Contents

Keywords univalent functions, starlike functions, convex functions, α- convex functions, subordination, differential subordination, strong differential subordination, differential superordination, strong differential superordination, Sălăgean differential operator, Ruscheweyh differential operator, Dziok-Srivastava linear operator, Hadamard product, Integral convolution, order of convolution consistence.

Introduction

A branch of mathematics with wide applications in various fields of science and technology, of which the Romanian school of mathematics has important contributions, is the Complex analysis. Complex analysis, dealing mainly with analytical functions of complex variable. As real and imaginary parts of analytic function must satisfy Laplaces equation, complex analysis is widely applied in the two-dimensional problems in physics.

Functions of one complex variable theory, combines geometric intuition and mathematical reasoning and it is a classic branch of mathematics which has roots in the 19th century and even earlier. Geometrical theory of analytic functions is based on the notion of comply representation which is the ideal model of geometric transformations in the plane. An important result as the basis for this theory is the theorem of Riemanns comply representation . Important names that have developed this discipline are Euler, Gauss, Riemann, Cauchy, Weierstress and many others in the 20th century.

Univalent functions proved to be most interesting for study, first necessary and sufficient conditions of univalency expressed by coefficients were obtained in 1931 by Gh.Călugăreanu. Around the year 1907 appears the first significant work that belongs to mathematician P. Koebe. In the geometric theory of functions a special role occupies the differential subordination known as method of admissible function, theory initiated by the S.S. Miller and P.T. Mocanu. Using differential subordinations have shown in a much simpler way some classical results in this area, their expansions, and even new results.

S.S. Miller and P.T. Mocanu recently introduced the notion of differential superordination, dual notion of the differential subordination.

The notion of strong subordination was introduced by J.A. Antonio and S. Romaguera, afterward the notion of strong superordination was introduced by Georgia Oros using as model the theory of differentiated subordination, in 2009.

This paper has five chapters; the first chapter presents concepts, definitions, properties and characterization theorems used during the whole work. The paragraphs of this first chapter present generalities, known results on class of univalent functions. As follows we enumerate properties of special univalent classes: starlike class functions, convex class functions, eight convex class functions, analytic functions with positive real part and functions whose derivative has positive real part. In the other paragraphs of the first chapter we presented notions as: subordination, differential subordination and strong subordinate, differential super ordination, strong super ordination with some known properties and characterization theorems.

The other four chapters contain original results already published or under publication. As the second chapter contains the results obtained in differential subordinations published in three papers. These original results were obtained using differential operators Sălăgean, Ruschewey and Dziok-Srivastava linear operator.

The third chapter shows the results obtained in the strong differential subordinations, which contains a paper published and dedicated to Professor Mr. Gr. Şt.Sălăgean coordinator of this work in the journal Studia University of Babes-Bolyai, Mathematica, at the age of 60 years. Sălăgean use differential operator for functions of class $A^*_{n\zeta}$. and get new hard superordination .

Chapter four illustrates three original works on the field strong superordination for different classes of univalent functions, already published or under publication. So we got differential strong superordination sort of first differential order, the best of their subordinate and subordinate chains. Chapter five we present other known results for analytic functions with negative coefficients, the characterization theorems, the notion of convolution or Hadamard product and the notion of consistency. For-

ward we mentioned the original results obtained with univ.Prof. Dr. Gr Şt. Sălăgean, scientific leader of the thesis, related to the order of consistency of analytic functions with negative coefficients This way I would like to present my sincere thanks, gratitude and esteem to univ. Prof. Dr. Gr Şt. Sălăgean for collaboration and guidance of scientific research in these years, for support and for the informations providing all this time. Also thank to the entire team for Complex Analysis of the Faculty of Mathematics and Computer Science, Babeş-Bolyai University of Cluj-Napoca, for comments and constructive participation and support for all my projects, my papers and presentations of these years. Thank you sincerely for constant encouragement, support all these years, the trust that was given, for current and future collaboration between Mr. univ. Prof. Dr. Gheorghe Oros, University of Oradea. Wish to thank colleagues in Oradea with whom I had and I hope will have a perfect scientific collaboration, Mrs. Univ. lect. Dr. Georgia Irina Oros, Mrs. Univ. lect. Adriana Cătaş and Mrs. as. drd. Roxana Şendrutiu. Over this period did not lack support of my children and my parents, who owe thousands of thanks for their support, understanding and help.

Chapter 1

Generalities

1.1 Univalent function. Definitions and properties

In this paragraph are set notions about the genre known univalent functions, defining, notations and class properties of Holomorphic and univalent functions in disk unit (U noted as S).

Denote:

$$U(z_0; r) = \{z \in \mathbb{C};\ |z - z_0| < r\}, \tag{1.1.1}$$

$r > 0$,

$$\dot{U}(z_0; r) = U(z_0; r) \setminus \{z_0\}, \tag{1.1.2}$$

$$\overline{U}(z_0; r) = \{z \in \mathbb{C};\ |z - z_0| \leq r\}, \tag{1.1.3}$$

and

$$\partial U(z_0; r) = \{z \in \mathbb{C};\ |z - z_0| = r\}. \tag{1.1.4}$$

For $a \in \mathbb{C}$ and $n \in \mathbb{N}^*$ denote

$$H[a, n] = \{f \in H(U):\ f(z) = a + a_n z^n + \ldots\}. \tag{1.1.5}$$

Let $H(U \times \overline{U})$ the class of analytic functions in $U \times \overline{U}$,

(1.1.6)
$$H^*[a,n,\xi] = \{f \in H(U \times \overline{U}) \mid f(z,\zeta) = a + a_n(\zeta)z^n + a_{n+1}(\zeta)z^{n+1} + \dots,\ z \in U,\ \zeta \in \overline{U}\},$$

with $a_k(\zeta)$ holomorphic functions in $\overline{U}$, $k \geq n$

(1.1.7)
$$A_n = \{f \in H(U) :\ f(z) = z + a_{n+1}z^{n+1} + \dots\},$$

$A = A_1$,

(1.1.8)
$$A^*_{n\zeta} = \{f \in H(U \times \overline{U}) \mid f(z,\zeta) = z + a_{n+1}(\zeta)z^{n+1} + \dots,\ z \in U,\ \zeta \in \overline{U}\},$$

with $a_k(\zeta)$ holomorphic functions in $\overline{U}$, $k \geq n$, for $n = 1$, $A^*_{n\zeta} = A^*_\zeta$,

(1.1.9)
$$S^*_\zeta = \{f \in H^*[a,n,\xi] :\ \operatorname{Re}\frac{zf'(z)}{f(z)} > 0,\ z \in U,\ \text{for all} \zeta \in \overline{U}\},$$

the class of starlike functions,

(1.1.10)
$$K^*_\zeta = \{f \in H^*[a,n,\xi] :\ \operatorname{Re}\frac{zf''(z,\zeta)}{f'(z,\zeta)} + 1 > 0,\ z \in U,\ \text{for all} \zeta \in \overline{U}\},$$

the class of convex functions,

(1.1.11)
$$S = \{f \in A :\ f \text{ is univalent function in } U\},$$

class of holomorphic and univalent functions, normalized by :

(1.1.12)
$$f(0) = 0, \quad f'(0) = 1,$$

with $f \in H_u(U)$ where

(1.1.13)
$$f(z) = z + a_2z^2 + \dots, \quad z \in U.$$

Study of meromorph and univalent functions can be in parallel with the S class.

