BABEŞ-BOLYAI UNIVERSITY OF CLUJ-NAPOCA
FACULTY OF MATHEMATICS AND INFORMATICS

PhD THESIS

ELEMENTS OF DYNAMICS AND GEOMETRY ON POISSON VECTOR SPACES

SUMMARY

Scientific adviser:
Prof. univ. dr. Dorin ANDRICA

PhD student:
ŞUŞOI Paul-Mihai

CLUJ- NAPOCA
2010

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INTRODUCTION

The theory of the dynamical systems has been rapidly developed due to its remarkable success in mathematical modelling of real phenomena and processes from physics, chemistry, biology, economics and various other scientific domains [35](Marsden and Raţiu, 1994), [18](Hirsch, Smale and Devaney, 2003), [41](Petrera, Phadlery and Surris, 2009), [43](Puta, 1993), [2](Andrica and Caşu, 2008). The analysis of the modelled processes may be studied using different geometrical, analytical and numerical methods. A class of dynamical systems is constituted from the HamiltonPoisson systems.

In order to dwell upon the dynamical properties of a HamiltonPoisson system, are investigated the following issues:

(1) existence of Poisson structure and of some Casimir functions;

(2) problem of stability in the stationary states and the existence of periodic orbits;

(3) determining of a Lax formulation and problem of numerical integration of the dynamical system, using geometrical integrators.

An extension of dynamical systems of the HamiltonPoisson type is represented by metriplectic systems, [27] (A. N. Kaufman, 1984),[40] (Ortega and Planas-Bielsa, 2004).

The scientific researches being carried during the last two decades have highlighted a major interest in study of non-linear systems with control on matrix Lie groups. All these appear naturally in various domains such as: robotics, elasticity, molecular dynamics, aeronautics and many others, [33](Leonard and Krishnaprasad, 1998), [28](Khalil, 2002).

The theory of finite-dimensional Lie groups has offered an open setting for to be explained and well-understood a series of phenomena from geometrical mechanics, theoretical physics and the control theory. In this respect there may be mentioned: the Lie group $SO(3)$ and the theory of free rigid body, the Lie group $SE(2, \mathbf{R})$ and the laser-matter dynamics, the matrix Lie groups and the control theory. The topic of the last chapter is focused upon this particular subject matter.

The present thesis has as its declared goal to bring in some geometrical and dynamical details of the dynamical systems on $\mathbf{R}^n$ *and to provide solutions to some problems in the area of geometrical mechanics.*

The work is divided into four chapters that to ensure the unitary character of its content and the relevance of the researched topics. The thesis is based on 53 bibliographical references.

Chapter 1, entitled "*Elements of geometrical mechanics*", is structured in four paragraphs and it has a monographic character. The principal objective is the presentation in a brief form of some results about symplectic structures, Hamiltonian mechanical systems, Poisson manifolds and Hamilton-Poisson mechanical systems.

In *Paragraph 1.1* are defined notions from symplectic geometry and those of Hamiltonian mechanical system. Are presented fundamental results linked of these [structure theorem of Darboux (Theorem 1.1.15), Hamilton's equations (Theorem 1.1.19), conservation energy principle (Proposition 1.1.21), Liouville's theorem (Proposition 1.1.22)].

The *Paragraph 1.2* contains notions and basic theorems from Poisson geometry [Definitions 1.2.1, 1.2.2, Proposition 1.2.3, structure theorem of Darboux-Lie-Weinstein (Theorem 1.2.11), Theorem 1.2.20, Propositions 1.2.21–1.2.23]. One proves that a bracket on the algebra $C^\infty(\mathbf{R}^n, \mathbf{R})$ which is $\mathbf{R}-$bilinear, skew-symmetric and satisfying Leibniz rule determines a Poisson structure on $\mathbf{R}^n$ if and only if Jacobi identity is verified for coordinate functions (Proposition 1.2.17). The third Section is dedicated to presentation of Poisson structures on the dual of Lie algebra (Propositions 1.2.29, 1.2.34).

The *Paragraph 1.3* deals with the Hamilton-Poisson mechanical systems on $\mathbf{R}^n$. Is defined the notion of Hamilton-Poisson realization for a system of differential equations on $\mathbf{R}^n$ and we present some important properties of those dynamical systems [conservation of energy (Proposition 1.3.2), the flow of an Hamilton-Poisson system preserves the Poisson structure (Proposition 1.3.3)].

In *Paragraph 1.4* gives two important examples of Hamilton-Poisson systems on $\mathbf{R}^3$, namely: the Euler's equations of the free rigid body and the equations of the dynamics of autonomous underwater vehicle.

Chapter 2, entitled "*Dynamical properties of the Hamilton-Poisson systems on* $\mathbf{R}^n$", is structured in three paragraphs and combines a monographic presentation with original elements, which can be found in the last two paragraphs.

This chapter is dedicated to present of the fundamental notions and methods frequently used for the qualitative study of the dynamics of an Hamilton-Poisson system on $\mathbf{R}^n$. As illustrative example we investigate the general Euler top system.

The *Paragraph 2.1* contains basic notions concerning the Hamilton-Poisson systems from theory of dynamical systems. Also the Lyapunov's results are presented (Theorems 2.1.3, 2.1.6, 2.1.9). Two methods for determination of the nature for nonlinear stability are highlighted [energy-Casimir method (Theorem 2.1.12), Arnold's method (Theorem 2.1.14)]. Also we present Lax formulation. In finally are given two methods of numerical integration for the approximation of solutions of a dynamical system using the geometrical integrators (Lie-Trotter integrator and Kahan integrator).

In *Paragraph 2.2* realizes a geometrical and dynamical study of general Euler top system. This system is described by a family of non-linear differential equations on $\mathbf{R}^3$ depending by a triple of reel parameters. The geometrical and dynamical properties of Euler top system are studied [Hamilton-Poisson realization (Propositions 2.2.3, 2.2.6), connection with pendulum dynamics (Proposition 2.2.11)] and stability problem (Propositions 2.2.13, 2.2.14, 2.2.16–2.2.19, Corollaries 2.2.15, 2.2.20)].

Original contributions of the author are included in Secţion 2.2.3 and refers to numerical integration of the Euler top dynamics, using the Lie-Trotter integrator (**Propositions 2.2.21, 2.2.22**, **Corollaries 2.2.23-2.2.25**) and Kahan integrator (**Proposition 2.2.26**, **Remark 2.2.27**). These results have been communicated to "**The 12th Symposium of Mathematics and its Applications, 5-7th November 2009, Timişoara**" and published in the paper [50] (Şuşoi, 2010).

In *Paragraph 2.3* we study the geometrical and dynamical properties of the metriplectic Euler top. Secţion 2.2.1 contains aspects with monographic character about metriplectic structures. The content of the two last sections is based on the author's results included in the paper [49] (Şuşoi and M. Ivan, 2009).

Original contributions refers to the construction of a metriplectic structure associated of Euler top system (**Proposition 2.3.4**) and to study of spectrally stability for the metriplectic Euler top system (**Propositions 2.3.8, 2.3.10–2.3.12**, **Corollary 2.3.13**). These results have been communicated to "**The International Conference on Theory and Applications of Mathematics and Informatics (ICTAMI 2009), 3-6th September 2009, Alba-Iulia**" [conference organized by "1 Decembrie 1918" University of Alba-Iulia and Institute of Mathematics "Simion Stoilow" of the Romanian Academy].

Chapter 3, entitled "*Two classical dynamical systems on* $\mathbf{R}^6$", is structured in three paragraphs. Original results are contained in the two last paragraphs.

In this chapter we establish some important geometrical and dynamical properties for two remarkable differential systems on $\mathbf{R}^6$, namely: *Goryachev-Chaplygin top system* and *Kowalevski top system*.

In *Paragraph 3.1* are presented the Lie-Poisson structures on the dual of Lie algebra $se(3, \mathbf{R})$.

In *Paragraph 3.2* we present a geometrical and dynamical study of the Goryachev-Chaplygin top system (3.2.1). This paragraph contains original contributions of the author and these have been published in the paper [5] (Aron, Puta and Suşoi, 2005). More precisely, these refers to: Hamilton-Poisson formulation of the a dynamics (3.2.1) (**Proposition 3.2.1**), Lax formulation (**Proposition 3.2.6**), stability problem for G-C top (**Propositions 3.2.8-3.2.13**), existence of periodic solutions (**Proposition 3.2.15**) and numerical integration of the dynamics (3.2.1) via Lie-Trotter integrator (**Propositions 3.2.16, 3.2.17**).

In *Paragraph 3.3* realizes a geometrical and dynamical study of the Kowalevski top system (3.3.1). The content of this paragraph bases on the author's results included in the paper [6] (Aron, Puta, Şuşoi et al., 2006). These contributions refers to: Hamilton-Poisson formulation of the dynamics (3.3.1) (**Proposition 3.3.1**), Lax formulation (**Proposition 3.3.4**, **Corollary 3.3.5**), stability problem for Kowalevski top

(**Propositions 3.3.6-3.2.11**), existence of periodic solutions (**Proposition 3.3.13**), numerical integration of the dynamics (3.3.1), using the Lie-Trotter integrator (**Propositions 3.3.14, 3.3.15**) and Kahan integrator (**Propositions 3.3.16, 3.3.17**).

Chapter 4, entitled "*Control dynamical systems on Lie group $SO(4)$*", is structured in two paragraphs. Original results of the author are included in Paragraph 4.2 and have been published in the papers [42](Pop, Puta and Şuşoi, 2005) and [7](Aron, Pop, Puta and Şuşoi, 2006).

In *Paragraph 4.1* we present the principal definitions and properties concerning the systems with control on matrix Lie groups (Theorems 4.1.2, 4.1.5). One gives two examples of controllable left invariant systems on the Lie group $SE(2, \mathbf{R})$ [resp. $SO(3)$] which describes the dynamics of the robot Hilare [resp. dynamics of spacecraft]. For each from these models is studied an optimal control problem. For the study of properties of dynamical systems (4.1.19) and (4.1.24) are used the results obtained in Chapter 2 about Euler top system. For these dynamical systems are investigated stability problem (**Proposition 4.1.7**) [resp., **Proposition 4.1.11**] and numerical integration via Lie-Trotter integrator (**Propositions 4.1.8, 4.1.9**)[resp., **Propositions 4.1.12, 4.1.13**].

Paragraph 4.2 is dedicated of controllable systems on Lie group $SO(4)$. Original results obtained refers to: the study of an optimal control problem with three controls for the system (4.2.4) (**Proposition 4.2.3**), stability problem (**Propositions 4.2.5-4.2.12**), Lax formulation and the complete integrability (**Propositions 4.2.14, 4.2.16**, **Corollary 4.2.15**) and numerical integration [**the Lie-Trotter integrator given by the system of recurrent equations (4.2.11)** and **Proposition 4.2.17**].

The authors original work has been published in six scientific papers which are cited in the bibliography section. Four are realized in collaboration with Professor Mircea Puta and his collaborators [([5], [6], [7], [42]], and one paper is written in collaboration with dr. Mihai Ivan ([49]).

I wish to express my gratitude for my first doctoral adviser, Professor Mircea Puta, because without his help and support I could not have written this thesis.

I also wish to bring my deepest thanks to Professor Dorin Andrica for having accepted to be my doctoral adviser after the tragic disappearance of Professor Mircea Puta. He gave me a substantial support for to complete this scientific demarche. Relevant remarks and suggestions of his have contributed to the current version of the thesis.

Finally, but not at last, I would like to express my sincere thanks to the staff of the Department of Geometry of the BabeşBolyai University from Cluj-Napoca, who have given me their moral support and trust, which was essential for me.

Chapter 1

Elements of geometrical mechanics

In this summary, for definitions, propositions, theorems etc., we keep the numbering used in the thesis.

Also to ensure the coherence and unity of content we recall some notions and basic results that are needed in the following.

Chapter 1 is structured in four paragraphs and it has a monographic character. The principal objective is the presentation in a brief form of some results about symplectic structures, Hamiltonian mechanical systems, Poisson manifolds and Hamilton-Poisson mechanical systems.

1.1 Elements of symplectic geometry on $\mathbf{R}^n$

The principal notations used are:

M – a differential manifold of dimension n of class C^∞;

$\mathcal{X}(M)$ – the reel Lie algebra of vector fields on M;

TM (resp. T^*M) – the total space of tangent (resp. cotangent) bundle on M;

$C^\infty(M, \mathbf{R})$ – the algebra of reels functions of class C^∞ definite on M.

The concepts from theory of differential manifolds and associated geometric structures are those from works of differential geometry, see [14](M. Craioveanu, 2008).

In this paragraph we present some elements from theory of symplectic manifolds. The principal objectives addressed are: symplectic vector space, symplectic map, property of characterization of a symplectic form (Proposition 1.1.6), symplectic structure, symplectomorphism, Hamiltonian mechanical system, conservation of energy for an Hamiltonian system (Proposition 1.1.21). The principal bibliographic sources used are: [1] (Abraham and Marsden, 1979), [43] (Puta, 1993).

Let M be a manifold of dimension n and $A^2(M)$ the reel vector space of exterior differential forms defined on M. An element $\omega \in A^2(V)$ is called $2-$*form* on M.

Definition 1.1.8 (a) By a *symplectic structure* or *symplectic form* on the manifold M, we mean a closed non-degenerate $2-$ form ω on M.

(b) The pair (M, ω) is called *symplectic manifold.* □

Definition 1.1.10 Let be the symplectic manifolds (M_1, ω_1) and (M_2, ω_2). A map $\varphi \in C^\infty(M_1, M_2)$ is said to be *symplectomorphism*, if $\varphi^* \omega_2 = \omega_1$. □

A differentiable map $c : I \to M$ ($I \subset \mathbf{R}$ is open interval containing 0) is a *integral curve* of a vector field $X \in \mathcal{X}(M)$ with initial condition x, if:

$$\frac{dc(t)}{dt} = X(c(t)) \quad \text{and} \quad c(0) = x.$$

A family $\{\varphi_t\}_{t \in I}$, where $\varphi_t : M \to M$ is a differentiable map with property that $\varphi_t(x) = c(t)$, is called *flow* of vector field X.

Let Q be a manifold of dimension n and T^*Q its cotangent manifold. The local coordinates $(q^1, q^2, ..., q^n)$ on Q induces the local coordinates $(q^1, q^2, ..., q^n, p_1, p_2, ..., p_n)$ on T^*Q, called *canonical cotangent coordinates.*

Define on T^*Q a $1-$ form θ, called *Liouville form*, given by:

$$\theta = p_1 dq^1 + p_2 dq^2 + ... + p_n dq^n. \tag{1.1.1}$$

Using the Liouville form θ, define a $2-$ form ω on T^*Q, given by:

$$\omega = d\theta = dp_1 \wedge dq^1 + dp_2 \wedge dq^2 + ... + dp_n \wedge dq^n. \tag{1.1.2}$$

Proposition 1.1.13 $(T^*Q, \omega = d\theta)$ *is a symplectic manifold.* □

The symplectic $\omega = d\theta$ given by (1.1.2) is called *canonical symplectic form* on T^*Q.

The local structure of a symplectic manifold is given in Darboux's theorem(Theorem 1.1.15). More precisely:

on each symplectic manifold (M, ω) *in the neighborhood of all point* $x \in M$ *there exist the local coordinates* $(q^1, q^2, ..., q^n)$ *in which* ω *has the expression (1.1.2). In other words, every symplectic manifold is, locally, of the form* T^*Q.

The local coordinates (q^i, p_i) from (1.1.2) are called *symplectic coordinates* on M.

Definition 1.1.17 Let (M, ω) be a symplectic manifold ($\dim M = 2n$) and $H \in C^\infty(M, \mathbf{R})$. The triple (M, ω, H) is said to be *Hamiltonian mechanical system.* The vector field $X_H \in \mathcal{X}(M)$ determined by the condition:

$$i_{X_H} \omega + dH = 0, \tag{1.1.3}$$

is called *Hamiltonian vector field* with *energy* or *Hamiltonian* H. □

Proposition 1.1.18 *Let* (M, ω) *be a symplectic manifold (*$\dim M = 2n$*),* $H \in C^{\infty}(M, \mathbf{R})$ *and* $(q^1, q^2, ..., q^n, p_1, p_2, ..., p_n)$ *the symplectic coordinates on* M*. The Hamiltonian vector field* X_H *has the local expression:*

$$X_H = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p_i} \right). \tag{1.1.4}$$

If (M, ω, H) is an Hamiltonian system, its *dynamics* is described by the integral curves of vector field X_H.

In Theorem 1.1.19 is given the *Hamilton's equations* of a vector field X_H.

As examples are determined the *Hamilton's equations of* $1-$*dimensional harmonic oscillator* [Example 1.1.20(i)] and *Hamilton's equations of pendulum* [Example 1.1.20(ii)].

În **Liouville theorem** gives characteristic properties of a Hamiltonian system (M, ω, H) [*if* $\{\varphi_t\}$ *is the flow of* X_H*, then* φ_t *is a symplectic map; the flow of* X_H *preserves the canonical volume form*] (Proposition 1.1.22).

Jacobi theorem gives and necessary and sufficient condition such that $f \in \mathcal{D}iff(M)$ to be a symplectodiffeomorphism of a symplectic manifold (M, ω) (Proposition 1.2.23).

