI* for mixed monotone operators in the framework of ordered sets;
- contains the *MMI* for increasing and for decreasing operators as a special case;
- establishes for the first time a comprehensive study of the fixed points for mixed monotone operators

based on the *MMI*, in the setting of ordered metric spaces;
- investigates for the first time the *MMI* for mixed monotone operators in the framework of ordered linear spaces, by means of *Thompson's metric*;
- contains a comprehensive analysis of *Thompson's metric* and its properties, with many new results;
- relies both on the recently published research and on the classical (sometimes, *forgotten*) results;
- points out several directions for future research;
- applies the newly derived results to several illustrative non-linear problems, including systems of non-linear equations;

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§1. With a few exceptions, all the fixed point results from our area of interest that have been published so far are given in the framework of *ordered Banach spaces*, although the theory of mixed monotone operators can be well established in a more general context.

Motivated by this, we begin to study the fixed points of mixed monotone operators in the general context of *ordered sets* and establish several important results (see *Sections 3.1* and *3.2*). First, we show that it is possible to express the approximating sequences $(x_n), (y_n)$ in (6) using the functional powers of A with respect to some associative composition law. Clearly, if $A, B : X^2 \to X$, then the usual composition $B \circ A$ (or $A \circ B$) is not possible, hence we define (see Definition 3.1.1) a new composition law (which we call *the mirror composition*, or simply *the m-composition*) of A and B by

$$(B * A)(x, y) = B(A(x, y), A(y, x)) \quad (x, y \in X),$$

and prove that it is associative and has as unit element the canonical projection of X^2 on X (Proposition 3.1.2). It is important to note that if A and B are mixed monotone, then $B * A$ is also mixed monotone (Proposition 3.1.17). Using this new concept, (x_n) and (y_n) defined in (6) can be expressed as

$$\begin{cases} x_n = A^n(x_0, y_0) \\ y_n = A^n(y_0, x_0) \end{cases} \quad (n \in \mathbb{N}),$$

where the functional powers of A are understood with respect to $*$.

With no need for a topological or metrical structure, we prove in Section 3.2 that it is possible to formulate an *approximation scheme* for the fixed points of a mixed monotone operator only in terms of the ordering. In the conditions previously expressed in the construction of *MMI* for heterotonic (i.e., mixed monotone) operators, we show (among other results) that:

1. If $\bigcap_{n\geq 0} [x_n, y_n] = \emptyset$, then (1) has no solutions in $[x_0, y_0]$ (see Corollary 3.2.3).
2. If $\bigcap_{n\geq 0} [x_n, y_n] = \{x^*\}$, then x^* is the unique solution of (1) in the order interval $[x_0, y_0]$ and for any $u_0, v_0 \in [x_0, y_0]$ such that $u_0 \leq x^* \leq v_0$, the sequences $(u_n), (v_n)$ defined by
$$u_n = A^n(u_0, v_0), \ v_n = A^n(v_0, u_0) \ (n \in \mathbb{N}), \quad (9)$$
satisfy $\bigcap_{n\geq 0} [u_n, v_n] = \{x^*\}$ (we describe this behavior in Theorem 3.2.1, by introducing the notion of *weakly order-attractive fixed point* – see Definitions 3.2.4 and 3.2.9).
3. If $\sup x_n$ and $\inf y_n$ exist and
$$\sup x_n = \inf y_n = x^*,$$
then $\bigcap_{n\geq 0} [x_n, y_n] = \{x^*\}$ (hence, x^* is the unique solution of (1) in $[x_0, y_0]$) and, for any u_0, v_0 in $[x_0, y_0]$ such that $u_0 \leq x^* \leq v_0$, the sequences $(u_n), (v_n)$ defined by (9) satisfy
$$\sup u_n = \inf v_n = x^*$$
(we describe this behavior in Theorem 3.2.1, by introducing the notion of *order-attractive fixed point* – see Definitions 3.2.7 and 3.2.9).

We also discuss in detail what happens when the initial *guess* (x_0, y_0) is not a lower-upper solution of (1). In this case, we show that it is possible to develop the *MMI* with no help from the method of lower and upper solutions (see Lemma 3.2.1, Theorem 3.2.16, Corollary 3.2.17). The definitions of the approximating sequences $(x_n), (y_n)$ remain unchanged, but the monotonicity is lost. In particular, if $x_0 \leq y_0$, then we recover that $x_n \leq y_n$ for any $n \in \mathbb{N}$. We also study the cases when, after a number of iterations, we obtain a pair (x_k, y_k) that is a lower-upper solution of (1), hence the monotonicity is (partially) recovered (see Theorem 3.2.19).

§2. We further develop (in *Section 3.3*) the *MMI* for mixed monotone operators, by setting the analysis in an abstract *ordered metric space*, endowed with a complete extended[1] metric d and study the convergence of the approximating sequences $(x_n), (y_n)$ in this new setting. Our results go in a slightly different direction than that recently pursued by Gnana Bhaskar and Lakshmikantham [30], Lakshmikantham and Ćirić [54].

We assume that d is connected to the ordering structure by the following properties (which are satisfied in any ordered Banach space with a normal cone by the norm-distance and by the so called *Thompson's extended metric*):

(i) d is interval-semi-monotone, i.e., there exists $\gamma \geq 1$ such that $d(x', y') \leq \gamma d(x, y)$ for any $x, x', y, y' \in X$ with $x \leq x' \leq y' \leq y$ (see Definition 3.3.1);

(ii) any order interval in X is closed with respect to monotone (increasing or decreasing) sequences.

Since we are interested in obtaining a unique fixed point for A in the order interval $[x_0, y_0]$ as the limit (with respect to d) of (x_n) and (y_n), we prove that in order for $(x_n), (y_n)$ to be convergent and have the same limit, it is

[1] The only difference in the definition of an extended metric with that of a (regular) metric is the fact that the extended metric may also take the value $+\infty$ (see *Section 1.4*). One such extended metric is *Thompson's (extended) metric*, that will be studied in detail in *Chapter 2* and used to obtain fixed point results for mixed monotone operators in ordered linear spaces and ordered Banach spaces.

necessary and sufficient (see Theorem 3.3.24) that

$$\lim_{n\to\infty} d(x_n, y_n) = 0, \tag{10}$$

or, equivalently,

$$\lim_{n\to\infty} (d * A^n)(x_0, y_0) = 0.$$

In the affirmative case, the common limit x^* is the unique fixed point of A in $[x_0, y_0]$ and, for any $x, y \in [x_0, y_0]$, the sequence $(A^n(x, y))$ is convergent to x^* (we describe this situation using the notion of m-Picard operator, see Definition 3.3.16). Moreover, $\sup x_n = \inf y_n = x^*$, hence all the previous conclusions from the context of ordered sets remain valid. Note that no continuity or compactness of A is required.

We also investigate the situation when (x_0, y_0) is not a lower-upper solution of (1) and show that this restrictive condition is not essential, if the monotonicity of the approximating sequences is not required and continuity of A is assumed (see Theorem 3.3.20, Corollary 3.3.21). We also consider the case when, after a number of iterations, we arrive at a lower-upper solution of (1) (in Theorem 3.3.26).

Essentially, all the fixed point theorems in this section express conditions on the operator A that are sufficient for (10) to be satisfied. In this direction, we study the case when A is of contractive type with respect to the m-composition, i.e.,

$$d(A(x, y), A(y, x)) \le \Psi(x, y)$$

or, equivalently,

$$(d * A)(x, y) \le \Psi(x, y)$$

with Ψ satisfying certain conditions (see Theorem 3.3.29, and Corollary 3.3.38). In particular we show in Corollary 3.3.39 that if A satisfies a condition of Meir-Keeler type (see [63]), then (10) is satisfied for any lower-upper solution (x_0, y_0) with $d(x_0, y_0) < \infty$.

A special case when Ψ is a radial function is also considered, i.e., when A satisfies a Φ-contraction condition, similar to that of Boyd and Wong [15] for univariate operators:

$$(d * A)(x, y) \le \Phi(d(x, y)). \tag{11}$$

We prove (in Theorem 3.3.47) that if (11) is satisfied and

($*$) for any $t > 0$, there exists $\tau > 0$ such that $\Phi(s) < t$ for any $s \in [t, t + \tau)$

then (10) is satisfied for any lower-upper solution (x_0, y_0) with $d(x_0, y_0) < \infty$.

The class of functions Φ satisfying ($*$) is further investigated (see Propositions 3.3.48, 3.3.49, 3.3.50, 3.3.54, Corollary 3.3.52, Example 3.3.46). For example, if $\Phi(t) < t$ for any $t > 0$ and Φ is right upper semicontinuous, then Φ satisfies ($*$).

In the closing of this topic, we study the fixed points of mixed monotone operators via extremal coupled fixed points, by restating the results of Opoĭtsev [74, Theorem 1.2 and Lemma 1.1], Guo and Lakshmikantham [34, Theorem 1], from the case of ordered Banach spaces to that of ordered metric spaces.

A brief comparison of our results with some recent fixed point theorems for mixed monotone operators in ordered metric spaces (see Agarwal et al. [1], Ćirić et al. [25], Gnana Bhaskar and Lakshmikantham [30], Lakshmikantham and Ćirić [54], Nieto and Rodríguez-López [69] and [70]) can be found in Section 3.5.

§3. An important part of the thesis consists of fixed point theorems for mixed monotone operators in ordered linear spaces. The approach idea was to use the newly derived results (obtained in ordered metric spaces) with a suitable (extended) metric defined in terms of the ordered linear structure. Our choice was *Thompson's metric* (cf. Thompson [102]), which is an extended (semi)metric on the cone of any ordered linear space and which possesses good properties. The idea of using this particular metric in order to prove fixed point results for mixed monotone operators is not new and can be found, for example, in the papers of Opoĭtsev [74] and Chen [22].

If (X, K) is an ordered linear space (see Section 1.6 for terminology and properties in ordered linear spaces) and $x, y \in K$, then Thompson's extended semimetric between x and y is defined by

$$\rho(x, y) = \inf \{s \ge 0 : \mathrm{e}^{-s} x \le y \le \mathrm{e}^{s} x\},$$

with the usual convention that $\inf \emptyset = +\infty$. If $\rho(x, y)$ is finite, then x and y are said to be *linked* (cf. Thompson [102]) - this is equivalent to the existence of $0 < \mu \le \lambda$ such that $\mu x \le y \le \lambda x$. This relation is an equivalence which splits K into disjoint components (called *parts*), hence ρ is a semimetric on any of the parts of K; additionally, if K is almost Archimedean, then ρ is a metric on each part of K (see Proposition 2.1.2).

Chapter 2 is dedicated entirely to the study of Thompson's metric, with a special interest on its completeness. One conclusion that can be drawn from this separate study is that it is possible to apply the fixed point results from Section 3.3 to the ordered metric spaces (K, ρ), provided that K is Archimedean and self-complete (see Definition 2.1.2).

In this direction, the remaining concern is to express the conditions in the fixed point theorems without using ρ explicitly. We show, for example, that if Y is a *part* of K and $A : Y^2 \to K$ is a mixed monotone operator, then the Φ-contraction condition (11) with respect to ρ, together with ($*$), can be obtain from the assumptions that

$$A(\lambda x, x) \ge \varphi(\lambda) A(x, \lambda x) \text{ for any } \lambda \in (0, 1), x \in Y \tag{12}$$

where $\varphi : (0, 1) \to (0, 1]$ is such that

($*'$) for any $t \in (0, 1)$, there exists $\tau > 0$ such that $\varphi(s) > t$ for any $s \in (t - \tau, t]$.

Note that a result involving similar conditions, in the framework of ordered Banach spaces, can be found in the paper of Chen [22].

The class of functions satisfying ($*'$) is also studied (Propositions 3.4.18, 3.4.21, 3.4.22, 3.4.23, 3.4.25, Corollary 3.4.24). For example, if $\varphi(t) > t$ for any $t \in (0, 1)$ and φ is left lower semicontinuous, then φ satisfies ($*'$). It is worth mentioning that it is possible to weaken ($*'$) by requiring only that $\varphi(t) > t$ for any $t \in (0, 1)$, provided that we replace (12) with the stronger assumption

$$\begin{gathered} A(\lambda x, y) \geq \varphi(\lambda) A(x, \lambda y) \\ \text{for any } \lambda \in (0,1),\, x, y \in Y \text{ linearly dependent.} \end{gathered} \tag{13}$$

In this way, we come close to some of the results of Opoĭtsev [74] which involve similar types of conditions and also use Thompson's metric in their proofs.

We point to the concluding results (pp. 23–23) for a quick and self-contained insight on the fixed point theory that we develop in the framework of ordered linear spaces.

§4. All the fixed point results obtained in ordered linear spaces using Thompson's metric remain valid in the case of ordered Banach spaces with a normal cone, since the normality of the cone ensures that ρ is complete, by Theorem 2.4.6. In addition, the ρ-convergence is stronger than the norm-convergence (by Theorem 2.4.6), meaning that m-Picard operators with respect to ρ are also m-Picard with respect to the norm.

In this way, we rediscover the results of Opoĭtsev [74], [75], Guo [32], Liang et al. [57], Liu et al. [59], Xu and Jia [113], Xu and Yuan [111], [112], Wu and Liang [110], Li et al. [56] as particular cases or weaker versions of our results.

In the same context, one can also apply the fixed point results obtained in ordered metric spaces to any ordered Banach space with a normal cone, by letting d to be the metric induced by the norm, since the normality of the cone is enough to ensure that d is semi-monotone. In this way, we obtain fixed point criteria which are not restricted to positive fixed points.

§5. While the fixed point theory for mixed monotone operators can be further developed, we show in Section 3.5 that it is possible to easily extend it to a larger class of operators, in the framework of ordered linear spaces. This idea appears, in a less general form, in the papers of Shuvar [96], Guo and Lakshmikantham [34].

If (X, K) is an ordered linear space, $A, B : X^2 \to X$, then define

$$B \circledast A := B * A - B + P : X^2 \to X,$$

where P is the canonical projection of X^2 on X (i.e., $P(x, y) = x$). We prove in Proposition 3.5.4 that A and $B \circledast A$ have the same (coupled) fixed points, provided that B is m-injective (see Definition 3.5.3). Assuming further that $B \circledast A$ is mixed monotone, it follows that the study of the fixed points of A can be reduced to that for a mixed monotone operator. Also, when B is mixed monotone, then any coupled lower-upper fixed point of A is a coupled lower-upper fixed point of $B \circledast A$ (Proposition 3.5.7), while the converse is true when B has a left inverse with respect to $*$, which is mixed monotone.

For example, by letting $B(x, y) = \alpha x$ ($\alpha > 0$) which satisfies all the above conditions, we see that any of the fixed point theorems (for mixed monotone operators) in this thesis remain valid if we require that the operator

$$B \circledast A = \alpha A(x, y) + (1 - \alpha)x$$

is mixed monotone, instead of asking that A is mixed monotone, while the conditions on A are replaced with identical conditions on $B \circledast A$. Clearly, $B \circledast A = A$, when $B := P$, hence the original case is already contained in this more general one. More examples can be found in Section 3.5.

§6. As a general application of the theory developed in Chapter 3, we study in *Section 4.3* the abstract system

$$\begin{gathered} x^i = T_i(x^1, x^2, \ldots, x^N), \quad i \in \{1, 2, \ldots, N\} \\ x = (x^1, x^2, \ldots, x^N) \in U \end{gathered}$$

assuming that $(X_i, \leq)$ is an ordered set, $U_i \subseteq X_i$, $U := U_1 \times U_2 \times \ldots \times U_N$ and $T_i : U \to X_i$ is monotone (increasing or decreasing) with respect to each variable independently, for any $i \in \{1, 2, \ldots, N\}$. Clearly, the above system is equivalent to the fixed point problem for the operator

$$T = (T_1, T_2, \ldots, T_N) : U \to X = X_1 \times X_2 \times \cdots \times X_N$$

The idea is to prove, in a constructive way, that T is a heterotonic operator, by constructing a mixed monotone operator $A : U^2 \to X$ such that $A(x, x) = T(x)$ for any $x \in U$, where X is ordered with respect to the relation given by $x = (x^1, x^2, \ldots, x^N) \leq (y^1, y^2, \ldots, y^N) = y$ *iff* $x^i \leq y^i$ for any $i \in \{1, 2, \ldots, N\}$. In this way, we can establish an equivalence between the above system and the fixed point problem for A on U, hence this problem can be approached by any of the available techniques and results for mixed monotone operators. Clearly, any fixed point result for T is to be expressed in terms of the operators T_i, rather than using the mixed monotone operator A, which should be considered only as an auxiliary tool. It is not our intention to develop here a full theory, but merely to give an idea of what kind of conditions and results are to be expected.

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The thesis is structured as follows.

In **Chapter 1**, we establish most of the notions and notations that will be used throughout and list, without proof, the most important results that will be assumed known. The main topics covered in this preliminary chapter are: orderings, (extended) semimetric spaces and ordered metric spaces, linear spaces and ordered linear (normed) spaces, order-units seminorms and the order-bound topology of an ordered linear space. By gathering all the basic notions, notations and results into one place, we provide a central point of reference and a unitary treatment for all the topics of the entire thesis.

The main references for this chapter are the papers of Amann [4], Andô [5], Blumenthal [14], Chen [20], Deimling [27], Guo et al. [31], Hyers et al. [36], Jameson [39] and [40], Jung [41], Krasnosel′skiĭ [46], Namioka [66], Ng [67] and [68], Nussbaum [71], Schaefer [93], Wong [106], [107] and [108].

In **Chapter 2**, we make a thorough analysis of the metric introduced by A. C. Thompson in 1963 (cf. [102]). The study of this important tool is mainly motivated by the important role it plays in the fixed point results in Chapter 3, as well as by the lack (with the exception of a paper of Nussbaum [71]) of a similar attempt.

The chapter is structured in five sections, each dealing with different aspects of Thompson's metric. While *Section 2.1* is an introduction to the subject, in *Section 2.2* we study the properties of Thompson's metric, by grouping them in several categories: monotonicity properties, convexity properties and topological properties. We also study the connections between Thompson's metric and order-unit seminorms, which will provide useful in *Section 2.3*, where we give a full characterization of the completeness of Thompson's metric in the general framework of ordered linear spaces. Since Thompson's metric is defined in the context of a generic ordered linear space, with no need of an underlying topological structure, one expects to express its completeness in terms of properties of the ordering, with respect to the linear structure. To the best of our knowledge, this has not been done so far.

In order to express the *completeness* of Thompson's metric in pure algebraic terms, we introduce and study the concept of *self-completeness* of a cone, and we prove, among others, that the two properties are equivalent. In the particular case of ordered Banach spaces, covered in *Section 2.4*, we show that the completeness of Thompson's metric and the normality of the cone are equivalent, a result similar to that of Nussbaum [71] for Hilbert's projective metric.

The chapter ends with a brief exposition of the generalizations of Thompson's metric and some directions for future research, in *Section 2.5*.

To the best of our knowledge, the majority of the properties in this chapter are entirely new, while the previously known results have been properly referred to their authors and papers. The most important personal results are Theorems 2.3.21, 2.4.5, 2.4.6, Propositions 2.2.4, 2.2.30, and Corollaries 2.2.7, 2.2.16, 2.3.23, 2.3.24, 2.4.7. We also introduce and study some new concepts like *self-bounded sequences* and *self-complete sets* in a cone.

The main reference papers on this subject are those of Chen [23], Nussbaum [71], Nussbaum and Walsh [72], Thompson [102]. Further generalizations were proposed by Bauer, Bear and Weiss (cf. [9], [10], [11], [12]) and by Turinici (cf. [103], [104]).

Chapter 3 is entirely dedicated to the fixed point theory for mixed monotone operators that we have previously described so far in this introduction.

The chapter is structured in five sections. *Section 3.1* contains the basic concepts and results used for developing the entire theory. *Section 3.2* constructs the method of monotone iterations for mixed monotone operators in a general context, that of ordered sets, while *Section 3.3* further develops the ideas from the previous section in the setting of ordered metric spaces. *Section 3.4* deals with the case of ordered linear spaces, where the results are derived using the fixed point theorems from the previous section in the particular case of Thompson's metric. In particular, the case of ordered Banach spaces is shortly described in *Section 3.5*, together with several other important aspects, including possible directions for future research and some extensions of the theory to a larger class of operators, in the framework of ordered linear spaces.

We a few exceptions (properly referred to their authors), most of the results in this chapter are new. There are a large number of important personal results, like Theorems 3.2.13, 3.2.14, 3.2.18, 3.2.19, 3.3.24, 3.3.26, 3.3.47, 3.4.19, 3.4.20, 3.4.26, 3.4.28, 3.4.29, 3.4.30, 3.4.31, 3.4.32, 3.4.33, 3.4.34, 3.5.1, Propositions 3.3.17, 3.3.49, 3.4.21, Lemma 3.2.1 and Corollaries 3.3.21, 3.3.38, 3.3.39, 3.3.52, 3.5.5. Some of these results (or similar ones) have already been published (see [89], [90], [91], [92]).

Chapter 4 contains several applications of the methods and results from Chapter 3 to nonlinear integral equations and second-order two-point boundary value problems, with concrete examples (*Sections 4.1 and 4.2*).

We also consider (in *Section 4.3*) an abstract application to the study of a system of equations involving partial monotone operators, i.e., multivariate operators that are monotone (increasing or decreasing) with respect to each variable independently.

The most important personal results in this chapter are Theorems 4.1.9, 4.1.10, 4.2.1, 4.2.2, 4.2.5, 4.2.6. Some of these results (or similar ones) have already been published (see [91], [92]).

1

Preliminary concepts and results

In this chapter, we establish some notions and notations which will be used throughout and list, without proof, the most important results that will be assumed known. The main purpose of this preliminary chapter is to provide a central point of reference for a unitary treatment of all the topics in the entire thesis. Most of the topics in this chapter are covered (to some extent) by Amann [4], Blumenthal [14], Deimling [27], Guo et al. [31], Edelstein [29], Jameson [40], Jung [41], Krasnosel′skiĭ [46], Luxemburg [61], Namioka [66], Nussbaum [71], Schaefer [93], Thompson [102], Wong [107].