We noted with Σ the class of meromorph functions φ with the single pole (simple) $\zeta = \infty$ and univalent in the outside of disk unit $U^- = \{\zeta \in \mathbb{C}_\infty \mid \zeta > 1\}$ who have shaped the development of Laurent series as:

$$\varphi(\zeta) = \zeta + \alpha_0 + \frac{\alpha_1}{\zeta} + \dots + \frac{\alpha_n}{\zeta^n} + \dots, \quad |\zeta| > 1.$$

Theorem 1.1.1 (Area theorem) [26] *If* $\varphi(\zeta) = \zeta + \sum_{n=0}^{\infty} \frac{\alpha_n}{\zeta^n}$ *is a function from class* Σ*, then area of* $E(\varphi)$ *where*

$$E(\varphi) = \mathbb{C} \setminus \varphi(U^-) \tag{1.1.14}$$

in sense Lebesgue bidimensional area is :

$$E(\varphi) = \pi \left(1 - \sum_{n=1}^{\infty} n|\alpha_n|^2\right) \geq 0 \tag{1.1.15}$$

then $\sum_{n=1}^{\infty} n|\alpha_n|^2 \leq 1$.

Theorem 1.1.2 (Bieberbach Theorem about a_2 coefficient) [26]
If $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in S$ *then* $|a_2| \leq 2$. *Equality* $|a_2| = 2$ *takes place if and only if* f *is the form*

$$K_\sigma(z) = \frac{z}{(1 + e^{i\sigma} z)^2} \tag{1.1.16}$$

(K_σ *is Koebe funcion).*

Conjecture 1.1.1 (Bieberbach conjecture) [26] *If function* $f(z) = z + a_2 z^2 + \ldots$ *is in class* S*, then* $|a_n| \leq n$, $n = 2, 3, \ldots$.

Theorem 1.1.3 [26] *If* $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, $f \in S$*, then* $|a_3 - a_2^2| \leq 1$*, delimitation been sharp.*

If f *is odd,* $|a_3| \leq 1$*, but equality takes place if and only if* f *is the form*

$$f(z) = \frac{z}{1 + e^{i\sigma} z^2}, \quad \sigma \in \mathbb{R}.$$

Theorem 1.1.4 (Koebe, Bieberbach theorem) [15] *Let* $f \in S$*. Then* $f(U) \supseteq U_{1/4}$.

Corolary 1.1.1 [15] *Class* S *is compact subset of* $H(U)$.

1.2 The class of starlike functions

Definition 1.2.1 [26] Let $f \in H(U)$ a function with properties $f(0) = 0$. Function f is starlike in U with respect to origin (or starlike) if f is a univalent function in U and $f(U)$ is a starlike domain with respect to the origin.

Theorem 1.2.1 (univalency theorem on border) [26] *Let D a set $D \subset \mathbb{C}$ and $f \in H(D)$ is a continuous function of $\overline{D}$. If f is a function injective of ∂D then f is injective of D.*

Theorem 1.2.2 (The characterization of analytic starlikeness theorem) [26] *let $f \in H(U)$ with $f(0) = 0$. Then function f is starlike if and only if $f'(0) \neq 0$ and*

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > 0, \quad z \in U. \tag{1.2.1}$$

Definition 1.2.2 [26] We denote S^* class of functions $f \in A$ are starlike and normalized in unit disc:

$$S^* = \left\{ f \in A : \operatorname{Re} \frac{zf'(z)}{f(z)} > 0, \ z \in U \right\}. \tag{1.2.2}$$

Theorem 1.2.3 (Theorem for determining the coefficient functions of S^*) *If $f(z) = z + a_2 z^2 + \ldots + a_n z^n + \ldots$ is a function of S^*, then*

$$|a_n| \leq n, \quad n = 2, 3, \ldots$$

Equality takes place if and only if f is Koebe function.

1.3 The class of convex functions

Definition 1.3.1 [26] Function $f \in H(U)$ is convex in U (or convex) if f is univalent in U and $f(U)$ is a convex domain.

Theorem 1.3.1 (The characterization of analytic convexity theorem) [26] *If $f \in H(U)$, function f is convex if and only if $f'(0) \neq 0$ and*

$$\operatorname{Re} \frac{zf''(z)}{f'(z)} + 1 > 0, \quad z \in U. \tag{1.3.1}$$

Theorem 1.3.2 (Duality theorem of Alexander) *Function f is convex in U if and only if function $F(z) = zf'(z)$ is starlike in U.*

Definition 1.3.2 [26] K is class of convex functions $f \in A$ and normalized in unit disc,

$$K = \left\{ f \in A : \operatorname{Re} \frac{zf''(z)}{f'(z)} + 1 > 0, \ z \in U \right\}. \tag{1.3.2}$$

Theorem 1.3.3 (Theorem for determining the coefficient functions of K) [26] *If function $f(z) = z + a_2 z^2 + \ldots + a_n z^n + \ldots$ is in K class, then*

$$|a_n| \leq 1, \quad n = 2, 3, \ldots$$

Equality takes place if and only if f has the form

$$f(z) = \frac{z}{1 + e^{i\sigma} z}, \quad \sigma \in \mathbb{R}. \tag{1.3.3}$$

1.4 The class of alfa-convex functions (Mocanu Functions)

Intending to find a connection between the notions of convexity and stellar P.T. Mocanu introduced in 1969 the notion of alpha-convex function.

Definition 1.4.1 [26],[25] Let $f \in A$ a function with condition

$$\frac{f(z)f'(z)}{z} \neq 0, \quad z \in U$$

and let number $\alpha \in \mathbb{R}$. Function f is α-convex in unit disc U (or α-convex) if $\operatorname{Re} J(\alpha, f; z) > 0$, $z \in U$ then:

$$J(\alpha, f; z) = (1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left(\frac{zf''(z)}{f'(z)} + 1 \right). \tag{1.4.1}$$

Definition 1.4.2 [26] We define the set

$$M_\alpha = \left\{ f \in A : \ \frac{f(z)f'(z)}{z} \neq 0, \ \operatorname{Re} J(\alpha, f; z) > 0, \ z \in U \right\}, \tag{1.4.2}$$

the class of functions α-convexe in unit disc$U$.

Theorem 1.4.1 (Starlikeness theorem of α-convex function)

1. *Let $\alpha \in \mathbb{R}$, $f \in M_\alpha$. Then $f \in S^*$, and*

$$M_\alpha \subset S^*.$$

2. *If $\alpha, \beta \in \mathbb{R}$ such that $0 \le \dfrac{\beta}{\alpha} < 1$, then*

$$M_\alpha \subset M_\beta.$$

3. *$M_\infty = \{id\}$, where $id(z) = z$, $z \in U$.*

1.5 Analytic function with positive real part

Properties of analytic functions with positive real part have an important role in the following paragraphs being closely related to the notion of subordination what will be presented in the chapters that follow.