Definition 1.1.24 Let (M, ω) be a symplectic manifold. The *Poisson bracket* of functions $f, g \in C^{\infty}(M, \mathbf{R})$ is the function $\{f, g\}_{\omega} \in C^{\infty}(M, \mathbf{R})$, given by:

$$\{f, g\}_{\omega} = -\omega(X_f, X_g), \tag{1.1.5}$$

where X_f, resp. X_g is the Hamiltonian vector field with energy f, resp. g. □

Proposition 1.1.27 *Let* (M, ω) *be a symplectic manifold (*$\dim M = 2n$*) and* $f, g \in C^{\infty}(M, \mathbf{R})$*. Then in the symplectic coordinates* (q^i, p_i) *on* M*, the Poisson bracket* $\{f, g\}_{\omega}$ *has the following expression:*

$$\{f, g\}_{\omega} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} \right). \tag{1.1.6}$$

Proposition 1.1.30 *Let* (M, ω) *be a symplectic manifold (*$\dim M = 2n$*). The map* $\{\cdot, \cdot\}_{\omega} : C^{\infty}(M, \mathbf{R}) \times C^{\infty}(M, \mathbf{R}) \to C^{\infty}(M, \mathbf{R})$*:*

(a) *is* $\mathbf{R}-$ *bilinear and skew-symmetric;*

(b) *verifies the Jacobi identity, i.e.*

$$\{\{f, g\}_{\omega}, h\}_{\omega} + \{\{g, h\}_{\omega}, f\}_{\omega} + \{\{h, f\}_{\omega}, g\}_{\omega} = 0, \quad (\forall)\ f, g, h \in C^{\infty}(M, \mathbf{R});$$

(c) *verifies the Leibniz identity, i.e.*

$$\{fg, h\}_{\omega} = f\{g, h\}_{\omega} + g\{f, h\}_{\omega}, \quad (\forall)\ f, g, h \in C^{\infty}(M, \mathbf{R}). \quad \square$$

1.2 Poisson manifolds. Poisson structures on $\mathbf{R}^n$

We will review some notions and basic results concerning Poisson geometry and Hamilton-Poisson systems. The content of this paragraph is based on the following bibliographic sources: [1] (Abraham and Marsden, 1979), [53](Weinstein, 1983), [35] (Marsden and Raţiu, 1994), [2] (Andrica and Caşu, 2008), [19](Holm et al., 1985).

Definition 1.2.1 (i) A *Poisson structure* or *Poisson bracket* on a manifold P is a map $\{\cdot,\cdot\}: C^\infty(P,\mathbf{R}) \times C^\infty(P,\mathbf{R}) \longrightarrow C^\infty(P,\mathbf{R})$ satisfying the following properties:

$(P1)$ $\{\cdot,\cdot\}$ is $\mathbf{R}-$ bilinear; $(P2)$ $\{\cdot,\cdot\}$ is skew-symmetric;

$(P3)$ $\{\cdot,\cdot\}$ satisfies *Leibniz rule*; $(P4)$ $\{\cdot,\cdot\}$ satisfies *Jacobi identity.*

(ii) The manifold P endowed with a Poisson structure $\{\cdot,\cdot\}$ on $C^\infty(P,\mathbf{R})$ is called *Poisson manifold.* A Poisson manifold is denoted with $(P,\{\cdot,\cdot\})$. □

We observe that *a Poisson bracket on the manifold P is a Lie bracket* $\{\cdot,\cdot\}$ *on* $C^\infty(P,\mathbf{R})$ (i.e. $(C^\infty(P,\mathbf{R}),\{\cdot,\cdot\})$ is a Lie algebra) *which satisfies Leibniz rule.*

Definition 1.2.2 Let $(P_1,\{\cdot,\cdot\}_1))$ and $(P_2,\{\cdot,\cdot\}_2)$ two Poisson manifolds. A *Poisson map* is a differentiable map $\varphi: P_1 \to P_2$ with the property:

$$\varphi^*\left(\{f,g\}_2\right) = \{\varphi^* f, \varphi^* g\}_1, \quad (\forall)\ f,g \in C^\infty(P_2,\mathbf{R}).$$

Proposition 1.2.3 *Every symplectic manifold is a Poisson manifold. More precisely, if (M,ω) is a symplectic manifold, then $(M,\{\cdot,\cdot\}_\omega)$ is a Poisson manifold, where the Poisson structure $\{\cdot,\cdot\}_\omega$ is given by the relation* (1.1.6).

Proposition 1.2.4 *Let $(P,\{\cdot,\cdot\})$ be a Poisson manifold. If $H \in C^\infty(P,\mathbf{R})$, then there exists an unique vector field $X_H \in \mathcal{X}(P)$ such that:*

$$X_H(f) = \{f,H\}, \quad (\forall)\ f \in C^\infty(P,\mathbf{R}). \tag{1.2.1}$$

The vector field X_H given by (1.2.1) is called *Hamiltonian vector field* with energy H associated to Poisson manifold $(P,\{\cdot,\cdot\})$.

Remark 1.2.7 Every symplectic manifold (M,ω) is a Poisson manifold. The question arises, when may be define a symplectic structure on a Poisson manifold? The answer was given by Jost ([23], 1964) and it was enunciated in Proposition 1.2.8.

Proposition 1.2.8 *If the Poisson structure $\{\cdot,\cdot\}$ defined on the manifold P is non-degenerate, then the symplectic structure ω defined on P is given by:*

$$\omega(X_f, X_g) = -\{f,g\}. \tag{1.2.2}$$

The locally structure of Poisson manifolds is more complex than of symplectic manifold. More precisely the following theorem holds.

Theorem 1.2.10 (**Kirilov**). *Every Poisson manifold is an smooth union of symplectic manifolds (called* **symplectic leaves***), not necessarily of same dimension.*

In finally we present the **Darboux-Lie-Weinstein theorem** (Theorem 1.2 11).

Definition 1.2.12 A function $C \in C^\infty(P,\mathbf{R})$ is *a Casimir* for the configuration $(P,\{\cdot,\cdot\})$, if $\{C,f\} = 0, \quad (\forall) f \in C^\infty(P,\mathbf{R})$.

Poisson structures on $\mathbf{R}^n$

The most part of thesis is dedicated to mechanical systems on $\mathbf{R}^n$. From this reason we will present in details some aspects in theory of Poisson structures on $\mathbf{R}^n$.

We denote the coordinate functions on $\mathbf{R}^n$ with $(x^1, x^2, ..., x^n)$.

Proposition 1.2.16 *Let $\{\cdot,\cdot\}$ be a Poisson structure on $\mathbf{R}^n$. Then:*

$$\{f,g\} = \{x^i, x^j\}\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^j}, \qquad (\forall)\ f,g \in C^\infty(\mathbf{R}^n, \mathbf{R}), \quad i,j = \overline{1,n}. \tag{1.2.3}$$

The matrix Π defined by:

$$\Pi = \left(\{x^i, x^j\}\right), \quad i,j = \overline{1,n}, . \tag{1.2.4}$$

is called *structure matrix* of Poisson manifold $(\mathbf{R}^n, \{\cdot,\cdot\})$.

The relation (1.2.3) reads in equivalent form:

$$\{f,g\} = (\nabla f)^T \cdot \Pi \cdot \nabla g, \tag{1.2.5}$$

where $\nabla\varphi$, is the gradient of $\varphi \in C^\infty(\mathbf{R}^n, \mathbf{R})$.

Proposition 1.2.17 *Let $\{\cdot,\cdot\}$ be a composition law on $C^\infty(\mathbf{R}^n, \mathbf{R})$ which satisfies the conditions $(P1)-(P3)$. Then it verifies Jacobi's identity if and only if it is verified by the coordinate functions x^i, $i = \overline{1,n}$.*

Remark. By Proposition 1.2.16, every Poisson structure on $\mathbf{R}^n$ determines a structure matrix $\Pi = (\{x^i, x^j\})$. Ourselves the following question:

In what conditions a matrix $\Pi = (\pi^{ij}(x))_{1\leq i,j\leq n}$ whose elements are functions is a structure matrix for a Poisson structure $\{\cdot,\cdot\}$ on $\mathbf{R}^n$?

The answer is given in the following theorem, [39](Olver, 1993).

Theorem 1.2.20 *Let be a matrix $\Pi = (\pi^{ij}(x))_{1\leq i,j\leq n}$, where $\pi^{ij}(x) \in C^\infty(\mathbf{R}^n, \mathbf{R})$. Then $\Pi(x)$ is a structure matrix for a Poisson bracket given by the relation (1.2.5), if and only if the following conditions hold:*

(i) Π *is skew-symmetric for all $x \in \mathbf{R}^n$;*

(ii) $\pi^{ij}(x)$ *verifies the Jacobi equations:*

$$\pi^{i\ell}\frac{\partial \pi^{jk}}{\partial x^\ell} + \pi^{j\ell}\frac{\partial \pi^{ki}}{\partial x^\ell} + \pi^{k\ell}\frac{\partial \pi^{ij}}{\partial x^\ell} = 0, \quad i,j,k,\ell = \overline{1,n}. \tag{1.2.6}$$

For $n = 3$, if we choose the functions $\pi^{12}(x), \pi^{23}(x)$ and $\pi^{13}(x)$ it is easy to see that: *Jacobi equation reduces to a single equation*, namely:

$$\pi^{12}\frac{\partial \pi^{31}}{\partial x^1} - \pi^{31}\frac{\partial \pi^{12}}{\partial x^1} + \pi^{23}\frac{\partial \pi^{12}}{\partial x^2} - \pi^{12}\frac{\partial \pi^{23}}{\partial x^2} + \pi^{31}\frac{\partial \pi^{23}}{\partial x^3} - \pi^{23}\frac{\partial \pi^{31}}{\partial x^3} = 0. \tag{1.2.7}$$

In Propositions 1.2.22 and 1.2.23 gives **two general methods for to construct Poisson structures on $\mathbf{R}^3$.**

Proposition 1.2.22 *Let* $A = (A^{ij}) \in \mathcal{M}_3(\mathbf{R})$ *be a skew-symmetric matrix. Define the matrix* $\Pi = (\pi^{ij})_{1 \le i,j \le 3}$*, where*

$$\pi^{ij} = A^{ij} x^k, \quad k \neq i \ \text{ şi } \ k \neq j.$$

Then $\Pi(x) = (\pi^{ij}(x))$ *is a structure matrix for the* $\{\cdot, \cdot\}$ *on* $\mathbf{R}^3$*.*

If in Proposition 1.2.22, we consider the skew-symmetric matrix

$$A = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}, \quad a, b, c \in \mathbf{R},$$

one obtains the structure matrix

$$\Pi_{(a,b,c)} = \begin{pmatrix} 0 & -cx^3 & bx^2 \\ cx^3 & 0 & -ax^1 \\ -bx^2 & ax^1 & 0 \end{pmatrix}, \quad a, b, c \in \mathbf{R}. \tag{1.2.8}$$

If in the relation (1.2.8) we consider the matrix $\Pi_{(a,b,c)}$ with $abc \neq 0$, we will say that this generates a *Poisson structure*$\{\cdot, \cdot\}$ *on* $\mathbf{R}^3$ *of so*$(3)-$*type.*

If in the relation (1.2.8) we consider the matrix Π of the form

$$\Pi_{(0,b,c)} = \begin{pmatrix} 0 & -cx^3 & bx^2 \\ cx^3 & 0 & 0 \\ -bx^2 & 0 & 0 \end{pmatrix}, \quad b, c \in \mathbf{R}, \ bc \neq 0 \tag{1.2.9}$$

we will say that this generates a *Poisson structure* $\{\cdot, \cdot\}$ *on* $\mathbf{R}^3$ *of se*$(2)-$*type.*

Proposition 1.2.23 *Let* $F \in C^\infty(\mathbf{R}^3, \mathbf{R})$ *be a given function.*

(i) *The algebraic operation* $\{\cdot, \cdot\}_F : C^\infty(\mathbf{R}^3, \mathbf{R}) \times C^\infty(\mathbf{R}^3, \mathbf{R}) \to C^\infty(\mathbf{R}^3, \mathbf{R})$ *given by:*

$$\{f, g\}_F = -\nabla F \cdot (\nabla f \times \nabla g), \quad (\forall) \ f, g \in C^\infty(\mathbf{R}^3, \mathbf{R}), \tag{1.2.10}$$

defines a Poisson structure on $\mathbf{R}^3$*.*

(ii) $F \in C^\infty(\mathbf{R}^3, \mathbf{R})$ *is a Casimir for the configuration* $(\mathbf{R}^3, \{f, g\}_F)$*.* $\square$

The relation (1.2.10) reads in the equivalent form:

$$\{f, g\}_F = (\nabla f)^T \cdot \Pi_F \cdot \ \nabla g, \quad \text{where} \ \ \Pi_F = \begin{pmatrix} 0 & -\dfrac{\partial F}{\partial x^3} & \dfrac{\partial F}{\partial x^2} \\ \dfrac{\partial F}{\partial x^3} & 0 & -\dfrac{\partial F}{\partial x^1} \\ -\dfrac{\partial F}{\partial x^2} & \dfrac{\partial F}{\partial x^1} & 0 \end{pmatrix}. \tag{1.2.11}$$

Remark 1.2.26 $C \in C^\infty(\mathbf{R}^n, \mathbf{R})$ *is a Casimir of configuration* $(\mathbf{R}^n, \{\cdot, \cdot\})$ *iff:*

$$\Pi \cdot \nabla C = 0.$$

Poisson structures on the dual of a Lie algebra

Let $\mathcal{G}$ be the Lie algebra of a finite dimensional Lie group G and $\mathcal{G}^*$ its dual space.

The Lie group G acts on $\mathcal{G}$ by the action $Ad : G \times \mathcal{G} \to \mathcal{G}$, called *adjoint action* of G on $\mathcal{G}$. For $g \in G$, the map $Ad_g : \mathcal{G} \to \mathcal{G}$ is given by:

$$Ad_g(\xi) = T_e(R_{g^{-1}} \circ L_g)(\xi), \quad (\forall)\xi \in \mathcal{G}.$$

The Lie group G acts on $\mathcal{G}^*$ by the action $Ad^* : G \times \mathcal{G}^* \to \mathcal{G}^*$, called *co-adjoint action* of G on $\mathcal{G}^*$. For $g \in G$, the map $Ad^*_{\mathcal{G}} : \mathcal{G}^* \to \mathcal{G}^*$ is given by:

$$< Ad^*_g\mu, \xi >=< \mu, Ad_g(\xi) >, \quad (\forall)\mu \in \mathcal{G}^*, \ \xi \in \mathcal{G}.$$

Let $\mu \in \mathcal{G}^*$. The *co-adjoint orbit* of μ is defined by:

$$\mathcal{O}_\mu = \{Ad^*_g\mu \mid g \in G\}.$$

The infinitesimal generator $\xi_{\mathcal{G}^*}$ of co-adjoint action of G on $\mathcal{G}^*$ is given by:

$$< \xi_{\mathcal{G}^*}(\mu), \eta >=< \mu, [\xi, \eta] >, \quad (\forall)\mu \in \mathcal{G}^*, \ \xi, \eta \in \mathcal{G}.$$

Proposition 1.2.28 (**Kirilov-Kostant-Souriau**).*Let G be a Lie group and $\mathcal{O} \subset \mathcal{G}^*$ a co-adjoint orbit. Then $\mathcal{O}$ is a symplectic manifold. In other words, there exists an unique symplectic form $\omega_{\mathcal{O}}$ on $\mathcal{O}$ such that*

$$\omega_{\mathcal{O}}(\xi_{\mathcal{G}^*}, \eta_{\mathcal{G}^*}) = - < \mu, [\xi, \eta] >, \quad (\forall)\mu \in \mathcal{O}, \ \xi, \eta \in \mathcal{G}. \tag{1.2.12}$$

Define a bracket on algebra $C^\infty(\mathcal{G}^*, \mathbf{R})$ by:

$$\{f, g\}^+_{LP} =< \theta, [df(\theta), dg(\theta)] >, \quad \forall \ f, g \in C^\infty(\mathcal{G}^*, \mathbf{R}), \ \theta \in \mathcal{G}^*. \tag{1.2.13}$$

Proposition 1.2.29 (**Lie-Poisson**) *The dual space $\mathcal{G}^*$ of Lie algebra $\mathcal{G}$ endowed with bracket $\{f, g\}^+_{LP}$ given by (1.2.13) has a (non-canonical) structure of Poisson manifold, called the* **plus Lie-Poisson structure** *on $\mathcal{G}^*$.*

Similarly, we define the **plus Lie-Poisson structure** on $\mathcal{G}^*$.

Corollary 1.2.30 *There exist two (non-canonical) Poisson structures on the dual $\mathcal{G}^*$ of Lie algebra $\mathcal{G}$, called the* **plus-minus Lie-Poisson structure** *and denoted, respectively with $\{\cdot, \cdot\}_\pm$. In consequence, $(\mathcal{G}^*, \{\cdot, \cdot\}_\pm)$ are Poisson manifolds.*

In the following proposition is established a connection between the Lie-Poisson structures on the dual of a Lie algebra and Kirilov-Kostant-Souriau symplectic form.

Proposition 1.2.31 *Let G be a Lie group and $\mathcal{O} \subset \mathcal{G}^*$ a co-adjoint orbit. For all $f, g \in C^\infty(\mathcal{G}^*, \mathbf{R})$ and $\mu \in \mathcal{O}$, we have:*

$$\{f, g\}_+(\mu) = \{f_{|\mathcal{O}}, g_{|\mathcal{O}}\}_{\omega_{\mathcal{O}}}. \tag{1.2.14}$$

Remark 1.2.32 *The symplectic leaves of $\mathcal{G}^*$ are exactly the co-adjoint orbits of $\mathcal{G}^*$.*

Proposition 1.2.33 *A function $f \in C^\infty(\mathcal{G}^*, \mathbf{R})$ is a Casimir of configuration $(\mathcal{G}^*, \{\cdot,\cdot\}_\pm)$ if and only if it is constant on each co-adjoint orbit.*

Proposition 1.2.34 *Let $\mathcal{G}$ be a Lie algebra of dimension n and $\mathcal{G}^*$ its dual. Then the Poisson structures $\{\cdot,\cdot\}_\pm$ on $\mathcal{G}^*$ are given by:*

$$\{f, g\}_\pm (m) = \pm c_{ij}^k \frac{\partial f}{\partial m_i} \frac{\partial g}{\partial m_j} m_k, \tag{1.2.15}$$

where c_{ij}^k, $i, j, k = \overline{1, n}$ are structure constants of Lie algebra $\mathcal{G}$.