1.1 Orderings

1.2 Semimetric spaces

Let (X, d) be a semimetric space. X is said to be *ε-chainable* (cf. Edelstein [29]), for some $\varepsilon > 0$, if for any $x, y \in X$, there exists an ε-chain, i.e., $x = x_0, x_1, \ldots, x_n = y$ (n may depend on x and y) such that $d(x_k, x_{k+1}) \leq \varepsilon$ for all $k \in \{0, 1, \ldots, n-1\}$. X is said to be *chainable* if it is ε-chainable for any $\varepsilon > 0$.

Assume next that d is a metric. X is said to be *metrically convex* (or *a Menger convex metric space*) if for any distinct points $x, y \in X$, there exists $z \in X \setminus \{x, y\}$ such that $d(x, y) = d(x, z) + d(z, y)$.

1.3 Extended semimetric spaces

The difference between an extended semimetric and a (regular) semimetric is that the extended semimetric may also take the value ∞. The topology induced by an extended semimetric is defined in the similar way as that of a semimetric, using as sub-base the family of all open balls (with finite radius). If (X, d) is an extended semimetric space, then the relation $\asymp$ defined by $x \asymp y$ *iff* $d(x, y) < \infty$ is an equivalence which splits X into components (called *(semi)metric components*). If $x \in X$, then $X(x)$ will denote the semimetric component of X that contains x. An extended semimetric space is, therefore, obtained by joining a family of mutually disjoint semimetric spaces, while considering the distance to be infinite between elements of different spaces. Also, any two component semimetric spaces are disconnected from one another in the topology of the extended semimetric. The completeness of an extended semimetric space is also defined as that of a semimetric space. By Jung [41, p. 114],

P 6 An extended semimetric space is complete *iff* all of its semimetric components are complete.

Any other concept defined for metric or semimetric spaces can be easily adapted for extended (semi)metric spaces. One important example of an extended semimetric is that of, so called, *Thompson's metric* (studied in Chapter 2).

1.4 Ordered metric spaces

Usually, *an ordered metric space* is considered to be, simply, a metric space (X, d) on which an ordering $\leq$ is defined (or, equivalently, an ordered set $(X, \leq)$ with a metric d), and is referred to by $(X, d, \leq)$ or by $(X, \leq, d)$. It is clear that this concept can be generalized, to include also the extended metric spaces. Throughout this paper, by *an ordered metric space* we will understand an extended metric space with an ordering. There is no general acceptance on what properties should link d and the ordering $\leq$. Most of these properties are concerned with the closedness of the ordering, like:

(C_1) $\leq$ is closed in X^2, i.e., if $(x_n), (y_n)$ are convergent sequences in X such that $x_n \leq y_n$ for any $n \in \mathbb{N}$, then $\lim\limits_{n\to\infty} x_n \leq \lim\limits_{n\to\infty} y_n$;

(C_2) $[x)$ and $(x]$ are closed, for any $x \in X$, i.e., if $x \in X$ and (x_n) is a convergent sequence such that $x_n \geq x$ for any $n \in \mathbb{N}$ (or $x_n \leq x$ for any $n \in \mathbb{N}$), then $\lim\limits_{n\to\infty} x_n \geq x$ (or $\lim\limits_{n\to\infty} x_n \leq x$, respectively);

(C_3) Any order interval in X is closed, i.e, if $x, y \in X$ such that $x \leq y$ and (x_n) is a convergent sequence from $[x, y]$, then $\lim\limits_{n\to\infty} x_n \in [x, y]$;

or weaker versions (denoted here by (C_1'), (C_2'), (C_3')) that require closedness only through monotone sequences.

We also consider the following additional properties which may relate the ordering and the metric:

(C_4') Any increasing (decreasing) and convergent sequence of X is majorised (minorised) by its limit.

(C_5') Any monotone and convergent sequence of X is order-bounded.

P 8 $(C_1') \Leftrightarrow (C_2') \Leftrightarrow \big((C_3') \text{ and } (C_5')\big) \Rightarrow (C_4') \Rightarrow (C_5')$.

1.5 Linear spaces

The linear spaces which appear in this paper are always over the scalar field $\mathbb{R}$ of the real numbers. The zero element will normally be denoted by θ and the standard notations are assumed.

If $x \in X$ such that $-x + U$ is absorbing, then x is called *an internal* (or *a core*) *point* of U and U is said to be *radial at* x. The set of all internal points of U is denoted by U° and is called *the lineal interior* (or *the core*) of U. The set $X \setminus (X \setminus U)^\circ$ is called *the lineal closure* of U and is denoted by U^c. If $U = U^\circ$, then U is said to be *lineally open*. If $U = U^c$, then U is said to be *lineally closed*.

A subset ℓ of X is called *a line* if there exists $x \in X$ such that $x + \ell$ is a one-dimensional linear subspace of X. U is said to be *lineally bounded* if $\ell \cap U$ is bounded (in ℓ) for any line ℓ in X. Also, U is said to be *lineless* if it does not include any line.

U is said to be *a star*, *a pointed star*, *balanced*, *positive-homogeneous*, *strictly positive-homogeneous*, or *homogeneous* if the inclusion $\lambda U \subseteq U$ holds for any λ in, respectively, $[0, 1]$, $(0, 1]$, $[-1, 1]$, $[0, \infty)$, $(0, \infty)$, $\mathbb{R}$.

1.6 Ordered linear spaces

Let X be a linear space. A nonempty subset K of X is called *a wedge* if it is convex and positive-homogeneous. Additionally, if $K \cap (-K) = \{\theta\}$, then K is called *a cone*. The semi-ordering of X associated with a wedge K is the relation defined by: $x \leq y$ *iff* $y - x \in K$. Additionally, if K is a cone, then $\leq$ is an ordering.

Assume next that (X, K) is an ordered linear space. The cone K is said to be:

1. *lineally solid* if $K^\circ \neq \emptyset$;
2. *generating* (or *reproducing*) if $X = K - K$;
3. *Archimedean* if $x \leq \theta$ whenever there exists $y \in X$ such that $nx \leq y$ for all $n \in \mathbb{N}$.
4. *almost Archimedean* if $x = \theta$ whenever there exists $y \in X$ such that $-y \leq nx \leq y$ for all $n \in \mathbb{N}$.

If K is lineally solid, then any element of K° is called *an order-unit*.

P 20 If K is lineally solid, then it is generating.

P 24 K is almost Archimedean *iff* any order-bounded set is lineally bounded *iff* K is lineless *iff* K^c is a cone.

P 25 K is Archimedean *iff* K is lineally closed.

If Y is a linear subspace of X and K is a cone (wedge) in X, then $Y \cap K$ is a cone (wedge) in Y such that its associated (semi)ordering on Y is the restriction of the (semi)ordering from X. Additionally, if Y is order-convex, then Y is called *an order-closed linear subspace*.

Two elements x, y in K are said to be *linked* (cf. Thompson [102]) if there exist $0 < \mu \leq \lambda$ such that $\mu x \leq y \leq \lambda x$, and we denote this relation by $x \sim y$. Clearly, this is an equivalence which splits K into disjoint components (called *parts*) and denote by $K(u)$ the equivalence class of $u \in K$.

P 26 $K(\theta) = \{\theta\}$.

P 27 If K is lineally solid, then K° is a part of K.

P 28 Any part of K is convex, order-convex, closed under addition and strictly positive-homogeneous.

1.7 Ordered normed spaces

Assume next that $(X, K, \|\cdot\|)$ is an ordered normed space. With respect to the norm topology, $\mathring{U}$ and $\overline{U}$ denote, respectively, the interior and the closure of the subset U of X.

The cone K is said to be:

1. *solid* if $\mathring{K} \neq \emptyset$;
2. *total* if $X = \overline{K - K}$;
3. *regular* if any increasing sequence which is majorised is convergent;
4. *fully regular* if any increasing sequence which is norm-bounded is convergent;

5. *normal* if the norm is semi-monotone, i.e., there exists $\gamma > 0$ such that $\|x\| \leq \gamma \|y\|$ for any $x, y \in K$, $x \leq y$.

P 32 If K is solid, then it is lineally solid and $\overset{\circ}{K} = K^{\circ}$.
P 33 If K is generating, then it is total.
P 34 If K is fully regular, then it is regular.
P 35 If K is regular, then it is normal.

The concept of *normal cone* can be introduced in the more general context of ordered topological linear spaces and is one of the most importance. In connection with the convergence of increasing (or decreasing) Cauchy sequences, two important concepts are introduced and studied by Wong [107]. X is said to be *fundamentally σ-order complete* (respectively, *monotonically sequential complete*) if any increasing Cauchy sequence in X has supremum (respectively, has limit). For more details on these topics we refer to Wong [107].

1.8 The order-bound topology of an ordered linear space

The order-bound topology τ_b of an ordered linear space (X, K) is the locally convex absorbing topology generated by the family of all intervals. In other words, a convex set is a τ_b-neighborhood of θ *iff* it absorbs each interval.

P 44 The order-bound topology is the largest locally convex topology making all intervals topologically bounded.
P 45 If K is lineally solid, then the order-bound topology for X is the seminormable topology induced by any of its *order-unit seminorms* $|\cdot|_u$ ($u \in K^{\circ}$) (see Section 1.9) and $\overset{\circ}{K} = K^{\circ}$.
P 46 If K is generating and τ_0 is a complete, metrizable topology for X such that K is closed, then $\tau_b \subseteq \tau_0$.

Assume next that $(X, K, \|\cdot\|)$ is an ordered normed space. The connection between the norm topology (denoted here by τ) and the order-bound topology τ_b is established in the following direct consequences of (*P 44*)–(*P 46*).

P 47 K is normal *iff* $\tau \subseteq \tau_b$.
P 48 If $(X, \|\cdot\|)$ is complete and K is generating, then $\tau_b \subseteq \tau$.
P 49 If $(X, \|\cdot\|)$ is complete and K is a normal, solid cone, then $\tau = \tau_b$ and the norm on X is equivalent to any of the order-unit seminorms. Conversely, if $\tau = \tau_b$ and K is lineally solid, then K is normal and solid.

1.9 Order-unit seminorms

Let (X, K) be an ordered linear space. Fix $u \in K \setminus \{\theta\}$ and let $X_u = \{x \in X : [-u, u] \text{ absorbs } x\}$, $K_u = X_u \cap K$.

P 50 X_u is an order-closed linear subspace of (X, K) containing $[-u, u]$ and having u as order-unit.
P 52 If $v \in K \setminus \{\theta\}$, then $X_v = X_u$ *iff* $v \sim u$.
P 53 $X_u = X$ *iff* $u \in K^{\circ}$.
P 54 $K_u = \bigcup\{[\theta, \lambda u] : \lambda \geq 0\}$
P 55 ${K_u}^{\circ} = K(u)$.

The functional $|\cdot|_u$ defined on X_u by $|x|_u = \inf\{\lambda \geq 0 : -\lambda u \leq x \leq \lambda u\}$ is the Minkowski functional associated to the convex, balanced, absorbing set $[-u, u]$, consequently, is a seminorm on X_u, called *the order-unit seminorm with respect to u*, or, simply, *the u-seminorm*.

P 56 If $v \sim u$, then $|\cdot|_u$ and $|\cdot|_v$ are equivalent.
P 58 $|\cdot|_u$ is monotone, i.e., $\theta \leq x \leq y$ implies $|x|_u \leq |y|_u$.
P 59 K_u is generating and normal.
P 60 $|\cdot|_u$ is a norm on X_u *iff* K_u is almost Archimedean.
P 61 K_u is closed in X_u *iff* K_u is lineally closed *iff* K_u is Archimedean.

Assume next that K_u is Archimedean. Some direct consequences are:

P 62 $(X_u, K_u, |\cdot|_u)$ is an ordered normed space.
P 64 $[-u, u]$ is the unit closed ball in X_u.
P 65 K_u is $|\cdot|_u$-solid, hence $\overset{\circ}{K_u} = {K_u}^{\circ} = K(u)$.
P 66 The u-norm topology on X_u is the order-bound topology.
P 67 If K is Archimedean, then each part Q of $K \setminus \{\theta\}$ has an associated order-closed linear subspace of (X, K) which is an ordered normed space with respect to any of the (equivalent) u-norms ($u \in Q$) and the cone is solid (hence generating) and normal. Moreover, Q is the (topological and lineal) interior of the positive cone in the corresponding normed space.
P 68 If K is Archimedean and lineally solid, then (X, K) becomes an ordered normed space with respect to any of the u-norms, with $u \in K^{\circ}$. Additionally, the topology on X is the order-bound topology, K is normal (any u-norm is monotone) and solid ($\overset{\circ}{K} = K^{\circ}$).

2

Thompson's metric

The purpose of this chapter is to make a detailed study of the metric introduced by A. C. Thompson in 1963 (cf. [102]), the so–called *Thompson's (part) metric*. The study of this important tool is motivated mainly by the role it plays in the fixed point results from Chapter 3. With the exception of Nussbaum [71], no other attempt of this kind has been made.

While Thompson defined in [102] the part metric for any closed cone of an ordered real Banach space, it is clear that one may define this metric in any ordered linear space (over the reals) with no need for an underlying topology, provided that the cone is almost Archimedean. Later on, Bauer, Bear and Weiss (cf. [9], [10], [11], [12]) extended Thompson's idea by associating a metric to any convex set that includes no line, in an arbitrary linear space (over the reals), while Thompson's metric became a particular case of the so–called Bauer-Bear-Weiss metric.

To the best of our knowledge, most of the properties in this chapter are new, while the previously known results have been properly referred to their authors and papers. There are, also, some definitions which introduce new notions. Note that since Thompson's metric is closely related to Hilbert's projective metric (see Birkhoff [13], Nussbaum [71]), it is also possible that some of the properties of Thompson's metric may be deduced by analogue results for Hilbert's projective metric. In this direction, we refer to Nussbaum [71].

The main references for this chapter are the papers of Amann [4], Andô [5], Bauer and Bear [9], Bear [10] and [11], Bear and Weiss [12], Birkhoff [13], Chen [23], Deimling [27], Jameson [40], Krasnosel′skiĭ [46], Krause and Nussbaum [51], Namioka [66]), Nussbaum [71], Nussbaum and Walsh [72], Ng [68], Opoĭtsev [75], Peressini [76], Schaefer [93], Stecenko [98], Thompson [102], Turinici [103] and [104], Wong [107], Zabreĭko et al. [115].

In what follows, X will stand for an ordered linear space with the cone K, if not stated otherwise.

2.1 Basic concepts and results

Thompson's metric (cf. [102]) (denoted here by ρ) is defined between any two linked elements $x, y \in K$ by

$$\rho(x, y) = \inf \{s \geq 0 : \mathrm{e}^{-s} x \leq y \leq \mathrm{e}^{s} x\}. \tag{2.1.1}$$

Proposition 2.1.2 *ρ is a semimetric on each part of K. Moreover, ρ is a metric on each part of K* iff *K is almost Archimedean.*

Remark 2.1.3 It is convenient to define ρ for any pair of elements in K, by setting $\rho(x, y) = \infty$ for any x, y not lying in the same part of K (cf. Krause [51]), and by the usual convention that $\inf \emptyset = \infty$, (2.1.1) is still valid. In this way, ρ becomes an *extended (semi)metric* on K and $x \sim y$ *iff* $\rho(x, y) < \infty$. Though ρ is not a usual (semi)metric on the whole cone, we will continue to call ρ a metric. Thompson's metric is also called, by some authors, *the part metric (of the cone K)*.

Remark 2.1.5 The definition of $\rho(x, y)$ depends only on the ordering of the linear subspace spanned by $\{x, y\}$. This ensures that if x and y are seen as elements of some linear subspace Y of X, then $\rho(x, y)$ is the same in X and in Y (assuming, of course, that Y inherits the ordering from X).

Example 2.1.6 If $X = \mathbb{R}$ and $K = \mathbb{R}_+$, then the parts of K are $\{0\}$ and $(0, \infty)$, while $\rho(x, y) = |\ln x - \ln y|$.

2.2 Properties of Thompson's metric

In this section we study the basic properties of Thompson's metric in the general case of ordered linear spaces. Note that most of the following results are true without the assumption of an Archimedean-type property for K.

2.2.1 Monotonicity properties

Proposition 2.2.4 *Let $x, x', y, y' \in K$ such that $x \leq x'$ and $y \geq y'$. Then*

$$\rho(x', x' + y') \leq \rho(x, x + y).$$

Corollary 2.2.6 *Let $x, x', y, y' \in K$ such that $x \leq y \leq x' \leq y'$ and $y' - x' \leq y - x$. Then*

$$\rho(x', y') \leq \rho(x, y).$$

Corollary 2.2.7 *Let $x, x', y, y' \in K$ such that $x \leq x' \leq y' \leq y$. Then*

$$\rho(x', y') \leq \rho(x, y).$$

The following two results are due to Bauer and Bear [9].

Corollary 2.2.9 (Bauer and Bear [9]) *Let $x, x', y, y' \in K$ and $\lambda, \mu > 0$. Then*

$$\rho(\lambda x + \mu y, \lambda x' + \mu y') \leq \max \left\{\rho(x, x'), \rho(y, y')\right\}.$$

Corollary 2.2.10 (Bauer and Bear [9]) *Let $x, y, z \in K$. Then*

$$\rho(x + z, y + z) \leq \rho(x, y).$$

2.2.2 Convexity properties

Proposition 2.2.11 *ρ is quasiconvex with respect to each of its arguments.*

Proposition 2.2.13 *Let $u \in K$, $x, y \in K(u)$, and $t \in [0, 1]$. Then*

$$\rho((1-t)x + ty, u) \leq \ln\left((1-t)\mathrm{e}^{\rho(x,u)} + t\mathrm{e}^{\rho(y,u)}\right). \tag{2.2.1}$$

Remark 2.2.15 In other terms, Proposition 2.2.13 states that e^{ρ} is convex with respect to each of its arguments.

Corollary 2.2.16 *Let $x, y \in K$ such that $x \sim y$, and $s, t \in [0, 1]$. Then*

$$\rho((1-t)x + ty, (1-s)x + sy) \leq \ln\left(|t-s|\,\mathrm{e}^{\rho(x,y)} + 1 - |t-s|\right). \tag{2.2.2}$$

Using Corollary 2.2.16, we rediscover a result of Chen.

Proposition 2.2.17 (Chen [23]) *Any part of K is chainable with respect to ρ.*

The next result is a slight extension of a result of Nussbaum [71, Proposition 1.12].

Proposition 2.2.18 (Nussbaum [71]) *Any part of K is metrically convex with respect to ρ.*

For more details on the geometry of Thompson's metric, we refer to Nussbaum [71], Nussbaum and Walsh [72].

2.2.3 Topological properties

The following three result have been proved by Bauer and Bear [9] in the context of Bauer-Bear-Weiss metric (see Section 2.5). Our proofs are slightly different, since they are formulated in terms of the original definition of Thompson.

Proposition 2.2.21 (Bauer and Bear [9]) *The operators $(\lambda, x) \mapsto \lambda x$ from $(0, \infty) \times K$ into K and $(x, y) \mapsto x + y$ from K^2 into K are continuous with respect to ρ.*

Proposition 2.2.22 (Bauer and Bear [9]) *Let Q be a part of K. Then the operator $(\lambda, x, y) \mapsto \lambda x + (1-\lambda)y$ from $[0, 1] \times Q^2$ into Q is continuous with respect to ρ.*

Corollary 2.2.23 (Bauer and Bear [9]) *The parts of K are its connected components.*

Proposition 2.2.24 *If K is Archimedean and $x \in K$, then $[x)$ and $[\theta, x]$ are ρ-closed.*

Remark 2.2.25 Proposition 2.2.24 states that the ordered metric space $(K, \rho, \leq)$ satisfies property (C_2) (see Section 1.4).

Proposition 2.2.26 *Let $x, y \in K$ such that $x \leq y$. Then the interval $[x, y]$ is*

1. *ρ-bounded* iff *$x \sim y$;*
2. *ρ-closed if K is Archimedean.*

2.2.4 The connection between Thompson's metric and order-unit seminorms

Proposition 2.2.30 *Let $u \in K \setminus \{\theta\}$ and $x, y \in K(u)$. Then*

1. $\rho(x, y) = \ln \max \{|x|_y, |y|_x\}$;
2. $\rho(x, y) \geq |\ln |x|_u - \ln |y|_u|$;
3. $|x|_u \leq e^{\rho(x,y)} |y|_u$ *(and* $|y|_u \leq e^{\rho(x,y)} |x|_u$*)*;
4. $e^{-\rho(x,u)} \leq |x|_u \leq e^{\rho(x,u)}$ *(and* $e^{-\rho(x,u)} \leq |u|_x \leq e^{\rho(x,u)}$*)*;
5. $|u|_x \leq e^{\rho(x,y)} |u|_y$ *(and* $|u|_y \leq e^{\rho(x,y)} |u|_x$*)*;
6. $\rho(x, y) \leq \ln \left(1 + |x - y|_u \cdot \max \{|u|_x, |u|_y\}\right)$;
7. $\left(e^{\rho(x,y)} - 1\right) \cdot \min \{|u|_x^{-1}, |u|_y^{-1}\} \leq |x - y|_u \leq \left(2e^{\rho(x,y)} - e^{-\rho(x,y)} - 1\right) \cdot \min\{|x|_u, |y|_u\}$;
8. $\left(1 - e^{-\rho(x,y)}\right) \cdot \max \left\{|u|_x^{-1}, |u|_y^{-1}\right\} \leq |x - y|_u$;
9. $|x - y|_x \geq 1 - e^{-\rho(x,y)}$ *(and* $|x - y|_y \geq 1 - e^{-\rho(x,y)}$*)*.

The following result is a direct consequence of Proposition 2.2.30. A less general version of this result is mentioned by Opoĭtsev [75] and Stecenko [98] without any proof or reference. A similar result is due to Bauer and Bear [9, Theorem 4] for the more general case of Bauer-Bear-Weiss metric (see Section 2.5).