Definition 1.5.1 [26] 1. The Carathéodory class of functions (functions with positive real part) is a class

$$P = \{p \in H(U) : \ p(0) = 1, \ \operatorname{Re} p(z) > 0, \ z \in U\}.$$

2. The Schwarz functions class is a class

$$B = \{\varphi \in H(U) : \ \varphi(0) = 0, \ |\varphi(z)| < 1, \ z \in U\}.$$

Theorem 1.5.1 (Carathéodory theorem about coefficients of class P) [26] *If $p(z) = 1 + p_1 z + p_2 z^2 + \ldots + p_n z^n + \ldots$ is in class P then $|p_n| \le 2$, $n \ge 1$, equality takes place for function $p(z) = \dfrac{1+\lambda z}{1-\lambda z}$, $|\lambda| = 1$.*

1.6 Subordination

Definition 1.6.1 [26] Let $f, g \in H(U)$. The function f is subordinate to g written $f \prec g$ or $f(z) \prec g(z)$, if there exist a function $w \in H(U)$ with $w(0) = 0$ and $|w(z)| < 1$, $z \in U$ or $w \in B$ such that

$$f(z) = g[w(z)], \quad z \in U.$$

Theorem 1.6.1 [26] *Let $f, g \in H(U)$ and suppose that g is univalent in U. Then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subseteq g(U)$.*

Corolary 1.6.1 (Principle of subordination of Lindelöf) [26] *Let functions $f, g \in H(U)$ such that g is univalent in U.*

1. If $f(0) = g(0)$ and $f(U) \subseteq g(U)$ then $f(\overline{U}_r) \subseteq g(\overline{U}_r)$, $0 < r < 1$.

2. Equality $f(\overline{U}_r) = g(\overline{U}_r)$ for one $r < 1$ takes place if and only if $f(U) = g(U)$ (or $f(z) = g(\lambda z)$, $|\lambda| = 1$).

1.7 Functions whose derivative has positive real part

Theorem 1.7.1 (The criteria of univalency Noshiro, Warschawschi, Wolff) [26] *If function f is holomorphic in convex domain $D \subset \mathbb{C}$ and if there exist a number $\gamma \in \mathbb{R}$ such that*

$$\mathrm{Re}\,[e^{i\gamma} f'(z)] > 0, \quad z \in D$$

then function f is univalent in D.

Definition 1.7.1 [26] We denote R class of normal functions usually standardized which derivative is positive in disk unit,

$$R = \{f \in A;\ \mathrm{Re}\, f'(z) > 0,\ z \in U\}.$$

Theorem 1.7.2 (Deformation theorem for class R) [26] *If function*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in U,$$

is in class R, then

$$|a_n| \leq \frac{2}{n}$$

$$\frac{1-r}{1+r} \leq |f'(z)| \leq \frac{1+r}{1-r}, \quad |z| = r$$

$$-r + 2\log(1+r) \leq |f(z)| \leq -r - 2\log(1-r), \quad |z| = r.$$

The extremal function has the form

$$f(z) = -z - \frac{2}{\lambda}\log(1-\lambda z), \quad |\lambda| = 1.$$

1.8 Differential subordination

Definition 1.8.1 [26] 1. Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and let function h univalent in U. If function $p \in H[a,n]$ verifies

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z), \quad z \in U \tag{1.8.1}$$

then function p is called (a,n) a solution of the differential subordination (1.8.1) or more simply, solution of the differential subordination (1.8.1).

2. Subordination (1.8.1) is called second order differential subordination, and function q univalent in U, is called (a,n) dominant of the solution of the differential subordination (1.8.1), or more simply, dominant of the differential subordination (1.8.1), if $p(z) \prec q(z)$ for all p satisfying (1.8.1).

3. A dominant $\widetilde{q}$ such that $\widetilde{q}(z) \prec q(z)$ for all dominants q for (1.8.1) is said to be the best (a,n) dominant, or more simply the best dominant of the a differential subordination (1.8.1).

Lemma 1.8.1 (I. S. Jack, S. S. Miller, P. T. Mocanu, lemma's) [26] *Let* $z_0 = r_0e^{i\theta_0}$ *with* $0 < r_0 < 1$ *and let* $f(z) = a_nz^n + a_{n+1}z^{n+1} + \ldots$ *continue function in* $\overline{U}(0;r_0)$ *and analytic in* $U(0;r_0) \cup U\{z_0\}$ *with* $f(z) \not\equiv 0$ *and* $n \geq 1$. *If*

$$|f(z_0)| = \max\{|f(z)| : \ z \in \overline{U}(0;r_0)\}$$

then there exist a real number m, $m \geq n$, *such that*

(i) $\dfrac{z_0f'(z_0)}{f(z_0)} = m$

and

(ii) $\operatorname{Re}\dfrac{z_0f''(z_0)}{f'(z_0)} + 1 \geq m.$

Definition 1.8.2 [26] We denote by Q the set of functions q that are holomorphic and injective on the set $\overline{U} \setminus E(q)$, where

$$E(q) = \left\{ \zeta \in \partial U : \lim_{z \to \zeta} q(z) = \infty \right\}$$

and $q'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(q)$.

The set $E(q)$ is called exception set.

Functions $q_1(z) = z$ and $q_2(z) = \dfrac{1+z}{1-z}$ is examples for two these cases.

Lemma 1.8.2 (S. S. Miller, P. T. Mocanu) [21], [26] *Let $q \in Q$ with $q(0) = a$ and let function $p \in H[a,n]$, $p(z) \not\equiv a$ and $n \geq 1$. If $p(z) \not\prec q(z)$ then there exist points $z_0 = r_0 e^{i\theta_0}$ and $\zeta_0 \in \partial U \setminus E(q)$ and a number $m \geq n \geq 1$ such that $p(U(0;r_0)) \subset q(U)$ and*

(i) $p(z_0) = q(\zeta_0)$

(ii) $z_0 p'(z_0) = m\zeta_0 q'(\zeta_0)$

(iii) $\mathrm{Re}\, \dfrac{z_0 p''(z_0)}{p'(z_0)} + 1 \geq m \mathrm{Re}\, \dfrac{\zeta_0 q''(\zeta_0)}{q'(\zeta_0)} + 1.$

Definition 1.8.3 [26], [24] Let $\Omega \subset \mathbb{C}$, let function $q \in Q$ and $n \in \mathbb{N}$, $n \geq 1$. We denote by $\Psi_n[\Omega, q]$ the class of function $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ that satisfy the admissibility conditions

$$\text{(A)} \qquad \psi(r, s, t; z) \notin \Omega$$

whenever

$$r = q(\zeta), \quad s = m\zeta q'(\zeta), \quad \mathrm{Re}\left[\frac{t}{s} + 1\right] \geq m\mathrm{Re}\left[\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right],$$

where $z \in U$, $\zeta \in \partial U \setminus E(q)$ and $m \geq n$.

The set $\Psi_n[\Omega, q]$ is called by admissibility functions class, but (A) condition is called admissibility condition.

Theorem 1.8.1 [26], [19], [24] *Let univalent function $h \in H_u(U)$ and let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$. Suppose that the differential equation*

$$\psi(p(z), zp'(z), z^2 p''(z); z) = h(z) \tag{1.8.2}$$

has a solution q, with $q(0) = a$, *and one of the following conditions is satisfied:*

(i) $q \in Q$ *and* $\psi \in \Psi[h, q]$;

(ii) q is univalent in U and $\psi \in \Psi[h, q_\rho]$ *for some* $\rho \in (0, 1)$;

(iii) q is univalent in U and there exists $\rho_0 \in (0, 1)$ *such that* $\psi \in \Psi[h_\rho, q_\rho]$ *for all* $\rho \in (\rho_0, 1)$.

If function $p \in H[a, 1]$ *and function*

$$\psi(p(z), zp'(z), z^2p''(z); z) \in H(U)$$

then

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z) \Rightarrow p(z) \prec q(z)$$

and function q is the best dominant of the subordination.