This paragraph ends with the presentation of the plus-minus Lie-Poisson structures on the dual of some from the following classical Lie algebras of dimension 3, namely: Lie algebra $(\mathbf{R}^3, \times)$, Lie algebra $so(3)$ and Lie algebra $se(2, \mathbf{R})$. More precisely:

- The plus-minus Lie-Poisson structures on the dual of Lie algebra $(\mathbf{R}^3, \times)$ are generated by the de matrices:

$$\Pi_- = \begin{pmatrix} 0 & -m_3 & m_2 \\ m_3 & 0 & -m_1 \\ -m_2 & m_1 & 0 \end{pmatrix} \quad \text{and} \quad \Pi_+ = \begin{pmatrix} 0 & m_3 & -m_2 \\ -m_3 & 0 & m_1 \\ m_2 & -m_1 & 0 \end{pmatrix}; \tag{1.2.16}$$

- The plus-minus Lie-Poisson structures on the dual $(so(3))^*$ of Lie algebra $so(3)$ are generated by the matrices Π_- and Π_+ given in the relation (1.2.16), where

$$so(3) = \left\{ \begin{pmatrix} 0 & -a & b \\ a & 0 & -c \\ -b & c & 0 \end{pmatrix} \mid a, b, c \in \mathbf{R} \right\};$$

- The plus-minus Lie-Poisson structures on the dual $(se(2, \mathbf{R}))^*$ of Lie algebra

$$se(2, \mathbf{R}) = \left\{ X(a, v_1, v_2) = \begin{pmatrix} 0 & -a & v_1 \\ a & 0 & v_2 \\ 0 & 0 & 0 \end{pmatrix} \mid a, v_1, v_2 \in \mathbf{R} \right\}.$$

are generated by the matrices $\Pi_{e2,-}$ and $\Pi_{e2,+}$, where

$$\Pi_{e2,-} = \begin{pmatrix} 0 & -m_3 & m_2 \\ m_3 & 0 & 0 \\ -m_2 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \Pi_{e2,+} = \begin{pmatrix} 0 & m_3 & -m_2 \\ -m_3 & 0 & 0 \\ m_2 & 0 & 0 \end{pmatrix}.$$

1.3 Hamilton-Poisson mechanical systems on $\mathbf{R}^n$

Definition 1.3.1 An *Hamilton-Poisson system* is a triple $(P, \{\cdot,\cdot\}, H)$ where $\{\cdot,\cdot\}$ is a Poisson structure on P and $H \in C^\infty(P, \mathbf{R})$ is the *Hamiltonian* or *energy* of system.

Let X_H be the Hamiltonian vector field with energy function H, that is:

$$X_H(f) = \{f, H\}, \quad (\forall)\ f \in C^\infty(P, \mathbf{R}).$$

The dynamics of the Hamilton-Poisson system $(P, \{\cdot,\cdot\}, H)$ is described by:

$$\frac{dx^i(t)}{dt} = X_H(x^i(t)), \quad t \in \mathbf{R} \quad \text{or equivalent} \quad \dot{x}^i = \{x^i, H\},\ i = \overline{1,n}. \tag{1.3.1}$$

Applying (1.2.5), the system (1.3.1) can be written in the equivalent form

$$\dot{X} = \Pi \cdot \nabla H, \tag{1.3.2}$$

where $\dot{X} = \begin{pmatrix} \dot{x}^1 & \dot{x}^2 & \ldots & \dot{x}^n \end{pmatrix}^T$, and $\Pi = (\{x^i, x^j\})$ is its associated matrix.

Let $(P, \{\cdot,\cdot\}, H)$ be an Hamilton-Poisson system and $\{\varphi_t\}$ the flow of Hamiltonian vector field X_H. Then for all $f \in C^\infty(P, \mathbf{R})$ and all $t \in \mathbf{R}$, we have:

– $H \circ \varphi_t = H$ *(***conservation of energy***);* $\quad \dfrac{d}{dt}(f \circ \varphi_t) = \{f, H\} \circ \varphi_t = \{f \circ \varphi_t, H\}$*;*

– $\{\varphi_t\}$ *preserves the Poisson structure* $\{\cdot,\cdot\}$*, i.e.* $\varphi_t^*\{f, g\} = \{\varphi_t^* f, \varphi_t^* g\}$ (Propositions 1.3.2 and 1.3.3)

We say that $f \in C^\infty(\mathbf{R}^n, \mathbf{R})$ is a *first integral* or *constant of motion* for the system (1.3.1), if its derivative along the trajectories of system is null.

Proposition 1.3.5 *Let $(\mathbf{R}^n, \{\cdot,\cdot\}, H)$ be an Hamilton-Poisson system. Then $f \in C^\infty(\mathbf{R}^n, \mathbf{R})$ is a first integral of the system (1.3.1) if and only if $\{f, H\} = 0$.*

Proposition 1.3.6 *Let $(\mathbf{R}^n, \{\cdot,\cdot\}, H)$ be an Hamilton- Poisson system and $C \in C^\infty(\mathbf{R}^n, \mathbf{R})$ a Casimir. Then C and H are first integrals for (1.3.2).*

The following result holds: *every Hamiltonian system on $\mathbf{R}^{2n}$ is an Hamilton-Poisson system* (Proposition 1.3.8).

Definition 1.3.9 If a system of differential equations of the form:

$$\dot{x}^i = f_i(x^1, x^2, \ldots, x^n), \quad f_i \in C^\infty(\mathbf{R}^n, \mathbf{R}),\ i = \overline{1,n} \tag{1.3.3}$$

can be written in the form (1.3.1), we say that $(\mathbf{R}^n, \{\cdot,\cdot\}, H)$ is an *Hamilton-Poisson realization* for (1.3.3); denoted also with $(\mathbf{R}^n, \Pi, H)$, where Π is the structure matrix.

1.4 Examples of Hamilton-Poisson systems

- **Free rigid body as Hamilton-Poisson system**

In mechanical processes, an important role is played by the free rigid body, [35] (Marsden and Raţiu, 1994).

The Euler equations which describe the dynamics of free rigid body are:

$$\dot{m}_1 = \left(\frac{1}{I_3} - \frac{1}{I_2}\right) m_2 m_3, \quad \dot{m}_2 = \left(\frac{1}{I_1} - \frac{1}{I_3}\right) m_1 m_3, \quad \dot{m}_3 = \left(\frac{1}{I_2} - \frac{1}{I_1}\right) m_1 m_2, \tag{1.4.1}$$

where $m = (m_1, m_2, m_3) \in C^\infty(\mathbf{R}^3, \mathbf{R})$ represents the angular velocities vector, and I_1, I_2, I_3 are the components of inertia tensor of body. We suppose that $I_1 > I_2 > I_3 > 0$. a

Proposition 1.4.1 [19](**Holm et al., 1985**). *Dynamical system* (1.4.1) *has the Hamilton-Poisson realization* $(\mathbf{R}^3, \{\cdot,\cdot\}_{RB}, H_{RB})$ *with the Casimir* C_{RB}*, where* $\{\cdot,\cdot\}_{RB}$ *is the Poisson structure defined by* (1.2.10)*, and* $H_{RB}, C_{RB} \in C^\infty(\mathbf{R}^3, \mathbf{R})$ *are given by:*

$$H_{RB}(m) = \frac{1}{2}\left(\frac{1}{I_1}m_1^2 + \frac{1}{I_2}m_2^2 + \frac{1}{I_3}m_3^2\right), \quad C_{RB}(m) = \frac{1}{2}\left(m_1^2 + m_2^2 + m_3^2\right). \qquad (1.4.2)$$

The Poisson structure $\{\cdot,\cdot\}_{RB}$ is generated by the matrix

$$\Pi_{RB} = \begin{pmatrix} 0 & -m_3 & m_2 \\ m_3 & 0 & -m_1 \\ -m_2 & m_1 & 0 \end{pmatrix} \qquad (1.4.3)$$

and is in fact the minus Lie-Poisson structure on the dual $(so(3))^* \cong \mathbf{R}^3$.

- **Equations of the motion of an underwater vehicle**

We refer now to the system of equations which models the dynamics of an autonomous underwater vehicle, [20] (Holmes et al., 1998).

The motions of an underwater vehicle in the subspace $\mathcal{S} \subset \mathbf{R}^6$ defined by $\pi_2 = 0, \pi_3 = 0, p_1 = 0$ are described by the equations:

$$\dot{\pi}_1 = \left(\frac{1}{m_3} - \frac{1}{m_2}\right)p_2p_3, \quad \dot{p}_2 = \frac{1}{I_1}p_3\pi_1, \quad \dot{p}_3 = -\frac{1}{I_1}p_2\pi_1. \qquad (1.4.4)$$

Proposition 1.4.4 [46],**(Puta et al., 2008**). *The triple* $(\mathbf{R}^3, \Pi_{vs}, H_{vs})$ *is an Hamilton-Poisson realization of the dynamics* (1.4.5) *with the Casimir* $C_{vs} \in C^\infty(\mathbf{R}^3, \mathbf{R})$*, where*

$$\Pi_{vs} = \begin{pmatrix} 0 & -p_3 & p_2 \\ p_3 & 0 & 0 \\ -p_2 & 0 & 0 \end{pmatrix},$$

$$H_{vs}(\pi_1, p_2, p_3) = \frac{1}{2}\left(\frac{1}{I_1}\pi_1^2 + \frac{1}{m_2}p_2^2 + \frac{1}{m_3}p_3^2\right), \quad C_{vs}(\pi_1, p_2, p_3) = \frac{1}{2}\left(p_2^2 + p_3^2\right). \qquad (1.4.5)$$

The Poisson structure $\{\cdot,\cdot\}_{vs}$ generated by the matrix Π_{vs} is in fact the minus Lie-Poisson structure on the dual $(se(2,\mathbf{R}))^* \cong \mathbf{R}^3$ of Lie algebra $se(2,\mathbf{R})$.

Remark. In Chapter 2, paragraph 2.2 will be give other Hamilton-Poisson realizations for the dynamics (1.4.1) and (1.4.4).

Chapter 2

Dynamical properties of the Hamilton-Poisson systems on $\mathbf{R}^n$

This chapter is dedicated to presentation of fundamental notions and methods frequently used for the qualitative study of the dynamics of an Hamilton-Poisson system on $\mathbf{R}^n$. As illustrative example we investigate the dynamics of general Euler top system.

The chapter is structured in three paragraphs and combines a monographic presentation with original elements, which can be found in the last two paragraphs.

2.1 Qualitative study of the dynamical system associated to a vector field

This paragraph contains basic notions concerning the Hamilton-Poisson systems. The principal objectives addressed are: nonlinear stable equilibrium state, Lyapunov function, Lax formulation, Lyapunov,s theorems, methods for determination of the nature for nonlinear stability (Theorems 2.1.12, 2.1.14), the existence of periodic orbits (Theorem 2.1.15) and the problem of numerical integration using geometric integrators. The principal bibliographic sources used are: [1] (Abraham and Marsden, 1979), [19](Holm, Marsden, Raţiu and Weinstein, 1985), [43] (Puta, 1993), [35](Marsden and Raţiu, 1994), [18](Hirsch, Smale and Devaney, 2004), [2](Andrica and Caşu, 2008).

Let be the system of differential equations associated to vector field $X \in \mathcal{X}(\mathbf{R}^n)$:

$$\dot{x} = X(x), \quad x \in \mathbf{R}^n. \tag{2.1.1}$$

The system (2.1.1) can be written in the equivalent form:

$$\dot{x}_i = f_i(x_1, x_2, \ldots, x_n), \quad i = \overline{1, n}, \tag{2.1.2}$$

where $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$, $X(x) = (f_1(x), \ldots, f_n(x))$ with $f_i \in C^\infty(D, \mathbf{R}), D \subseteq \mathbf{R}^n$.

A point $x_e \in D \subseteq \mathbf{R}^n$ is called *equilibrium state* of the system (2.1.1), if:

$$X(x_e) = 0, \quad \text{equivalent} \quad f_i(x_e) = 0, \quad i = \overline{1, n}. \tag{2.1.3}$$

Definition 2.1.1 (i) A equilibrium state x_e is *nonlinear stable* or *Lyapunov stable*, if for all neighborhood U of x_e from $\mathbf{R}^n$ there exists a neighborhood V of x_e with $V \subset U$ such that all trajectories $x(t)$ with the initial condition in V is included in U, or equivalent, for all $\varepsilon > 0$, there exists $\delta > 0$ such that:

$$\text{for} \quad \|x(0) - x_e\| < \delta \quad \Rightarrow \quad \|x(t) - x_e\| < \varepsilon, \quad (\forall)\ t > 0.$$

(ii) x_e is *asymptotic stable* if the neighborhood V can be chosen such that is satisfied the supplementary condition $\lim\limits_{t \to \infty} x(t) = x_e$.

(iii) x_e is *unstable*, if x_e is not nonlinear stable. □

Definition 2.1.5 Let $D \subseteq \mathbf{R}^n$ be an open subset and $L : U \to \mathbf{R}$, a function defined on a neighborhood $U \subset D$ of $x_e \in D$. We say that L is a *Lyapunov function* for the system (2.1.1), if the following conditions are verified:

(i) L and its partial derivatives are continuous;

(ii) L is positive definite (resp. negative definite), that is:

$$L(x_e) = 0 \quad \text{şi} \quad L(x) > 0 \text{ (resp. } L(x) < 0), \quad (\forall)\ x \in U \setminus \{x_e\};$$

(iii) The derivative of L along the trajectories of the system (2.1.1) is negative semi-definite (resp. positive semi-definite), that is:

$$\dot{L}(x) \leq 0 \text{ (resp. } \dot{L}(x) \geq 0), \quad (\forall)\ x \in U.$$

Theorem 2.1.6 [34](**Lyapunov**) *Let $x_e \in D$ be an equilibrium state for* (2.1.1).

(i) *If there exists a Lyapunov function L defined on a neighborhood U of $x_e \in D$, then x_e is nonlinear stable;*

(ii) *If there exists a Lyapunov function $L \in C^\infty(\mathbf{R}^n, \mathbf{R})$ such that*

$$\dot{L}(x) < 0 \ (\textit{resp. } \dot{L}(x) > 0) \quad (\forall)\ x \in \mathbf{R}^n,\ x \neq x_e,$$

then x_e is asymptotic stable.

The function L satisfying the conditions of Definition 2.1.5 is called *Lyapunov function associated to x_e*.

We call *linear part* of the dynamical system (2.1.1) in the equilibrium state x_e, the following system of differential equations:

$$\dot{X} = A(x_e)X, \quad \text{where}$$

$$A(x_e) = \begin{pmatrix} \dfrac{\partial f_1}{\partial x_1}(x_e) & \dfrac{\partial f_1}{\partial x_2}(x_e) & \dots & \dfrac{\partial f_1}{\partial x_n}(x_e) \\ \vdots & \vdots & \dots & \vdots \\ \dfrac{\partial f_n}{\partial x_1}(x_e) & \dfrac{\partial f_n}{\partial x_2}(x_e) & \dots & \dfrac{\partial f_n}{\partial x_n}(x_e) \end{pmatrix}. \tag{2.1.4}$$

Theorem 2.1.9 (Lyapunov) *The equilibrium state x_e of the system* (2.1.1) *is:*

(i) asymptotic stable (nonlinear stable), if $Re(\lambda_i) < 0$, for all eigenvalue of the matrix $A(x_e)$, that is it have all eigenvalues with strictly negative reel parts.

(ii) unstable, if $Re(\lambda_i) > 0$, for at least eigenvalue of the matrix $A(x_e)$, that is it has an eigenvalue with strictly positive real part.

Let us we present two practical methods for to determine the nonlinear stability for a system which admits more first integrals.

Theorem 2.1.12 [19]**(Energy-Casimir method)** *Let $(\mathbf{R}^n, \{\cdot,\cdot\}, H)$ be an Hamilton-Poisson system, x_e an equilibrium state and $\mathcal{C}$ a family of first integrals. If there exists $C \in \mathcal{C}$ such that the following conditions hold:*

(i) $D(H+C)(x_e) = 0$;

(ii) $D^2(H+C)(x_e)$ is positive (resp. negative) definite,

then x_e is nonlinear stable.

Theorem 2.1.14 [3]**(Arnold's method)** *Let be the first integrals $C_1, ..., C_k \in C^\infty(\mathbf{R}^n, \mathbf{R})$ for the dynamics* (2.1.1) *and x_e an equilibrium point. Let be the functions $F_i \in C^\infty(\mathbf{R}^n \times \mathbf{R}^{k-1}, \mathbf{R})$ given by:*

$$F_i(x, \lambda_1, \dots, \hat{\lambda}_i, \dots, \lambda_k) = C_i(x) - \lambda_1 C_1(x) - \dots - \lambda_i \hat{C}_i(x) - \dots - \lambda_k C_k(x), \quad i = \overline{1,k},$$

here $\hat{C}_i$ means that C_i is omitted.