Theorem 2.2.31 *Thompson's metric and the u-(semi)norm are topologically equivalent on $K(u)$.*

Remark 2.2.32 Though topologically equivalent on $K(u)$, Thompson's metric and the u-seminorm are not, in general, metrically equivalent. Let $U = [\theta, u] \cap K(u)$. Then $U \subseteq K(u)$ and $|x - u|_u \leq 1$ for any $x \in U$, but U is not ρ-bounded because $x_n := e^{-n} u \in U$ and $\rho(x_n, u) = n$ for any $n \in \mathbb{N}$.

2.3 The completeness of Thompson's metric

Since ρ is defined in the context of a generic ordered linear space, with no need of an underlying topological structure, one expects to express its completeness in terms of properties of the ordering, with respect to the linear structure. To the best of our knowledge, this has not been done yet.

As seen in the previous section, Thompson's metric and order-unit (semi)norms are strongly related and share important properties, as both are defined in terms of the ordered linear structure. While ρ and $|\cdot|_u$ are topological (but *not* metrical) equivalent on K_u, we will prove that the completeness is a common feature.

In order to state the main results, we need to define some new notions and establish some properties.

2.3.1 Self-bounded sequences and self-complete sets in a cone

Definition 2.3.1 A sequence (x_n) in K is said to be:

1. *self order-bounded from above* (or *upper self-bounded*) if for any $\lambda > 1$ exists $k \in \mathbb{N}$ such that $x_n \leq \lambda x_k$ for any $n \geq k$.
2. *self order-bounded from below* (or *lower self-bounded*) if for any $\mu \in (0, 1)$ exists $k \in \mathbb{N}$ such that $\mu x_k \leq x_n$ for any $n \geq k$.
3. *self order-bounded* (or, simply, *self-bounded*) if it is self order-bounded from above and self order-bounded from below.

Remark 2.3.2 The self-boundedness properties of a sequence depend only on the ordering of the linear subspace spanned by its values. In other words, if (x_n) is a sequence of a linear subspace Y of X, then (x_n) is upper (or lower) self-bounded in X *iff* (x_n) is upper (or lower) self-bounded in Y (where Y inherits the ordering from X).

Remark 2.3.3 If (x_n) is an increasing sequence in K, then it is upper self-bounded *iff* for any $\lambda > 1$ exists $k \in \mathbb{N}$ such that λx_k is an upper bound for (x_n). Also, if (x_n) is decreasing, then it is lower self-bounded *iff* for any $\mu \in (0, 1)$ exists $k \in \mathbb{N}$ such that μx_k is a lower bound for (x_n).

Proposition 2.3.4

1. *Any ρ-Cauchy sequence in K is self-bounded.*
2. *Any increasing sequence in K is lower self-bounded.*
3. *Any decreasing sequence in K is upper self-bounded.*
4. *An increasing sequence in K is upper self-bounded* iff *it is ρ-Cauchy.*
5. *A decreasing sequence in K is lower self-bounded* iff *it is ρ-Cauchy.*

Definition 2.3.7 A nonempty subset U of K is said to be:

1. *self order-complete from above* (or *upper self-complete*) if any increasing, upper self-bounded sequence in U has supremum and $\sup x_n \in U$.
2. *self order-complete from below* (or *lower self-complete*) if any decreasing, lower self-bounded sequence in U has infimum and $\inf x_n \in U$.
3. *self order-complete* (or, simply, *self-complete*) if it is both self order-complete from above and self order-complete from below.

If we do not require the supremum (respectively, the infimum) to be in U, we say that U is *quasi* (*upper/lower*) *self-complete*.

Before further investigating this new concepts, we show that there is a duality between increasing upper self-bounded sequences and decreasing lower self-bounded sequences, together with some important consequences.

Proposition 2.3.8 *Let (x_n) be an increasing, upper self-bounded sequence in K and (t_k) a decreasing sequence of real numbers, convergent to* 1. *Then there exists a subsequence $\left(x_{n_k}\right)$ of (x_n) such that the following conditions are satisfied:*

1. *The sequence (y_k) given by $y_k = t_k x_{n_k}$ ($k \in \mathbb{N}$) is decreasing and lower self-bounded.*
2. *$x_n \leq y_k$ for any $n, k \in \mathbb{N}$.*
3. *If K is Archimedean, x is an upper bound of (x_n) and y is a lower bound for (y_k), then $y \leq x$.*
4. *If K is Archimedean and (x_n) lies in a linear subspace Y of X, then the following statements are equivalent:*

   (a) (x_n) has supremum.
   (b) (x_n) has supremum in Y.
   (c) (y_k) has infimum.
   (d) (y_k) has infimum in Y.
   (e) there exists $x \in K$ such that $x_n \leq x \leq y_k$ for any $n, k \in \mathbb{N}$.

   In the affirmative case, $\sup x_n = \sup\limits_Y x_n = \inf y_k = \inf\limits_Y y_k = x$ *and* $y_k \xrightarrow{\rho} x$, $x_n \xrightarrow{\rho} x$.

A similar result can be stated for decreasing, lower self-bounded sequences.

The duality between increasing upper self-bounded sequences and decreasing lower self-bounded sequences has the following important consequence:

Theorem 2.3.10 *Assume that K is Archimedean and let U be an order-convex, strictly positive-homogeneous, nonempty subset of K. Then all of the six completeness properties given in Definition 2.3.7 are equivalent on U.*

Remark 2.3.11 All the results proven so far can be restated into *local* versions, by replacing X with X_u, hence K with K_u (where $u \in K \setminus \{\theta\}$). In this way, we can weaken the Archimedean condition by requiring only that K_u is Archimedean. In this case, the conditions “has supremum”, respectively “has infimum” must be understood with respect to X_u. Consequently, a subset U of K_u can be self-complete in X_u, but may not be self-complete in X (yet, this cannot happen when K is Archimedean - see the next corollary). Note that the definition of Thompson's metric is not affected by this change (see Remark 2.1.5).

The following result is a direct consequence of Proposition 2.3.8 and Theorem 2.3.10.

Corollary 2.3.12 *Let Y be an order-closed linear subspace of X, and let U be an order-convex, strictly positive-homogeneous, nonempty subset of $Y \cap K$. Assume that K is Archimedean. Then U is self-complete in X* iff *U is self-complete in Y.*

We show next that the self-completeness of a lineally, solid, Archimedean cone and that of its lineal interior are equivalent.

Proposition 2.3.13 *Assume that K is Archimedean.*

1. *K is self-complete* iff *any part of K is self-complete.*
2. *If K is lineally solid and K° is self-complete, then K is self-complete.*

2.3.2 Some properties of monotone sequences with respect to the order-unit norms

Fix $u \in K \setminus \{\theta\}$ and assume that K_u is Archimedean.

Proposition 2.3.20 *Let (x_n) be a $|\cdot|_u$-Cauchy sequence in K_u.*

1. *If there exist $\delta > 0$ and a subsequence (x_{n_k}) of (x_n) such that $x_{n_k} \geq \delta u$ for any $k \in \mathbb{N}$, then (x_n) is self-bounded.*
2. *If (x_n) is increasing and there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} \in K(u)$, then (x_n) is self-bounded.*

2.3.3 The main results

The following important result shows that the completeness of Thompson's metric on $K(u)$ and that of the u-norm on X_u are equivalent when K_u is Archimedean (Remark 2.2.32 makes this result non-trivial) and also reduces the completeness to the convergence of the monotone Cauchy sequences. We also show that the completeness of ρ on $K(u)$ is equivalent to several order-completeness conditions in X_u. Note that some parts of the proof are, essentially, an extension of the arguments advanced by Andô in [5, Lemma 2] and by Zabreĭko et al. in [115, Lemma 1].

Theorem 2.3.21 *Let $u \in K \setminus \{\theta\}$ such that K_u is Archimedean (in X_u). Then the following conditions are equivalent:*

1. *$K(u)$ is ρ-complete.*
2. *$K(u)$ is self-complete in X_u.*
3. *K_u is self-complete in X_u.*
4. *X_u is fundamentally σ-order complete.*
5. *X_u is monotonically sequential complete.*
6. *X_u is $|\cdot|_u$-complete.*

*Additionally, if K is Archimedean, then condition **2** and **3** can be replaced by the stronger versions:*

2a. *$K(u)$ is self-complete (in X).*
3b. *K_u is self-complete (in X).*

Remark 2.3.22 The equivalence between **4**, **5** and **6** can be obtained also from some more general results of Jameson [40, pp. 111–119] and Wong [107]. Other completeness conditions for the order-unit norms, using the notion of *order-summable sequence* and *l^1-summable sequence* can be found in Ng [68], Schaefer [93, p. 231] and Wong [107]. There are also other order-related types of completeness which can be associated to an ordered topological vector space (see, for example, Andô [5], Jameson [40], Peressini [76], Wong [107]).

The following corollaries follow from the combining of Proposition 2.3.13 and Theorem 2.3.21.

Corollary 2.3.23 *If K is Archimedean, then ρ is complete* iff *K is self-complete.*

Corollary 2.3.24 *If K is Archimedean and lineally solid, then the following conditions are equivalent:*

1. *ρ is complete.*
2. *K is self-complete.*
3. *K° is self-complete.*
4. *K° is ρ-complete.*
5. *The ordered-bound normed topology on X is complete.*

2.4 Thompson's metric in ordered Banach spaces

When X is an ordered Banach space, the completeness of X_u with respect to the u-norm is in a strong connection with the normality of K. This topic is partially covered by Krasnosel′skiĭ [46] and will be extended with new results in this section.

It is assumed next that $(X, K, \|\cdot\|)$ is an ordered Banach space. A first result concerning the completeness of ρ in this framework is the following result of Thompson [102].

Theorem 2.4.2 (Thompson [102]) *If K is normal, then any part Q of K is a complete metric space with respect to ρ. Furthermore, if a sequence (x_n) in Q is ρ-convergent to $x \in Q$, then it is norm-convergent to x.*

The last results in this chapter are extensions of Theorem 2.3.21 (and its corollaries) for the case of ordered Banach spaces, and contain several partial results by Amann [4, Theorem 2.3], Chen [22], Deimling [27, Proposition 19.9], Jameson [40, pp. 111–119], Krasnosel′skiĭ [46, Theorem 1.3], Nussbaum [71, Remark 1.3] and Thompson [102, Lemma 3]. Note that there is a similar result concerning the completeness of Hilbert's projective metric (see Birkhoff [13]), due to Zabreiko et al. [115].

Theorem 2.4.5 *Let $u \in K \setminus \{\theta\}$. Then the following conditions are equivalent:*

1. *ρ is complete on $K(u)$.*
2. *$|\cdot|_u$ is complete on X_u.*
3. *The embedding of X_u in X is continuous.*
4. *$[\theta, x]$ is norm-bounded for any $x \in K(u)$.*
5. *$[\theta, u]$ is norm-bounded.*
6. *Any sequence (x_n) in $K(u)$ which is ρ-convergent to $x \in K(u)$, is also norm-convergent to x.*

Theorem 2.4.6 *The following conditions are equivalent:*

1. *ρ is complete.*

2. K *is self–complete.*
3. K *is normal.*
4. *The norm-topology on* K *is weaker than the topology of* ρ.

Corollary 2.4.7 *If* K *is solid, then the following conditions are equivalent:*

1. ρ *is complete.*
2. ρ *is complete on* $\overset{\circ}{K}$.
3. $[\theta, u]$ *is norm-bounded for some (or, for all)* $u \in \overset{\circ}{K}$.
4. *Any sequence* (x_n) *in* $\overset{\circ}{K}$*, which is* ρ*-convergent to* $x \in \overset{\circ}{K}$*, is also norm-convergent to* x.
5. K *is normal.*
6. *The ordered-bound (normed) topology on* X *is complete.*
7. *The ordered-bound topology on* X *is finer than the norm-topology.*

2.5 Extensions of Thompson's metric

2.5.1 The metric generated by a semigroup of increasing operators

An extension of Thompson's metric to a metric generated by a semigroup of increasing operators was given by Turinici in [103] and [104].

Let $S : [0, \infty) \times K \to K$ and let $S(t)$ denote the operator $x \in K \mapsto S(t, x) \in K$ $(t \geq 0)$. Assume that S verifies the following properties:

(S1) $S(0)x = x$ for any $x \in K$.
(S2) $S(t + s) = S(t)S(s)$ for any $t, s \geq 0$.
(S3) The operator $S(t) : K \to K$ is increasing, for any $t \geq 0$.
(S4) $S(t)x \geq x$ for any $t \geq 0$ and $x \in K$.

Then S is called *a semigroup of increasing operators on* K. Any such semigroup induces a partition of the cone with respect to the equivalence relation:

$$x \overset{S}{\sim} y \text{ iff there exists } t \geq 0 \text{ such that } x \leq S(t)y \text{ and } y \leq S(t)x$$

and on any component Q of K with respect to $\overset{S}{\sim}$, one can define the semimetric

$$\rho_S(x, y) = \inf \{t \geq 0 : x \leq S(t)y \text{ and } y \leq S(t)x\} \quad (x, y \in Q).$$

By letting $\rho_S(x, y) = \infty$ for any non-equivalent $x, y \in K$, one obtains an extended semimetric on K.

With the additional assumption that S is Archimedean, i.e.,

(S5) If $x, y \in K$ such that $x \leq S(t)y$ for any $t > 0$, then $x \leq y$

ρ_S becomes a metric on each component of K, hence an extended metric on K.

Thompson's metric is obtained for the particular semigroup $S(t, x) = \mathrm{e}^t x$.

An interesting subject for a future research would be to study in details the properties of ρ_S in a similar manner to that of Thompson's metric and identify which properties of ρ are preserved for ρ_S with no additional assumptions on S. Another idea of research would be to find additional properties for ρ_S, when $S : \mathbb{R} \times K \to K$ is a group of increasing operators (this is the case of the (semi)group which generates ρ).

2.5.2 The Bauer-Bear-Weiss part metric

The *Bauer-Bear-Weiss part metric* (see [9], [10], [11], [12]) is a natural extension of Thompson's metric. It is possible to redefine ρ only in terms of the linear structure, without an ordering, while the cone can be replaced by any lineless convex set (or, more generally, by any convex set, if we are satisfied with a semimetric, instead of a metric)

Let X be a linear space and K a nonempty convex subset of X. Two elements $x, y \in K$ are considered equivalent (denoted here by $x \backsim y$) if $x = y$ or if x, y are contained in some open linear segment within K. This defines an equivalence relation on K which splits it into components. When K is a cone, then $x \backsim y$ *iff* x and y are linked, hence the components of a cone with respect to $\backsim$ are its parts.

On any component Q of K one can define the following (semi)metric which is identical to Thompson's metric when K is a cone (hence, we will use the same notation):

$$\rho(x, y) = \inf \left\{\log\left(1 + t^{-1}\right) : t > 0, x + t(x - y) \in K, y + t(y - x) \in K\right\}.$$

If K is lineless (when K is a cone, this is equivalent to K being almost Archimedean), then ρ is a metric on each component and can be generalized to an extended metric on K.

An important subject for future research is the study of the properties of this metric, by following a similar pattern to that from Thompson's metric.

3

Fixed point theory for mixed monotone operators

In this chapter, we develop a detailed and unitary study of the fixed points of mixed monotone operators, based on the Method of Monotone Iterations (*MMI* for short), from the most general setting (that of ordered sets) to the most commonly used framework (that of ordered Banach spaces). The focus is put on the fixed point results that are based on the *MMI* and which provide both the existence and uniqueness (in some predefined set) of the fixed point, in a constructive manner. We are less concerned with the study of fixed points that are not the limit of an iterative scheme, where other methods behave better. We also exclude any nonconstructive results based on the axiom of choice, that employ Zorn's Lemma to prove the existence of fixed points.

It is *not* our intention to make a survey (even a partial one) of the fixed point results for mixed monotone operators, nor to list a large collection of already available results, but merely to follow some particular directions of research in this field and establish their fundamental concepts and results, both within the available published papers and by providing new ideas and new methods of investigation.

Since the fixed point theory for mixed monotone operators has the advantage that it contains both the theory for increasing and for decreasing operators in one unitary approach, we can obtain the classical results of the two particular cases, while also obtaining new results. Clearly, it is an elementary task to rewrite the fixed point results from the more general case of mixed monotone operators to the mentioned particular cases, hence we omit any details in this direction.

3.1 Basic concepts and results

Throughout this section, U, V, W will be nonempty sets and $A : U^2 \to V$, $B : V^2 \to W$ arbitrary bivariate operators, if not stated otherwise.

3.1.1 The mirror composition of bivariate operators

While the usual composition of A and B makes no sense, it is possible to define a similar associative composition law (called *the mirror composition* or, simply, *the m-composition* and denoted by $*$).

Definition 3.1.1 The m-composition of A and B is defined by

$$B * A : U^2 \to W, \quad (B * A)(x, y) = B(A(x, y), A(y, x)) \quad (x, y \in U). \tag{3.1.1}$$

Proposition 3.1.2 *The m-composition is associative.*

We can also find an analogue to the identity operator, acting as unit element for the m-composition. For any nonempty set X, denote by P_X the projection operator

$$P_X : X^2 \to X, \quad P(x, y) = x \quad (x, y \in X) \tag{3.1.2}$$

Note that we will usually write P instead of P_X, when X is clear from the context.

Proposition 3.1.3 $A * P_U = P_V * A = A$.

Remark 3.1.4 When A is a bivariate self-map of U, i.e., $A : U^2 \to V \subseteq U$, a direct consequence of Proposition 3.1.2 is that one can define the functional powers of A by $A^{n+1} = A * A^n = A^n * A$ ($n \in \mathbb{N}$), with $A^0 = P_U$.

Using the standard algebraic terminology, A is called *m-left invertible* when there exists $A_l^{-1} : V^2 \to U$ (called *a m-left inverse*) such that $A_l^{-1} * A = P_U$. Similarly, A is called *m-right invertible* if $A * A_r^{-1} = P_V$ for some

$A_r^{-1} : V^2 \to U$ called *a m-right inverse*. If A is both left and right invertible (with respect to $*$), then A is called *m-invertible*. It is a standard result that, in this case, both the left and the right inverses are unique and equal, and their common value will be denoted by A^{-1} (called *the m-inverse of* A).

Proposition 3.1.5 *A is m-invertible* iff *for any $(u, v) \in V^2$ the system*

$$\begin{cases} A(x, y) = u \\ A(y, x) = v \end{cases} \tag{3.1.3}$$

has a unique solution $(x, y) \in U^2$. In the affirmative case, the inverse $A^{-1} : V^2 \to U$ satisfies

$$\begin{cases} A^{-1}(u, v) = x \\ A^{-1}(v, u) = y \end{cases},$$

where (x, y) is the solution of (3.1.3).

3.1.2 Fixed points and coupled fixed points for bivariate operators

Assume next that $U \cap V \neq \emptyset$. Note that the notions and results that follow are, usually, considered in one of the cases: $U \subseteq V$, $V \subseteq U$ or $U = V$.

Definition 3.1.7 An element $x \in U \cap V$ is called *a fixed point of A* if $A(x, x) = x$.

In order to study the fixed points of a bivariate operator, we first investigate a more general concept, called *coupled fixed point*.

Definition 3.1.9 (Guo and Lakshmikantham [34]) A pair $(x, y) \in (U \cap V)^2$ is called *a coupled fixed point of A* if

$$\begin{cases} x = A(x, y) \\ y = A(y, x) \end{cases}.$$

Remark 3.1.10 Obviously, (x, y) is a coupled fixed point (of A) *iff* (y, x) is a coupled fixed point. Also, x is a fixed point *iff* (x, x) is a coupled fixed point.

For convenience, we denote the set of coupled fixed points of A by $\mathrm{cfp}(A)$.

3.1.3 Extremal coupled fixed points of bivariate operators

From this point forward in this section U, V, W are ordered sets ($\leq$ denotes the ordering on all sets).

Since we are interested mostly in the fixed points, rather than the coupled fixed points for a given bivariate operator A, it is often desirable that A has additional properties that make (almost) any coupled fixed point to be obtained from a fixed point. In this sense, will consider the following property:

(μ) If $(x, y) \in \mathrm{cfp}(A)$ such that x and y are comparable, then $x = y$.

If U_1 is a nonempty subset of U and the restriction of A to U_1^2 satisfies (μ), then we will say that A has property (or, satisfies) (μ) on U_1.

Definition 3.1.13 If $U_1 \subseteq U$ and $(x, y) \in \mathrm{cfp}(A)$, then (x, y) is said to be *extremal in U_1^2* if

$$x, y \in U_1, \quad x \leq y, \quad \mathrm{cfp}(A) \cap U_1^2 \subseteq [x, y]^2.$$

The combination of property (μ) and the existence of an extremal coupled fixed point gives our first elementary result concerning the existence and uniqueness of fixed points for bivariate operators.

Proposition 3.1.14 *If A has property (μ) on $U_1 \subseteq U$ and (x, y) is an extremal coupled fixed point of A in U_1^2, then $x = y$, (x, x) is the unique coupled fixed point of A in U_1^2 and x is the unique fixed point of A in U_1.*

3.1.4 Mixed monotone operators

Definition 3.1.15 (Guo and Lakshmikantham [34]) A is said to be *mixed monotone*, or is said to have *the mixed monotone property*, if $A(x_1, y_1) \leq A(x_2, y_2)$ for any $x_1, x_2, y_1, y_2 \in U$ such that $x_1 \leq x_2$, $y_1 \geq y_2$.

Proposition 3.1.16 *A is mixed monotone* iff *$A(\cdot, y) : U \to V$ is increasing for any $y \in U$ and $A(x, \cdot) : U \to V$ is decreasing for any $x \in U$.*

Proposition 3.1.17 *If A and B are mixed monotone, then $B * A$ is mixed monotone.*

Corollary 3.1.18 *If $V \subseteq U$, $n \in \mathbb{N}$ ($n \geq 1$) and A is mixed monotone, then A^n is mixed monotone.*

3.1.5 Coupled lower-upper fixed points of bivariate operators

In order to develop an abstract monotone iterative method to study the (coupled) fixed points of mixed monotone operators (or, in general, of any bivariate operator), we need an analogue of the notions of *lower fixed point* and *upper fixed point* from the univariate case to the bivariate case, so assume next that U, V, W are subsets of an ordered set $(X, \leq)$ (meaning that U, V, W share the same ordering).