Theorem 1.8.2 [26], [19], [24] *Let univalent function* $h \in H_u(U)$ *and let* $\psi : \mathbb{C}^3 \to \mathbb{C}$. *Suppose that the differential equation*

$$\psi(q(z), nzq'(z), n(n-1)zq'(z) + n^2z^2q''(z)) = h(z) \tag{1.8.3}$$

has a solution q, with $q(0) = a$ *and one of the following conditions is satisfied:*

(i) $q \in Q$ *and* $\psi \in \Psi_n[h, q]$;

(ii) q is univalent in U and $\psi \in \Psi_n[h, q_\rho]$ *for some* $\rho \in (0, 1)$;

(iii) q is univalent in U and there exists $\rho_0 \in (0, 1)$ *such that* $\psi \in \Psi_n[h_\rho, q_\rho]$ *for all* $\rho \in (\rho_0, 1)$.

If function $p \in H[a, n]$ *and function* $\psi(p(z), zp'(z), z^2p''(z); z) \in H(U)$ *then*

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z) \Rightarrow p(z) \prec q(z)$$

and function q is the best (a, n) *dominant of the subordination.*

1.9 Strong differential subordination. Definitions and properties

$H_u(U, \overline{U}) = \{f \in H^*[a, n; \xi] : f(z, \xi)$ univalent in U for $\xi \in \overline{U}\}$ is the class of univalent functions in U for all $\xi \in \overline{U}$ (see (1.1.6)).

Definition 1.9.1 [37] Let $H(z, \xi)$ analytic in $U \times \overline{U}$ and $f(z, \xi)$ analytic in $U \times \overline{U}$ for all $\xi \in \overline{U}$ and $f(z, \xi) \in H_u(U)$.

Function $H(z, \xi)$ is strongly subordinate to $f(z, \xi)$ written $H(z, \xi) \prec\prec f(z, \xi)$, if for every $\xi \in \overline{U}$, $H(z, \xi)$ is subordinate to $f(z, \xi)$, the function of z.

1.10 Differential superordinations. Generalities and proprerties

Definition 1.10.1 [6] Let $\varphi : \mathbb{C}^3 \times U \to \mathbb{C}$ and let h analytic in U. If p and $\varphi(p(z), zp'(z), z^2p''(z); z)$ is univalent in U and satisfies the second-order strong differential subordination

$$h(z) \prec \varphi(p(z), zp'(z), z^2p''(z); z), \tag{1.10.1}$$

then p is called a solution of the strong differential subordination. Let analytic function q is called a subordinant of the solution of the strong differential subordination,or more simply subordinant if $q \prec p$ for all p satisfying (1.10.1). A univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all subordinant q of (1.10.1) is said to be the best subordinant. The best subordinant is unique up to a rotation of U.

Theorem 1.10.1 [6] *Let $\Omega \subset \mathbb{C}$, let $q \in H[a, n]$ and let $\varphi \in \phi_n[\Omega, q]$. If $p \in Q(a)$ and $\varphi(p(z), zp'(z), z^2p''(z); z)$ is univalent in U, then*

$$\Omega \subset \{\varphi(p(z), zp'(z), z^2p''(z); z) : \ z \in U\} \tag{1.10.2}$$

implies $q(z) \prec p(z)$.

Theorem 1.10.2 [6] *Let $q \in H[a, n]$, let h analytic and $\varphi \in \phi_n[h, q]$. If $p \in Q(a)$ and $\varphi(p(z), zp'(z), z^2p''(z); z)$ is univalent in U, then*

$$h(z) \prec \varphi(p(z), zp'(z), z^2p''(z); z) \tag{1.10.3}$$

implies $q(z) \prec p(z)$.

Theorem 1.10.3 [6] *Let h analytic in U and $\varphi : \mathbb{C}^3 \times U \to \mathbb{C}$. Suppose that the differential equation*

$$\varphi(p(z), zp'(z), z^2p''(z); z) = h(z) \tag{1.10.4}$$

has a solution $q \in Q(a)$. If $\varphi \in \phi[h, q]$, $p \in Q(a)$ and $\varphi(p(z), zp'(z), z^2p''(z); z)$ is univalent in U, then

$$h(z) \prec \varphi(p(z), zp'(z), z^2p''(z); z) \tag{1.10.5}$$

implies $q(z) \prec p(z)$ and q is the best subordinant.

1.11 Strong superordinations. Definitinos and properties

Definition 1.11.1 [41] (see Definition 1.9.1) Let $H(z, \xi)$ an analytic function in $U \times \overline{U}$ and let $f(z)$ an analytic function and univalent in U. Function $f(z)$ is said to be strongly subordinate to $H(z, \xi)$,or $H(z, \xi)$ is said to be strongly superordinate to $f(z)$, written $f(z) \prec\prec H(z, \xi)$, if $f(z)$ is subordinate to $H(z, \xi)$ the function of z, for all $\xi \in \overline{U}$. If $H(z, \xi)$ ia a univalent function in U, for all $\xi \in \overline{U}$, then $f(z) \prec\prec H(z, \xi)$ if and only if $f(0) = H(0, \xi)$ for all $\xi \in \overline{U}$ and $f(U) \subset H(U \times \overline{U})$.

Definition 1.11.2 [41] Let $\varphi : \mathbb{C}^3 \times U \times \overline{U} \to \mathbb{C}$ and h a analytic function in U. If p and $\varphi(p(z), zp'(z), z^2p''(z); z, \xi)$ is univalent function in U for all $\xi \in \overline{U}$ and satisfy strong differential superordination (of second order)

$$h(z) \prec\prec \varphi(p(z), zp'(z), z^2p''(z); z, \xi) \tag{1.11.1}$$

then function p is called the solution of the a strong differential superordination. Analytic function q is called the subordinant of a solution of a strong differential superordination, or more simply subordinant if $q \prec p$ for all p satisfying (1.11.1). A univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all subordinant q of (1.11.1) is said to be the best subordinant. Note the best subordinant is unique up to a rotation of U.

Chapter 2

Differential subordination

2.1 The study of a class of univalent functions defined by Sălăgean differential operator

By using the operator $S^n f(z)$, $z \in U$, we introduce a class of holomorphic function $S_n(\beta)$, and obtained some subordination results.

Lemma 2.1.1 [10] *Let h be convex function, with $h(0) = a$ and let $\gamma \in \mathbb{C}^*$ be a complex number with* $\mathrm{Re}\,\gamma \geq 0$. *If $p \in H[a, n]$ and*

$$p(z) + \frac{1}{\gamma} zp'(z) \prec h(z), \quad z \in U$$

then

$$p(z) \prec q(z) \prec h(z), \quad z \in U$$

where

$$q(z) = \frac{\gamma}{nz^{\gamma/n}} \int_0^z h(t) t^{\frac{\gamma}{n}-1} dt, \quad z \in U.$$

Function q is convex in U and is the best dominant.

Lemma 2.1.2 [30] *Let* $\mathrm{Re}\, r > 0$ *and let*

$$\omega = \frac{k^2 + |r|^2 - |k^2 - r^2|}{4k \mathrm{Re}\, r}.$$

Let h be an analytic function in U with $h(0) = 1$ and suppose that

$$\mathrm{Re}\left(\frac{zh''(z)}{h'(z)} + 1\right) > -\omega.$$

If

$$p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \dots$$

is analytic in U and

$$p(z) + \frac{1}{r} zp'(z) \prec h(z),$$

then $p(z) \prec q(z)$, where q is solution of the differential equation

$$q(z) + \frac{n}{r} zq'(z) = h(z), \quad q(0) = 1,$$

given by

$$q(z) = \frac{r}{nz^{r/n}} \int_0^z t^{\frac{r}{n}-1} h(t)dt.$$

Moreover q is the best dominant.