If there exist the constants $\lambda_1^*, \dots, \hat{\lambda}_i^*, \dots, \lambda_k^* \in \mathbf{R}$ such that:

(i) $\nabla_x F_i(x_e, \lambda_1^*, \dots, \hat{\lambda}_i^*, \dots, \lambda_k^*) = 0$, for all $i = 1, ..., k$;

(ii) $\nabla^2_{xx} F_i(x_e, \lambda_1^*, \dots, \hat{\lambda}_i^*, \dots, \lambda_k^*)|_{W \times W}$ is positive or negative definite on $W \times W$, where

$$W = \bigcap_{j=1, j\neq i}^{k} Ker\, dC_j(x_e),$$

then x_e is nonlinear stable.

In certain hypothesis, the existence of periodic orbits for a system of differential equations can be establish with aid of Moser's theorem, [36](Moser, 1976).

Theorem 2.1.15 (J. Moser) *Let be the system of differential equations*

$$\dot{x} = f(x), \quad \text{where} \quad x \in \mathbf{R}^n \quad \text{and} \quad f \in C^\infty(\mathbf{R}^n, \mathbf{R}^n). \tag{2.1.5}$$

If x_e is an equilibrium state for the system (2.1.5) *such that* 0 *is not an eigenvalue for the matrix of linear part of the system* (2.1.4) *in x_e and if there exists $K \in C^\infty(\mathbf{R}^n, \mathbf{R})$*

such that:

(i) K *is a first integral;* *(ii)* $K(x_e) = 0$;

(iii) $dK(x_e) = 0$; *(iv)* $d^2K(x_e)$ *is positive definite,*

then for all ε *sufficiently small any integral manifold*

$$K(x) = \varepsilon^2$$

contains at least one periodic solution whose periods are closed to those of the corresponding linear system in x_e.

Lax formulation and integrability

A efficiently method for the study of the integrable dynamical systems is "Lax formulation". The connection between the Lax formulation and the integrability of an Hamiltonian system is given by the fact that the Lax formulation furnishes first integrals of dynamical system.

Definition 2.1.16 We say that the dynamical system (2.1.1) admits a *Lax formulation Lax*, if there exists a pair of matrices (L, B), where $L = L(t)$ and $B = B(t)$ are matrices of type $n \times n$ whose components are functions of class C^1 such that

$$\dot{L} = [L, B] = LB - BL, \quad \text{where} \quad \dot{L} = \frac{dL}{dt}. \tag{2.1.6}$$

Theorem 2.1.17 (Flaschka's theorem). *The flow of the equations* $\dot{L} = [L, B]$ *is iso-spectral, that is the eigenvalues of the matrix* $L(t)$ *are independently of* t.

If the system (2.1.5) has the Lax formulation (2.1.6), then the eigenvalues of the matrix $L(t)$ are constants of motion (Remarks 2.1.18, 2.1.19).

Sometimes, the motion equations of some dynamical systems cannot be integrated by elementary functions. Their solutions can be expressed with elliptic functions ([31]).

Numerical methods for the approximation of solution of a dynamical system

The geometric integrators are methods of numerical integration for simulation on computer of dynamical processes described by differential equations. The geometric integrators preserves the properties of the system (energy, symplectic structure, volume).

Let $(\mathbf{R}^n, \Pi, H)$ be an Hamilton-Poisson system whose dynamics is described by the system of differential equations:

$$\dot{x} = \Pi(x) \cdot \nabla H(x), \quad x \in \mathbf{R}^n. \tag{2.1.7}$$

Definition 2.1.20 By a *(geometric) integrator on* $\mathbf{R}^n$, we mean a family of smooth maps $\varphi_t : \mathbf{R}^n \to \mathbf{R}^n$, which differentiable depends by $t \in \mathbf{R}$; φ_t is said to be:

(i) *Poisson integrator* if the relation holds:

$$(D\varphi_t(x))^T \cdot \Pi(x) \cdot D\varphi_t(x) = \Pi(\varphi_t(x)), \quad (\forall)\ x \in \mathbf{R}^n;$$

(ii) *energy-integrator* if it preserves the energy H of the system (2.1.7), i.e.

$$H(\varphi_t(x)) = H(x);$$

(iii) *Casimir integrator* if preserves a Casimir $C \in C^\infty(\mathbf{R}^n, \mathbf{R})$, i.e.

$$C(\varphi_t(x)) = C(x);$$

(iv) if $n = 2m$ and ω is a symplectic form on $\mathbf{R}^{2m}$, then the integrator $\{\varphi_t\}$ is called *symplectic integrator*, if:

$$\varphi_t^*\omega = \omega, \quad (\forall)\ t \in \mathbf{R}, \quad \text{or equivalent}$$

$$(D\varphi_t(x))^T \cdot \Pi_{can} \cdot D\varphi_t(x) = \Pi_{can}, \quad \text{where} \quad \Pi_{can} = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}. \quad \square$$

For a given integrator we shall use the notation

$$x^{k+1} = \varphi_t(x^k), \quad \text{where} \quad x^k(t) = x(kt),\ k \in \mathbf{N}.$$

We present now two geometric integrators which are used in this work.

Lie-Trotter integrator. Let $(\mathbf{R}^n, \Pi, H)$ be an Hamilton-Poisson system. The Lie-Trotter integrator ([51]) is applied when H can be written in the form:

$$H = H_1 + H_2,$$

such that the generated dynamics by H_1 and H_2 may be explicitly integrated.

Let $exp(tX_{H_1})$ (resp. $exp(tX_{H_2})$) be the integral curve associated to vector field X_{H_1} (resp. X_{H_2}). Then the Lie-Trotter integrator ([17]) is given by the formula:

$$\varphi_t(x) = exp(tX_{H_2})exp(tX_{H_1})(x) = exp(tX_H)(x) + \mathcal{O}(t^2). \tag{2.1.8}$$

Proposition 2.1.21 ([45]) *The Lie-Trotter integrator* (2.1.8) *has the following properties:*

(i) *φ_t is a Poisson integrator;*

(ii) *the restriction to symplectic foliation of the Poisson vector space* $(\mathbf{R}^n, \Pi)$ *defines a symplectic integrator.*

Kahan integrator. Let be the system of differential equations:

$$\dot{x} = X(x),\ x(0) = x_0,\ x \in \mathbf{R}^n, \tag{2.1.9}$$

with property that X is greatest quadratic in x, that is:

$$X(x) = A(x, x) + Bx + b,$$

$A(\cdot, \cdot)$ is a symmetric tensor, B is a matrix, and b is a constant vector.

The Kahan integrator ([26]) for (2.1.9) is given by:

$$\frac{x^{k+1} - x^k}{h} = A(x^{k+1}, x^k) + B\frac{x^{k+1} + x^k}{2} + b, \qquad h \in \mathbf{R}_+. \tag{2.1.10}$$

2.2 Dynamics of the general Euler top system

In this paragraph are studied various geometrical and dynamical properties of the Euler top system, stability problem, the link between the dynamics Euler top and the dynamics of pendulum and numerical integration problem. Original contributions are contained in Section 2.2.3 and these was published in the cited paper [50] (Şuşoi, 2010).

The general Euler top system is described by a family of differential equations on $\mathbf{R}^3$ which depends by a triple of reel parameters. A remarkable representant is the free rigid body [35]. For different values given of parameters one obtains dynamical systems, for example: Lagrange system [48](Takhtajan, 1994), the equations of underwater vehicle dynamics [20], Rabinovich system [12](Chiş and Puta, 2008) etc.

2.2.1 Poisson geometry of the Euler top system

The *(general) Euler top system* is described by the following set of differential equations on $\mathbf{R}^3$ ([41]):

$$\frac{dx_1}{dt} = \alpha_1 x_2(t)x_3(t), \quad \frac{dx_2}{dt} = \alpha_2 x_1(t)x_3(t), \quad \frac{dx_3}{dt} = \alpha_3 x_1(t)x_2(t), \tag{2.2.1}$$

where $\alpha_1, \alpha_2, \alpha_3 \in \mathbf{R}$ are parameters such that $\alpha_1\alpha_2\alpha_3 \neq 0$ and t is the time.

Remark 2.2.1 For $\alpha_1 = \dfrac{1}{I_3} - \dfrac{1}{I_2}$, $\alpha_2 = \dfrac{1}{I_1} - \dfrac{1}{I_3}$, $\alpha_3 = \dfrac{1}{I_2} - \dfrac{1}{I_1}$ [respectively, $\alpha_1 = \dfrac{1}{m_3} - \dfrac{1}{m_2}$, $\alpha_2 = \dfrac{1}{I_1}$, $\alpha_3 = -\dfrac{1}{I_1}$], the system (2.2.1) reduces to equations of free rigid body (1.4.1) [respectively, equations (1.4.4) of underwater vehicle dynamics]. $\square$

If in (2.2.1) we replace the parameters α_i with the corresponding values one obtains:

$$\dot{x}_1 = x_2x_3, \quad \dot{x}_2 = -x_3x_1, \quad \dot{x}_3 = -k^2x_1x_2, \quad \text{cu} \;\; 0 < k^2 < 1. \tag{2.2.2}$$

(*equations of Tzitzeica-Lorentz gradient flow* [15]);

$$\dot{x}_1 = x_2x_3, \quad \dot{x}_2 = -x_1x_3, \quad \dot{x}_3 = x_1x_2 \quad (\textit{Rabinovich system}); \tag{2.2.3}$$

$$\dot{x}_1 = x_2 x_3, \quad \dot{x}_2 = x_1 x_3, \quad \dot{x}_3 = x_1 x_2 \quad (\textit{Lagrange system}). \tag{2.2.4}$$

We denote the vector of parameters involved in (2.2.1) with $\alpha = (\alpha_1, \alpha_2, \alpha_3)$.

Consider the functions $H^\alpha, C^\alpha \in C^\infty(\mathbf{R}^3, \mathbf{R})$ given by:

$$H^\alpha(x_1, x_2, x_3) = \frac{1}{2}(\alpha_2 x_1^2 - \alpha_1 x_2^2) \text{ and } C^\alpha(x_1, x_2, x_3) = \frac{1}{2}(\frac{\alpha_3}{\alpha_2} x_2^2 - x_3^2). \tag{2.2.5}$$

Proposition 2.2.2 *The functions H^α and C^α given by* (2.2.5), *are constant of motion (first integrals) for the dynamics* (2.2.1).

Proposition 2.2.3 *An Hamilton-Poisson realization of the Euler top system* (2.2.1) *is* $(\mathbf{R}^3, P^\alpha, H^\alpha)$ *with the Casimir C^α, where H^α, C^α are given by* (2.2.5), *and P^α is:*

$$P^\alpha = \begin{pmatrix} 0 & -x_3 & -\frac{\alpha_3}{\alpha_2} x_2 \\ x_3 & 0 & 0 \\ \frac{\alpha_3}{\alpha_2} x_2 & 0 & 0 \end{pmatrix}. \tag{2.2.6}$$

The Poisson geometry of the system (2.2.1) is generated by a matrix of $se(2)-$type.

Remark 2.2.5 Since H^α and C^α are first integrals, it follows that: *the trajectories of motion of the Euler top system are intersections of the surfaces*:

$$\frac{1}{2}(\alpha_2 x_1^2 - \alpha_1 x_2^2) = \textit{constant} \qquad \text{and} \qquad \frac{1}{2}(\frac{\alpha_3}{\alpha_2} x_2^2 - x_3^2) = \textit{constant}. \quad \square$$

Define the functions $C_{ab}^\alpha, H_{cd}^\alpha \in C^\infty(\mathbf{R}^3, \mathbf{R})$ given by:

$$C_{ab}^\alpha = aC^\alpha + bH^\alpha, \quad H_{cd}^\alpha = cC^\alpha + dH^\alpha, \quad a, b, c, d \in \mathbf{R} \qquad \text{that is} \tag{2.2.7}$$

$$\begin{cases} C_{ab}^\alpha(x_1, x_2, x_3) & = & \frac{1}{2}\left(b\alpha_2 x_1^2 + (a\frac{\alpha_3}{\alpha_2} - b\alpha_1) x_2^2 - a x_3^2\right) \\ H_{cd}^\alpha(x_1, x_2, x_3) & = & \frac{1}{2}\left(d\alpha_2 x_1^2 + (c\frac{\alpha_3}{\alpha_2} - d\alpha_1) x_2^2 - c x_3^2\right) \end{cases} \tag{2.2.8}$$

Proposition 2.2.6 *The Euler top system* (2.2.1) *has an infinite number of Hamilton-Poisson realizations. More precisely,* $(\mathbf{R}^3, \{\cdot, \cdot\}_{ab}^\alpha, H_{cd}^\alpha)$, *where:*

$$\{f, g\}_{ab}^\alpha = -\nabla C_{ab}^\alpha \cdot (\nabla f \times \nabla g), \quad (\forall) f, g \in C^\infty(\mathbf{R}^3, \mathbf{R}) \tag{2.2.9}$$

and $a, b, c, d \in \mathbf{R}$ such that $ad - bc = 1$, is an Hamilton-Poisson realization.

The Poisson structure given by (2.2.9) is generated by the matrix

$$P_{ab}^\alpha = \begin{pmatrix} 0 & a x_3 & (a\frac{\alpha_3}{\alpha_2} - b\alpha_1) x_2 \\ -a x_3 & 0 & -b\alpha_2 x_1 \\ -(a\frac{\alpha_3}{\alpha_2} - b\alpha_1) x_2 & b\alpha_2 x_1 & 0 \end{pmatrix},$$

and C^{α}_{ab} is a Casimir for the configuration $(\mathbf{R}^3, \{\cdot,\cdot\}^{\alpha}_{ab})$.

Remark 2.2.8 Proposition 2.2.6 assures that the equations (2.2.1) are invariant, if H^{α} and C^{α} are replaced by linear combinations with coefficients modulo $SL(2,\mathbf{R})$. In consequence, the trajectories of motion of the Euler top system remain unchanged. □

If in Proposition 2.2.3 are replaced α_i with corresponding values we obtains Hamilton-Poisson realizations for the systems (1.4.1), (1.4.4), (2.2.2), (2.2.3) and (2.2.4) (Corollary 2.2.9).

Remark 2.2.10 The Euler equations of the free rigid body (1.4.1) have two Hamilton-Poisson realizations, namely: the first of $so(3)-$type and the second of $se(2)-$type.

The systems (2.2.2) – (2.2.4) have Hamilton-Poisson realizations of $se(2)-$type. □

For certain restrictions on parameters α_i, the motion of Euler top system reduces to motion on the surface described by the conservation law:

$$x_1^2 - \frac{\alpha_1}{\alpha_2} x_2^2 = 2H, \quad \text{where} \;\; H = constant. \tag{2.2.10}$$

More precisely: *if* $\alpha_1\alpha_2 < 0$, *then the dynamics of Euler top system* (2.2.1) *can be reduced to pendulum dynamics* (Proposition 2.2.11)

The solutions of Euler top system restricted to constant level surface (2.2.10) are:

$$x_1(t) = \sqrt{2H}\cdot\cos\frac{\theta(t)}{2}, \quad x_2(t) = \sqrt{2H}\sqrt{-\frac{\alpha_2}{\alpha_1}}\cdot\sin\frac{\theta(t)}{2}, \quad x_3(t) = \frac{1}{2\alpha_2}\sqrt{-\frac{\alpha_2}{\alpha_1}}\cdot\dot{\theta}(t),$$

where θ is a solution of the pendulum equation:

$$\ddot{\theta}(t) = 2H\alpha_2\alpha_3\cdot\sin\theta(t).$$

Since the pendulum equation may be integrated by elliptic functions ([31]), it follows that the solutions of Euler top system restricted to the surface (2.2.10) can be written using elliptic functions (a similar result obtain in the case $\alpha_2\alpha_3 < 0$).

2.2.2 Stability problem for the Euler top dynamics

The equilibrium states of the Euler top system (2.2.1) are
$e_0 = (0,0,0)$, $e_1^m = (m,0,0)$, $e_2^m = (0,m,0)$ and $e_3^m = (0,0,m)$ for all $m \in \mathbf{R}^*$.

In Proposition 2.2.14 is established the nature of spectral stability for the equilibrium states of Euler top system. We get the following results:

– e_1^m, $m \in \mathbf{R}^*$ *is spectrally stable if* $\alpha_2\alpha_3 < 0$ *and unstable if* $\alpha_2\alpha_3 > 0$;
– e_2^m, $m \in \mathbf{R}^*$ *is spectrally stable if* $\alpha_1\alpha_3 < 0$ *and unstable if* $\alpha_1\alpha_3 > 0$;
– e_3^m, $m \in \mathbf{R}^*$ *is spectrally stable if* $\alpha_1\alpha_2 < 0$ *and unstable if* $\alpha_1\alpha_2 > 0$;
– e_0 *is spectrally stable.*

Indeed, the matrix of the linear part of the system (2.2.1) is

$$A(x) = \begin{pmatrix} 0 & \alpha_1 x_3 & \alpha_1 x_2 \\ \alpha_2 x_3 & 0 & \alpha_2 x_1 \\ \alpha_3 x_2 & \alpha_3 x_1 & 0 \end{pmatrix}.$$

The characteristic polynomial of the matrix $A(e_1^m)$ is $p_{A(e_1^m)}(\lambda) = -\lambda(\lambda^2 - \alpha_2\alpha_3 m^2)$ which has the roots $\lambda_1 = 0$, $\lambda_{2,3} = \pm m\sqrt{\alpha_2\alpha_3}$.