Definition 3.1.19 (Guo and Lakshmikantham [34]) A pair $(x, y) \in U^2$ is called *a coupled lower-upper fixed point of* A if $x \leq y$, $x \leq A(x, y)$ and $y \geq A(y, x)$.

For convenience, the set of coupled lower-upper fixed points of A will be denoted by $\Lambda(A)$.

3.2 The method of monotone iterations for mixed monotone operators in ordered sets and order-attractive fixed points

Throughout this section, $(U, \leq)$ is an ordered set and A is a mixed monotone self-map of U, if not stated otherwise.

Also, let $x_0, y_0 \in U$ such that $x_0 \leq y_0$, define the sequences $(x_n), (y_n)$ recursively by

$$x_{n+1} = A(x_n, y_n), \quad y_{n+1} = A(y_n, x_n) \qquad (n \in \mathbb{N}), \tag{3.2.1}$$

and let $U_0 := \bigcap\limits_{n \geq 0} [x_n, y_n]$. Using the powers of A (with respect to the m-composition), $(x_n), (y_n)$ can be written as

$$x_n = A^n(x_0, y_0), \quad y_n = A^n(y_0, x_0) \qquad (n \in \mathbb{N}). \tag{3.2.2}$$

The basic properties of $(x_n), (y_n)$ and U_0 are gathered in the following Lemma. Though elementary, the following results are fundamental for the study of the (coupled) fixed points of mixed monotone operators.

Lemma 3.2.1 *The following properties take place:*

1. $x_n \leq y_n$ *and* $A\left([x_n, y_n]^2\right) \subseteq [x_{n+1}, y_{n+1}]$ *for any* $n \in \mathbb{N}$.
2. U_0 *is order-convex (possibly empty) and*
$$\mathrm{cfp}(A) \cap [x_0, y_0]^2 \subseteq U_0^2. \tag{3.2.3}$$
3. *Assume that* $x_n \leq y_m$ *for any* $m, n \in \mathbb{N}$. *If* $\sup x_n$ *exists, then* $\sup x_n$ *is the least element of* U_0. *Symmetrically, if* $\inf y_n$ *exists, then* $\inf y_n$ *is the greatest element of* U_0. *If both* $\sup x_n$ *and* $\inf y_n$ *exist, then* $\sup x_n \leq \inf y_n$ *and*
$$U_0 = [\sup x_n, \inf y_n]. \tag{3.2.4}$$
4. *If* $(x_0, y_0) \in \Lambda(A)$, *then* (x_n) *is increasing,* (y_n) *is decreasing,* $x_n \leq y_m$ *for any* $m, n \in \mathbb{N}$, $(x_n, y_n) \in \Lambda(A)$ *and* $A\left([x_n, y_n]^2\right) \subseteq [x_n, y_n]$ *for any* $n \in \mathbb{N}$.

Corollary 3.2.3 *If* $U_0 = \emptyset$, *then* A *has no coupled fixed points in* $[x_0, y_0]^2$ *(hence, no fixed points in* $[x_0, y_0]$*).*

Motivated by Lemma 3.2.1, we give the following definitions and prove some results concerning the new notions.

Definition 3.2.4 A point $x^* \in U$ is said to be (x_0, y_0)-*weakly order-attractive for* A if $U_0 = \{x^*\}$. In this case, we say that x^* *weakly order-attracts* (x_0, y_0) *through* A, or that (x_0, y_0) *is weakly order-attracted by* x^* *through* A and we denote this by $x^* \underset{w}{\overset{A}{\Subset}} (x_0, y_0)$.

Remark 3.2.5 Obviously, if $x^* \underset{w}{\overset{A}{\Subset}} (x_0, y_0)$, then $x^* \in [x_0, y_0]$.

Definition 3.2.7 A point $x^* \in U$ is said to be (x_0, y_0)-*order-attractive for* A if $\sup x_n = \inf y_n = x^*$. In this case, we say that x^* *order-attracts* (x_0, y_0) *through* A, or that (x_0, y_0) *is order-attracted by* x^* *through* A and we denote this by $x^* \overset{A}{\Subset} (x_0, y_0)$.

Proposition 3.2.8 $x^* \overset{A}{\Subset} (x_0, y_0)$ iff $x^* \underset{w}{\overset{A}{\Subset}} (x_0, y_0)$ *and* $\sup x_n, \inf y_n$ *exist.*

Definition 3.2.9 A point $x^* \in U$ is said to be *(weakly) order-attractive for* A *on* $[x_0, y_0]$ if $x^* \in [x_0, y_0]$ and $x^* \overset{A}{\Subset} (u_0, v_0)$ (respectively, $x^* \underset{w}{\overset{A}{\Subset}} (u_0, v_0)$) for any $u_0, v_0 \in [x_0, y_0]$ such that $u_0 \leq x^* \leq v_0$. In this case, we say that x^* *(weakly) order-attracts* $[x_0, y_0]$ *through* A, or that $[x_0, y_0]$ *is (weakly) order-attracted by* x^* *through* A and denote this by $x^* \overset{A}{\Subset} [x_0, y_0]$ (respectively, by $x^* \underset{w}{\overset{A}{\Subset}} [x_0, y_0]$).

Proposition 3.2.10 *If* $x^* \overset{A}{\underset{w}{\Subset}} [x_0, y_0]$*, then* $x^* \overset{A}{\underset{w}{\Subset}} (x_0, y_0)$ *and* $x^* \overset{A}{\underset{w}{\Subset}} [u_0, v_0]$ *for any* $u_0, v_0 \in [x_0, y_0]$ *such that* $u_0 \leq x^* \leq v_0$.

Proposition 3.2.11 *If* $x^* \overset{A}{\Subset} [x_0, y_0]$*, then* $x^* \overset{A}{\Subset} (x_0, y_0)$ *and* $x^* \overset{A}{\Subset} [u_0, v_0]$ *for any* $u_0, v_0 \in [x_0, y_0]$ *such that* $u_0 \leq x^* \leq v_0$.

Proposition 3.2.12 *If* $x^* \overset{A}{\Subset} [x_0, y_0]$*, then* $x^* \overset{A}{\underset{w}{\Subset}} [x_0, y_0]$.

Theorem 3.2.13 *Let* $x^* \in U$*. The following statements are equivalent:*

1. $x^* \overset{A}{\underset{w}{\Subset}} [x_0, y_0]$.
2. x^* *is a fixed point of* A *and* $x^* \overset{A}{\underset{w}{\Subset}} (x_0, y_0)$.

In the affirmative case, (x^*, x^*) *is the unique coupled fixed point of* A *in* $[x_0, y_0]^2$ *(hence,* x^* *is the unique fixed point of* A *in* $[x_0, y_0]$*).*

Theorem 3.2.14 *Let* $x^* \in U$*. The following statements are equivalent:*

1. $x^* \overset{A}{\Subset} [x_0, y_0]$.
2. x^* *is a fixed point of* A *and* $x^* \overset{A}{\Subset} (x_0, y_0)$.

In the affirmative case, (x^*, x^*) *is the unique coupled fixed point of* A *in* $[x_0, y_0]^2$ *(hence,* x^* *is the unique fixed point of* A *in* $[x_0, y_0]$*).*

We conclude with the main results of this section.

Theorem 3.2.16 *Let* $x^* \in U$ *and assume that there exists* $k \geq 1$ *such that* $x^* \in \bigcap_{n=0}^{k-1} [x_n, y_n]$ *and* $x^* \overset{A}{\underset{w}{\Subset}} (x_k, y_k)$*. Then* $x^* \overset{A}{\underset{w}{\Subset}} [x_n, y_n]$ *for any* $n \in \{0, 1, \ldots, k\}$*,* (x^*, x^*) *is the unique coupled fixed point of* A *in* $\bigcup_{n=0}^{k} [x_n, y_n]^2$ *and* x^* *is the unique fixed point of* A *in* $\bigcup_{n=0}^{k} [x_n, y_n]$*. Additionally, if* $x^* \overset{A}{\Subset} (x_k, y_k)$*, then* $x^* \overset{A}{\Subset} [x_n, y_n]$ *for any* $n \in \{0, 1, \ldots, k\}$.

Corollary 3.2.17 *If* $x^* \in [x_0, y_0]$ *such that* $x^* \overset{A}{\underset{w}{\Subset}} (x_1, y_1)$*, then* $x^* \overset{A}{\underset{w}{\Subset}} [x_0, y_0]$*,* $x^* \overset{A}{\underset{w}{\Subset}} [x_1, y_1]$*,* (x^*, x^*) *is the unique coupled fixed point of* A *in* $[x_0, y_0]^2 \cup [x_1, y_1]^2$ *and* x^* *is the unique fixed point of* A *in* $[x_0, y_0] \cup [x_1, y_1]$*. Additionally, if* $x^* \overset{A}{\Subset} (x_1, y_1)$*, then* $x^* \overset{A}{\Subset} [x_0, y_0]$ *and* $x^* \overset{A}{\Subset} [x_1, y_1]$.

Theorem 3.2.18 *Assume that* $(x_0, y_0) \in \Lambda(A)$*. If* $x^* \overset{A}{\underset{w}{\Subset}} (x_0, y_0)$*, then* (x^*, x^*) *is the unique coupled fixed point of* A *in* $[x_0, y_0]^2$*,* x^* *is the unique fixed point of* A *in* $[x_0, y_0]$ *and* $x^* \overset{A}{\underset{w}{\Subset}} [x_0, y_0]$*. In particular, if* $x^* \overset{A}{\Subset} (x_0, y_0)$*, then* $x^* \overset{A}{\Subset} [x_0, y_0]$ *and all the other conclusions remain valid.*

Theorem 3.2.19 *If* $k \in \mathbb{N}$ *and* $x^* \in [x_0, y_0]$ *are such that* $(x_k, y_k) \in \Lambda(A)$ *and* $x^* \overset{A}{\underset{w}{\Subset}} (x_k, y_k)$*, then* (x^*, x^*) *is the unique coupled fixed point of* A *in* $\bigcup_{n=0}^{k} [x_n, y_n]^2$*,* x^* *is the unique fixed point of* A *in* $\bigcup_{n=0}^{k} [x_n, y_n]$ *and* $x^* \overset{A}{\underset{w}{\Subset}} [x_n, y_n]$ *for any* $n \in \{0, 1, \ldots, k\}$*. Additionally, if* $x^* \overset{A}{\Subset} (x_k, y_k)$*, then* $x^* \overset{A}{\Subset} [x_n, y_n]$ *for any* $n \in \{0, 1, \ldots, k\}$.

3.3 Fixed point theorems for mixed monotone operators in ordered metric spaces

Throughout this section, $(X, d, \leq)$ will be an ordered metric space, with d being an extended metric, U a nonempty order-convex subset of X and A a mixed monotone self-map of U, if not stated otherwise. Note that, in this case, the supremum with respect to (the larger set) X is denoted by sup and that with respect to (the subset) U is denoted by $\sup\limits_{U}$ (a similar remark for the infimum). For example, this distinction is necessary when we speak of order-attractive points with respect to A: if $x^* \overset{A}{\Subset} (x_0, y_0)$ $(x_0, y_0 \in U, x_0 \leq y_0)$, then it is understood that $x^* \in U$ and $x^* = \sup\limits_{U} x_n = \inf\limits_{U} y_n$.

3.3.1 Properties of ordered metric spaces with interval-semi-monotone extended distance

Definition 3.3.1 d is said to be *interval-semi-monotone* (with respect to $\leq$) if there exists $\gamma \geq 1$, called *semi-monotonicity constant*, such that $d(x', y') \leq \gamma d(x, y)$ for any $x, x', y, y' \in X$ with $x \leq x' \leq y' \leq y$. If $\gamma = 1$, then d is said to be *interval-monotone*.

Example 3.3.2 If (X, K) is an ordered linear space and K is almost Archimedean, then Thompson's metric ρ is an interval-monotone extended metric on K.

Example 3.3.3 If $(X, K, \|\cdot\|)$ is an ordered normed space and K is normal, then X is an ordered metric space whose distance (induced by the norm) is semi-monotone.

From this point forward, in this subsection, assume that d is interval-semi-monotone, with γ denoting the semi-monotonicity constant.

Proposition 3.3.4 *The following properties take place:*

1. *Any metric component of X is order-convex.*
2. *Any interval of a metric component of X is bounded.*

Proposition 3.3.5 *If (C_3') is satisfied and (x_n) is an increasing sequence in X having a convergent, majorised subsequence (x_{n_k}), then (x_n) is convergent, has supremum and* $\lim\limits_{n\to\infty} x_n = \sup x_n = \lim\limits_{k\to\infty} x_{n_k}$.

An analogue to Proposition 3.3.5 can be easily stated for decreasing sequences.

Proposition 3.3.7 *If $(x_n), (y_n), (z_n)$ are sequences in X such that $x_n \leq y_n \leq z_n$ for any $n \in \mathbb{N}$ and $(x_n), (z_n)$ both converge to x, then (y_n) also converges to x.*

Proposition 3.3.8 *Let (x_n) be an increasing sequence and (y_n) a decreasing sequence in X such that $x_n \leq y_n$ for any $n \in \mathbb{N}$. Consider the following statements:*

(i) $(x_n), (y_n)$ *are convergent and have the same limit.*

(ii) $\lim\limits_{n\to\infty} d(x_n, y_n) = 0$.

(iii) $\inf\limits_{n} d(x_n, y_n) = 0$.

*Then **(ii)** and **(iii)** are equivalent, and are implied by **(i)**. Additionally, if d is complete, then the three statements are equivalent.*

3.3.2 Attractive fixed points and m-Picard mixed monotone operators

Let $x_0, y_0 \in U$ and define the sequences (x_n), (y_n) recursively by (3.2.1) or, equivalently, by (3.2.2).

Definition 3.3.10 $x^* \in X$ is said to be (x_0, y_0)-*attractive for* A (with respect to d) if $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} y_n = x^*$. In this case, we say that x^* *attracts* (x_0, y_0) *through* A, or that (x_0, y_0) *is attracted by* x^* *through* A and we denote this by $x^* \underset{d}{\overset{A}{\Subset}} (x_0, y_0)$.

Proposition 3.3.13 *If A is continuous and $x^* \in U$ such that $x^* \underset{d}{\overset{A}{\Subset}} (x_0, y_0)$, then x^* is a fixed point for A.*

Definition 3.3.14 Let V be a nonempty subset of U. A point $x^* \in X$ is said to be *attractive for* A on V if $x^* \underset{d}{\overset{A}{\Subset}} (u_0, v_0)$ for any $u_0, v_0 \in V$. In this case, we say that x^* *attracts* V *through* A, or that V *is attracted by* x^* *through* A and denote this by $x^* \underset{d}{\overset{A}{\Subset}} V$.

Definition 3.3.16 Let V be a nonempty subset of U. A is called a *mirror Picard operator* (shortly, m-*Picard*) on V if

$$\mathbf{A} : U^2 \to U^2, \quad \mathbf{A}(x, y) = (A(x, y), A(y, x))$$

is a *Picard operator* on V^2 (i.e., $\mathbf{A}$ has a unique fixed point in V^2 and $(\mathbf{A}^n(x, y))$ is convergent to the unique fixed point for any $(x, y) \in V^2$).

Proposition 3.3.17 *Let V be a nonempty subset of U. The following statements are equivalent:*

1. *A is m-*Picard *on V.*
2. *A is* Picard *on V with respect to the m-composition (i.e., A has a unique fixed point x^* in V and $(A^n(x, y))$ is convergent to x^* for any $(x, y) \in V^2$).*

3. *A has a unique fixed point x^* in V and $x^* \overset{A}{\underset{d}{\Subset}} V$.*
4. *A has a unique coupled fixed point (x^*, x^*) in V^2 and $x^* \overset{A}{\underset{d}{\Subset}} V$.*

In the affirmative case, (x^*, x^*) is the unique fixed point of $\mathbf{A}$ in V^2.

Proposition 3.3.18 *If d is interval-semi-monotone, $x_0 \le y_0$ and $x^* \in X$ such that $x^* \overset{A}{\underset{d}{\Subset}} (x_0, y_0)$, then $x^* \overset{A}{\underset{d}{\Subset}} [x_n, y_n]$ for any $n \in \mathbb{N}$.*

Remark 3.3.19 If $x^* \overset{A}{\underset{d}{\Subset}} V_1$ and $x^* \overset{A}{\underset{d}{\Subset}} V_2$, then it is *not* generally true that $x^* \overset{A}{\underset{d}{\Subset}} V_1 \cup V_2$.

The following results show that attractive fixed points are order-attractive, hence locally unique.

Theorem 3.3.20 *Assume that (C_2) is satisfied. If $x_0 \le y_0$ and x^* is a fixed point of A in $[x_0, y_0]$ such that $x^* \overset{A}{\underset{d}{\Subset}} (x_0, y_0)$, then $x^* = \sup x_n = \inf y_n$, (x^*, x^*) is the unique coupled fixed point of A in $\bigcup_{n\ge 0} [x_n, y_n]^2$, x^* is the unique fixed point of A in $\bigcup_{n\ge 0} [x_n, y_n]$ and $x^* \overset{A}{\Subset} [x_n, y_n]$ for any $n \in \mathbb{N}$.*

Additionally, if d is interval-semi-monotone, then A is m-Picard on $[x_n, y_n]$ for any $n \in \mathbb{N}$.

Corollary 3.3.21 *Assume that (C_2) is satisfied and A is continuous. If $x_0 \le y_0$ and $x^* \in [x_0, y_0]$ such that $x^* \overset{A}{\underset{d}{\Subset}} (x_0, y_0)$, then x^* is a fixed point of A and the conclusions of Theorem 3.3.20 hold.*

Remark 3.3.22 When $(x_n), (y_n)$ are order-bounded, then (C_2) can be replaced by (C_3) in Theorem 3.3.20 and Corollary 3.3.21.

Remark 3.3.23 In general, order-attractive fixed points are not attractive (with respect to the metric), even when (C_2) is satisfied, the metric is interval-semi-monotone and the mixed monotone operator is continuous.

The following results are essential for the upcoming fixed point theorems.

Theorem 3.3.24 *Assume that $(x_0, y_0) \in \Lambda(A)$ and consider the following statements:*

(i) $x^* \overset{A}{\underset{d}{\Subset}} (x_0, y_0)$ *for some* $x^* \in X$.
(ii) $\lim_{n\to\infty} d(x_n, y_n) = 0$.
(iii) $\inf_n d(x_n, y_n) = 0$.

Then:

1. *If (C_3') is satisfied, then **(i)** implies that x^* is the unique fixed point of A in $[x_0, y_0]$, $x^* \overset{A}{\Subset} [x_0, y_0]$ and A is m-Picard on $[x_0, y_0]$.*
2. *If d is complete and interval-semi-monotone, then the three statements are equivalent.*

Theorem 3.3.26 *Assume that (C_3') is satisfied, d is complete and interval-semi-monotone.*

If $x_0 \le y_0$ and there exists $k \in \mathbb{N}$ such that $(x_k, y_k) \in \Lambda(A)$, then the three statements in Theorem 3.3.24 are equivalent. In the affirmative case, x^* is the unique fixed point of A in $[x_k, y_k]$, $x^* \overset{A}{\Subset} [x_k, y_k]$, $x^* \overset{A}{\underset{d}{\Subset}} [x_n, y_n]$ for any $n \in \{0, 1, \dots, k\}$ and A is m-Picard on $[x_k, y_k]$.

Additionally, if $x^* \in [x_0, y_0]$, then x^* is the unique fixed point of A in $\bigcup_{n=0}^{k} [x_n, y_n]$ and A is m-Picard on $[x_n, y_n]$ for any $n \in \{0, 1, \dots, k\}$.

3.3.3 Existence and uniqueness of attractive fixed points for m-contractive mixed monotone operators via coupled lower-upper fixed points

Definition 3.3.27 $(x_0, y_0) \in \Lambda(A)$ is said to be:

1. *proper for A* if there exists $n \in \mathbb{N}$ such that $(d * A^n)(x_0, y_0) < \infty$;
2. *improper for A* if it is not proper, i.e., if $(d * A^n)(x_0, y_0) = \infty$ for any $n \in \mathbb{N}$.

Definition 3.3.28 A is called *proper* if $\Lambda(A)$ is nonempty and any of its coupled lower-upper fixed points is proper.

In all the results that follow, we assume that (C_3') is satisfied, d is complete and interval-semi-monotone, while $\gamma \geq 1$ denotes the semi-monotonicity constant of d. For convenience, let

$$W = \{(x, y) \in \Lambda(A) : x \neq y \text{ and } d(x, y) < \infty\}$$

and assume that W is nonempty.

Theorem 3.3.29 *Let $\Psi : W \to [0, \infty)$ satisfying the following property:*

(π_1) For any increasing sequence (x_n) and any decreasing sequence (y_n) from the same metric component of U such that $(x_n, y_n) \in \Lambda(A)$ for all $n \in \mathbb{N}$ and $\inf_n d(x_n, y_n) > 0$, there exists $k \in \mathbb{N}$ such that $\Psi(x_k, y_k) < \inf_n d(x_n, y_n)$.

If

$$(d * A)\,(x, y) \leq \Psi(x, y) \text{ for any } (x, y) \in W, \tag{3.3.1}$$

then the following properties take place:

1. *$(d * A)\,(x, y) < d(x, y)$ for any $(x, y) \in W$.*
2. *A satisfies property (μ) on each metric component of U.*
3. *If $(x_0, y_0) \in \Lambda(A)$ is proper, then A has a unique fixed point x^* in $[x_0, y_0]$, $x^* \overset{A}{\Subset} [x_0, y_0]$ and A is m-Picard on $[x_0, y_0]$.*
4. *If A is proper, then A satisfies property (μ) and the conclusions in* **3** *hold for any $(x_0, y_0) \in \Lambda(A)$.*

In order to find suitable functions Ψ that connect to A by (3.3.1) and verify (π_1), it is imperative to provide clear criteria for (π_1). For convenience, let $D = d(W) \subseteq (0, \infty)$. We will use D'_+ to denote the set of right limit points of D in $(0, \infty)$, i.e.,

$$D'_+ = \{t > 0 : (t, t + \tau) \cap D \neq \emptyset \text{ for any } \tau > 0\}$$

and $\overline{D}_+$ to denote the right closure of D in $(0, \infty)$, i.e.,

$$\overline{D}_+ = D \cup D'_+ = \{t > 0 : [t, t + \tau) \cap D \neq \emptyset \text{ for any } \tau > 0\}.$$

Clearly, these notations apply to any subset D of $(0, \infty)$, if necessary.