Definition 2.1.1 [49] For $f \in A$, $n \in \mathbb{N} = 0, 1, 2, ...$, the operator $S^n f$ is defined by $S^n : A \to A$

$$S^0 f(z) = f(z)$$
$$S^1 f(z) = zf'(z)$$
$$\dots$$
$$S^{n+1} f(z) = z[S^n f(z)]', \ z \in U.$$

Remark 2.1.1 [30] If $f \in A$,

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j$$

then

$$S^n f(z) = z + \sum_{j=2}^{\infty} j^n a_j z^j, \quad z \in U.$$

Definition 2.1.2 [30] If $0 \leq \beta < 1$ and $n \in \mathbb{N}$, we let $S_n(\beta)$ denote the class of functions $f \in A$ which satisfy the inequality:

$$\mathrm{Re}\,(S^n f)'(z) > \beta, \quad z \in U.$$

Theorem 2.1.1 [57] *The set $S_n(\beta)$ is convex.*

Theorem 2.1.2 [57] *Let q be a convex function in U, with $q(0) = 1$ and let*

$$h(z) = q(z) + \frac{1}{c+2} zq'(z), \quad z \in U,$$

where c is a complex number, with $\operatorname{Re} c > -2$.

If $f \in S_n(\beta)$ and $F = I_c(f)$, where

$$F(z) = I_c(f)(z) = \frac{c+2}{z^{c+1}} \int_0^z t^c f(t)dt, \quad \operatorname{Re} c > -2 \tag{2.1.1}$$

then

$$[S^n f(z)]' \prec h(z), \quad z \in U \tag{2.1.2}$$

implies

$$[S^n F(z)]' \prec q(z), \quad z \in U,$$

and this results is sharp.

Theorem 2.1.3 [57] *Let $\operatorname{Re} c > -2$ and let*

$$w = \frac{1 + |c+2|^2 - |c^2 + 4c + 3|}{4\operatorname{Re}(c+2)}. \tag{2.1.3}$$

Let h be an analytic function in U, with $h(0) = 1$ and suppose that

$$\operatorname{Re} \frac{zh''(z)}{h'(z)} + 1 > -w.$$

If $f \in S_n(\beta)$ and $F = I_c(f)$, where F is defined by (2.1.1), then

$$[S^n f(z)]' \prec h(z), \quad z \in U \tag{2.1.4}$$

implies

$$[S^n F(z)]' \prec q(z), \quad z \in U,$$

where q is the solution of the differential equation

$$q(z) + \frac{1}{c+2} zq'(z) = h(z), \quad h(0) = 1,$$

given by

$$q(z) = \frac{c+2}{z^{c+2}} \int_0^z t^{c+1} h(t)dt, \quad z \in U.$$

Moreover q is the best dominant.

2.2 Differential subordinations obtained using the Dziok-Srivastava linear operator

By using the properties of the Dziok-Srivastava linear operator we obtain differential subordinations using functions from class A.

For two functions of A class

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad \text{şi} \quad g(z) = z + \sum_{k=2}^{\infty} b_k z^k,$$

the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) := z + \sum_{k=2}^{\infty} a_k b_k z^k.$$

For $\alpha_i \in \mathbb{C}$, $i = 1, 2, 3, \ldots, l$ şi $\beta_j \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$, $j = 1, 2, \ldots, m$, the generalized hypergeometric function is defined by

$${}_lF_m(\alpha_1, \alpha_2, \ldots, \alpha_l; \beta_1, \beta_2, \ldots, \beta_m; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_l)_n}{(\beta_1)_n \ldots (\beta_m)_n} \cdot \frac{z^n}{n!}$$

$$(l \leq m + 1,\ m \in \mathbb{N}_0 = \{0, 1, 2, \ldots\})$$

where $(a)_n$ is the Pochhammer symbol defined by

$$(a)_n : \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & n = 0 \\ a(a+1)\ldots(a+n-1), & n \in \mathbb{N} := \{1, 2, \ldots\} \end{cases}$$

Corresponding to the function

$$h(\alpha_1, \alpha_2, \ldots, \alpha_l; \beta_1, \beta_2, \ldots, \beta_m; z) = z \cdot {}_lF_m(\alpha_1, \alpha_2, \ldots, \alpha_l; \beta_1, \beta_2, \ldots, \beta_m; z).$$

The Dziok-Srivastava operator ([7], [8], [44]) is

$$H_m^l(\alpha_1, \alpha_2, \ldots, \alpha_l; \beta_1, \beta_2, \ldots, \beta_m; z)$$

$$= h(\alpha_1, \alpha_2, \ldots, \alpha_l; \beta_1, \beta_2, \ldots, \beta_m; z) * f(z)$$

$$= z + \sum_{n=2}^{\infty} \frac{(\alpha_1)_{n-1}(\alpha_2)_{n-1} \ldots (\alpha_l)_{n-1}}{(\beta_1)_{n-1}(\beta_2)_{n-2} \ldots (\beta_l)_{n-1}} \cdot a_n \cdot \frac{z^n}{(n-1)!}.$$

For simplicity, we write

$$\alpha_1' = (\alpha_2, \ldots, \alpha_l; \beta_1, \beta_2, \ldots, \beta_m)$$

and we denote

$$H_m^l[\alpha_1, \alpha_1']f(z) = H_m^l(\alpha_1, \alpha_2, \ldots, \alpha_l; \beta_1, \beta_2, \ldots, \beta_m; z).$$

It is well known [19] that

$$\alpha_1 H_m^l[\alpha_1 + 1, \alpha_1']f(z) = z\{H_m^l[\alpha_1, \alpha_1']f(z)\}' + (\alpha_1 - 1)H_m^l[\alpha_1, \alpha_1']f(z). \tag{2.2.1}$$

Theorem 2.2.1 [58] *Let $l, m \in \mathbb{N}$, $l \le m+1$, $\alpha_i \in \mathbb{C}$, $i = 1, 2, \ldots, l$ and $\beta_j \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$, $j = 1, 2, 3, \ldots, m$, $f \in A$ and the Dziok-Srivastava linear operator $H_m^l[\alpha_1, \alpha_1']f(z)$ is given by (2.2.1).*

If it is verified the differential subordination

$$\{H_m^l[\alpha_1 + 1, \alpha_1']f(z)\}' \prec h(z), \quad z \in U, \ \operatorname{Re}\alpha_1 > 0, \tag{2.2.2}$$

then h is a convex function, then

$$[H_m^l[\alpha_1, \alpha_1']f(z)]' \prec q(z),$$

where

$$q(z) = \frac{\alpha_1}{z^{\alpha_1}} \int_0^z h(t)t^{\alpha_1 - 1}dt,$$

q is a convex function and the best dominant.

Theorem 2.2.2 [58] *Let $l, m \in \mathbb{N}$, $l \le m+1$, $\alpha_i \in \mathbb{C}$, $i = 1, 2, \ldots, l$, $\beta_j \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$, $j = 1, 2, \ldots, m$ let $f \in A$ and $H_m^l[\alpha_1, \alpha_1']f(z)$ Dziok-Srivastava linear operator given by (2.2.1).*

If we denote

$$H_m^l[\alpha_1, \alpha_1']f(z) = q(z),$$

then

$$q'(z) \prec h(z)$$

and it is verified the differential subordination

$$\{H_m^l[\alpha_1, \alpha_1']f(z)\}' \prec h(z), \quad z \in U, \ \operatorname{Re} \alpha_1 > 0, \tag{2.2.3}$$

implies

$$\frac{q(z)}{z} \prec \frac{1}{z} \int_0^z h(t)dt,$$

then

$$\frac{H_m^l[\alpha_1, \alpha_1']f(z)}{z} \prec q(z) = \frac{1}{z} \int_0^z h(t)dt.$$

2.3 The study of a class of univalent functions defined by Ruscheweyh differential operator

By using a certain operator D^n, we introduce a class of holomorphic functions $M_n(h)$, h convex function and obtain some subordination results. We also show that, for $h(z) \equiv \alpha$, $0 \leq \alpha < 1$ and $z \in U$, the set $M_n(\alpha)$ is convex and obtain some new differential subordinations related to certain integral operators.