We have $\lambda_1 = 0$ and $\lambda_{2,3} = \pm m\sqrt{\alpha_2\alpha_3}$, if $\alpha_2\alpha_3 > 0$ and $\lambda_{2,3} = \pm im\sqrt{-\alpha_2\alpha_3}$, if $\alpha_2\alpha_3 < 0$. Then, by Theorem 2.1.9 (Lyapunov), it follows that e_1^m is spectrally stable for $\alpha_2\alpha_3 < 0$ and unstable for $\alpha_2\alpha_3 > 0$.

We replace in Proposition 2.2.2 the parameters α_i with the corresponding values and one obtains the spectral stability of the equilibrium states for Lagrange system etc. (Corollary 2.2.15).

Proposition 2.2.16 *If $\alpha_1\alpha_2 < 0$ (resp. $\alpha_1\alpha_3 < 0$; resp. $\alpha_2\alpha_3 < 0$), then the state e_0 of Euler top system* (2.1.1) *is nonlinear stable.*

For demonstration it is easy to prove that L^α is a Lyapunov function, where

$$L^\alpha(x_1, x_2, x_3) = \frac{1}{2}(\alpha_2 x_1^2 - \alpha_1 x_2^2).$$

In the following propositions is studied nonlinear stability of the equilibrium states e_1^m (if $\alpha_2\alpha_3 < 0$), e_2^m (if $\alpha_1\alpha_3 < 0$) and e_3^m (if $\alpha_1\alpha_2 < 0$), where $m \in \mathbf{R}^*$.

Proposition 2.2.17 *If $\alpha_1\alpha_2 < 0$, then e_3^m, $m \in \mathbf{R}^*$, is nonlinear stable.*

Proof. Let the function $F_\lambda^\alpha \in C^\infty(\mathbf{R}^3, \mathbf{R})$, $\lambda \in \mathbf{R}$, given by:

$$F_\lambda^\alpha(x_1, x_2, x_3) = H^\alpha(x_1, x_2, x_3) - \lambda C^\alpha(x_1, x_2, x_3), \quad \text{that is}$$

$$F_\lambda^\alpha(x_1, x_2, x_3) = \frac{1}{2}\left(\alpha_2 x_1^2 - \alpha_1 x_2^2\right) - \frac{\lambda}{2}\left(\frac{\alpha_3}{\alpha_2}x_2^2 - x_3^2\right).$$

Then we have successively:

(i) $\nabla F_\lambda^\alpha(e_3^m) = 0$ if and only if $\lambda = 0$;

(ii) $W := ker\ dC^\alpha(e_3^m) = span_{\mathbf{R}}\left((1,\,0,\,0)^T, (0,\,1,\,0)^T\right)$;

(iii) For all $v \in W$, i.e. $v = (a, b, 0)^T$, $a, b \in \mathbf{R}$, follows:

$$v^T \cdot \nabla^2 F_0^\alpha(e_3^m) \cdot v = \alpha_2 a^2 - \alpha_1 b^2$$

ant so $\nabla^2 F_0^\alpha(e_3^m)\Big|_{W\times W}$ is pozitive definite if $\alpha_1 < 0$, $\alpha_2 > 0$ and negative definite if $\alpha_1 > 0$, $\alpha_2 < 0$.

Therefore via Arnold's method (Theorem 2.1.14), we conclude that e_3^m, is nonlinear stable. □

In Propositions 2.2.18, 2.2.19 one prove that:

– *if* $\alpha_1\alpha_3 < 0$ *(resp.,* $\alpha_2\alpha_3 < 0$ *), then* e_2^m, $m \in \mathbf{R}^*$ *(resp.,* e_1^m, $m \in \mathbf{R}^*$*), is nonlinear stable.*

As consequences of the above propositions obtain nonlinear stability of equilibrium states for the systems (1.4.1), (1.4.4), (2.2.1) – (2.2.3) (Corollary 2.2.20).

2.2.3 Numerical integration of the Euler top system

We shall discuss the numerical integration of the Euler top dynamics (2.2.1), using the Lie-Trotter integrator [51] and Kahan integrator[26]. The results included in this section was published in the cited paper [50] (Şuşoi, 2010).

The vector field X_{H^α} associated to Hamiltonian H^α of the dynamics (2.2.1) reads:

$$X_{H^\alpha} = X_{H_1^\alpha} + X_{H_2^\alpha}, \qquad \text{where}$$

$$H_1^\alpha(x_1, x_2, x_3) = \frac{1}{2}\alpha_2 x_1^2, \quad H_2^\alpha(x_1, x_2, x_3) = -\frac{1}{2}\alpha_1 x_2^2.$$

The corresponding integral curves are, respectively, given by:

$$X(t) = A_i \cdot X(0), \quad i = 1, 2,$$

where $X(t) = (x_1(t), x_2(t), x_3(t))^T$ and A_i is the matrix of operator $\exp(tX_{H_i^\alpha}), i = 1, 2.$

Determine the matrix A_1 of operator $\exp(tX_{H_1^\alpha})$. We have

$$\dot{X} = P^\alpha \cdot \nabla H_1^\alpha = AX \ \text{ where } \ A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \alpha_2 a \\ 0 & \alpha_3 a & 0 \end{pmatrix} \ \text{ and } \ a = x_1(0).$$

The characteristic polynomial of the matrix At is $p_{At}(\lambda) = -\lambda(\lambda^2 - \alpha_2\alpha_3 a^2 t^2)$. If $\alpha_2\alpha_3 < 0$, then the roots of polynomial $p_{At}(\lambda)$ are $\lambda_1 = 0$ and $\lambda_{2,3} = \pm iat\sqrt{-\alpha_2\alpha_3} = \pm ia\gamma t$, where $\gamma = \sqrt{-\alpha_2\alpha_3}$. We have

$\exp(At) = I_3 + \frac{\sin\gamma at}{\gamma a} \cdot A + \frac{1-\cos\gamma at}{\gamma^2 a^2} \cdot A^2 = A_1$, where

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(a\gamma t) & \frac{\alpha_2}{\gamma}\sin(a\gamma t) \\ 0 & \frac{\alpha_3}{\gamma}\sin(a\gamma t) & \cos a(\gamma t) \end{pmatrix}, \qquad a = x_1(0).$$

In the same manner one determine the matrix A_2 of operator $\exp(tX_{H_2}^\alpha)$ and we have:

$$A_2 = \begin{pmatrix} 1 & 0 & b\alpha_1 t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad b = x_2(0).$$

Then via [51], the Lie-Trotter integrator is given by:

$$X(n+1) = A_1 A_2 X(n). \tag{2.2.11}$$

Proposition 2.2.21 *If* $\alpha_2\alpha_3 < 0$ *and* $\gamma = \sqrt{-\alpha_2\alpha_3}$*, then the Lie-Trotter integrator of Euler top dynamics* (2.2.1) *is given by:*

$$\begin{cases} x_1(n+1) &= x_1(n) + t\alpha_1 x_2(0)\cdot x_3(n) \\ x_2(n+1) &= \cos(t\gamma x_1(0))\cdot x_2(n) + \frac{\alpha_2}{\gamma}\sin(t\gamma x_1(0))\cdot x_3(n) \\ x_3(n+1) &= \frac{\alpha_3}{\gamma}\sin(t\gamma x_1(0))\cdot x_2(n) + \cos(t\gamma x_1(0))\cdot x_3(n) \end{cases} \tag{2.2.12}$$

Proposition 2.2.22 *The Lie-Trotter integrator* (2.2.12) *has the following properties:*
(i) *it is a Poisson integrator;* (ii) *it is a Casimir integrator;*
(iii) *it doesn't a energy-integrator.*
Applying Proposition 2.2.21 one obtain successively the Lie-Trotter integrator for the systems (1.4.4), (2.2.2) and (2.2.3) (Corollaries 2.2.23-2.2.25).

Proposition 2.2.26 *For the Euler top system* (2.2.1)*, the Kahan integrator is given by the system of recurrent equations:*

$$\begin{cases} x_1^{k+1} - x_1^k = \frac{h\alpha_1}{2}(x_2^{k+1}x_3^k + x_3^{k+1}x_2^k), \\ x_2^{k+1} - x_2^k = \frac{h\alpha_2}{2}(x_1^{k+1}x_3^k + x_3^{k+1}x_1^k), \\ x_3^{k+1} - x_3^k = \frac{h\alpha_3}{2}(x_2^{k+1}x_1^k + x_1^{k+1}x_2^k), \end{cases} \quad \textit{where} \quad x^k = x_0 + k\cdot h. \tag{2.2.13}$$

Remark 2.2.27 Replacing in (2.2.13) the parameters α_i with corresponding values one obtain the Kahan integrator for the systems (1.4.1), (1.4.4), (2.2.2) − (2.2.4).

2.3 Metriplectic Euler top system

In this paragraph we study the geometrical and dynamical properties of the metriplectic Euler top system. The content of the last two sections focuses on the author's results contained in the cited paper [49] (Şuşoi and M. Ivan, 2009).

In Section 2.3.1 one presents notions and results concerning the metriplectic systems [40] (Ortega and Plannas-Bielsa, 2004), [22](Gh. Ivan and Opriş, 2006). In Section 2.3.2 we construct the metriplectic structure associated to Euler top system. Section 2.3.3 is dedicated to study of spectral stability for the metriplectic Euler top system.

First present the **construction of the metriplectic system associated to a Hamilton-Poisson system** (Section 2.3.1).

A *Leibniz bracket* on the differential manifold M of dimension n, is a bilinear map $[\cdot,\cdot] : C^\infty(M)\times C^\infty(M) \to C^\infty(M)$ which satisfies the *Leibniz rules* :

$$[fg, h] = [f,h]g + f[g,h] \quad \text{and} \quad [f, gh] = [f,g]h + g[f,h], \quad f,g,h \in C^\infty(M).$$

A *Leibniz manifold* is a pair $(M, [\cdot,\cdot])$, where $[\cdot,\cdot]$ is a Leibniz bracket.

Let P and $\mathbf{g}$ two 2−contravariant tensor fields on M. Define the map $[\cdot,(\cdot,\cdot)] : C^\infty(M) \times (C^\infty(M) \times C^\infty(M)) \to C^\infty(M)$ given by:

$$[f,(h_1,h_2)] = P(df,dh_1) + \mathbf{g}(df,dh_2), \quad (\forall)\ f,h_1,h_2 \in C^\infty(M). \tag{2.3.1}$$

One prove that the map $[[\cdot,\cdot]] : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ given by:

$$[[f,h]] = [f,(h,h)], \quad (\forall)\ f,h \in C^\infty(M), \tag{2.3.2}$$

is a Leibniz bracket and $(M, P, \mathbf{g}, [[\cdot,\cdot]])$ is a Leibniz manifold.

A Leibniz manifold $(M, P, \mathbf{g}, [[\cdot,\cdot]])$ such that P is a skew-symmetric tensor and $\mathbf{g}$ is a symmetric tensor is called *metriplectic manifold.*

Let $(M, P, \mathbf{g}, [[\cdot,\cdot]])$ be a metriplectic manifold. In the paper [22] was proved that , if there exist the functions $h_1, h_2 \in C^\infty(M)$ such that $P(df, dh_2) = 0$ and $\mathbf{g}(df, dh_1) = 0$ for all $f \in C^\infty(M)$, then the bracket $[[\cdot,\cdot]]$ given by (2.3.2) satisfy the relation:

$$[[f, h_1 + h_2]] = [f,(h_1,h_2)], \quad (\forall)\ \ f \in C^\infty(M). \tag{2.3.3}$$

In these hypothesis, the vector field $X_{h_1h_2}$ given by:

$$X_{h_1h_2}(f) = [[f, h_1 + h_2]] \quad (\forall)\ \ f \in C^\infty(M), \tag{2.3.4}$$

is called the *Leibniz field* associated to pair (h_1, h_2) on M.

Applying the relations (2.3.1) − (2.3.3), it follows that $X_{h_1h_2}$ is given by:

$$X_{h_1h_2}(f) = P(df,dh_1) + \mathbf{g}(df,dh_2), \quad (\forall)\ \ f \in C^\infty(M). \tag{2.3.5}$$

In local coordinates $(x^i), i = \overline{1,n}$ on M, the following system:

$$\dot{x}^i = X_{h_1h_2}(x^i) = P^{ij}\frac{\partial h_1}{\partial x^j} + G^{ij}\frac{\partial h_2}{\partial x^j},\ \ i,j = \overline{1,n}, \tag{2.3.6}$$

with $P^{ij} = P(dx^i, dx^j)$ şi $G^{ij} = \mathbf{g}(dx^i, dx^j)$, is called *metriplectic system* on M associated to field $X_{h_1h_2}$ with the bracket $[[\cdot,\cdot]]$.

We present now a method to obtain metriplectic systems which consist in adding of a dissipation therm to an Hamilton-Poisson system [10] (Birtea, Puta et al,, 2007).

Let $\{\cdot,\cdot\}$ be a Poisson structure on $\mathbf{R}^n$ generated by the skew-symmetric matrix $P = (P^{ij})$, a function $H \in C^\infty(\mathbf{R}^n)$ and $C_1, \ldots, C_k \in C^\infty(\mathbf{R}^n)$ a complete set of functionally independent Casimir functions. Let G be a smooth function from $\mathbf{R}^n$ to the vector space of symmetric matrices of type $n \times n$.

Definition 2.3.1([10])A *metriplectic system* on $\mathbf{R}^n$ is a system of differential equations of the following form:

$$\dot{x}(t) = P(x(t)) \cdot \nabla H(x(t)) + G(x(t)) \cdot \nabla \varphi(C_1, \dots, C_k)(x(t)), \tag{2.3.7}$$

where $\varphi \in C^\infty(\mathbf{R}^k)$ such that the following conditions hold:

$(i) \quad P(x) \cdot \nabla C_i(x) = 0, \quad i = \overline{1,k}, \qquad (ii) \quad G(x) \cdot \nabla H(x) = 0;$

$(iii) \qquad (\nabla \widetilde{C}(x))^T \cdot G(x) \cdot \nabla \widetilde{C}(x)) \leq 0, \quad$ where $\widetilde{C} = \varphi(C_1, \dots, C_k)$. □

The metriplectic system (2.3.7), denoted with $(\mathbf{R}^n, P, H, G, \widetilde{C})$, can be regarded as a "perturbation" of the Hamilton-Poisson system

$$\dot{x}(t) = P(x(t)) \cdot \nabla H(x(t))$$

with the dissipation therm $G(x) \cdot \nabla \varphi(C_1, \dots, C_k)(x)$. We say that *metriplectic system* (2.3.7) *is associated to Hamilton-Poisson system* $(\mathbf{R}^n, P, H)$.

If $(\mathbf{R}^n, P, H)$ is a Hamilton-Poisson system, then we determine a symmetric tensor $\mathbf{g}$ on $\mathbf{R}^n$, generated by the matrix $G = (G^{ij})$, where:

$$G^{ii}(x) = -\sum_{k=1,\ k \neq i}^{n} (\frac{\partial h_1}{\partial x^k})^2 \quad \text{and} \quad G^{ij}(x) = \frac{\partial h_1}{\partial x^i}\frac{\partial h_1}{\partial x^j}, \quad \text{for} \quad i \neq j. \tag{2.3.8}$$

Definition 2.3.3 A differential system on $\mathbf{R}^n$ of the form:

$$\dot{x}^i = \varphi^i(x^1, x^2, ..., x^n), \quad \text{where} \quad \varphi^i \in C^\infty(\mathbf{R}^n), \ i = \overline{1,n} \tag{2.3.9}$$

has a *metriplectic realization on* $\mathbf{R}^n$, if there exists a metriplectic structure $(\mathbf{R}^n, P, H, G, \widetilde{C})$ such that (2.3.9) can be written in the form (2.3.7).

As illustrative example we construct the **metriplectic system associated to Euler top system** (2.2.1) (Section 2.3.2). For this we use the Hamilton-Poisson realization $(\mathbf{R}^3, P^\alpha, H^\alpha)$ given in Proposition 2.2.3.

We apply now the relations (2.3.8) for $h_1 = H^\alpha \in C^\infty(\mathbf{R}^3)$ given by (2.2.5).

The symmetric tensor $\mathbf{g}$ is generated by the matrix $G^\alpha = (G^{ij}_\alpha)$, where

$$G^\alpha = \begin{pmatrix} -\alpha_1^2 x_2^2 & -\alpha_1\alpha_2 x_1 x_2 & 0 \\ -\alpha_1\alpha_2 x_1 x_2 & -\alpha_2^2 x_1^2 & 0 \\ 0 & 0 & -\alpha_2^2 x_1^2 - \alpha_1^2 x_2^2 \end{pmatrix}. \tag{2.3.10}$$

We consider $H = H^\alpha$ and $C = C^\alpha$ given by the relations (2.2.5), the skew-symmetric tensor $P = P^\alpha$ given by (2.2.6) and the symmetric tensor $\mathbf{g}$ given by (2.3.10).