Proposition 3.3.36 *If $\Psi : W \to [0, \infty)$ satisfies:*

(π_2) For any $t \in \overline{D}_+$, there exists $\tau > 0$ such that

$$(x, y) \in W \text{ and } d(x, y) \in [t, t + \tau) \text{ imply } \Psi(x, y) < t; \tag{3.3.2}$$

then Ψ satisfies (π_1).

Remark 3.3.37 If $t > 0$ such that $t \notin \overline{D}_+$, then $[t, t + \tau) \cap D = \emptyset$ for some $\tau > 0$, hence (3.3.2) is automatically satisfied (since there is nothing to verify). With this remark, (π_2) is equivalent to:

(π_2') For any $t > 0$, there exists $\tau > 0$ such that (3.3.2) is satisfied.

Corollary 3.3.38 *If $\Psi : W \to [0, \infty)$ satisfies (π_2) and (3.3.1), then the conclusion of Theorem 3.3.29 hold.*

Corollary 3.3.39 *If for any $t \in \overline{D}_+$, there exists $\tau > 0$ such that*

$$(x, y) \in W \text{ and } d(x, y) \in [t, t + \tau) \text{ imply } (d * A)(x, y) < t, \tag{3.3.3}$$

then the conclusion of Theorem 3.3.29 hold.

Remark 3.3.40 When A is an univariate operator, i.e., $A(x, y) = A(x)$ or $A(x, y) = A(y)$, the condition in Corollary 3.3.39 is identical to that of Meir and Keeler [63] for the univariate case (except that no monotonicity is required).

Motivated by the previous remark, we give the following definition.

Definition 3.3.41 We say that A is a *weakly uniformly strict m-contraction* if A satisfies the conditions in Corollary 3.3.39.

It is possible to simplify the conditions in (π_1) and in (π_2), by considering only radial function, i.e., functions that depend only on the distance between the arguments. In this direction, first, we define and study some new concepts.

Let $\Phi : D \to [0, \infty)$ be an arbitrary function.

Definition 3.3.42 We say that A is a *Φ–m-contraction* (or that A is *Φ–m-contractive*) if

$$(d * A)\,(x, y) \leq \Phi(d(x, y)) \text{ for any } (x, y) \in W. \tag{3.3.4}$$

Remark 3.3.43 In the univariate case, the concept of Φ-contraction was introduced by Boyd and Wong [15], in order to extend the well-known fixed point result of Banach to a larger class of operators.

The following concept is a slight modification of a similar notion (called *L-function*) introduced by Lim [58] to express Meir–Keeler contraction condition (cf. [63]) in a similar manner to the that of Boyd and Wong (cf. [15]).

Definition 3.3.44 We say that Φ *is an M–function* if the following property is satisfied:

(M_1) For any $t \in \overline{D}_+$, there exists $\tau > 0$ such that $\Phi(s) < t$ for any $s \in [t, t+\tau) \cap D$.

If Φ is defined on a larger set that includes D and the restriction of Φ to D is an M–function, then we say that Φ *is an M–function on D*.

Remark 3.3.45 It is clear that the conditions in (M_1) are verified for any $t > 0$ such that $t \notin \overline{D}_+$, since the set $[t, t+\tau) \cap D$ is empty for some $\tau > 0$. With this remark, (M_1) is equivalent to:

(M_1') For any $t > 0$, there exists $\tau > 0$ such that $\Phi(s) < t$ for any $s \in [t, t+\tau) \cap D$.

Example 3.3.46 If $\alpha \in [0, 1)$ and $\Phi(t) = \alpha t$ for any $t \in D$, then Φ is an M–function.

The following fixed point theorem is a direct consequence of Corollary 3.3.38 in the case when Ψ is radial.

Theorem 3.3.47 *If A is a Φ–m-contraction and Φ is an M-function, then the conclusion of Theorem 3.3.29 hold.*

For related results, we refer to Agarwal et al. [1], Ćirić et al. [25], Gnana Bhaskar and Lakshmikantham [30], Lakshmikantham and Ćirić [54], Nieto and Rodríguez-López [69] and [70]. For a comparative analysis between our results and the results in the mentioned papers, we refer to Section 3.5.

In order to see the range of applicability for Theorem 3.3.47, it is necessary to show that the class of M–functions on a set is rich enough. We show this by first giving a full characterization of this class of functions and, further, by giving sufficient conditions for a function to be in this class.

We consider the following properties:

(M_2) For any $t \in D'_+$, there exists $\tau > 0$ such that $\Phi(s) < t$ for any $s \in (t, t+\tau) \cap D$.

(M_3) For any sequence (t_n) in D such that $\lim_{n\to\infty} t_n = t > 0$, there exists $k \in \mathbb{N}$ such that $\Phi(t_k) < t$.

(M_4) For any sequence (t_n) in D such that $\inf t_n = t > 0$, there exists $k \in \mathbb{N}$ such that $\Phi(t_k) < t$.

(M_5) For any decreasing sequence (t_n) in D such that $\lim_{n\to\infty} t_n = t > 0$, there exists $k \in \mathbb{N}$ such that $\Phi(t_k) < t$.

(M_6) For any strictly decreasing sequence (t_n) in D such that $\lim_{n\to\infty} t_n = t > 0$, there exists $k \in \mathbb{N}$ such that $\Phi(t_k) < t$.

Proposition 3.3.48 *The following statements are equivalent:*

1. *Φ is an M–function.*
2. *Φ satisfies (M_2) and $\Phi(t) < t$ for any $t \in D$.*
3. *Φ satisfies (M_3).*
4. *Φ satisfies (M_4).*
5. *Φ satisfies (M_5).*
6. *Φ satisfies (M_6) and $\Phi(t) < t$ for any $t \in D$.*

The following conditions (similar to that used by Boyd and Wong in [15]) are sufficient for Φ to be an M–function.

Proposition 3.3.49 *If $\Phi(t) < t$ for any $t \in D$ and $\limsup_{s\to t^+} \Phi(s) < t$ for any $t \in D'_+$, then Φ is an M–function.*

Assume next that the domain of Φ is extended to $\overline{D}_+$, hence $\Phi : \overline{D}_+ \to [0, \infty)$.

Proposition 3.3.50 *If $\Phi(t) < t$ for any $t \in \overline{D}_+$ and $\limsup_{s\to t^+,\, s\in D} \Phi(s) \le \Phi(t)$ for any $t \in D'_+$, then Φ is an M–function on D.*

Proposition 3.3.51 *If $\Phi(t) < t$ for any $t \in \overline{D}_+$ and Φ is right upper semicontinuous, then Φ is an M–function on D.*

Corollary 3.3.52 *If $\Phi(t) = \alpha(t)t$ for any $t \in \overline{D}_+$, where $\alpha : \overline{D}_+ \to [0, 1)$ is right upper semicontinuous, then Φ is an M–function on D.*

For convenience, fix $\tau > 0$ (any value, as small as needed, will suffice) and let

$$D_\tau = (D - D'_+) \cap (0, \tau) = \{s - t : t \in D'_+, s \in (t, t+\tau) \cap D\}$$

and assume next that $\Phi : \overline{D}_+ \cup D_\tau \to [0, \infty)$.

Proposition 3.3.54 *If the following conditions are satisfied:*

(i) $\Phi(t) < t$ *for any* $t \in \overline{D}_+$;

(ii) $\Phi(t) \le t$ *for any* $t \in D_\tau$;

(iii) $\Phi(s - t) \ge \Phi(s) - \Phi(t)$ *for any* $t \in D'_+$ *and any* $s \in (t, t+\tau) \cap D$;

then Φ is an M–function on D.

3.3.4 Fixed point theorems for mixed monotone operators via extremal coupled fixed points

We assume next that d is a (regular) metric and that (C_3') is satisfied. Recall that A is said to be *completely continuous* if it is continuous and maps bounded sets to relatively compact ones.

The following theorem is an extension of the results of Opoĭtsev [74, Theorem 1.2 and Lemma 1.1], Guo and Lakshmikantham [34, Theorem 1], from the case of ordered Banach spaces to that of ordered metric spaces.

Theorem 3.3.55 *Let $(x_0, y_0) \in \Lambda(A)$ and define the sequences $(x_n), (y_n)$ by (3.2.1) (or, equivalently, by (3.2.2)). Assume that one of the following two situations takes place:*

(i) A is continuous on $[x_0, y_0]^2$ and any monotone sequence in $[x_0, y_0]$ is convergent.

(ii) A is completely continuous on $[x_0, y_0]^2$ and d is interval-monotone.

Then there exists (x^*, y^*) an extremal coupled fixed point of A in $[x_0, y_0]^2$ and $x_n \overset{d}{\to} x^*$, $y_n \overset{d}{\to} y^*$ as $n \to \infty$.

Remark 3.3.56 It is possible to weaken the continuity condition on A in Theorem 3.3.55, by weakening the topology on $A([x_0, y_0]^2)$.

If τ is a Hausdorff topology on $A([x_0, y_0]^2)$ that is weaker than the original topology τ_d induced by d, then the continuity of A in the new setting (which we will call *weak continuity*) is also weaker than in the original case. Assuming that A is weakly continuous on $[x_0, y_0]^2$, then, in the proof of Theorem 3.3.55, (x_n) τ-converges to $A(x^*, y^*)$ and (y_n) τ-converges to $A(y^*, x^*)$. Also, $\tau \subseteq \tau_d$ ensures that (x_n) τ-converges to x^* and (y_n) τ-converges to y^*, hence $x^* = A(x^*, y^*)$ and $y^* = A(y^*, x^*)$, by using that τ is Hausdorff.

The following result is an extension of the results of Opoĭtsev [74, Theorem 1.3].

Corollary 3.3.57 *If the conditions in Theorem 3.3.55 are satisfied and A has property (μ) on $[x_0, y_0]$, then A has a unique fixed point x^* in $[x_0, y_0]$, $x^* \overset{A}{\in} [x_0, y_0]$ and A is m-Picard on $[x_0, y_0]$.*

We close this section with a simple result (inspired by Theorem 2.2 in the paper of Buică and Precup [19]) that provides sufficient conditions for A to satisfy property (μ). First, let i_X denote the identity operator on X.

Proposition 3.3.58 *Let X be an ordered linear space and assume that for any $x, y \in U$ with $x \leq y$, there exists a operator $L(x, y) : X \to X$ such that $i_X - L(x, y)$ is inversely positive, i.e.,*

$$L(x, y)(z) \leq z \quad (z \in X) \quad \textit{implies} \quad z \geq \theta, \tag{3.3.5}$$

and satisfies

$$A(x, y) \geq A(y, x) + L(x, y)(x - y). \tag{3.3.6}$$

Then A has property (μ).

Remark 3.3.59 By examining the proof of Proposition 3.3.58, it is clear that the result is true for any bivariate map A, not necessarily mixed monotone and for any ordered linear space X (there is no need for the metric).

3.4 Fixed point theorems for mixed monotone operators in ordered linear spaces

Throughout this section, (X, K) will be an ordered linear space, with K being an Archimedean and self-complete cone, while $\leq$ denotes the ordering induced by K on X. Using the results from *Chapter 2*, it follows that Thompson's metric ρ is a complete extended metric on K (by Proposition 2.1.2, Remark 2.1.3 and Corollary 2.3.23) which is interval-monotone (by Corollary 2.2.7). Additionally, the ordered metric space $(K, \rho, \leq)$ satisfies (C_2) (by Proposition 2.2.24), hence also (C_3) and (C_i'), with $i = \overline{1, 5}$. Concluding, one may apply any of the results from the previous section for the ordered metric space $(K, \rho, \leq)$. Note that one of the particularities of using Thompson's metric is that we are restricted to K, hence we can only obtain information about positive fixed points.

We refer to Sections 1.5, 1.6 and Chapter 2 for details on ordered linear spaces and Thompson's metric.

3.4.1 Preliminary results, with conditions expressed in terms of Thompson's metric

Let U be a nonempty, order-convex subset of K and A a mixed monotone self-map of U. In order to be able to successfully apply the techniques and results from Section 3.3, one needs to rewrite the conditions involving Thompson's metric, without using it explicitly. For convenience, we use the same notations from Subsection 3.3.3, i.e.,

$$W = \{(x, y) \in \Lambda(A) : x \neq y \text{ and } x \sim y\}, \quad D = \rho(W) \subseteq (0, \infty)$$

and assume, from this point forward, that W is nonempty.

Consider, also, the projective components of W,

$$W_1 = \{x \in U : \text{ there exists } y \in U \text{ such that } (x, y) \in W\}$$
$$W_2 = \{y \in U : \text{ there exists } x \in U \text{ such that } (x, y) \in W\}$$

and let

$$E = \left\{ e^{-\rho(x,y)} : (x, y) \in W \right\} = e^{-D} \subseteq (0, 1)$$
$$E'_- = \{\lambda \in (0, 1) : (\lambda - \varepsilon, \lambda) \cap E \neq \emptyset \text{ for any } \varepsilon > 0\} = e^{-D'_+} \subseteq (0, 1)$$
(the set of left limit points of E in $(0, 1)$)
$$\overline{E}_- = \{\lambda \in (0, 1) : (\lambda - \varepsilon, \lambda] \cap E \neq \emptyset \text{ for any } \varepsilon > 0\} = e^{-\overline{D}_+} = E \cup E'_- \subseteq (0, 1)$$
(the left closure of E in $(0, 1)$)
$$E_\delta = \left\{ \frac{\mu}{\lambda} : \lambda \in E'_-, \ \mu \in (\delta\lambda, \lambda) \right\} = e^{-D_{\ln \delta^{-1}}} \subseteq (\delta, 1)$$
for some fixed $\delta \in (0, 1)$ (any value, as close to 1 as needed, will suffice).

The following result is a direct consequence of Theorem 3.3.29, in terms of Thompson's metric.

Theorem 3.4.2 *Let $\psi : W \to (0, 1]$ satisfying the following property:*

(ρ_1) *For any $\lambda \in \overline{E}_-$ and any sequences (x_n), (y_n) from the same part of U such that (x_n) is increasing, (y_n) is decreasing, $(x_n, y_n) \in \Lambda(A)$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \rho(x_n, y_n) = \ln \lambda^{-1}$, there exists $k \in \mathbb{N}$ such that $\psi(x_k, y_k) > \lambda$.*

If

$$A(x, y) \geq \psi(x, y)A(y, x) \quad \textit{for any } (x, y) \in W, \tag{3.4.1}$$

then the following properties take place:

1. *$(\rho * A)(x, y) < \rho(x, y)$ for any $(x, y) \in W$.*
2. *A satisfies property (μ) on each part of U.*
3. *If $(x_0, y_0) \in \Lambda(A)$ is proper, then A has a unique fixed point x^* in $[x_0, y_0]$, $x^* \overset{A}{\Subset} [x_0, y_0]$ and A is m-Picard on $[x_0, y_0]$ with respect to ρ.*
4. *If A is proper, then A has property (μ) and the conclusions in **3** hold for any $(x_0, y_0) \in \Lambda(A)$.*

3.4.2 Main results

The major (practical) inconvenience of Theorem 3.4.2 is the explicit use of Thompson's metric in expressing (ρ_1), which we will circumvent in the following results. Consider next that V is a subset of K that includes U and let $A : V^2 \to K$ be a mixed monotone bivariate operator such that $A(U^2) \subseteq U$. Note that we keep the notations introduced in the previous subsection ($W, W_1, W_2, D, E, \ldots$) for the restriction of A to U^2.

In order to state the following results, we first give some definitions.

Definition 3.4.6 An increasing sequence (x_n) from W_1 is said to be *W–paired* if there exists a decreasing sequence (y_n) such that $(x_n, y_n) \in W$ and $x_0 \sim y_0$. The sequence (y_n) is called *a decreasing W–pair of (x_n)*.

Symmetrically, a decreasing sequence (y_n) from W_2 is said to be *W–paired* if there exists an increasing sequence (x_n) such that $(x_n, y_n) \in W$ and $x_0 \sim y_0$. The sequence (x_n) is called *an increasing W–pair of (y_n)*.

Remark 3.4.7 If (y_n) is a decreasing W–pair of an increasing W–paired sequence (x_n) from W_1, then (y_n) is a W–paired sequence from W_2 and (x_n) is an increasing W–pair of (y_n). Also, $(x_n), (y_n)$ are from the same part of U.

Symmetrically, if (x_n) is an increasing W–pair of a decreasing W–paired sequence (y_n) from W_2, then (x_n) is a W–paired sequence from W_1 and (y_n) is a decreasing W–pair of (x_n). Also, $(x_n), (y_n)$ are from the same part of U.

Theorem 3.4.8 *Let $\phi : E \times W_1 \to (0, 1]$ satisfying the following property:*

(ρ_2) *For any W–paired, increasing sequence (x_n) from W_1 and any increasing sequence (λ_n) from E that converges to some $\lambda \neq 1$, there exists $k \in \mathbb{N}$ such that $\phi(\lambda_k, y_k) > \lambda$.*

If

$$\lambda^{-1} W_1 \subseteq V \quad \textit{for any } \lambda \in E \tag{3.4.2}$$
$$A(x, \lambda^{-1}x) \geq \phi(\lambda, x)A(\lambda^{-1}x, x) \quad \textit{for any } \lambda \in E, x \in W_1, \tag{3.4.3}$$

then the conclusions of Theorem 3.4.2 hold.

Theorem 3.4.9 *Let $\phi : E \times W_2 \to (0, 1]$ satisfying the following property:*

(ρ_3) *For any W–paired, decreasing sequence (y_n) from W_2 and any increasing sequence (λ_n) from E that converges to some $\lambda \neq 1$, there exists $k \in \mathbb{N}$ such that $\phi(\lambda_k, y_k) > \lambda$.*

If

$$\lambda W_2 \subseteq V \quad \textit{for any } \lambda \in E \tag{3.4.4}$$

$$A(\lambda y, y) \geq \phi(\lambda, y) A(y, \lambda y) \quad \textit{for any } \lambda \in E, y \in W_2, \tag{3.4.5}$$

then the conclusions of Theorem 3.4.2 hold.

Remark 3.4.10 The decision of using either Theorem 3.4.8 or Theorem 3.4.9 may be influenced by the *shape* of V. If $\lambda U \subseteq V$ for any $\lambda \in (0,1]$, then (3.4.4) is automatically verified, hence Theorem 3.4.9 would be the best choice. If $\mu U \subseteq V$ for any $\mu \geq 1$, then we would prefer Theorem 3.4.8 instead.

Remark 3.4.11 If $\phi : E \times W_1 \to (0,1]$ satisfies (ρ_2), then $\phi(\lambda, x) > \lambda$ for any $\lambda \in E$, $x \in W$. Similarly, if $\phi : E \times W_2 \to (0,1]$ satisfies (ρ_3), then $\phi(\lambda, y) > \lambda$ for any $\lambda \in E$, $y \in W_2$.

If ϕ can be chosen independently of the second argument, i.e., $\phi(\lambda, \cdot) = \varphi(\lambda) \in (0,1]$ for any $\lambda \in E$, then (ρ_2) and (ρ_3) simplify, similarly to the case when Ψ (in Subsection 3.3.3) was a radial function. Following the same steps as in Subsection 3.3.3, we first define and study a new concept, called *N–function* (an analogue to that of an M–function, previously defined). Let $\varphi : E \to (0,1]$.

Definition 3.4.14 We say that φ *is an N–function* if the following property is satisfied:
(N_1) For any $\lambda \in \overline{E}_-$, there exists $\varepsilon > 0$ such that $\varphi(\mu) > \lambda$ for any $\mu \in (\lambda - \varepsilon, \lambda] \cap E$.
If φ is defined on a larger subset of $(0,1)$ that includes E and the restriction of φ to E is an N–function, then we say that φ *is an N–function on E*.

Remark 3.4.15 It is clear that the conditions in (N_1) are verified for any $\lambda \notin \overline{E}_-$, since the set $(\lambda - \varepsilon, \lambda] \cap E$ is empty for some $\varepsilon > 0$.

The following result shows that there is a duality between M–functions and N–functions.

Proposition 3.4.16 *φ is an N–function* iff $\Phi : D \to [0, \infty)$ *is an M–function, where*

$$D = \ln E^{-1} = \{\ln \lambda^{-1} : \lambda \in E\} \quad \textit{(equivalently, } E = \mathrm{e}^{-D} = \left\{\mathrm{e}^{-t} : t \in D\right\}\textit{)}$$

$$\Phi(t) = \ln(\varphi(\mathrm{e}^{-t}))^{-1},\ t \in D \quad \textit{(equivalently, } \varphi(\lambda) = \mathrm{e}^{-\Phi(\ln \lambda^{-1})},\ \lambda \in E\textit{)}.$$

Example 3.4.17 If $\varphi(\lambda) = \lambda^\alpha$ for any $\lambda \in E$, where $\alpha \in [0,1)$, then φ is an N–function, since the corresponding function $\Phi(t) = \ln(\varphi(\mathrm{e}^{-t}))^{-1} = \alpha t$ is an M–function (see Example 3.3.46).

Related to (N_1), consider the following similar properties:
(N_2) For any $\lambda \in E'_-$, there exists $\varepsilon > 0$ such that $\varphi(\mu) > \lambda$ for any $\mu \in (\lambda - \varepsilon, \lambda) \cap E$.
(N_3) For any sequence (λ_n) in E such that $\lim_{n\to\infty} \lambda_n = \lambda < 1$, there exists $k \in \mathbb{N}$ such that $\varphi(\lambda_k) > \lambda$.
(N_4) For any sequence (λ_n) in E such that $\sup \lambda_n = \lambda < 1$, there exists $k \in \mathbb{N}$ such that $\varphi(\lambda_k) > \lambda$.
(N_5) For any increasing sequence (λ_n) in E such that $\lim_{n\to\infty} \lambda_n = \lambda < 1$, there exists $k \in \mathbb{N}$ such that $\varphi(\lambda_k) > \lambda$.
(N_6) For any strictly increasing sequence (λ_n) in E such that $\lim_{n\to\infty} \lambda_n = \lambda < 1$, there exists $k \in \mathbb{N}$ such that $\varphi(\lambda_k) > \lambda$.