Lemma 2.3.1 [1, Lema 1.4] *Let q be convex function in U with $q(0) = 1$ and let* $\operatorname{Re} c > 0$. *Let*

$$h(z) = q(z) + \frac{n}{c} zq'(z).$$

If $p(z) = 1 + p_n z^n + p_{n_1} z^{n+1} + \ldots$ is analytic in U and

$$p(z) + \frac{1}{c} zp'(z) \prec h(z),$$

then

$$p(z) \prec q(z)$$

and q is the best dominant.

Definition 2.3.1 (St. Ruscheweyh [48]) For $f \in A$, $n \in \mathbb{N}$, the operator D^n is defined by $D^n : A \to A$

$$D^0 f(z) = f(z)$$

$$(n+1)D^{n+1} f(z) = z[D^n f(z)]' + nD^n f(z), \ z \in U,$$

this is Ruscheweyh differential operator.

Remark 2.3.1 [29] If $f \in A$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, Then

$$D^n f(z) = z + \sum_{j=2}^{\infty} C_{n+j-1}^{n} a_j z^j, \quad z \in U.$$

Definition 2.3.2 For $h \in K$ and $n \in \mathbb{N}$, we let $M_n(h)$ denote the class of functions $f \in A$ which satisfy the subordination:

$$[D^n f(z)]' \prec h(z), \quad z \in U.$$

If $h(z) = h_\alpha(z) = \dfrac{1 + (2\alpha - 1)z}{1 + z}$, Then we denote $M_n(\alpha)$ the class $M_n(h_\alpha)$.

Theorem 2.3.1 [59] *The set* $M_n(\alpha)$ *is convex,* $0 \le \alpha < 1$.

Theorem 2.3.2 [59] *Let* q *be a convex function in* U*, with* $q(0) = 1$ *and let*

$$h(z) = q(z) + \frac{1}{c+2} z q'(z), \quad z \in U$$

where c *is a complex number, with* $\operatorname{Re} c > -2$.

If $f \in M_n(h)$ *and* $F = I_c(f)$*, where*

$$F(z) = I_c(f)(z) = \frac{c+2}{z^{c+1}} \int_0^z t^c f(t) dt, \quad \operatorname{Re} c > -2, \tag{2.3.1}$$

then

$$[D^n f(z)]' \prec h(z), \quad z \in U, \tag{2.3.2}$$

implies

$$[D^n F(z)]' \prec q(z), \quad z \in U,$$

and this result is sharp.

Theorem 2.3.3 [59] *Let* c *a complex number with* $\operatorname{Re} c > -2$ *and let*

$$w = \frac{1 + |c+2|^2 - |c^2 + 4c + 3|}{4 \operatorname{Re}(c+2)}. \tag{2.3.3}$$

Let h be an analytic function in U, with $h(0) = 1$ and suppose that

$$\operatorname{Re} \frac{zh''(z)}{h'(z)} + 1 > -w.$$

If $f \in M_n(h)$ and $F = I_c(f)$, where the function F is defined by (2.3.1), then

$$[D^n f(z)]' \prec h(z), \quad z \in U, \tag{2.3.4}$$

implies

$$[D^n F(z)]' \prec q(z), \quad z \in U,$$

where q is the solution of the differential equation

$$q(z) + \frac{1}{c+2} zq'(z) = h(z), \quad h(0) = 1,$$

given by

$$q(z) = \frac{c+2}{z^{c+2}} \int_0^z t^{c+1} h(t) dt, \quad z \in U.$$

Moreover q is the best dominant.

Chapter 3

Strong subordinations

3.1 Some strong differential subordinations obtained by Sălăgean differential operator

By using the Sălăgean differential operator we introduce a class of holomorphic functions denoted by $S_n^m(\alpha)$ and obtain some strong differential subordinations results .

Lemma 3.1.1 *[18, page 71] Let $h(z,\zeta)$ be a convex function with $h(0,\zeta)=a$ for every $\zeta\in\overline{U}$ and let $\gamma\in\mathbb{C}^*$ be a complex number with $\operatorname{Re}\gamma\geq 0$. If $p\in H^*[a,n,\zeta]$ şi*

$$p(z,\zeta)+\frac{1}{\gamma}zp'(z,\zeta)\prec\prec h(z,\zeta) \tag{3.1.1}$$

then $p(z,\zeta)\prec\prec q(z,\zeta)\prec\prec h(z,\zeta)$ where

$$g(z,\zeta)=\frac{\gamma}{nz^{\gamma/n}}\int_0^z h(t,\zeta)t^{(\gamma/n)-1}dt. \tag{3.1.2}$$

Function $g(z,\zeta)$ is convex and is the best dominant.

Lemma 3.1.2 [17] *Let$q(z,\xi)$ be a convex function in în U, for all $\xi\in\overline{U}$ and let*

$$h(z,\xi)=q(z,\xi)+n\alpha q'(z,\xi), \tag{3.1.3}$$

where $\alpha>0$ and n is a positive integer. If

$$p(z,\xi)=q(0,\xi)+p_n(\xi)z^n+\dots$$

is holomorphic in U*, for all* $\xi \in \overline{U}$ *and*

$$p(z,\xi) + \alpha z p'(z,\xi) \prec\prec h(z,\xi) \tag{3.1.4}$$

then

$$p(z,\xi) \prec\prec q(z,\xi) \tag{3.1.5}$$

and this result is sharp.

Definition 3.1.1 [49] For $f \in A^*_\xi$, $n \in \mathbb{N}^* \cup \{0\}$, the operator $S^n f$ is defined by:

$$\begin{aligned}
&S^n : A^*_\xi \to A^*_\xi \\
&S^0 f(z,\xi) = f(z,\xi) \\
&\ldots \\
&S^{n+1} f(z,\xi) = z[S^n f(z,\xi)]', \ z \in U, \ \xi \in \overline{U}.
\end{aligned}$$

Definition 3.1.2 [60] If $\alpha < 1$ şi $m, n \in \mathbb{N}$, let $S^n_m(\alpha)$ denote the class of functions $f \in A^*_{n\xi}$ which satisfy the inequality

$$\operatorname{Re}[S^m f(z,\xi)]' > \alpha. \tag{3.1.6}$$

Theorem 3.1.1 [60] *If* $\alpha < 1$ *and* $m, n \in \mathbb{N}$*, then*

$$S^{m+1}_n(\alpha) \subset S^m_n(\delta) \tag{3.1.7}$$

where

$$\delta = \delta(\alpha, n, m) = (2\alpha - 1) + 1 - (2\alpha - 1)\frac{1}{n}\beta\left(\frac{1}{n}\right)$$

$$\beta(x) = \int_0^1 \frac{t^{x-1}}{1+t} dt. \tag{3.1.8}$$

Theorem 3.1.2 [60] *Let* $q(z,\xi)$ *be a convex function with* $q(0,\xi) = 1$ *and let* $h(z,\xi)$ *be a function such that*

$$h(z,\xi) = q(z,\xi) + z q'(z,\xi). \tag{3.1.9}$$

If $f \in A^*_{n\xi}$ *and verifies the strong differential subordination*

$$[S^{m+1}f(z,\xi)]' \prec\prec h(z,\xi) \tag{3.1.10}$$

then

$$[S^m f(z,\xi)]' \prec\prec q(z,\xi). \tag{3.1.11}$$

Theorem 3.1.3 [60] *Let* $h \in H^*[a,n,\xi]$, *with* $h(0,\xi) = 1$, $h'(0,\xi) \neq 0$ *which verifies the inequality*

$$\operatorname{Re}\left[1 + \frac{zh''(z,\xi)}{h'(z,\xi)}\right] > -\frac{1}{2(m+1)}, \quad m \geq 0. \tag{3.1.12}$$