For the function $\widetilde{C}^\alpha = \beta C^\alpha$ with $\beta \in \mathbf{R}$, the dynamical system (2.3.7) reads:

$$\begin{cases} \dot{x}_1 &= \alpha_1 x_2(x_3 - \beta\alpha_3 x_1 x_2) \\ \dot{x}_2 &= \alpha_2 x_1(x_3 - \beta\alpha_3 x_1 x_2) \\ \dot{x}_3 &= \alpha_3 x_1 x_2 + \beta x_3(\alpha_2^2 x_1^2 + \alpha_1^2 x_2^2) \end{cases} \tag{2.3.11}$$

Proposition 2.3.4 $(\mathbf{R}^3, P^\alpha, H^\alpha, G^\alpha, \widetilde{C}^\alpha)$ *is a metriplectic realization for* (2.3.11).
The system (2.3.11) is called the *metriplectic Euler top system.*
If $\beta = 0$, the system (2.3.11) reduces to Hamilton-Poisson system (2.2.1).
Proposition 2.3.6 *The function H^α given by* (2.2.5) *is a constant of motion of the metriplectic system* (2.3.11).
For $\beta \neq 0, \widetilde{C}^\alpha = \beta C^\alpha$ is not a constant of motion for (2.3.11).
If in (2.3.11) we take $\alpha = (1, 1, 1)$, one obtain the *metriplectic Lagrange system*:

$$\dot{x}_1 = x_2 x_3 - \beta x_1 x_2^2,\ \dot{x}_2 = x_1 x_3 - \beta x_1^2 x_2,\ \dot{x}_3 = x_1 x_2 + \beta x_3(x_1^2 + x_2^2). \tag{2.3.12}$$

Finally we deal with the **study of spectral stability for the dynamics** (2.3.11) (Section 2.3.3).
The Euler top system (2.2.1) *and metriplectic Euler top system* (2.3.11) *have the same equilibrium states.*
Proposition 2.3.10 (*i*) *For $\beta \neq 0$, the equilibrium states e_1^m, $m \in \mathbf{R}^*$ of metriplectic system* (2.3.11), *have the following behavior:*

(1) if $\beta\alpha_2(\alpha_2 - \alpha_3) \leq 0$, then e_1^m is spectrally stable;

(2) if $\beta\alpha_2(\alpha_2 - \alpha_3) > 0$, then e_1^m is unstable.
(ii) For $\beta = 0$, the equilibrium state e_1^m, $m \in \mathbf{R}^*$ of Euler top system (2.2.1), *is spectrally stable if $\alpha_2\alpha_3 < 0$ and unstable if $\alpha_2\alpha_3 > 0$.*
Proposition 2.3.11 (*i*) *For all $\beta \neq 0$, the equilibrium states e_2^m, $m \in \mathbf{R}^*$ of metriplectic system* (2.3.11), *have the following behavior:*

(1) if $\beta\alpha_1(\alpha_1 - \alpha_3) \leq 0$, then e_2^m is spectrally stable;

(2) if $\beta\alpha_1(\alpha_1 - \alpha_3) > 0$, then e_2^m is unstable.
(ii) For $\beta = 0$, the equilibrium state e_2^m, $m \in \mathbf{R}^*$ of Euler top system (2.2.1), *is spectrally stable if $\alpha_1\alpha_3 < 0$ and unstable if $\alpha_1\alpha_3 > 0$.*
Proposition 2.3.12 *(i) For all $\beta \in \mathbf{R}^*$ and $m \in \mathbf{R}$, the equilibrium state e_3^m of the system* (2.3.11) *is spectrally stable if $\alpha_1\alpha_2 < 0$ and unstable if $\alpha_1\alpha_2 > 0$.*
(ii) For all $\beta \in \mathbf{R}$, the equilibrium state $e_0 = (0, 0, 0)$ is spectrally stabile.
Corollary 2.3.13 (*i*) *If $\beta \neq 0$ and $m \in \mathbf{R}^*$, then e_0, e_1^m, e_2^m of metriplectic Lagrange system* (2.3.13), *are spectrally stable and e_3^m, $m \in \mathbf{R}^*$ is unstable.*
(ii) For $\beta = 0$, the states e_1^m, e_2^m, e_3^m for $m \in \mathbf{R}^*$ of Lagrange system (2.2.4) *are unstable and e_0 is spectrally stable.*

Chapter 3

Two classical dynamical systems on $\mathbf{R}^6$

In this chapter one establish some important geometrical and dynamical properties for two remarkable differential systems on $\mathbf{R}^6$, namely: the *Goryachev-Chaplygin top system* and the *Kowalevski top system*. The chapter is structured in three paragraphs. Original contributions of the author are contained in the last two paragraphs.

3.1 Lie-Poisson structure on the dual of Lie algebra $se(3)$

Let $SE(3, \mathbf{R}) = SO(3) \times \mathbf{R}^3-$ be the special Euclidean group of order 3. This is a Lie group with Lie algebra $se(3, \mathbf{R})$ which it can identified with $so(3) \times \mathbf{R}^3$, [2](Andrica and Caşu, 2008). We have:

$$se(3, \mathbf{R}) = \{ \begin{pmatrix} \hat{x} & y \\ 0 & 0 \end{pmatrix} \mid \hat{x} \in so(3), y \in \mathbf{R}^3 \}, \quad \hat{x} = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}. \qquad (3.1.1)$$

The plus-minus Lie-Poisson structures on the dual $(se(3, \mathbf{R}))^*$ of Lie algebra $se(3, \mathbf{R})$ are generated by the matrices $\Pi_{e3,-}$, respectively $\Pi_{e3,+}$ (Proposition 3.1.3).

3.2 Study of the dynamics Goryachev- Chaplygin top

In Paragraph 3.2 gives an Hamilton-Poisson realization of the dynamics Goryachev-Chaplygin top (3.2.1) and studies the Lax formulation, stability problem, existence of periodic solutions and numerical integration. This paragraph contains the original contributions and these was published in [5] (Aron, Puta and Şuşoi, 2005).

The Goryachev- Chaplygin top (or shorter, G-C top) was introduced by Goryachev in 1900 ([16]) and was integrated in terms of hyper-elliptic integrals by Chaplygin in 1948 ([11]). G-C top is a rigid body rotating about a fixed point with principal moments of inertia I_1, I_2, I_3 satisfying $I_1 = I_2 = 4I_3 = I$, and with center of mass lying in the equatorial plane. For simplicity, we consider $I = 1$.

The dynamical variables are components m_1, m_2, m_3 of angular momentum and components $\gamma_1, \gamma_2, \gamma_3$ of the center mass vector in the system related to the principal axes of the body.

The dynamics of Goryachev-Chaplygin top is described by the differential system:

$$\begin{cases} \dot{m}_1 = 3m_2m_3, & \dot{m}_2 = -3m_1m_3 - 2\gamma_3, \quad \dot{m}_3 = 2\gamma_2, \\ \dot{\gamma}_1 = 4\gamma_2m_3 - \gamma_3m_2, & \dot{\gamma}_2 = \gamma_3m_1 - 4\gamma_1m_3, \quad \dot{\gamma}_3 = \gamma_1m_2 - \gamma_2m_1. \end{cases} \tag{3.2.1}$$

Proposition 3.2.1 *An Hamilton-Poisson realization of* (3.2.1) *is* $(\mathbf{R}^6, \Pi, H)$*, where*

$$\Pi = \begin{pmatrix} 0 & -m_3 & m_2 & 0 & -\gamma_3 & \gamma_2 \\ m_3 & 0 & -m_1 & \gamma_3 & 0 & -\gamma_1 \\ -m_2 & m_1 & 0 & -\gamma_2 & \gamma_1 & 0 \\ 0 & -\gamma_3 & \gamma_2 & 0 & 0 & 0 \\ \gamma_3 & 0 & -\gamma_1 & 0 & 0 & 0 \\ -\gamma_2 & \gamma_1 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.2.2}$$

$$H(m_1, m_2, m_3, \gamma_1, \gamma_2, \gamma_3) = \frac{1}{2}(m_1^2 + m_2^2 + 4m_3^2) - 2\gamma_1 \tag{3.2.3}$$

The Poisson structure generated by the matrix Π on $\mathbf{R}^6$ is in fact the minus Lie-Poisson structure on $(se(3, \mathbf{R}))^* \cong \mathbf{R}^6$ generated by the matrix $\Pi_{e3,-}$.

Proposition 3.2.3 *The configuration* $(\mathbf{R}^6, \Pi)$ *has the Casimirs* $C_1, C_2 \in C^\infty(\mathbf{R}^6, \mathbf{R})$*:*

$$C_1(m, \gamma) = m_1\gamma_1 + m_2\gamma_2 + m_3\gamma_3, \quad C_2(m, \gamma) = \frac{1}{2}(\gamma_1^2 + \gamma_2^2 + \gamma_3^2).$$

The functions $H, C_1, C_2 \in C^\infty(\mathbf{R}^6, \mathbf{R})$ *are constants of motion for* (3.2.1).

Remark 3.2.5 *On co-adjoint orbit* $(\mathcal{O}_{0,1}, \omega_{\mathcal{O}_{0,1}})$ *, where*

$$\mathcal{O}_{0,1} = \{(m, \gamma) \in \mathbf{R}^6 \mid \begin{cases} m_1\gamma_1 + m_2\gamma_2 + m_3\gamma_3 = 0 \\ \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 2 \end{cases} \}$$

$$\omega_{\mathcal{O}_{0,1}} = \frac{1}{2\gamma_3}(dm_2 \wedge d\gamma_1 - dm_1 \wedge d\gamma_2) + \frac{m_3}{2\gamma_3^2} d\gamma_1 \wedge d\gamma_2,$$

the system (3.2.1) *has another first integral, namely:*

$$K(m,\gamma) = m_3(m_1^2 + m_2^2) + 2m_1\gamma_3. \tag{3.2.4}$$

Moreover, the Hamiltonian system $(\mathcal{O}_{0,1}, \omega_{\mathcal{O}_{0,1}}, H)$ *is completely integrable with both independent first integrals in involution* H *and* K*.*

G-C top system and Lax formulation

Proposition 3.2.6 *The Dynamics G-C top* (3.2.1) *has a Lax formulation, that is:*

$$\dot{L} = [L, B], \quad \textit{where}$$

$$L = \begin{pmatrix} 0 & m_2 - im_1 & m_1 + im_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -m_2 + im_1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -m_1 - im_2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -im_3 & m_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & im_3 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -m_3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma_3 & \gamma_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\gamma_3 & 0 & \gamma_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\gamma_1 & -\gamma_2 & 0 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 0 & -2i\gamma_3 & 2\gamma_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2i\gamma_3 & 0 & 3m_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2\gamma_3 & -3m_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\gamma_2 & 2i\gamma_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2\gamma_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2i\gamma_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4m_3 & -m_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4m_3 & 0 & -m_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & m_1 & m_2 & 0 \end{pmatrix}.$$

Corollary 3.2.7 *The flow of the dynamics G-C top* (3.2.1) *is iso-spectral.*

Stability problem and periodic solutions for the dynamics (3.2.1)

Proposition 3.2.8 *The dynamics G-C top has the following equilibrium states:*

$$e_{12}=(M,N,0,0,0,0),\quad e_{14}=(M,0,0,N,0,0),\quad e_{1346}=(M,0,N,-\frac{3M^2}{8},0,-\frac{3MN}{2}),$$

for all $M,N\in\mathbf{R}$.

In Propositions 3.2.9 - 3.2.11 is studied the spectral stability of equilibrium states for the dynamics (3.2.1). These states have the following behavior:

- e_{1346} is spectrally stable if $M^2<2N^2$, and unstable if $M^2\geq 2N^2$;
- e_{14} is spectrally stable if $N>0$, and unstable if $N\leq 0$;
- $e_{12}, M,N\in\mathbf{R}$ is spectrally stable.

Nonlinear stability of states e_{14} *and* e_{1346} *is analyzed in the following propositions.*

Proposition 3.2.12 e_{14} *for* $M,N\in\mathbf{R}, N>0$*, is nonlinear stable.*

Proposition 3.2.13 e_{1346} *is nonlinear stable, if* $M,N\in\mathbf{R}, M^2<2N^2, M<0$.

Remark 3.2.14. It is an open problem to decide the nonlinear stability for the equilibrium states $e_{12}, M,N\in\mathbf{R}$ and $e_{1346}, M,N\in\mathbf{R}, M^2<2N^2, M\geq 0$.

The system (3.2.1) reduced to the co-adjoint orbit $\mathcal{O}_{M,N}$, where

$$\mathcal{O}_{M,N}=\{(m,\gamma)\in\mathbf{R}^6\mid\left\{\begin{array}{l} m_1\gamma_1+m_2\gamma_2+m_3\gamma_3=MN\\ \gamma_1^2+\gamma_2^2+\gamma_3^2=N^2\end{array}\right.\}$$

gives rise to a classical Hamiltonian system. Then the following proposition holds.

Proposition 3.2.15 *Near to* $e_{14}, M,N\in\mathbf{R}, N>0$*, the reduced system has for each sufficiently small value of the reduced energy at least two periodic solutions.*

Numerical integration of the dynamics (3.2.1)

We shall discuss the numerical integration of the dynamics G-C top, using the Lie-Trotter integrator ([51]).

The vector field X_H associated to Hamiltonian H of the dynamics (3.2.1) splits as:

$$X_H=X_{H_1}+X_{H_2}+X_{H_3}+X_{H_4},\quad \text{where}$$

$$H_1(m,\gamma)=\frac{1}{2}m_1^2,\quad H_2(m,\gamma)=\frac{1}{2}m_2^2,\quad H_3(m,\gamma)=2m_3^2,\quad H_4(m,\gamma)=-2\gamma_1.$$

Their corresponding integral curves are respectively given by:

$$\begin{pmatrix} m_1(t)\\ m_2(t)\\ m_3(t)\\ \gamma_1(t)\\ \gamma_2(t)\\ \gamma_3(t)\end{pmatrix}=A_i\cdot\begin{pmatrix} m_1(0)\\ m_2(0)\\ m_3(0)\\ \gamma_1(0)\\ \gamma_2(0)\\ \gamma_3(0)\end{pmatrix},$$

where A_i is the matrix of operator $\exp(tX_{H_i})$, for $i = \overline{1,4}$.

Determine the matrix A_i of operator $\exp(tX_{H_i})$ for $i = \overline{1,4}$ and we obtains:

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos m_1(0)t & \sin m_1(0)t & 0 & 0 & 0 \\ 0 & -\sin m_1(0)t & \cos m_1(0)t & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos m_1(0)t & \sin m_1(0)t \\ 0 & 0 & 0 & 0 & -\sin m_1(0)t & \cos m_1(0)t \end{pmatrix}, \tag{3.2.5}$$

$$A_2 = \begin{pmatrix} \cos m_2(0)t & 0 & -\sin m_2(0)t & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \sin m_2(0)t & 0 & \cos m_2(0)t & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos m_2(0)t & 0 & -\sin m_2(0)t \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \sin m_2(0)t & 0 & \cos m_2(0)t \end{pmatrix}, \tag{3.2.6}$$

$$A_3 = \begin{pmatrix} \cos 4m_3(0)t & \sin 4m_3(0)t & 0 & 0 & 0 & 0 \\ -\sin 4m_3(0)t & \cos 4m_3(0)t & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos 4m_3(0)t & \sin 4m_3(0)t & 0 \\ 0 & 0 & 0 & -\sin 4m_3(0)t & \cos 4m_3(0)t & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.2.7}$$

$$A_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -2t \\ 0 & 0 & 1 & 0 & 2t & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.2.8}$$

Then via [51], the Lie-Trotter integrator is given by:

$$\begin{pmatrix} m_1^{n+1} \\ m_2^{n+1} \\ m_3^{n+1} \\ \gamma_1^{n+1} \\ \gamma_2^{n+1} \\ \gamma_3^{n+1} \end{pmatrix} = A_1 A_2 A_3 A_4 \begin{pmatrix} m_1^{n} \\ m_2^{n} \\ m_3^{n} \\ \gamma_1^{n} \\ \gamma_2^{n} \\ \gamma_3^{n} \end{pmatrix}, \tag{3.2.9}$$

where $A_i, i = \overline{1,4}$ are given by the relations (3.2.5) - (3.2.8).

Proposition 3.2.16 *The Lie-Trotter integrator* (3.2.9) *has the following properties:*

(i) *it is a Poisson integrator and it preserves the Casimirs* C_1, C_2.

(ii) *it doesn't preserve the Hamiltonian* H.

Proposition 3.2.17 *The restriction of the Lie-Trotter integrator to the generic co-adjoint orbits:*

$$m_1\gamma_1 + m_2\gamma_2 + m_3\gamma_3 = constant \quad \text{and} \quad \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = constant$$

give rise to a symplectic integrator.

3.3 Study of the dynamics Kowalevski top

The contents of this paragraph is based on the author's results contained in [6] (Aron, Puta, Şuşoi et al., 2006).

The Kowalevski top [30] (Kowalevski, 1989) is a rigid body rotating about a fixed point with principal moments of inertia I_1, I_2, I_3 satisfying $I_1 = I_2 = 2I_3 = I$, and with center of mass lying in the equatorial plane. For simplicity, we consider $I = 1$.