Proposition 3.4.18 *The following statements are equivalent:*

1. *φ is an N–function.*
2. *φ satisfies (N_2) and $\varphi(\lambda) > \lambda$ for any $\lambda \in E$.*
3. *φ satisfies (N_3).*
4. *φ satisfies (N_4).*
5. *φ satisfies (N_5).*
6. *φ satisfies (N_6) and $\varphi(\lambda) > \lambda$ for any $\lambda \in E$.*

The following results are clear consequences of Theorems 3.4.8, 3.4.9 and Proposition 3.4.18.

Theorem 3.4.19 *If φ is an N–function, (3.4.2) is satisfied and*

$$A(x, \lambda^{-1} x) \geq \varphi(\lambda) A(\lambda^{-1} x, x) \quad \textit{for any } \lambda \in E, x \in W_1, \tag{3.4.6}$$

then the conclusions of Theorem 3.4.2 hold.

Theorem 3.4.20 *If φ is an N–function, (3.4.4) is satisfied and*

$$A(\lambda y, y) \geq \varphi(\lambda) A(y, \lambda y) \quad \textit{for any } \lambda \in E, y \in W_2, \tag{3.4.7}$$

then the conclusion of Theorem 3.4.2 hold.

In order for Theorems 3.4.19 and 3.4.20 to be *practical*, it is necessary to provide sufficient conditions for φ to be an N–function. The results are similar to those for M–functions and follow by applying Propositions 3.3.49, 3.3.50, 3.3.51, 3.3.54, Corollary 3.3.52 and the duality between M–functions and N–functions in Proposition 3.4.16.

Proposition 3.4.21 *If $\varphi(\lambda) > \lambda$ for any $\lambda \in E$ and $\liminf_{\mu \to \lambda^-} \varphi(\mu) > \lambda$ for any $\lambda \in E'_-$, then φ is an N–function.*

Proposition 3.4.22 *If $\varphi : \overline{E}_- \to (0,1]$ such that $\varphi(\lambda) > \lambda$ for any $\lambda \in \overline{E}_-$ and $\liminf_{\substack{\mu \to \lambda^- \\ \mu \in E}} \varphi(\mu) \geq \varphi(\lambda)$ for any $\lambda \in E'_-$, then φ is an N–function on E.*

Proposition 3.4.23 *If $\varphi : \overline{E}_- \to (0,1]$ such that $\varphi(\lambda) > \lambda$ for any $\lambda \in \overline{E}_-$ and φ is left lower semicontinuous, then φ is an N–function on E.*

Corollary 3.4.24 *If $\alpha : \overline{E}_- \to [0,1)$ is left upper semicontinuous (in particular, an increasing function), then $\varphi : \overline{E}_- \to (0,1]$ given by $\varphi(\lambda) = \lambda^{\alpha(\lambda)}$ is an N–function.*

Proposition 3.4.25 *If $\varphi : \overline{E}_- \cup E_\delta \to (0,1]$ satisfies:*

(i) $\varphi(\lambda) > \lambda$ for any $\lambda \in \overline{E}_-$;

(ii) $\varphi(\lambda) \geq \lambda$ for any $\lambda \in E_\delta$;

(iii) $\varphi\left(\frac{\mu}{\lambda}\right) \leq \frac{\varphi(\mu)}{\varphi(\lambda)}$ for any $\lambda \in E'_-$ and any $\mu \in (\delta\lambda, \lambda) \cap E$;

then φ is an N–function on E.

We show next that if either (3.4.6) or (3.4.7) is straightened, it is possible to weaken the requirements on φ.

Theorem 3.4.26 *Let $\varphi : \overline{E}_- \cup E_\delta \to (0,1]$ such that*

$$\varphi(\lambda) > \lambda \quad \text{for any } \lambda \in \overline{E}_- \tag{3.4.8}$$

$$\varphi(\lambda) \geq \lambda \quad \text{for any } \lambda \in E_\delta. \tag{3.4.9}$$

If

$$\lambda V \subseteq V \quad \text{for any } \lambda \in \overline{E}_- \cup E_\delta \tag{3.4.10}$$

$$A(\lambda x, y) \geq \varphi(\lambda) A(x, \lambda y) \quad \text{for any } \lambda \in \overline{E}_- \cup E_\delta \text{ and } x, y \in V \text{ linearly dependent,} \tag{3.4.11}$$

then the conclusions of Theorem 3.4.2 hold.

Theorem 3.4.27 *Let $\varphi : \overline{E}_- \cup E_\delta \to (0,1]$ satisfying (3.4.8) and (3.4.9).*

If

$$\lambda^{-1} V \subseteq V \quad \text{for any } \lambda \in \overline{E}_- \cup E_\delta \tag{3.4.12}$$

$$A(x, \lambda^{-1} y) \geq \varphi(\lambda) A(\lambda^{-1} x, y) \quad \text{for any } \lambda \in \overline{E}_- \cup E_\delta \text{ and } x, y \in V \text{ linearly dependent,}$$

then the conclusions of Theorem 3.4.2 hold.

In the more general case, when ϕ is not necessarily independent of the second variable, it is still possible to obtain similar results to those in Theorems 3.4.19, 3.4.20 and 3.4.26.

Theorem 3.4.28 *Let $\phi : E \times W_1 \to (0,1]$ satisfying the following property:*

(ρ_4) For any $(u,v) \in W$ with $u \sim v$, there exists an N–function $\varphi_{u,v} : E \to (0,1]$ such that

$$\phi(\lambda, x) \geq \varphi_{u,v}(\lambda) \quad \text{for any } \lambda \in E, x \in [u,v] \cap W_1.$$

If (3.4.2) and (3.4.3) are satisfied, then the conclusions of Theorem 3.4.2 hold.

Theorem 3.4.29 *Let $\phi : E \times W_2 \to (0,1]$ satisfying the following property:*

(ρ_5) For any $(u,v) \in W$ with $u \sim v$, there exists an N–function $\varphi_{u,v} : E \to (0,1]$ such that

$$\phi(\lambda, y) \geq \varphi_{u,v}(\lambda) \quad \text{for any } \lambda \in E, y \in [u,v] \cap W_2.$$

If (3.4.4) and (3.4.5) are satisfied, then the conclusions of Theorem 3.4.2 hold.

In a similar way, we can formulate more general versions of Theorems 3.4.26 and 3.4.27.

Theorem 3.4.30 *Let $\phi : \overline{E}_- \times V^2 \to (0,1]$ satisfying the following property:*

(ρ_6) For any two linked elements $u, v \in V$ with $u \leq v$, there exists a function $\varphi_{u,v} : \overline{E}_- \to (0,1]$ such that

$$\phi(\lambda, x, y) \geq \varphi_{u,v}(\lambda) > \lambda \quad \text{for any } \lambda \in \overline{E}_- \text{ and } x, y \in [u,v] \cap V \text{ linearly dependent.} \tag{3.4.13}$$

If V satisfies (3.4.10) and A verifies

$$A(\lambda x, y) \geq \phi(\lambda, x, y) A(x, \lambda y) \quad \text{for any } \lambda \in \overline{E}_- \text{ and } x, y \in V \text{ linearly dependent,} \tag{3.4.14}$$

and

$$A(\lambda x, y) \geq \lambda A(x, \lambda y) \quad \text{for any } \lambda \in E_\delta \text{ and } x, y \in V \text{ linearly dependent,} \tag{3.4.15}$$

then the conclusions of Theorem 3.4.2 hold.

Theorem 3.4.31 *Let $\phi : \overline{E}_- \times V^2 \to (0, 1]$ satisfying (ρ_6). If (3.4.12),*

$$A(x, \lambda^{-1} y) \geq \phi(\lambda, x, y) A(\lambda^{-1} x, y) \quad \text{for any } \lambda \in \overline{E}_- \text{ and } x, y \in V \text{ linearly dependent,}$$

$$A(x, \lambda^{-1} y) \geq \lambda A(\lambda^{-1} x, y) \quad \text{for any } \lambda \in E_\delta \text{ and } x, y \in V \text{ linearly dependent,}$$

then the conclusions of Theorem 3.4.2 hold.

3.4.3 Concluding results

We conclude this section with some important (more particular but more practical) results that are derived from the more general results in the previous subsection. The setting is unchanged, i.e., an ordered linear space (X, K), with K being an Archimedean and self-complete cone.

Theorem 3.4.32 *Let Y be a non-empty subset of K, $A : Y^2 \to K$ a mixed monotone operator and $(x_0, y_0) \in \Lambda(A)$ such that $[x_0, y_0] \subseteq Y$.*

Define the sequences $(x_n), (y_n)$ recursively by (3.2.1) and assume that there exists $k \in \mathbb{N}$, $\varepsilon \in (0, 1)$ and $\varphi : [\varepsilon, 1) \to (0, 1]$ such that the following conditions are satisfied:

(i) $x_k \geq \varepsilon y_k$;

(ii) $V := \bigcup\limits_{\lambda \in (0,1]} \lambda [x_k, y_k] \subseteq Y$;

(iii) $\varphi(\lambda) > \lambda$ *for any* $\lambda \in [\varepsilon, 1)$;

(iv) $A(\lambda x, y) \geq \varphi(\lambda) A(x, \lambda y)$ *for any* $\lambda \in [\varepsilon, 1)$ *and* $x, y \in V$ *linearly dependent.*

Then A has a unique fixed point x^* in $[x_0, y_0]$, $x^* \overset{A}{\Subset} [x_0, y_0]$ and A is m-Picard on $[x_0, y_0]$ with respect to ρ.

Theorem 3.4.33 *Let Y be a non-empty, order-convex subset of K such that $\lambda Y \subseteq Y$ for any $\lambda \in (0, 1)$ and $A : Y^2 \to K$ a mixed monotone operator. Assume that there exists $\varphi : (0, 1) \to (0, 1]$ such that*

$$\varphi(\lambda) > \lambda \quad \text{for any } \lambda \in (0, 1) \tag{3.4.16}$$

$$A(\lambda x, y) \geq \varphi(\lambda) A(x, \lambda y) \quad \text{for any } \lambda \in (0, 1), \ x, y \in Y. \tag{3.4.17}$$

If $(x_0, y_0) \in \Lambda(A)$ is proper, then A has a unique fixed point x^* in $[x_0, y_0]$, $x^* \overset{A}{\Subset} [x_0, y_0]$ and A is m-Picard on $[x_0, y_0]$ with respect to ρ.

Additionally, if $(x, y) \in \Lambda(A)$ is proper and $\lambda \in (0, 1)$ such that $x^* \in [x, y]$ and $\lambda^{-1} y \in Y$, then $(\lambda x, \lambda^{-1} y) \in \Lambda(A)$ is proper, x^* is the unique fixed point of A in $[\lambda x, \lambda^{-1} y]$, $x^* \overset{A}{\Subset} [\lambda x, \lambda^{-1} y]$ and A is m-Picard on $[\lambda x, \lambda^{-1} y]$ with respect to ρ. In particular, if $\lambda \in (0, 1)$ such that $\lambda^{-1} x^* \in Y$, then x^* is the unique fixed point of A in $[\lambda x^*, \lambda^{-1} x^*]$, $x^* \overset{A}{\Subset} [\lambda x^*, \lambda^{-1} x^*]$ and A is m-Picard on $[\lambda x, \lambda^{-1} x^*]$ with respect to ρ.

Theorem 3.4.34 *Let Y be a part of K and $A : Y^2 \to K$ a mixed monotone operator.*

If there exists $u \in Y$ such that $A(u, u) \in Y$ and $\varphi : (0, 1) \to (0, 1]$ such that (3.4.16) and

$$A(\lambda x, y) \geq \varphi(\lambda) A(x, \lambda y) \text{ for any } \lambda \in (0, 1) \text{ and } x, y \in Y \text{ linearly dependent,} \tag{3.4.18}$$

then $A(Y^2) \subseteq Y$, $\Lambda(A)$ is nonempty and A has a unique fixed point x^* in Y. Moreover, A is m-Picard on Y with respect to ρ and $x^* \overset{A}{\Subset} [x_0, y_0]$ for any $(x_0, y_0) \in \Lambda(A)$.

3.5 Some final notes and remarks

3.5.1 Fixed point theorems for mixed monotone operators in ordered Banach spaces

All the results obtained in the framework of ordered linear spaces using Thompson's metric (see Section 3.4) remain valid in the case of ordered Banach spaces with a normal cone (the normality of the cone ensures that ρ is complete, by Theorem 2.4.6). In addition, the ρ-convergence is stronger than the norm-convergence (by Theorem 2.4.6), hence the ρ-attractiveness of the fixed points ensures the norm-attractiveness, meaning that m-Picard operators with respect to ρ are also m-Picard with respect to the norm.

In this way, assuming that $(X, K, \|\cdot\|)$ is an ordered Banach space and K is normal (i.e., $\|\cdot\|$ is semi-monotone), Theorems 3.4.2, 3.4.8, 3.4.9, 3.4.19, 3.4.20, 3.4.26, 3.4.27, 3.4.28, 3.4.29, 3.4.30, 3.4.31, 3.4.32, 3.4.34, 3.4.33 remain valid, while their conclusion can be improved by adding the m-Picard property of A with respect to the norm, whenever the corresponding property with respect to ρ is present.

As a direct consequence, many known results (e.g., Opoĭtsev [74], [75], Guo [32], Liang et al. [57], Liu et al. [59], Xu and Jia [113], Xu and Yuan [111], [112], Wu and Liang [110], Li et al. [56], etc.) follow as particular cases or weaker versions of the fixed point theorems in Section 3.4.

Note that one can also apply the fixed point results from ordered metric spaces (see Section 3.3) to any ordered Banach space with a normal cone, by taking d to be the metric induced by the norm, since the normality of the cone is enough to ensure that d is interval semi-monotone. In this way, we obtain fixed point criteria which are not restricted to positive fixed points, as it would have been the case when using Thompson's metric.

Concluding, all the fixed point results from ordered linear spaces and ordered metric spaces remain valid in the present framework, provided the cone is normal.

3.5.2 The fixed point theory for increasing and for decreasing operators from the perspective of mixed monotone operators

The fixed point theory developed for mixed monotone operators has the advantage that it contains both the theory for increasing and for decreasing operators in one unitary approach. In this way, we obtain the classical results of the two particular cases, while also obtain new results, especially in the case of decreasing operators (which possesses more difficulties than the case of increasing operators).

3.5.3 The duality between increasing operators and mixed monotone operators

In what follows, we develop some ideas of Opoĭtsev [75] and Chen [21] and show that it is always possible to write a coupled fixed point problem for mixed monotone operators as a fixed point problem for increasing operators.

Let X be an ordered linear space with respect to the cone K and consider $\mathcal{X} = X^2$, $\mathcal{K} = K \times (-K)$. It is straightforward that $\mathcal{X}$ is an ordered linear space with respect to $\mathcal{K}$ such that

$$(x_1, y_1) \le (x_2, y_2) \textit{ iff } x_1 \le x_2, y_1 \ge y_2, \qquad (x_1, y_1), (x_2, y_2) \in \mathcal{X}.$$

For any $\mathbf{u} = (x, y) \in \mathcal{X}$, define $\mathbf{u}^* = (y, x)$. If U is a nonempty subset of X and $A : U^2 \to X$, define $\mathbf{A} : \mathcal{U} = U^2 \to \mathcal{X}$ by $\mathbf{A}(\mathbf{u}) = (A(\mathbf{u}), A(\mathbf{u}^*))$. Clearly,

- A is mixed monotone *iff* $\mathbf{A}$ is increasing.
- $\mathbf{u}_0 = (x_0, y_0)$ is a coupled lower-upper fixed point for A *iff* $\mathbf{u}_0$ is a lower fixed point of $\mathbf{A}$ *iff* $\mathbf{u}_0^*$ is an upper fixed point for $\mathbf{A}$.
- $\mathbf{u}^* = (x^*, v^*)$ is a coupled fixed point for A *iff* $\mathbf{u}^*$ is a fixed point for $\mathbf{A}$ *iff* $(\mathbf{u}^*)^*$ is a fixed point for $\mathbf{A}$.

These properties establish a duality between increasing operators and mixed monotone operators, such that a result about the coupled fixed points set of a mixed monotone operator A could be derived from a corresponding result for the fixed points set of increasing operators when applied to $\mathbf{A}$. However, this analogy is not at all *perfect*, since:

- $\mathbf{A}(\mathbf{u}^*) = (\mathbf{A}(\mathbf{u}))^*$ for any $\mathbf{u} \in \mathcal{U}$.
- $\mathbf{u} = (x, y) \ge \theta$ *iff* $x \le \theta \le y$.

Due to this symmetry of $\mathbf{A}$ and, consequently, of its fixed points set, many results for increasing operators may not be applied to $\mathbf{A}$, except in some trivial cases. Also, since the positivity is not preserved from X to $\mathcal{X}$, it is not possible to obtain results about positive fixed points of A from similar ones for $\mathbf{A}$. We also refer to the paper of Kolosov [45] as a possible alternative approach. This leaves open way for *future research.*

3.5.4 The equivalence of some fixed point theorems in ordered metric spaces

We believe that it is possible to show the equivalences between some of our fixed point results results in ordered metric spaces for m-contractive operators, by following the ideas and techniques from the univariate case (we refer here especially to Jachymski [38, Theorem 1], Lim [58, Theorem 1]). Also, other types of contractive properties may be considered. This leaves open way for *future research.*

3.5.5 A brief comparison between some fixed point theorems for mixed monotone operators in ordered metric spaces

Let $(X, \le, d)$ be an ordered metric space with d being a complete metric and $A : X^2 \to X$ a mixed monotone operator. A direct consequence of Theorem 3.3.47 and Proposition 3.3.49 is the following result.

Theorem 3.5.1 *Assume there exists a function* $\Phi : [0, \infty) \to [0, \infty)$ *with* $\Phi(t) < t$ *and* $\limsup_{s \to t+} \Phi(s) < t$ *for any* $t > 0$, *such that*

$$d(A(x, y), A(y, x)) \leq \Phi(d(x, y)) \quad \text{for any } x, y \in X \text{ such that } x \leq y. \tag{3.5.1}$$

Also suppose that X *satisfies property* (C_3') *and* d *is interval-semi-monotone. If there exists* $(x_0, y_0) \in \Lambda(A)$, *then* A *has a unique fixed point* x^* *in* $[x_0, y_0]$, $x^* \overset{A}{\in} [x_0, y_0]$ *and* A *is* m*-Picard on* $[x_0, y_0]$.

In comparison, one of the most recent and general results of a similar type is the following theorem (see also Agarwal et al. [1], Ćirić et al. [25], Gnana Bhaskar and Lakshmikantham [30], Nieto and Rodríguez-López [69] and [70] for similar, less general, results):

Theorem 3.5.2 (Lakshmikantham and Ćirić [54]) *Assume there exists a function* $\Phi : [0, \infty) \to [0, \infty)$ *with* $\Phi(t) < t$ *and* $\lim_{s \to t} \Phi(s) < t$ *for any* $t > 0$, *such that*

$$d(A(x, y), A(u, v)) \leq \Phi\left(\frac{d(x, u) + d(y, v)}{2}\right) \quad \text{for any } x, y, u, v \in X \text{ such that } x \leq u \text{ and } y \geq v.$$

Also suppose either:

1. A *is continuous, or*
2. X *satisfies property* (C_4').

If there exist $x_0, y_0 \in X$ *such that*

$$x_0 \leq A(x_0, y_0) \text{ and } y_0 \geq A(y_0, x_0),$$

then A *has a coupled fixed point. Furthermore, if* x_0, y_0 *are comparable, then* A *has a fixed point.*

Clearly, the Φ-contraction condition (the most restrictive among all the conditions in the above theorems) is more general in our result, while the conclusions are much stronger.

3.5.6 The fixed point problem for bivariate operators in ordered linear spaces via mixed monotone operators

We show here that it is possible to easily extend the fixed point methods for mixed monotone operators to a much larger class of operators, in the framework of ordered linear spaces. The following idea appears, in a less general form, in the papers of Shuvar [96], Guo and Lakshmikantham [34], and will be further developed.

Let U, V be nonempty subsets of a linear space X such that $U \cap V \neq \emptyset$ and $A : U^2 \to V$, $B : V^2 \to X$ arbitrary bivariate operators, if not stated otherwise. Note that, in this context, the bivariate operator

$$B \circledast A := B * A - B + P : (U \cap V)^2 \to X$$

is correctly defined, where P is the canonical projection of X^2 to X (see (3.1.2)).

Definition 3.5.3 B is said to be *mirror injective* (or, simply, m*-injective*) if, for any $x, x', y, y' \in U$,

$$\begin{cases} B(x, y) = B(x', y') \\ B(y, x) = B(y', x') \end{cases}$$

ensures that $x = x'$ and $y = y'$.

Proposition 3.5.4 $\operatorname{cfp}(A) \subseteq \operatorname{cfp}(B \circledast A)$ *and the converse takes place if* B *is* m*-injective.*

Corollary 3.5.5 *If* $\alpha, \beta \in \mathbb{R}$ *such that* $|\alpha| \neq |\beta|$ *and*

$$A_{\alpha,\beta} : U^2 \to X, \quad A_{\alpha,\beta}(x, y) = \alpha A(x, y) - \beta A(y, x) + (1 - \alpha)x + \beta y, \tag{3.5.2}$$

then $\operatorname{cfp}(A) = \operatorname{cfp}\left(A_{\alpha,\beta}\right)$.

Corollary 3.5.6 *If* $\alpha \neq 0$, *then* $\operatorname{cfp}(A) = \operatorname{cfp}(\alpha A + (1 - \alpha)P)$.

If X is an ordered linear space, the above results show that, when studying the (coupled) fixed points of some bivariate operator A (not necessarily mixed monotone), it is possible to replace it with another operator (i.e., with $B \circledast A$, in particular with $A_{\alpha,\beta}$) which may have the mixed monotone property, while preserving the set of (coupled) fixed points. In this way, one can apply the methods from the mixed monotone case to a more general class of operators, though first we may need to study what happens with the set of coupled lower-upper fixed points of A, if we replace A with $B \circledast A$.