If $f \in A^*_{n\xi}$ *and verifies the strong differential subordination*

$$[S^{m+1}f(z,\xi)]' \prec\prec h(z,\xi), \quad z \in U \tag{3.1.13}$$

then

$$[S^m f(z,\xi)]' \prec\prec q(z,\xi),$$

where

$$q(z,\xi) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z t^{\frac{1}{n}-1} h(t,\xi)dt.$$

The function g *is convex and is the best dominant.*

Theorem 3.1.4 [60] *Let* $q(z,\xi)$ *be a convex function with* $q(0,\xi) = 1$ *and*

$$h(z,\xi) = q(z,\xi) + zq'(z,\xi). \tag{3.1.14}$$

If $f \in A^*_{n\xi}$ *and verifies the strong differential subordination*

$$[S^m f(z,\xi)]' \prec\prec h(z,\xi), \quad z \in U,\ \xi \in \overline{U} \tag{3.1.15}$$

then

$$\frac{S^m f(z,\xi)}{z} \prec\prec q(z,\xi). \tag{3.1.16}$$

Chapter 4

Strong superordination

4.1 Best subordinants of the strong differential superordination

The aim of this paper is to obtain the best subordinants of the strong differential superordinations.

Lemma 4.1.1 [31, Teorema 2] *Let* $q \in H[a,n]$, *let* h *be analytic in* U *and* $\varphi \in \phi_n[h,q]$. *If* $p \in Q(a)$ *and* $\varphi(p(z), zp'(z), z^2p''(z); z, \xi)$ *is univalent in* U *for all* $\xi \in \overline{U}$,*then*

$$h(z) \prec\prec \varphi(p(z), zp'(z), z^2p''(z); z, \xi), \quad z \in U, \ \xi \in \overline{U}$$

implies

$$q(z) \prec p(z), \quad z \in U.$$

Theorem 4.1.1 [32] *Let* h *şi* q *univalent în* U, *with* $q(0) = a$, $q_\rho(z) = q(\rho z)$ *and* $h_\rho(z) = h(\rho z)$. *Let* $\varphi : \mathbb{C}^3 \times U \times \overline{U} \to \mathbb{C}$ *satisfy one of the following conditions:*

(i) $\varphi \in \phi_n[h, q_\rho]$, *for some* $\rho \in (0,1)$, *or*

(ii) there exists $\rho_0 \in (0,1)$ *such that* $\varphi \in \phi_n[h_\rho, q_\rho]$, *for all* $\rho \in (\rho_0, 1)$.

If $p \in H[a,n]$, $\varphi(p(z), zp'(z), z^2p''(z); z, \xi)$ *is univalent in* U *for all* $\xi \in \overline{U}$ *and*

$$h(z) \prec\prec \varphi(p(z), zp'(z), z^2p''(z); z, \xi), \quad z \in U, \ \xi \in \overline{U}, \tag{4.1.1}$$

then

$$q(z) \prec p(z), \quad z \in U.$$

Theorem 4.1.2 [32] *Let h univalent function in U and $\varphi : \mathbb{C}^3 \times U \times \overline{U} \to \mathbb{C}$. Suppose that the differential equation*

$$\varphi(q(z), zq'(z), z^2q''(z); z) = h(z) \tag{4.1.2}$$

has a solution q with $q(0) = a$ and one of the following conditions is satisfied:

(i) $q \in Q$ and $\varphi \in \phi[h, q]$, or

(i) q is univalent in U and $\varphi \in \phi[h, q_\rho]$ for some $\rho \in (0, 1)$, or

(iii) q is univalent function in U and there exist $\rho_0 \in (0, 1)$ such that $\varphi \in \phi[h_\rho, q_\rho]$ for all $\rho \in (\rho_0, 1)$.

If $p \in H[a, 1]$ and $\varphi(p(z), zp'(z), z^2p''(z); z, \xi)$ is univalent in U, for all $\xi \in \overline{U}$ and if p satisfies

$$h(z) \prec\prec \varphi(p(z), zp'(z), z^2p''(z); z, \xi), \quad z \in U,\ \xi \in \overline{U}, \tag{4.1.3}$$

then

$$q(z) \prec p(z), \quad z \in U,$$

and q is the best subordinant.

Theorem 4.1.3 [32] *Let h be univalent function in în U and $\varphi : \mathbb{C}^3 \times U \times \overline{U} \to \mathbb{C}$. Suppose that the differential equation*

$$\varphi(q(z), nzq'(z), n(n-1)zq'(z) + n^2z^{2n}q''(z)) = h(z) \tag{4.1.4}$$

has a solution q, with $q(0) = a$ and one of the following conditions is satisfied:

(i) $q \in Q$ and $\varphi \in \phi_n[h, q]$,

(ii) q is univalent function in U and $\varphi \in \phi_n[h, q_\rho]$, for some $\rho \in (0, 1)$,or

(iii) q is univalent function in U and there exists $\rho_0 \in (0, 1)$ such that $\varphi \in \phi_n[h_\rho, q_\rho]$ for all $\rho \in (\rho_0, 1)$.

If $p \in H[a,n]$, $\varphi(p(z), zp'(z), z^2p''(z); z, \xi)$ *is univalent function in* U *for all* $\xi \in \overline{U}$ *and* p *satisfies*

$$h(z) \prec\prec \varphi(p(z), zp'(z), z^2p''(z); z, \xi), \quad z \in U, \ \xi \in \overline{U}, \tag{4.1.5}$$

then

$$q(z) \prec p(z)$$

and q *is the best subordinant.*

4.2 On a new best subordinant of the strong differential superordination

In this section we present the best subordinant of a certain differential superordination.

Lemma 4.2.1 [34] *Let* $(q, \cdot, \xi) \in Q$ *with* $q(0,\xi) = a$ *and*

$$p(z,\xi) = a + a_n(\xi)z^n + a_{n+1}(\xi)z^{n+1} + \dots$$

be analytic function in $U \times \overline{U}$ *with* $p(z,\xi) \not\equiv a$ *and* $n \geq 1$. *If* $p(\cdot,\xi)$ *is not subordinated to* $q(\cdot,\xi)$, *then there exist points* $z_0 = r_0e^{i\theta_0} \in U$ *and* $\zeta_0 \in \partial U \setminus E(q)$ *and* $m \geq n \geq 1$ *for which* $p(U_{r_0} \times \overline{U}_{r_0}) \subset q(U \times \overline{U})$.