The dynamics Kowalevski top is described by the differential system:

$$\begin{cases} \dot{m}_1 = m_2 m_3, & \dot{m}_2 = -m_1 m_3 - \dfrac{1}{2}\gamma_3, & \dot{m}_3 = \dfrac{1}{2}\gamma_2, \\ \dot{\gamma}_1 = 2\gamma_2 m_3 - \gamma_3 m_2, & \dot{\gamma}_2 = \gamma_3 m_1 - 2\gamma_1 m_3, & \dot{\gamma}_3 = \gamma_1 m_2 - \gamma_2 m_1. \end{cases} \tag{3.3.1}$$

Proposition 3.3.1 $(\mathbf{R}^6, \Pi, H)$ *is an Hamilton-Poisson realization for* (3.3.1), *where*

$$\Pi = \begin{pmatrix} 0 & -m_3 & m_2 & 0 & -\gamma_3 & \gamma_2 \\ m_3 & 0 & -m_1 & \gamma_3 & 0 & -\gamma_1 \\ -m_2 & m_1 & 0 & -\gamma_2 & \gamma_1 & 0 \\ 0 & -\gamma_3 & \gamma_2 & 0 & 0 & 0 \\ \gamma_3 & 0 & -\gamma_1 & 0 & 0 & 0 \\ -\gamma_2 & \gamma_1 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.3.2}$$

$$H(m_1, m_2, m_3, \gamma_1, \gamma_2, \gamma_3) = \frac{1}{2}(m_1^2 + m_2^2 + 2m_3^2 - \gamma_1) \tag{3.3.3}$$

Proposition 3.3.2 *The configuration* $(\mathbf{R}^6, \Pi)$ *has the Casimirs* $C_1, C_2 \in C^\infty(\mathbf{R}^6, \mathbf{R})$:

$C_1(m, \gamma) = m_1\gamma_1 + m_2\gamma_2 + m_3\gamma_3, \qquad C_2(m, \gamma) = \dfrac{1}{2}(\gamma_1^2 + \gamma_2^2 + \gamma_3^2).$ □

One prove that:

– $H, C_1, C_2 \in C^\infty(\mathbf{R}^6, \mathbf{R})$ *are constants of motion for* (3.3.1)(Proposition 3.3.3);

– *the dynamics Kowalevski top* (3.3.1) *has a Lax formulation and the flow of the dynamics Kowalevski top is iso-spectral* (Proposition 3.3.4, Corollary 3.3.5).

Stability problem and the existence of periodic solutions. The dynamics Kowalevski top has the following equilibrium states:

$e_{12} = (M, N, 0, 0, 0, 0)$, $e_{14} = (M, 0, 0, N, 0, 0)$, $e_{1346} = (M, 0, N, M^2, 0, -2MN)$, for all $M, N \in \mathbf{R}$ (Proposition 3.3.6).

In Propositions 3.3.7 $-$ 3.3.11 one studies the stability problem and we find that:

- $e_{12}, M, N \in \mathbf{R}$ *is spectrally stable;*
- e_{1346} *is nonlinear stable if* $M^2 < 2N^2, M < 0$ *and unstable if* $M^2 \geq 2N^2$*;*
- e_{14} *is nonlinear stable if* $M, N \in \mathbf{R}, N > 0,$ *and unstable if* $N \leq 0.$

Remark 3.3.12 It is an open problem to decide stability for the equilibrium states $e_{12}, M, N \in \mathbf{R}$ and $e_{1346}, M, N \in \mathbf{R}, M^2 < 2N^2, M \geq 0$. □

The system (3.3.1) reduced to co-adjoint orbit $\mathcal{O}_{M,N}$, where

$$\mathcal{O}_{M,N} = \{(m,\gamma) \in \mathbf{R}^6 \mid \begin{cases} m_1\gamma_1 + m_2\gamma_2 + m_3\gamma_3 = MN \\ \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = N^2 \end{cases} \}$$

gives rise to a classical Hamiltonian system. Then the following proposition holds.

Proposition 3.3.13 *Near* $e_{14}, M, N \in \mathbf{R}, N > 0$*, the reduced system has for each sufficiently small value of the reduced energy at least two periodic solutions.*

Numerical integration of the dynamics (3.3.1) is realized by two methods.

The Lie-Trotter integrator of the dynamics (3.3.1) is given in the relations (3.3.13), and its properties are established in Propositions 3.3.14 and 3.3.15.

Proposition 3.3.16 *For the Kowalevski top system* (3.3.1)*, the Kahan integrator is given by the recurrent equations:*

$$\begin{cases} m_1^{k+1} - m_1^k &= \frac{h}{2}(m_2^{k+1}m_3^k + m_3^{k+1}m_2^k) \\ m_2^{k+1} - m_2^k &= -\frac{h}{2}(m_1^{k+1}m_3^k + m_3^{k+1}m_1^k) - \frac{h}{4}(\gamma_3^{k+1} - \gamma_3^k) \\ m_3^{k+1} - m_3^k &= \frac{h}{4}(\gamma_2^{k+1} - \gamma_2^k) \\ \gamma_1^{k+1} - \gamma_1^k &= h(\gamma_2^{k+1}m_3^k + m_3^{k+1}\gamma_2^k) - \frac{h}{2}(\gamma_3^{k+1}m_2^k + m_2^{k+1}\gamma_3^k) \\ \gamma_2^{k+1} - \gamma_2^k &= \dfrac{h}{2}(\gamma_3^{k+1}m_1^k + m_1^{k+1}\gamma_3^k) - h(\gamma_1^{k+1}m_3^k + m_3^{k+1}\gamma_1^k) \\ \gamma_3^{k+1} - \gamma_3^k &= \dfrac{h}{2}(\gamma_1^{k+1}m_2^k + m_2^{k+1}\gamma_1^k) - h(\gamma_2^{k+1}m_1^k + m_1^{k+1}\gamma_2^k) \end{cases} \tag{3.3.4}$$

where $m_i^k = m_i^0 + k \cdot h,\ \gamma_i^k = \gamma_i^0 + k \cdot h,\ i = 1, 2, 3.$

A long but straightforward computation or using, eventually, MATHEMATICA software leads to: *the Kahan integrator does not preserves the Poisson structure and not also the Casimirs* C_1, C_2 *and doesn't an energy-integrator* (Proposition 3.3.17).

Remark. It is easy to conclude by numerical simulation that the Lie-Trotter integrator and the Kahan integrator approximates only "small" portions the dynamics Kowalevski top. It is an open problem for to argument this behavior.

Chapter 4

Control dynamical systems on Lie group $SO(4)$

Chapter 4 is structured in two paragraphs. Original contributions of the author are included in Paragraph 4.2.

4.1 Control systems on matrix Lie groups

This paragraph contains basic definitions and properties concerning the control systems on Lie groups. We have analyzed an optimal control problem for the dynamics of mobile robot Hilare resp. the dynamics of spacecraft.

The main bibliographic sources used are: [25](Jurdjevic and Sussmann, 1972), [29] (Khrisnaprasad, 1993), [32](Leonard, 1994), [47](Struemper, 1997), [28](Khalil, 2002).

We start with recalling of some elements of optimal control on matrix Lie groups.

Let G be an $n-$dimensional matrix Lie group and $\mathbf{g}$ its Lie algebra. A *left invariant vector field* on G takes the form XA, with $X \in G$ and $A \in \mathbf{g}$. Let $\mathcal{B} = \{E_1, E_2, ..., E_n\}$ be the basis of constant matrices in the Lie algebra $\mathbf{g}$.

A *drift-free left invariant control system on* G is a system of the form:

$$\dot{X}(t) = X(t)U(t) = X(t)\sum_{i=1}^{m} u_i(t)A_i, \quad m \leq n, \tag{4.1.1}$$

where $X(t)$ is a curve in G, $U(t)$ is a curve in $\mathbf{g}$ and $\{A_1, A_2, ..., A_m\} \subseteq \mathcal{B}$. A choosing of set $\{A_1, A_2, ..., A_m\}$ is called *control authority* of system (4.1.1).

Remark. Left invariance signifies the fact that, if it is known the solution $X_{I_n}(t)$ of the system (4.1.1) with initial condition $X(0) = I_n$, then every solution $X(t)$ of the system (4.1.1) with initial condition X_0 is of the form $X(t) = X_0 \cdot X_{I_n}(t)$. □

Let $\mathcal{U}$ denote the *set of admissible controls*, that is the set of set of locally bounded, measurable functions defined on $[0, \infty]$ with values in $\mathbf{R}^m$.

Definition4.1.1 The left invariant system (4.1.1) is called *controllable*, if for all $X_0, X_f \in G$ there exist a time $t > 0$ and an admissible control

$$u(t) = (u_1(t), u_2(t), ..., u_m(t)) \in \mathcal{U}, \quad t \in [0, t_f],$$

such that the solution $X(t)$ of the system (4.1.1) satisfy the conditions:

$$X(0) = X_0 \quad \text{and} \quad X(t) = X_f. \tag{4.1.2}$$

The problem of controllability of a left invariant control system on G of the form (4.1.1) can be reduced by studying algebraic properties of the corresponding Lie algebra $\mathbf{g}$ and topological properties of the manifold G, [25](Jurdjevic and Sussmann, 1972).

Let $\mathcal{C}$ denote the set of Lie brackets generated by $\{A_1, A_2, ..., A_m\}$ and defined as:

$$\mathcal{C} = \{\eta \mid \eta = [\eta_{k+1}, [\eta_k, [..., [\eta_2, \eta_1]...]]], \ \eta_i \in \{A_1, A_2, ..., A_m\}, \ i = \overline{1, k+1}.$$

Theorem 4.1.2.([25]) (**Jurdjevic-Sussmann**). *Let S be a control system of the form* (4.1.1) *on a connected Lie group G. Then S is controllable iff span $\mathcal{C} = \mathbf{g}$.*

If the system (4.1.1) is controllable, then an interesting problem is to find the optimal controls. More exactly:

to determine the controls $u_i(t), i = \overline{1, m}$ that steer the system (4.1.1) *from* $X(0) = X_0$ *at $t = 0$ to $X(t_f) = X_f$ at $t = t_f$ and minimize a cost function:*

$$J(u_1, u_2, ..., u_m) = \frac{1}{2} \int_0^{t_f} [\sum_{i=1}^m c_i u_i^2(t)] dt, \quad c_i > 0, i = \overline{1, m}. \tag{4.1.3}$$

Theorem 1.1.5 ([29]) (**Krishnaprasad**) *Let a left invariant controllable system on Lie group G given by* (4.1.1) *with restrictions* (4.1.2). *The controls u_i, $i = \overline{1, m}$ which minimize the cost function J defined by* (4.1.3) *are given by the relations:*

$$u_i = \frac{1}{c_i} P_i, \quad i = \overline{1, m}, \tag{4.1.4}$$

where P_i are solutions of the reduced Hamilton's equations on $(\mathbf{g}^*, \{\cdot, \cdot\}_-)$, that is:

$$\dot{P}_i = \{P_i, H_{opt}\}_-, \quad i = \overline{1, m}, \tag{4.1.5}$$

where H_{opt} is the reduced (or optimal) Hamiltonian given by:

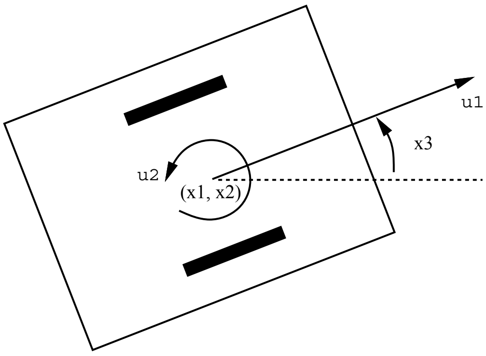
$$H_{opt}(P_1, P_2, ..., P_m) = \frac{1}{2} \sum_{i=1}^m \frac{1}{c_i} P_i^2. \tag{4.1.6}$$

- **Mobile robot Hilare as left invariant control system on $SE(2,\mathbf{R})$**

The space of the configurations is $\mathbf{R}^2 \times S^1$, and its dynamics is described by the system of differential equations ([52]):

$$\dot{x}_1 = u_1 \cos x_3, \quad \dot{x}_2 = u_1 \sin x_3, \quad \dot{x}_3 = u_2, \tag{4.1.7}$$

where (x_1, x_2) represent the position of the robot in the plane, and x_3 is its orientation, see the figure:



A basis in the Lie algebra $se(2,\mathbf{R})$ of the Lie group $SE(2,\mathbf{R})$ is $\{E_1, E_2, E_3\}$, see Paragraph 1.2. Choose the control authority $\{A_1, A_2\}$, where $A_1 = E_2,\ A_2 = E_1$.

The system (4.1.7) can be written in the equivalent form:

$$\dot{X} = X(A_1 u_1 + A_2 u_2), \quad \text{where} \quad X = \begin{pmatrix} \cos x_3 & -\sin x_3 & x_1 \\ \sin x_3 & \cos x_3 & x_2 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.1.8}$$

The system (4.1.8) is a controllable system on $SE(2,\mathbf{R})$ with the control authority $\{A_1, A_2\}$ (we apply Jurdjevic-Sussmann's Theorem).

Consider the cost function J, given by:

$$J(u_1, u_2) = \frac{1}{2}\int_0^{t_f} [c_1 u_1^2(t) + c_2 u_2^2(t)]dt, \quad c_1 > 0, c_2 > 0. \tag{4.1.9}$$

Proposition 4.1.6 *The controls which minimize the cost function J given by* (4.1.9) *and steers the system* (4.1.8) *from $X(0) = X_0$ at $t = 0$ to $X(t_f) = X_f$ at $t = t_f$ are given by $u_1 = \dfrac{P_1}{c_1}$, $u_2 = \dfrac{P_2}{c_2}$ where P_i, $i = \overline{1,3}$ are solutions of the system:*

$$\dot{P}_1 = -\frac{1}{c_2} P_2 P_3, \quad \dot{P}_2 = \frac{1}{c_1} P_1 P_3, \quad \dot{P}_3 = -\frac{1}{c_1} P_1 P_2. \tag{4.1.10}$$

The system (4.1.10) is a Euler top system (**we apply the results from Chap. 2**).

The system (4.1.10) has the Hamilton-Poisson realization $(\mathbf{R}^3, H_{rh}, P_{rh})$(v. Proposition 2.2.3), where:

$$P_{rh} = \begin{pmatrix} 0 & -P_3 & P_2 \\ P_3 & 0 & 0 \\ -P_2 & 0 & 0 \end{pmatrix} \quad \text{and} \quad H_{rh}(P_1, P_2, P_3) = \frac{1}{2}\left(\frac{P_1^2}{c_1} + \frac{P_2^2}{c_2}\right).$$

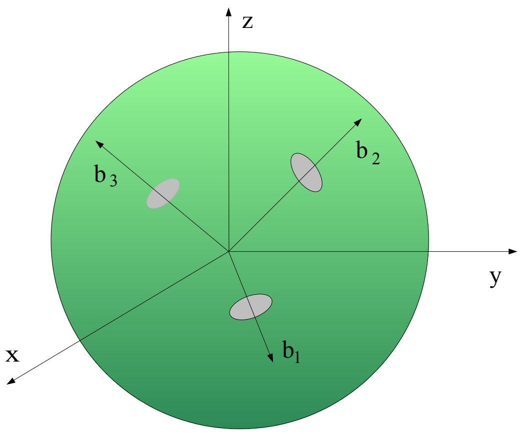
Proposition 4.1.7 *The equilibrium states* $e_0 = (0,0,0), e_1^m = (m,0,0), e_3^m = (0,0,m)$ *for* $m \in \mathbf{R}^*$ *are nonlinear stable, and* e_2^m, $m \in \mathbf{R}^*$ *is unstable.*

Proposition 4.1.8 *The Lie-Trotter integrator for the system* (4.1.10) *is given by:*

$$\left\{\begin{array}{lcl} x_1(n+1) & = & x_1(n) - \dfrac{t}{c_2}x_2(0)\cdot x_3(n) \\ x_2(n+1) & = & \cos(\dfrac{t}{c_1}x_1(0))\cdot x_2(n) + \sin(\dfrac{t}{c_1}x_1(0))\cdot x_3(n) \\ x_3(n+1) & = & -\sin(\dfrac{t}{c_1}x_1(0))\cdot x_2(n) + \cos(\dfrac{t}{c_1}x_1(0))\cdot x_3(n) \end{array}\right. \tag{4.1.11}$$

- **Spacecraft dynamics as left invariant control system on** $SO(3)$

We consider a spacecraft free to move in $\mathbf{R}^3$, [44] (Puta, 1997). Let (b_1, b_2, b_3) be an orthonormal frame fixed on the body and let $(r_1, r_2, r_3) = (x, y, z)$ define an inertial frame with the origin coincident with the origin of the body-fixed frame, see the figure:



We define a matrix $X(t) \in SO(3)$ such that

$$r_i = X(t)\cdot b_i, \quad i = \overline{1,3},$$

that is $X(t)$ determines the attitude of the spacecraft at time t.

Let $\{e_i | i = 1, 2, 3\}$ the canonical basis of $\mathbf{R}^3$ and define $E_i = \Phi(e_i)$, $i = \overline{1,3}$, where Φ, is the isomorphism between the Lie algebras $(\mathbf{R}^3, \times)$ and $(so(3), [\cdot, \cdot])$.

Then $X(t)$ satisfies the equation:

$$\dot{X} = X \cdot \hat{\omega}, \quad \hat{\omega} = \sum_{i=1}^{3} \omega_i(t) E_i, \tag{4.1.12}$$

where $\omega = (\omega_1, \omega_2, \omega_3)$ is the angular velocity of the spacecraft in the body-fixed coordinates.

If we let $u_i = \omega_i$, $i = \overline{1,3}$, that is we interpret the components of the angular velocity as our control, then the system (4.1.12) takes in the form:

$$\dot{X} = X \cdot \left(\sum_{i=1}^{3} u_i(t) E_i \right). \tag{4.1.13}$$

We consider the case when only two components of the angular velocity can be controlled. For example, if we can control the angular velocity about the b_1 and b_2 axes, then $X(t) \in SO(3)$ satisfy the system:

$$\dot{X} = X(A_1 u_1 + A_2 u_2), \quad \text{where} \quad A_1 = E_1, \ A_2 = E_2. \tag{4.1.14}$$

The system (4.1.14) is a left invariant controllable system on $SO(3)$ with control authority $\{A_1, A_2\}$ (we apply Jurdjevic-Sussmann's Theorem). This realization of the spacecraft dynamics is due to Leonard [32].

For the controllable system (4.1.14), we consider the cost function J, given by (4.1.9).