From this point forward in the current section, X will be an ordered linear space with respect to some cone K.

Proposition 3.5.7

1. *If B is mixed monotone, then $\Lambda(A) \cap V^2 \subseteq \Lambda(B \circledast A)$.*
2. *If B has a m-left inverse which is mixed monotone, then $\Lambda(A) \cap V^2 \supseteq \Lambda(B \circledast A)$.*

Corollary 3.5.8 *If $\alpha > 0$, then $\Lambda(A) = \Lambda(\alpha A + (1-\alpha)P)$.*

Remark 3.5.9 In connection to Corollaries 3.5.6, 3.5.8, by writing $\alpha = \frac{1}{M+1}$ $(M \geq 0)$, it is a simple exercise to show that $\lambda A + (1-\lambda)P$ is mixed monotone *iff*

$$A(x_2, y_2) \geq A(x_1, y_1) - M(x_2 - x_1)$$

for any $x_1, x_2, y_1, y_2 \in U$ such that $x_1 \leq x_2$, $y_1 \geq y_2$ (i.e., $A + MP$ is mixed monotone)

This particular case has been noted by Guo and Lakshmikantham in [34].

Corollary 3.5.10 *If $\alpha, \beta \geq 0$, then $\Lambda(A) \subseteq \Lambda(A_{\alpha,\beta})$, where $A_{\alpha,\beta}$ is defined by (3.5.2).*

Remark 3.5.11 In Corollary 3.5.10, if $\alpha \neq \beta$ $(\alpha, \beta \geq 0)$, then B is m-invertible and

$$B^{-1}(u, v) = \frac{\alpha}{\alpha^2 - \beta^2} u + \frac{\beta}{\alpha^2 - \beta^2} v.$$

Yet, B^{-1} is mixed monotone *iff* $\alpha(\alpha - \beta) \geq 0, \beta(\alpha - \beta) \leq 0$, which leads to $\alpha = 0$ or $\beta = 0$.

Concluding, B and B^{-1} are both mixed monotone *iff* either $\alpha = 0, \beta > 0$ or $\alpha > 0, \beta = 0$.

In order to obtain, by Proposition 3.5.7, that $\Lambda(A) = \Lambda(A_{\alpha,\beta})$, both B and B^{-1} must be mixed monotone, which happens in two situations: if $\beta = 0$, we obtain the result from Corollary 3.5.8; if $\alpha = 0$, then $\Lambda(A) = \Lambda(A_{0,\beta})$, where

$$A_{0,\beta}(x, y) = x + \beta y - \beta A(y, x) \quad (\beta > 0).$$

3.5.7 Some historical commentaries

Monotone iterative methods related to the technique of lower and upper solutions go back at least to E. Picard in the 1890s, in the study of the Dirichlet problem for nonlinear second order (ordinary and partial) differential equations (see, for example, [77], [78], [79]). Later on, in the the 1930s, the method of upper and lower solutions was further developed, without reference to these iteration schemes, by Scorza-Dragoni [95] and Nagumo [65]. Following Chaplygin and using the technique of lower and upper solutions, the Russian school studied the monotone methods in a systematic way. A first abstract formulation of the monotone iterative methods used by Picard was given by Kantorovich [42] in 1939. This was further developed during the following decades (mainly in the framework of ordered Banach spaces) by Collatz and Schröder [26], Schröder [94], Krasnosel′skiĭ [46], Krasnosel′skiĭ et al. [49], [50], Amann [3], [4], just to mention the early contributions. A detailed account of the development of the method of monotone iterations and the method of lower and upper solutions can be found in the paper of Cherpion et al. [24].

An important step in widening the range of applicability for these methods was done by considering the class of *heterotonic* operators (in the terminology of Opoĭtsev [74]), i.e., the operators T that can be expressed as $T(x) = A(x, x)$ with A being *mixed monotone*.

The concept of mixed monotone operator and the associated iterative techniques go back at least to Kurpel′ in the 1960s (see [52]), in the study of two-sided operator inequalities (*TSOI*) and their applications to the problems of constructing two-sided successive approximations to the solutions of integral, differential, integro-differential and finite (algebraic and transcendental) equations.

In its early stages of development, the method of monotone iterations for heterotonic (and mixed monotone) operators was used mainly for its approximation mechanism, rather than for studying the existence (and uniqueness) of the solution. We point in this direction to the monograph of Kurpel′ and Šuvar [53] (and the references therein), which represents a good survey on the theory and applications of *TSOI* for mixed monotone operators. Also, more than 100 papers in this direction and indexed by *AMS Mathematical Reviews* have appeared since 1980, when the book of Krupel′ and Šuvar was published.

Following the theory of increasing and concave operators developed by Krasnosel′skiĭ and his students during the 1960s (see [7], [8], [46], [47], [48], [49], [97], [98]), Opoĭtsev [73], [74], [75] studied, in the 1970s, a particular class of heterotonic operators (the, so called, *pseudoconcave* heterotonic operators) and established the first (to the best of our knowledge) fixed point results for this type of operators, in the framework of ordered Banach spaces.

In the past three decades, the results of Opoĭtsev have been "*rediscovered*" or extended by several authors (listed here in chronological order), like Moore [64], Khavanin and Lakshmikantham [44], [43], Guo and Lakshmikantham [34], Guo [32], [33], Chen [21], Sun [100], Sun and Liu [99], Zhang [116], [117], Liu and Wu [60], Liang et al. [57], Liu et al. [59], Xu and Jia [113], Xu and Yuan [111], [112], Li and Duan [55], Wu and Liang [110], Li et al. [56], Zhang and Wang [118], Wu [109]. We also refer to the monograph of Guo et al. [31], which contains many results published so far only in Chinese. Regrettably, none of these papers make any reference to the results of Opoĭtsev, although English translations of his articles have been available right after their initial publication in Russian.

4

Applications

In this chapter, we apply the fixed point results from Chapter 3 to several classes of nonlinear problems that can be expressed as fixed point equations for mixed monotone operators. Some of the results (or similar ones) in this chapter have already been published (see [91], [92]).

4.1 Continuous positive solutions of nonlinear Fredholm integral equations

Consider the nonlinear Fredholm integral equation

$$x(t) = \int_0^1 G(t,s) f(s, x(s), x(s))\,\mathrm{d}s, \quad t \in [0,1], \tag{4.1.1}$$

with the unknown x, where f may be singular in the second or third variable. We look for positive, continuous solutions, hence consider $X = \mathcal{C}([0,1])$ to be the linear space of real–valued continuous functions on $[0,1]$. Endowed with the norm $|x|_\infty = \sup_{t\in[0,1]} |x(t)|$ $(x \in X)$ and with respect to the cone $K = \mathcal{C}([0,1]; \mathbb{R}_+)$, X becomes an *ordered Banach space* such that the norm is monotone, hence K is *normal*. Recall that if $x \in K$, then $K(x)$ denotes the part of K containing x. We refer also to Precup [83] for details on the topic of nonlinear integral equations.

4.1.1 Preliminary results

First, assume that

(G_1) $G : [0,1] \times [0,1] \to [0,\infty)$ is continuous and non-identically zero

and consider the following sets and functions associated to G:

$$\begin{aligned} g(t) &= \sup_{s\in[0,1]} G(t,s) = |G(t,\cdot)|_\infty, \quad t \in [0,1] \\ \Gamma &= \{t \in [0,1] : G(t,s) = 0 \text{ for any } s \in [0,1]\} \\ J &= [0,1] \setminus \Gamma \end{aligned}$$

$$\begin{aligned} g_0(s) &= \inf_{t\in J} \frac{G(t,s)}{g(t)}, \quad s \in [0,1] \\ \Gamma_0 &= \{s \in [0,1] : g_0(s) = 0\} \\ J_0 &= [0,1] \setminus \Gamma_0. \end{aligned}$$

Further, we restrict our discussion to the cases when

(G_2) Γ and Γ_0 are finite

and consider the sets

$$\begin{aligned} U_r &= \{x \in K(g) : |x|_\infty \le r\} \quad \text{for any arbitrary } r \in (0,\infty] \\ V &= \left\{x \in \mathcal{C}(J) : x(s) > 0 \text{ for any } s \in J \text{ and } \int_0^1 x(s)\,\mathrm{d}s < \infty\right\} \end{aligned}$$

Clearly, $K(g) = U_\infty$. Also, the linear space $\mathcal{C}(J)$ is considered to be ordered by the cone of positive (i.e., ≥ 0) continuous functions on J.

Now, fix $r, r_0 \in (0,\infty]$ such that $r_0 \le r$. Note that, if $a < b = \infty$, then by $[a,b]$ we understand $[a,\infty)$ and by $(a,b]$ we mean (a,∞). Consider the following conditions on f, r and r_0:

(f_1) $f : J \times (0, r] \times (0, r] \to (0, \infty)$ is continuous.
(f_2) $f(s, u_1, v_1) \le f(s, u_2, v_2)$ for any $s \in J$ and $u_1, u_2, v_1, v_2 \in (0, r]$ such that $u_1 \le u_2$, $v_1 \ge v_2$.
(f_3) For any $\lambda \in (0, 1)$,

$$\inf \left\{ \frac{f(s, \lambda u, v)}{f(s, u, \lambda v)} : s \in J,\ u, v \in (0, r_0] \right\} > \lambda.$$

(f_4) $\int_0^1 f(s, u_0, v_0 g(s))\,\mathrm{d}s < \infty$ for some $v_0 \in \left(0, \dfrac{r_0}{|g|_\infty}\right]$ and $\begin{cases} \text{for all } u_0 \in [r_0, r], & \text{if } r_0 < \infty \\ \text{for some } u_0 > 0, & \text{if } r_0 = \infty \end{cases}$.

Lemma 4.1.3 *Under the assumptions (f_1)–(f_3), the function*

$$\varphi : (0, 1) \to \mathbb{R}, \quad \varphi(\lambda) = \inf \left\{ \frac{f(s, \lambda u, v)}{f(s, u, \lambda v)} : s \in J,\ u, v \in (0, r_0] \right\} \qquad (\lambda \in (0, 1))$$

is correctly defined and verifies

$$\lambda < \varphi(\lambda) \le 1 \quad \textit{for any } \lambda \in (0, 1)$$
$$f(s, \lambda u, v) \ge \varphi(\lambda) f(s, u, \lambda v) \quad \textit{for any } \lambda \in (0, 1),\ s \in J,\ u, v \in (0, r_0]. \tag{4.1.2}$$

Lemma 4.1.4 *If (f_1)–(f_4) are satisfied, then*

$$\int_0^1 f(s, u, v g(s))\,\mathrm{d}s < \infty \quad \textit{for any } u \in (0, r],\ v \in \left(0, \frac{r}{|g|_\infty}\right].$$

Lemma 4.1.5 *Assume (G_1), (G_2), (f_1), (f_2) and (f_4). If $x^* \in K$ is a solution of (4.1.1), then $x^* \in U_r$.*

Proposition 4.1.7 *Under the assumptions (G_1), (G_2), (f_1)–(f_4), the operator*

$$A : U_r^2 \to K(g), \quad A(x, y)(t) = \int_0^1 G(t, s) f(s, x(s), y(s))\,\mathrm{d}s, \quad (x, y \in U_r,\ t \in [0, 1])$$

is correctly defined, mixed monotone and

$$A(\lambda x, y) \ge \varphi(\lambda) A(x, \lambda y) \quad \textit{for any } \lambda \in (0, 1),\ x, y \in U_{r_0}.$$

4.1.2 The main results

Definition 4.1.8 If $x_0, y_0 \in K(g)$ satisfy

$$\begin{cases} x_0(t) \le y_0(t) \\ x_0(t) \le \int_0^1 G(t, s) f(s, x_0(s), y_0(s))\,\mathrm{d}s \\ y_0(t) \ge \int_0^1 G(t, s) f(s, y_0(s), x_0(s))\,\mathrm{d}s \end{cases} \quad \text{for any } t \in [0, 1], \tag{4.1.3}$$

then the pair (x_0, y_0) is called a *lower-upper quasi-solution* of (4.1.1).

Theorem 4.1.9 *Under the assumptions (G_1), (G_2), (f_1)–(f_4) with $r_0 = r = \infty$, the equation (4.1.1) has a unique solution $x^* \in K$.*

Moreover, $x^* \in K(g)$ and for any $x_0, y_0 \in K(g)$, the sequences $(x_n), (y_n)$ defined by

$$\begin{cases} x_{n+1}(t) = \int_0^1 G(t, s) f(s, x_n(s), y_n(s))\,\mathrm{d}s \\ y_{n+1}(t) = \int_0^1 G(t, s) f(s, y_n(s), x_n(s))\,\mathrm{d}s \end{cases}, \quad t \in [0, 1],\ n \in \mathbb{N} \tag{4.1.4}$$

are convergent to x^* with respect to both the infinity norm $|\cdot|_\infty$ and Thompson's metric ρ.

Additionally, if (x_0, y_0) is a lower-upper quasi-solution of (4.1.1), then $(x_n(t))$ is increasing, $(y_n(t))$ is decreasing and $x_n(t) \le x^*(t) \le y_n(t)$ for any $t \in [0, 1]$ and $n \in \mathbb{N}$.

Theorem 4.1.10 *Assume (G_1), (G_2), (f_1)–(f_4) with $r_0 < \infty$ and let $(x_0, y_0) \in U_r^2$ be a lower-upper quasi-solution of (4.1.1). Define the sequences $(x_n), (y_n)$ by (4.1.4) and assume that there exists $k \in \mathbb{N}$ such that $|y_k|_\infty \le r_0$. Then the equation (4.1.1) has a unique solution $x^* \in K$ such that $x_0(t) \le x^*(t) \le y_0(t)$ for any $t \in [0, 1]$. Moreover, the following properties are satisfied:*

1. *$(x_n(t))$ is increasing, $(y_n(t))$ is decreasing and $x_n(t) \le x^*(t) \le y_n(t)$ for any $t \in [0, 1]$ and $n \in \mathbb{N}$;*
2. *$(x_n), (y_n)$ are convergent to x^* with respect to both the infinity norm $|\cdot|_\infty$ and Thompson's metric ρ.*

Additionally, for any $u_0, v_0 \in U_r$ such that (u_0, v_0) is a lower-upper quasi-solution of (4.1.1) and

$$u_0(t) \le x^*(t) \le v_0(t) \quad \textit{for any } t \in [0, 1], \tag{4.1.5}$$

define

$$\begin{cases} u_{n+1}(t) = \int_0^1 G(t, s) f(s, u_n(s), v_n(s))\,\mathrm{d}s \\ v_{n+1}(t) = \int_0^1 G(t, s) f(s, v_n(s), u_n(s))\,\mathrm{d}s \end{cases}, \quad t \in [0, 1],\ n \in \mathbb{N}. \tag{4.1.6}$$

If there exists $k \in \mathbb{N}$ such that $|v_k|_\infty \le r_0$, then x^* is the unique solution of (4.1.1) in K that satisfies (4.1.5).

4.2 Positive solutions of second-order two-point boundary value problems

Consider the second-order two-point boundary value problem

$$\begin{cases} -x''(t) = f(t, x(t), x(t)), & t \in (0,1) \\ a_0 x(0) - b_0 x'(0) = 0 \\ a_1 x(1) + b_1 x'(1) = 0 \end{cases} \tag{4.2.1}$$

with

$$\begin{cases} a_0, a_1, b_0, b_1 \geq 0 \\ \delta := a_0 a_1 + a_0 b_1 + a_1 b_0 > 0 \end{cases}. \tag{4.2.2}$$

4.2.1 Preliminary results on Green's function

It is a standard result that the boundary value problem (4.2.1) can be reduced to the Fredholm integral equation (4.1.1), with G being the Green's function given by

$$G(t,s) = \frac{1}{\delta} \begin{cases} (a_0 t + b_0)(a_1(1-s) + b_1) & \text{if } 0 \leq t \leq s \leq 1 \\ (a_0 s + b_0)(a_1(1-t) + b_1) & \text{if } 0 \leq s < t \leq 1 \end{cases}. \tag{4.2.3}$$

By letting $u_0(t) = \dfrac{a_0 t + b_0}{a_0 + b_0}$ and $u_1(t) = \dfrac{a_1(1-t) + b_1}{a_1 + b_1}$ for any $t \in [0,1]$, we may rewrite

$$G(t,s) = \frac{(a_0 + b_0)(a_1 + b_1)}{\delta} \begin{cases} u_0(t) u_1(s) & \text{if } 0 \leq t \leq s \leq 1 \\ u_0(s) u_1(t) & \text{if } 0 \leq s < t \leq 1 \end{cases}.$$

It is easy to obtain that

$$g(s) = G(s,s) = \frac{(a_0 s + b_0)(a_1(1-s) + b_1)}{\delta} = \frac{(a_0 + b_0)(a_1 + b_1)}{\delta} u_0(s) u_1(s), \quad s \in [0,1] \tag{4.2.4}$$

$$g_0(t) = \min\{u_0(t), u_1(t)\}, \quad t \in [0,1].$$

A simple computation also shows that

$$\frac{(a_0 + b_0)(a_1 + b_1)}{\delta + a_0 a_1} g_0(t) \leq g(t) \leq \frac{(a_0 + b_0)(a_1 + b_1)}{\delta} g_0(t) \quad \text{for any } t \in [0,1]$$

proving that g and g_0 are linked. Note that $\Gamma = \Gamma_0 \subseteq \{0,1\}$.

4.2.2 The main results

Theorem 4.2.1 *Under the assumptions (4.2.2), (f_1)–(f_4) with $r_0 = r = \infty$, $J = (0,1)$, G and g given by (4.2.3)-(4.2.4), the problem (4.2.1) has a unique positive solution $x^* \in \mathcal{C}^2(0,1) \cap \mathcal{C}^1[0,1]$.*

Moreover, $x^* \in K(g)$ and for any $x_0, y_0 \in K(g)$, the sequences $(x_n), (y_n)$ defined by (4.1.4) are convergent to x^* with respect to both the infinity norm $|\cdot|_\infty$ and Thompson's metric ρ.

Additionally, if (x_0, y_0) is a lower-upper quasi-solution of (4.1.1), then $(x_n(t))$ is increasing, $(y_n(t))$ is decreasing and $x_n(t) \leq x^*(t) \leq y_n(t)$ for any $t \in [0,1]$ and $n \in \mathbb{N}$.

Theorem 4.2.2 *Assume (4.2.2), (f_1)–(f_4) with $r_0 < r \leq \infty$, $J = (0,1)$, G and g given by (4.2.3)-(4.2.4) and let $(x_0, y_0) \in U_r^2$ be a lower-upper quasi-solution of (4.1.1).*

If the sequences $(x_n), (y_n)$ given by (4.1.4) satisfy $|y_k|_\infty \leq r_0$ for some $k \in \mathbb{N}$, then the problem (4.2.1) has a unique positive solution $x^* \in \mathcal{C}^2(0,1) \cap \mathcal{C}^1[0,1]$ such that $x_0(t) \leq x^*(t) \leq y_0(t)$ for any $t \in [0,1]$.

Moreover, the following properties are satisfied:

1. *$(x_n(t))$ is increasing, $(y_n(t))$ is decreasing and $x_n(t) \leq x^*(t) \leq y_n(t)$ for any $t \in [0,1]$ and $n \in \mathbb{N}$;*
2. *$(x_n), (y_n)$ are convergent to x^* with respect to both the infinity norm $|\cdot|_\infty$ and Thompson's metric ρ.*

Additionally, if $u_0, v_0 \in U_r$ are such that (u_0, v_0) is a lower-upper quasi-solution of (4.1.1),

$$u_0(t) \leq x^*(t) \leq v_0(t) \quad \textit{for any } t \in [0,1], \tag{4.2.5}$$

and $|v_k|_\infty \leq r_0$ for some $k \in \mathbb{N}$, with $(u_n), (v_n)$ defined by (4.1.6), then x^* is the unique positive solution of (4.2.1) in $\mathcal{C}^2(0,1) \cap \mathcal{C}^1[0,1]$ that satisfies (4.1.5).

4.2.3 Some examples

Consider the second-order two-point boundary value problem

$$\begin{cases} -x'' = \mu(x^\alpha + x^{-\beta}), & t \in (0,1) \\ a_0 x(0) - b_0 x'(0) = 0 \\ a_1 x(1) + b_1 x'(1) = 0 \end{cases} \tag{4.2.6}$$

with

$$\begin{cases} a_0, a_1, b_0, b_1, \alpha, \beta \geq 0 \\ \mu > 0 \\ \delta := a_0 a_1 + a_0 b_1 + a_1 b_0 > 0 \end{cases}. \tag{4.2.7}$$

Let

$$f : (0, \infty) \times (0, \infty) \to (0, \infty), \quad f(x, y) = \mu(x^\alpha + y^{-\beta}).$$

Clearly, f is continuous and mixed monotone, hence (f_1) and (f_2) are satisfied, with $r = \infty$.

Lemma 4.2.3 *If $\beta < 1$, then (f_3) and (f_4) are satisfied, with $r = \infty$ and any r_0 such that*

1. $r_0 \leq \infty$, *if* $\alpha < 1$;
2. $r_0 < \infty$, *if* $\alpha = 1$;
3. $r_0 \leq \left(\dfrac{1-\beta}{\alpha - 1}\right)^{\frac{1}{\alpha+\beta}}$, *if* $\alpha > 1$.

When $\alpha, \beta < 1$, we rediscover a result of Zhao [119]. However, our result is slightly more general and the proof follows a different pattern.

Theorem 4.2.4 *Under the assumptions (4.2.7) and $\alpha, \beta < 1$, the problem (4.2.6) has a unique positive solution $x^* \in \mathcal{C}^2(0, 1) \cap \mathcal{C}^1[0, 1]$.*

Moreover, if G and g are defined by (4.2.3)-(4.2.4), then $x^* \in K(g)$ and for any $x_0, y_0 \in K(g)$, the sequences $(x_n), (y_n)$ given by

$$\begin{cases} x_{n+1}(t) = \mu \int_0^1 G(t, s) \left(x_n(s)^\alpha + y_n(s)^{-\beta}\right) \mathrm{d}s \\ y_{n+1}(t) = \mu \int_0^1 G(t, s) \left(y_n(s)^\alpha + x_n(s)^{-\beta}\right) \mathrm{d}s \end{cases}, \quad t \in [0, 1],\ n \in \mathbb{N} \tag{4.2.8}$$

are convergent to x^* with respect to both the infinity norm $|\cdot|_\infty$ and Thompson's metric ρ.