Proposition 4.1.10 *The controls which minimize the cost function J given by* (4.1.9) *and steers the system* (4.1.15) *from $X(0) = X_0$ at $t = 0$ to $X(t_f) = X_f$ at $t = t_f$ are given by* $u_1 = \dfrac{P_1}{c_1}$, $u_2 = \dfrac{P_2}{c_2}$, *where P_i, $i = \overline{1,3}$ are solutions of the system:*

$$\dot{P}_1 = -\frac{1}{c_2} P_2 P_3, \quad \dot{P}_2 = \frac{1}{c_1} P_1 P_3, \quad \dot{P}_3 = \left(\frac{1}{c_2} - \frac{1}{c_1} \right) P_1 P_2. \tag{4.1.15}$$

The system (4.1.15) is a Euler top system and has the Hamilton-Poisson realization $(\mathbf{R}^3, H_{ns}, P_{ns})$, where:

$$P_{ns} = \begin{pmatrix} 0 & -P_3 & \dfrac{c_2 - c_1}{c_2} P_2 \\ P_3 & 0 & 0 \\ \dfrac{c_1 - c_2}{c_2} P_2 & 0 & 0 \end{pmatrix} \quad \text{and} \quad H_{ns}(P_1, P_2, P_3) = \frac{1}{2} \left(\frac{P_1^2}{c_1} + \frac{P_2^2}{c_2} \right).$$

Proposition 4.1.11 *The equilibrium states* $e_0 = (0,0,0)$, $e_1^m = (m,0,0)$, $e_2^m = (0,m,0)$ *and* $e_3^m = (0,0,m)$ *for* $m \in \mathbf{R}^*$, *of the system* (4.1.15), *have the following behavior:*

(i) e_0, e_3^m *is nonlinear stable.*

(ii) if $c_1 < c_2$ *(resp.* $c_1 > c_2$*), then* e_1^m *(resp.* e_2^m*) is nonlinear stable (unstable), and* e_2^m *(resp.* e_1^m*) is unstable.*

Proposition 4.1.12 *If* $c_1 < c_2$ *and* $\delta = \sqrt{\dfrac{c_2 - c_1}{c_2}}$, *then the Lie-Trotter integrator of the system* (4.1.15) *is given by:*

$$\left\{\begin{array}{rcl} x_1(n+1) & = & x_1(n) - \dfrac{t}{c_2} x_2(0) \cdot x_3(n) \\ \\ x_2(n+1) & = & \cos(\dfrac{\delta}{c_1} t x_1(0)) \cdot x_2(n) + \dfrac{1}{\delta} \sin(\dfrac{\delta}{c_1} t x_1(0)) \cdot x_3(n) \\ \\ x_3(n+1) & = & -\delta \sin(\dfrac{\delta}{c_1} t x_1(0)) \cdot x_2(n) + \cos(\dfrac{\delta}{c_1} t x_1(0)) \cdot x_3(n). \end{array}\right. \tag{4.1.16}$$

4.2 Controllable systems on the Lie group $SO(4)$

The original results in this direction have published in the papers [42](Pop, Puta and Şuşoi, 2005) and [7](Puta, Şuşoi et al., 2006).

Let $SO(4)$ be the set of all matrices $A \in \mathcal{M}_{4\times4}(R)$ such that $A^T \cdot A = I_4$ and $\det(A) = 1$.

$SO(4)$ is a Lie group of dimension 6 with Lie algebra $so(4)$ given by:

$$so(4) = \left\{ \left(\begin{array}{cccc} 0 & a_1 & a_2 & a_3 \\ -a_1 & 0 & a_4 & a_5 \\ -a_2 & -a_4 & 0 & a_6 \\ -a_3 & -a_5 & -a_6 & 0 \end{array} \right) \middle| \; a_1, a_2, a_3, a_4, a_5, a_6 \in \mathbf{R} \right\}$$

Let $\{A_i | i = \overline{1,6}\}$ be the standard basis of Lie algebra $so(4)$.

A drift-free left invariant control system on Lie group $SO(4)$ *with fewer controls than state variables* can be written in the following form:

$$\dot{X} = X \left(\sum_{i=1}^{m} A_i u_i \right), \tag{4.2.1}$$

where $X \in SO(4)$, and $u_i, i = \overline{1,m}$ are the controls with $m < 6$.

Proposition 4.2.1 *There exists* 37 *drift-free left invariant controllable systems on* $SO(4)$ *with fewer controls than* 6.

Let us we shall study only the following drift-free left invariant controllable system on $SO(4)$ **with** 3 **controls**:

$$\dot{X} = X\left(A_1u_1 + A_2u_2 + A_3u_3\right) \tag{4.2.2}$$

Proposition 4.2.2 *The system* (4.2.2) *is controllable.*

Remark. An other drift-free left invariant controllable system on $SO(4)$ with control authority $\{A_2, A_3, A_4\}$ has studied in the paper [8](Aron et al., 2009).

4.2.1 An optimal control problem on $SO(4)$

Let J to be the cost function given by:

$$J(u_1,u_2,u_3) = \frac{1}{2}\int_0^{t_f}[c_1u_1^2(t) + c_2u_2^2(t) + c_3u_3^2(t)]dt, \quad c_1 > 0, c_2 > 0, c_3 > 0.$$

Proposition 4.2.3 *The controls that minimize the cost function* J *and steers the system (4.2.2) from* $X = X_0$ *at* $t = 0$ *to* $X = X_f$ *at* $t = t_f$ *are given by* $u_1 = \frac{1}{c_1}P_1,\ u_2 = \frac{1}{c_2}P_2,\ u_3 = \frac{1}{c_3}P_3,$ *where* $P_i,\ i = \overline{1,6}$ *are solutions of the system:*

$$\begin{cases} \dot{P}_1 = \dfrac{P_2P_4}{c_2} + \dfrac{P_3P_5}{c_3}, & \dot{P}_2 = -\dfrac{P_1P_4}{c_1} + \dfrac{P_3P_6}{c_3}, \quad \dot{P}_3 = -\dfrac{P_1P_5}{c_1} - \dfrac{P_2P_6}{c_2} \\ \dot{P}_4 = \left(\dfrac{1}{c_1} - \dfrac{1}{c_2}\right)P_1P_2, & \dot{P}_5 = \left(\dfrac{1}{c_1} - \dfrac{1}{c_3}\right)P_1P_3, \quad \dot{P}_6 = \left(\dfrac{1}{c_2} - \dfrac{1}{c_3}\right)P_2P_3 \end{cases} \tag{4.2.3}$$

Applying Krishnaprasad's theorem [29], it follows that the optimal Hamiltonian is:

$$H_{opt}(P_1,P_2,P_3,P_4,P_5,P_6) = \frac{1}{2}\left(\frac{P_1^2}{c_1} + \frac{P_2^2}{c_2} + \frac{P_3^2}{c_3}\right). \tag{4.2.4}$$

The minus Lie-Poisson structure on $(so(4))^* \simeq \mathbf{R}^6$ is generated by the matrix:

$$\Pi = \begin{pmatrix} 0 & P_4 & P_5 & -P_2 & -P_3 & 0 \\ -P_4 & 0 & P_6 & P_1 & 0 & -P_3 \\ -P_5 & -P_6 & 0 & 0 & P_1 & P_2 \\ P_2 & -P_1 & 0 & 0 & P_6 & -P_5 \\ P_3 & 0 & -P_1 & -P_6 & 0 & P_4 \\ 0 & P_3 & -P_2 & P_5 & -P_4 & 0 \end{pmatrix}.$$

Proposition 4.2.4 *C_1 and C_2 are Casimirs for $((so(4))^*, \Pi) \simeq (\mathbf{R}^6, \Pi)$, where:*

$$C_1(P) = \frac{1}{2}\sum_{i=1}^{6} P_i^2, \quad C_2(P) = P_1P_6 - P_2P_5 + P_3P_4.$$

4.2.2 Stability problem for the dynamics (4.2.5)

If we suppose that $c_1 = 1,\ c_2 = 1,\ c_3 = k, \quad k > 0$, the system (4.2.4) becomes:

$$\begin{cases} \dot{P}_1 = P_2P_4 + \dfrac{1}{k}P_3P_5, & \dot{P}_2 = -P_1P_4 + \dfrac{1}{k}P_3P_6, & \dot{P}_3 = -P_1P_5 - P_2P_6 \\ \\ \dot{P}_4 = 0, & \dot{P}_5 = \left(1 - \dfrac{1}{k}\right)P_1P_3, & \dot{P}_6 = \left(1 - \dfrac{1}{k}\right)P_2P_3 \end{cases} \tag{4.2.5}$$

Proposition 4.2.5 *The dynamics* (4.2.5) *has the following equilibrium states:*

$e_{12}^{MN} = (M, N, 0, 0, 0, 0), \quad e_{25}^{MN} = (0, M, 0, 0, N, 0), \quad e_{34}^{MN} = (0, 0, M, N, 0, 0),$

$e_{16}^{MN} = (M, 0, 0, 0, 0, N), \quad e_{345}^{MNP} = (0, 0, M, N, P, 0), \quad M, N, P \in \mathbf{R};$

$e_{1256}^{QMNP} = (Q, M, 0, 0, N, P), \quad$ *where* $\quad Q = -\dfrac{MP}{N} \quad$ *cu* $\ M, P \in \mathbf{R}$ *and* $N \in \mathbf{R}^*$. □

Proposition 4.2.6 [42](Pop, Puta and Şuşoi, 2006) *The equilibrium state e_{345}^{MNP}, $M^2 + bN^2 + bP^2 \neq 0$, where $b = \dfrac{1}{k}$ is spectrally stable.*

In Propositions 4.2.7 – 4.2.10 [[7](Aron, Pop. Puta and Şuşoi, 2006)] is studied the spectral stability for the system (4.2.5) and are obtained the following results:

(1)(*i*) *if $(1-k)M^2 < N^2$, then e_{16}^{MN} is spectral stable;*
(*ii*) *if $(1-k)M^2 > N^2$, then e_{16}^{MN} is unstable.*

(2)(*i*) *if $(1-k)(M^2+N^2) < 0$, then e_{12}^{MN} is spectrally stable;*
(*ii*) *if $(1-k)(M^2+N^2) > 0$, then e_{12}^{MN} is unstable.*

(3) *e_{1256}^{QMNP} is spectrally stable.*

(4)(*i*) *if $(1-k)M^2 < N^2$, then e_{25}^{MN} is spectrally stable;*
(*ii*) *if $(1-k)M^2 > N^2$, then e_{25}^{MN} is unstable.*

Proposition 4.2.11 [42] *$e_{34}^{MN}, M, N \in \mathbf{R}^*$ has the following behavior:*

(*i*) *if $k \in (0, 1]$, then e_{34}^{MN} is spectrally stable.*

(ii) if $k\in(1,\infty)$ *and* $a=\frac{2}{k}\sqrt{k-1},\ b=\frac{\sqrt{2}}{k}\sqrt{k-1}$, *then* e_{34}^{MN} *is spectrally stable for* $\frac{N}{M}\in(-\infty,-a]\cup\{-b,b\}\cup[a,\infty)$ *and unstable for* $\frac{N}{M}\in(-a,-b)\cup(-b,b)\cup(b,a)$.

Proposition 4.2.12 [7] *If* $\begin{cases}(-1+k)M^2\neq N^2\\(-1+k)(M^2-N^2)>0\end{cases}$, *then* e_{1256}^{QMNP} *for* $Q=-\frac{MP}{N}$ *with* $M,P\in\mathbf{R}$ *and* $N\in\mathbf{R}^*$, *is nonlinear stable.*

4.2.3 Lax formulation and complete integrability for the dynamics (4.2.5)

Proposition 4.2.14 [7] *The dynamics* (4.2.5) *has a Lax formulation, that is:*

$$\dot{L}=[L,B],\quad \textit{where}$$

$$L=\begin{pmatrix}0&\ell_{12}&\ell_{13}&0&0&0\\-\ell_{12}&0&\ell_{23}&0&0&0\\-\ell_{13}&-\ell_{23}&0&0&0&0\\0&0&0&0&\ell_{45}&\ell_{46}\\0&0&0&-\ell_{45}&0&\ell_{56}\\0&0&0&-\ell_{46}&-\ell_{56}&0\end{pmatrix},\quad B=\begin{pmatrix}0&b_{12}&b_{13}&0&0&0\\-b_{12}&0&b_{23}&0&0&0\\-b_{13}&-b_{23}&0&0&0&0\\0&0&0&0&b_{45}&b_{46}\\0&0&0&-b_{45}&0&b_{56}\\0&0&0&-b_{46}&-b_{56}&0\end{pmatrix},$$

and

$$\begin{aligned}
\ell_{12}&=2P_1\sqrt{3}-2P_2+P_5+P_6\sqrt{3}, & b_{12}&=P_1\sqrt{3}-P_2;\\
\ell_{13}&=-4P_3-2P_4, & b_{13}&=-2P_3-P_4;\\
\ell_{23}&=2P_1+2P_2\sqrt{3}-P_5\sqrt{3}+P_6, & b_{23}&=P_1+P_2\sqrt{3};\\
\ell_{45}&=-P_2-P_5, & b_{45}&=-P_2;\\
\ell_{46}&=P_3-P_4, & b_{46}&=P_3-P_4;\\
\ell_{56}&=P_1-P_6, & b_{56}&=P_1.
\end{aligned}$$

Corollary 4.2.15 *The flow of the dynamics* (4.2.5) *is iso-spectral.*

Consider now the co-adjoint orbit $\mathcal{O}_{M,N}$ for the Poisson configuration $((so(4))^*\cong(\mathbf{R}^6,\Pi)$ endowed with its Kirilov-Kostant-Souriau symplectic structure ω_{MN}, where

$$\mathcal{O}_{M,N}=\{P\in\mathbf{R}^6\mid \begin{cases}\sum_{i=1}^{6}P_i^2=M\\P_1P_6-P_2P_5+P_3P_4=N\end{cases}\}.$$

Then $(\mathcal{O}_{M,N}, \omega_{MN}, H_{|\mathcal{O}_{M,N}})$ *is a* $4-$*dimensional completely integrable Hamiltonian system* ([7], Proposition 4.2.16), where

$$H_{|\mathcal{O}_{M,N}}(P) = \frac{1}{2}\left(P_1^2 + P_2^2 + \frac{1}{k}P_3^2\right).$$

4.2.4 Numerical integration of the dynamics (4.2.3)

The results of this section have published in [42] (Pop, Puta and Şuşoi, 2006).

The vector field $X_{H_{opt}}$ associated to Hamiltonian H_{opt} of the dynamics (4.2.3) splits:

$$X_{H_{opt}} = X_{H_1} + X_{H_2} + X_{H_3}, \quad \text{where}$$

$$H_1(P) = \frac{1}{2c_1}P_1^2, \quad H_2(P) = \frac{1}{2c_2}P_2^2, \quad H_3(P) = \frac{1}{2c_3}P_3^2.$$

The Lie-Trotter integrator of (4.2.3) is given by the recurrent equations:

$$\left\{\begin{array}{lll}
P_1^{n+1} & = & P_1^n \cos a_2 t \cos a_3 t + P_4^n \sin a_2 t + P_5^n \cos a_2 t \sin a_3 t \\
P_2^{n+1} & = & P_1^n \cos a_3 t \sin a_1 t \sin a_2 t + P_2^n \cos a_1 t \cos a_3 t - P_4^n \cos a_2 t \sin a_1 t + \\
 & & \quad + P_5^n \sin a_1 t \sin a_2 t \sin a_3 t + P_6^n \cos a_1 t \sin a_3 t \\
P_3^{n+1} & = & P_1^n \sin a_1 t \sin a_3 t + P_2^n \cos a_1 t \sin a_2 t \sin a_3 t + P_3^n \cos a_1 t \cos a_2 t - \\
 & & \quad - P_5^n \cos a_3 t \sin a_1 t - P_6^n \cos a_1 t \cos a_3 t \sin a_2 t \\
P_4^{n+1} & = & -P_1^n \cos a_1 t \cos a_3 t \sin a_2 t + P_2^n \cos a_3 t \sin a_1 t + P_4^n \cos a_1 t \cos a_2 t - \\
 & & \quad - P_5^n \cos a_1 t \sin a_2 t \sin a_3 t + P_6^n \sin a_1 t \sin a_3 t \\
P_5^{n+1} & = & -P_1^n \cos a_1 t \sin a_3 t + P_2^n \sin a_1 t \sin a_2 t \sin a_3 t + P_3^n \cos a_2 t \sin a_1 t + \\
 & & \quad + P_5^n \cos a_1 t \cos a_3 t - P_6^n \cos a_3 t \sin a_1 t \sin a_2 t \\
P_6^{n+1} & = & -P_2^n \cos a_2 t \sin a_3 t + P_3^n \sin a_2 t + P_6^n \cos a_2 t \cos a_3 t
\end{array}\right. \tag{4.2.6}$$

where $a_1 = \dfrac{P_1(0)}{c_1}, \quad a_2 = \dfrac{P_2(0)}{c_2}, \quad a_3 = \dfrac{P_3(0)}{c_3}.$

The following proposition are proved.

Proposition 4.2.17 *Lie-Trotter integrator* (4.2.6) *has the following properties:*

(i) *it is a Poisson integrator, i.e. it preserve the Poisson structure generated by* Π.

(ii) *its restrictions to the co-adjoint orbits:*

$$\sum_{i=1}^{6} P_i^2 = constant, \qquad P_1P_6 - P_2P_5 + P_3P_4 = constant$$

give rise to a symplectic integrator.

(iii) *it doesn't preserve the energy* H_{opt} *of our system.*

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