Additionally, if

$$\begin{cases} x_0(t) \leq y_0(t) \\ x_0(t) \leq \mu \int_0^1 G(t, s) \left(x_0(s)^\alpha + y_0(s)^{-\beta}\right) \mathrm{d}s \\ y_0(t) \geq \mu \int_0^1 G(t, s) \left(y_0(s)^\alpha + x_0(s)^{-\beta}\right) \mathrm{d}s \end{cases} \quad \textit{for any } t \in [0, 1], \tag{4.2.9}$$

then $(x_n(t))$ is increasing, $(y_n(t))$ is decreasing and $x_n(t) \leq x^*(t) \leq y_n(t)$ for any $t \in [0, 1]$ and $n \in \mathbb{N}$.

Theorem 4.2.5 *Assume (4.2.7), $\alpha > 1$ and $\beta < 1$ and let G and g be defined by (4.2.3)-(4.2.4).*

If $(x_0, y_0) \in K(g)^2$ satisfies (4.2.9) and the sequences $(x_n), (y_n)$ given by (4.2.8) satisfy

$$|y_k|_\infty \leq \left(\frac{1-\beta}{\alpha - 1}\right)^{\frac{1}{\alpha+\beta}} \quad \textit{for some } k \in \mathbb{N},$$

then the problem (4.2.6) has a unique positive solution $x^* \in \mathcal{C}^2(0, 1) \cap \mathcal{C}^1[0, 1]$ such that $x_0(t) \leq x^*(t) \leq y_0(t)$ for any $t \in [0, 1]$.

Moreover, the following properties are satisfied:

1. *$(x_n(t))$ is increasing, $(y_n(t))$ is decreasing and $x_n(t) \leq x^*(t) \leq y_n(t)$ for any $t \in [0, 1]$ and $n \in \mathbb{N}$;*
2. *$(x_n), (y_n)$ are convergent to x^* with respect to both the infinity norm $|\cdot|_\infty$ and Thompson's metric ρ.*

Additionally, if $u_0, v_0 \in K(g)$ are such that

$$\begin{cases} u_0(t) \leq v_0(t) \\ u_0(t) \leq \mu \int_0^1 G(t, s) \left(u_0(s)^\alpha + v_0(s)^{-\beta}\right) \mathrm{d}s \\ v_0(t) \geq \mu \int_0^1 G(t, s) \left(v_0(s)^\alpha + u_0(s)^{-\beta}\right) \mathrm{d}s \end{cases} \quad \textit{for any } t \in [0, 1], \tag{4.2.10}$$

$$u_0(t) \leq x^*(t) \leq v_0(t) \quad \textit{for any } t \in [0, 1], \tag{4.2.11}$$

and

$$|v_k|_\infty \leq \left(\frac{1-\beta}{\alpha - 1}\right)^{\frac{1}{\alpha+\beta}} \quad \textit{for some } k \in \mathbb{N},$$

with $(u_n), (v_n)$ defined by

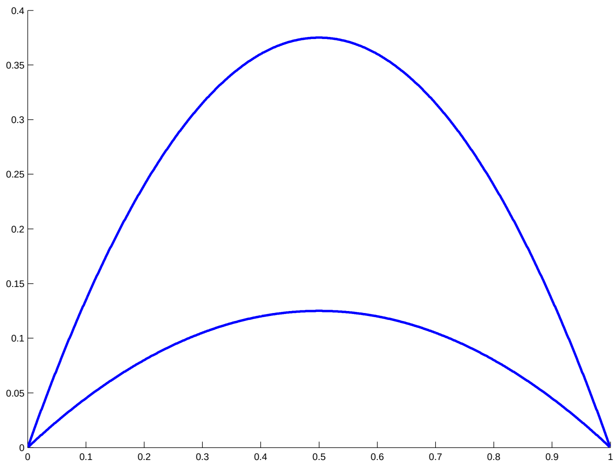
$$\begin{cases} u_{n+1}(t) = \mu \int_0^1 G(t, s) \left(u_n(s)^\alpha + v_n(s)^{-\beta}\right) \mathrm{d}s \\ v_{n+1}(t) = \mu \int_0^1 G(t, s) \left(v_n(s)^\alpha + u_n(s)^{-\beta}\right) \mathrm{d}s \end{cases}, \quad t \in [0, 1],\ n \in \mathbb{N},$$

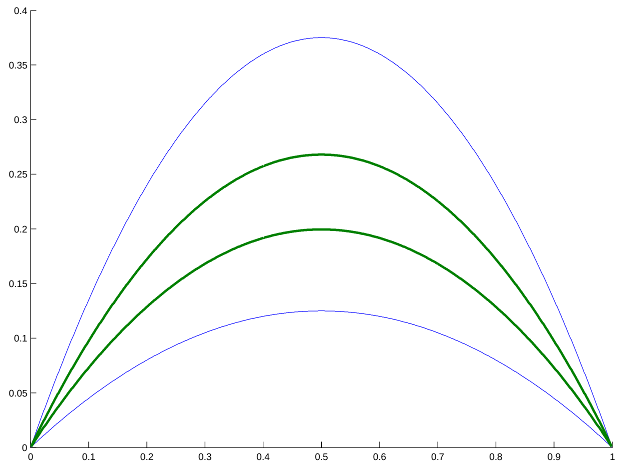
then x^* is the unique positive solution of (4.2.6) in $\mathcal{C}^2(0, 1) \cap \mathcal{C}^1[0, 1]$ that satisfies (4.2.11).

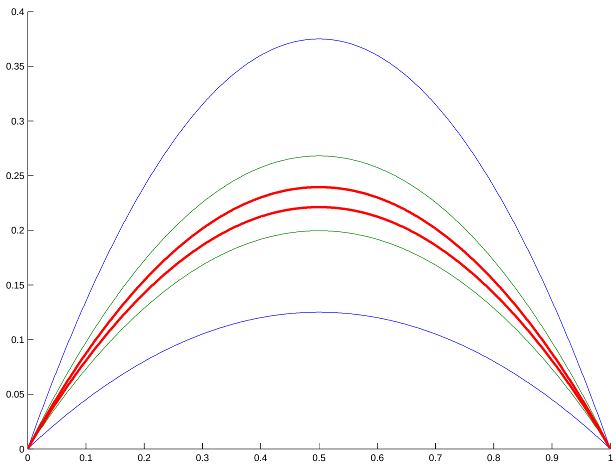
Theorem 4.2.6 *Under the assumptions (4.2.7) and $\alpha = 1$, $\beta < 1$, the problem (4.2.6) has a unique positive solution $x^* \in \mathcal{C}^2(0, 1) \cap \mathcal{C}^1[0, 1]$ and the conclusions of Theorem 4.2.4 hold.*

4.2.4 Some graphical representations

We illustrate our results by some graphical representations of the two-sided approximation of the solution in some concrete cases. One example is the following:


The initial pair (x_0, y_0)


The result after the 1st iteration


The result after the 2nd iteration

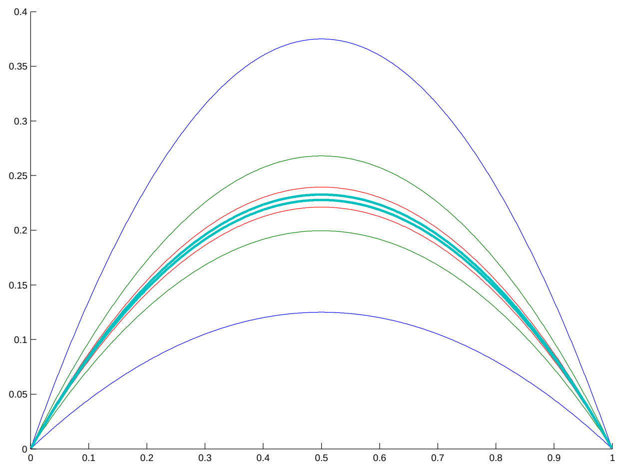

The result after the 3rd iteration

Figure 1: The two-sided approximation of the unique positive solution of

$$\begin{cases} -x'' = x^{\frac{1}{2}} + x^{-\frac{1}{5}}, & t \in (0,1) \\ x(0) = x(1) = 0 \end{cases}$$

where the initial *guess* is given by $x_0(t) = \frac{t(1-t)}{2}$ and $y_0(t) = \frac{3t(1-t)}{2}$ $(t \in [0,1])$

4.3 The fixed point problem for systems of partially monotone operators

The following is an application of the methods and results from Chapter 3 to an abstract problem. Let $N \geq 1$ be an integer, $\{(X_i, \leq) : i \in \{1, 2, \ldots, N\}\}$ a family of ordered sets, $X := X_1 \times X_2 \times \cdots \times X_N$ ordered with respect to

$$x = (x^1, x^2, \ldots, x^N) \leq (y^1, y^2, \ldots, y^N) = y \text{ iff } x^i \leq y^i \text{ for any } i \in \{1, 2, \ldots, N\},$$

$U_i \subseteq X_i$ $(i \in \{1, 2, \ldots, N\})$, $U := U_1 \times U_2 \times \ldots \times U_N \subseteq X$, $\{T_i : U \to X_i : i \in \{1, 2, \ldots, N\}\}$ a family of N-variate operators and $T = (T_1, T_2, \ldots, T_N) : U \to X$. We study the system

$$x^i = T_i(x^1, x^2, \ldots, x^N), \quad x = (x^1, x^2, \ldots, x^N) \in U, \; i \in \{1, 2, \ldots, N\} \tag{4.3.1}$$

under the assumption that T_i is a *partially monotone operator* for any $i \in \{1, 2, \ldots, N\}$, i.e., T_i is monotone (increasing or decreasing) with respect to each variable independently. Clearly, the system (4.3.1) can be viewed as a fixed point problem on U for the operator T. The idea we developed in [91] was to prove, in a constructive way, that T is a heterotonic operator, i.e., there exists a mixed monotone operator $A : U^2 \to X$ such that

$$A(x, x) = T(x) \quad \text{for any } x \in U. \tag{4.3.2}$$

In this way, we can establish an equivalence between (4.3.1) and the fixed point problem for A on U

$$x = A(x,x), \quad x \in U. \tag{4.3.3}$$

In order to construct such an operator A, we first associate to T_i (for any $i \in \{1,2,\dots,N\}$) an operator

$$\Sigma_i = (\Sigma_{i,1}, \Sigma_{i,2}, \dots, \Sigma_{i,N}) : U^2 \to U$$

whose components are defined by

$$\Sigma_{i,j} : U^2 \to U_j, \quad \Sigma_{i,j}(x,y) = \begin{cases} x_j, & \text{if } T_i \text{ is increasing in the } j\text{-th variable} \\ y_j, & \text{if } T_i \text{ is decreasing in the } j\text{-th variable} \end{cases} \quad (j \in \{1,2,\dots,N\}, \quad x,y \in U).$$

We also write $x \le_i y$ $(x,y \in X)$ when $\Sigma_i(x,y) \le \Sigma_i(y,x)$, i.e.,

$$\begin{cases} x_j \le y_j \text{ if } T_i \text{ is increasing in the } j\text{-th variable} \\ x_j \ge y_j \text{ if } T_i \text{ is decreasing in the } j\text{-th variable} \end{cases} \quad (j \in \{1,2,\dots,N\}). \tag{4.3.4}$$

This clearly defines an (alternative) order on X. In these conditions and using the notations from Chapter 3, we prove:

Lemma 4.3.1 *For any $i \in \{1,2,\dots,N\}$:*

1. $\Sigma_i(\Sigma_i(x,y), \Sigma_i(u,v)) = \Sigma_i(x,v)$ *for any $x,y,u,v \in U$.*
2. $\Sigma_i(x,x) = x$ *for any $x \in U$.*
3. $\Sigma_i^2 = P_U$ *, where the functional powers are considered with respect to the m-composition.*
4. Σ_i *is m-invertible (i.e., invertible with respect to the m-composition) and* $\Sigma_i^{-1} = \Sigma_i$.
5. $\Sigma_i(x,y) \le_i \Sigma_i(u,v)$ iff $\Sigma_i(x,v) \le \Sigma_i(u,y)$ *for any $x,y,u,v \in U$.*
6. $\Sigma_i(x,y) \le_i \Sigma_i(y,x)$ iff $x \le y$ *for any $x,y \in U$.*
7. $\Sigma_i : (U^2, \le) \to (U, \le)$ *is increasing, i.e., $\Sigma_i(x,y) \le \Sigma_i(u,v)$ if $x \le u$ and $y \le v$ $(x,y,u,v \in U)$.*
8. $\Sigma_i : (U^2, \le) \to (U, \le_i)$ *is mixed monotone, i.e., $\Sigma_i(x,y) \le_i \Sigma_i(u,v)$ if $x \le u$ and $y \ge v$ $(x,y,u,v \in U)$.*
9. $T_i : (U, \le_i) \to (X_i, \le)$ *is increasing, i.e., $T_i(x) \le T_i(y)$ if $x \le_i y$ $(x,y \in U)$.*

Theorem 4.3.2 *The operator*

$$A = (A_1, A_2, \dots, A_N) : U^2 \to X, \quad A_i = T_i \Sigma_i \quad (i \in \{1,2,\dots,N\}) \tag{4.3.5}$$

is mixed monotone and satisfies (4.3.2). Moreover,

$$(A_i * \Sigma_i)(x,y) = T_i(x) \quad \textit{for any } i \in \{1,2,\dots,N\} \textit{ and } x,y \in U.$$

Example 4.3.3 If $N = 3$ and T_1 is decreasing in the first two arguments and increasing in the last argument (and write $T_1 : \searrow\searrow\nearrow$ for short), then $\Sigma_1(x,y) = (y^1, y^2, x^3)$ and $A_1(x,y) = T_1(y^1,y^2,x^3)$ for any $x = (x^1,x^2,x^3)$, $y = (y^1,y^2,y^3) \in U$. Also, $x \le_1 y$ *iff* $x^1 \ge y^1, x^2 \ge y^2, x^3 \le y^3$.

Concluding, the system (4.3.1), under the assumptions that T_i are partially monotone operators, is equivalent to the fixed point problem for the mixed monotone operator A defined by Theorem 4.3.2, hence this problem can be approached by any of the available techniques and results for mixed monotone operators from the previous chapter. Clearly, any fixed point result for T is expected to be expressed in terms of the operators T_i, rather than using the mixed monotone operator A, which should be considered only as an auxiliary tool. It is not our intention to develop here a theory for systems of equations involving partially monotone operators (which we believe is an interesting topic for a *future research*), but merely to give an idea of what kind of conditions and results are to be expected.

4.3.1 A fixed point theorem in ordered metric spaces

Assume next that d_i is a complete metric on X_i that satisfies (C_3') and is interval semi-monotone, with γ_i denoting the semi-monotonicity constant; also, let $U_i := X_i$ for any $i \in \{1,2,\dots,N\}$, hence $U = X$. Clearly,

$$d : X^2 \to [0,\infty), \quad d(x,y) = \max_i d_i(x_i, y_i) \quad (x,y \in X)$$

is a complete metric on X that verifies (C_3') and is interval semi-monotone, with $\max_i \gamma_i$ the semi-monotonicity constant.

A direct consequence of Theorems 3.5.1 and 4.3.2 is the following result.

Theorem 4.3.4 *Let $\Phi : [0,\infty) \to [0,\infty)$ be a function satisfying $\Phi(t) < t$ and $\limsup_{s\to t+} \Phi(s) < t$ for any $t > 0$. Suppose that*

$$d_i(T_i(x), T_i(y)) \le \Phi\left(\max_j d_j(x_j, y_j)\right) \quad \textit{for any } x,y \in X \textit{ with } x \le_i y \textit{ and } i \in \{1,2,\dots,N\}. \tag{4.3.6}$$

If there exists $(x_0, y_0) \in X^2$ such that

$$\begin{cases} x_0^i \le y_0^i \\ x_0^i \le T_i\Sigma_i(x_0, y_0) \\ y_0^i \ge T_i \Sigma_i(y_0, x_0) \end{cases} \quad (i \in \{1,2,\dots,N\}), \tag{4.3.7}$$

then T is a Picard operator on $[x_0, y_0]$. Moreover, if x^* is the unique fixed point of T (i.e., the unique solution of (4.3.1)) in $[x_0, y_0]$, then the sequences (x_n), respectively (y_n) defined by

$$\begin{cases} x^i_{n+1} = T_i \Sigma_i(x_n, y_n) \\ y^i_{n+1} = T_i \Sigma_i(y_n, x_n) \end{cases} \quad (i \in \{1, 2, \ldots, N\},\ n \in \mathbb{N}) \tag{4.3.8}$$

are increasing, respectively decreasing, and convergent to x^*.

4.3.2 A fixed point theorem in ordered linear spaces

In what follows, assume that X_i is an ordered linear space with respect to the Archimedean and self-complete cone K_i (for any $i \in \{1, 2, \ldots, N\}$) and that U_i is a part of K_i. Clearly, X is an ordered linear space with respect to the cone $K = K_1 \times K_2 \times \cdots \times K_N$. It is easy to check that U is a part of K and that K is Archimedean and self-complete. In particular, one may consider that $(X_i, K_i, |\cdot|_i)$ is an ordered Banach space with a normal cone for any $i \in \{1, 2, \ldots, N\}$.

The following result follows by Theorem 3.4.34.

Theorem 4.3.5 *Assume there exists $\varphi : (0, 1) \to (0, 1]$ such that $\varphi(\lambda) > \lambda$ for any $\lambda \in (0, 1)$ and*

$$T_i \Sigma_i(\lambda x, x) \geq \varphi(\lambda) T_i \Sigma_i(x, \lambda x) \text{ for any } \lambda \in (0, 1),\ x \in U \text{ and } i \in \{1, 2, \ldots, N\}. \tag{4.3.9}$$

If there exists $u_0 \in U$ such that $T(u_0) \in U$, then $T(U) \subseteq U$, T is Picard on U with respect to Thompson's metric and there exists $(x_0, y_0) \in U^2$ satisfying (4.3.7). Moreover, for any $(x_0, y_0) \in U^2$ satisfying (4.3.7), the sequences (x_n), (y_n) defined by (4.3.8) are increasing, respectively decreasing, and convergent to the unique fixed point of T in U.

4.3.3 An abstract example

Consider (4.3.1) for $N = 3$: $\begin{cases} x^1 = T_1(x^1, x^2, x^3) \\ x^2 = T_2(x^1, x^2, x^3) \\ x^3 = T_3(x^1, x^2, x^3) \end{cases}$ and assume that $T_1 : \nearrow\searrow\searrow$, $T_2 : \searrow\nearrow\searrow$ and $T_3 : \searrow\searrow\searrow$. In this case, $\begin{cases} \Sigma_1(x, y) = (x^1, y^2, y^3) \\ \Sigma_2(x, y) = (y^1, x^2, y^3) \\ \Sigma_3(x, y) = (y^1, y^2, y^3) \end{cases}$ and $A(x, y) = \begin{pmatrix} T_1(x^1, y^2, y^3) \\ T_2(y^1, x^2, y^3) \\ T_3(y^1, y^2, y^3) \end{pmatrix}^t$ for any $x = (x^1, x^2, x^3)$ and $y = (y^1, y^2, y^3)$.
Assume also the framework from the previous subsection. In this case, (4.3.9) becomes

$$\begin{cases} T_1(\lambda x^1, x^2, x^3) & \geq & \varphi(\lambda) T_1(x^1, \lambda x^2, \lambda x^3) \\ T_2(x^1, \lambda x^2, x^3) & \geq & \varphi(\lambda) T_2(\lambda x^1, x^2, \lambda x^3) \\ T_3(x^1, x^2, x^3) & \geq & \varphi(\lambda) T_2(\lambda x^1, \lambda x^2, \lambda x^3) \end{cases}, \tag{4.3.10}$$

(4.3.7) is explicitly written as

$$\begin{cases} x_0 \leq y_0 & \\ x^1_0 \leq T_1(x^1_0, y^2_0, y^3_0) & y^1_0 \geq T_1(y^1_0, x^2_0, x^3_0) \\ x^2_0 \leq T_2(y^1_0, x^2_0, y^3_0) & y^2_0 \geq T_2(x^1_0, y^2_0, x^3_0) \\ x^3_0 \leq T_3(y^1_0, y^2_0, y^3_0) & y^3_0 \geq T_3(x^1_0, x^2_0, x^3_0) \end{cases}. \tag{4.3.11}$$

and the sequences (x_n), respectively (y_n) in (4.3.8) are

$$\begin{cases} x^1_{n+1} = T_1(x^1_n, y^2_n, y^3_n) & y^1_{n+1} = T_1(y^1_n, x^2_n, x^3_n) \\ x^2_{n+1} = T_2(y^1_n, x^2_n, y^3_n) & y^2_{n+1} = T_2(x^1_n, y^2_n, x^3_n) \\ x^3_{n+1} = T_3(y^1_n, y^2_n, y^3_n) & y^3_{n+1} = T_3(x^1_n, x^2_n, x^3_n) \end{cases}. \tag{4.3.12}$$

Theorem 4.3.5 rewrites as follows:

Theorem 4.3.6 *Assume there exists $\varphi : (0, 1) \to (0, 1]$ such that $\varphi(\lambda) > \lambda$ for any $\lambda \in (0, 1)$ and (4.3.10) for any $\lambda \in (0, 1)$ and $x = (x^1, x^2, x^3) \in U$. If there exists $u_0 \in U$ such that $T(u_0) \in U$, then $T(U) \subseteq U$, T is Picard on U with respect to Thompson's metric and there exists $(x_0, y_0) \in U^2$ satisfying (4.3.11). Moreover, for any $(x_0, y_0) \in U^2$ satisfying (4.3.11), the sequences (x_n), (y_n) defined by (4.3.12) are increasing, respectively decreasing, and convergent to the unique fixed point of T in U.*

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