(i) $p(z_0,\xi) = q(z_0,\xi)$

(ii) $z_0p'(z_0,\xi) = m\zeta_0q'(\zeta_0,\xi)$ *and*

(iii) $\operatorname{Re}\dfrac{z_0p''(z_0,\xi)}{p'(z_0,\xi)} + 1 \geq m\left[\operatorname{Re}\dfrac{\zeta_0q''(\zeta_0,\xi)}{q'(\zeta_0,\xi)} + 1\right]$.

Theorem 4.2.1 [33] *Let* $\Omega_\xi \in \mathbb{C}$, *let* $q(\cdot,\xi) \in H^*[a,n,\xi]$ *and let* $\varphi \in \phi_n[\Omega_\xi, q(\cdot,\xi)]$. *If* $p(\cdot,\xi) \in Q(a)$ *and* $\varphi(p(z,\xi), zp'(z,\xi), z^2p''(z,\xi); z, \xi)$ *is univalent in* U *for all* $\xi \in \overline{U}$, *then*

$$\Omega_\xi \subset \{\varphi(p(z,\xi), zp'(z,\xi), z^2p''(z,\xi); z, \xi)\}, \tag{4.2.1}$$

implies

$$q(z,\xi) \prec\prec p(z,\xi), \quad z \in U, \ \xi \in \overline{U}.$$

We next consider the special situation when $h(z,\xi)$ is analytic on $U\times\overline{U}$ and $h(U\times\overline{U})=\Omega_\xi\neq\mathbb{C}$. then the Theorem 4.2.1 becomes

Theorem 4.2.2 [33] *Let $q(z,\xi)\in H[a,n,\xi]$, let $h(z,\xi)$ analytic in $U\times\overline{U}$ and let $\varphi\in\phi_n[h(z,\xi),q(z,\xi)]$. If $p(z,\xi)\in Q(a)$ şi $\varphi(p(z,\xi),zp'(z,\xi),z^2p''(z,\xi);z,\xi)$ is univalent in U for all $\xi\in\overline{U}$, then*

$$h(z,\xi)\prec\prec\varphi(p(z,\xi),zp'(z,\xi),z^2p''(z,\xi);z,\xi)$$

implies

$$q(z,\xi)\prec\prec p(z,\xi),\quad z\in U,\ \xi\in\overline{U}.$$

Theorem 4.2.3 [33] *Let $h(z,\xi)$ and $q(z,\xi)$ be univalent functions in U for all $\xi\in\overline{U}$, with $q(0,\xi)=a$, $q_\rho(z,\xi)=q(\rho z,\xi)$ and $h_\rho(z,\xi)=h(\rho z,\xi)$. Let $\varphi:\mathbb{C}^3\times U\times\overline{U}\to\mathbb{C}$ satisfy one of conditions*

(i) $\varphi\in\phi_n[h(z,\xi),q_\rho(z,\xi)]$, for some $\rho\in(0,1)$, or

(ii) there exists $\rho_0\in(0,1)$ such that $\varphi\in\phi_n[h_\rho(z,\xi),q_\rho(z,\xi)]$ for all $\rho\in(\rho_0,1)$.

If $p(z,\xi)\in H^*[a,n,\xi]$, $\varphi(p(z,\xi),zp'(z,\xi),z^2p''(z,\xi);z,\xi)$ is univalent function in U for all $\xi\in\overline{U}$ and

$$h(z,\xi)\prec\prec\varphi(p(z,\xi),zp'(z,\xi),z^2p''(z,\xi);z,\xi),\quad z\in U,\ \xi\in\overline{U},\tag{4.2.2}$$

then

$$q(z,\xi)\prec\prec p(z,\xi),\quad z\in U,\ \xi\in\overline{U}.$$

Theorem 4.2.4 [33] *Let $h(z,\xi)$ univalent function in U for all $\xi\in\overline{U}$ and let $\varphi:\mathbb{C}^3\times U\times\overline{U}\to\mathbb{C}$. Suppose that the differential equation*

$$\varphi(q(z,\xi),zq'(z,\xi),z^2q''(z,\xi);z,\xi)=h(z,\xi),\quad z\in U,\ \xi\in\overline{U}\tag{4.2.3}$$

has a solution $q(z,\xi)$, with $q(0,\xi)=a$ and one of the following conditions is satisfied:

(i) $q(z,\xi)\in Q$ and $\varphi\in\phi[h(z,\xi),q(z,\xi)]$

(ii) $q(z,\xi)$ is univalent in U for all $\xi\in\overline{U}$ and $\varphi\in\phi[h(z,\xi),q_\rho(z,\xi)]$, for some $\rho\in(0,1)$ or

(iii) $q(z,\xi)$ *is univalent function in* U *for all* $\xi \in \overline{U}$ *and there exists* $\rho_0 \in (0,1)$ *such that*

$$\varphi \in \phi[h_\rho(z,\xi) q_\rho(z,\xi)] \text{ pentru toţi } \rho \in (\rho_0, 1).$$

If $p(z,\xi) \in H^*[a,1,\xi]$ *and* $\varphi(p(z,\xi), zp'(z,\xi), z^2p''(z,\xi); z,\xi)$ *is univalent in* U *for all* $\xi \in \overline{U}$ *and*

$$(4.2.4) \qquad h(z,\xi) \prec\prec \varphi(p(z,\xi), zp'(z,\xi), z^2p''(z,\xi); z,\xi), \quad z \in U,\ \xi \in \overline{U},$$

then

$$q(z,\xi) \prec\prec p(z,\xi), \quad z \in U,\ \xi \in \overline{U}$$

and $q(z,\xi)$ *is the best subordinant.*

Theorem 4.2.5 [33] *Let function* $h(z,\xi)$ *univalent in* U *and let* $\varphi : \mathbb{C}^3 \times U \times \overline{U} \to \mathbb{C}$. *Suppose that the differential equation*

$$(4.2.5) \qquad \varphi(q(z,\xi), nzq'(z,\xi), n(n-1)zq'(z,\xi) + n^2z^{2n}q''(z,\xi)) = h(z,\xi)$$

has a solution $q(z,\xi)$, *with* $q(0,\xi) = a$ *and one of the following conditions is satisfied:*

(i) $q(z,\xi) \in Q$ *and* $\varphi \in \phi_n[h(z,\xi), q(z,\xi)]$

(ii) $q(z,\xi)$ *is univalent in* U *for all* $\xi \in U$ *and* $\varphi \in \phi_n[h(z,\xi), q_\rho(z,\xi)]$ *for some* $\rho \in (0,1)$, *or*

(iii) $q(z,\xi)$ *univalent function in* U *for all* $\xi \in \overline{U}$ *and there exists* $\rho_0 \in (0,1)$ *such that* $\varphi \in \phi_n[h_\rho(z,\xi), q_\rho(z,\xi)]$ *for all* $\rho \in (\rho_0, 1)$.

If $p(z,\xi) \in H^*[a,n,\xi]$, $\varphi(p(z,\xi), zp'(z,\xi), z^2p''(z,\xi); z,\xi)$ *is univalent in* U *for all* $\xi \in \overline{U}$ *şi* $p(z,\xi)$ *satisfies*

$$(4.2.6) \qquad h(z,\xi) \prec\prec \varphi(p(z,\xi), zp'(z,\xi), z^2p''(z,\xi); z,\xi), \quad z \in U,\ \xi \in \overline{U}$$

then

$$q(z,\xi) \prec\prec p(z,\xi), \quad z \in U,\ \xi \in \overline{U}$$

and $q(z,\xi)$ *is the best subordinant.*

4.3 First-order strong differential superordinations

In this paper we study the special case of first order strong differential superordinations.

Lemma 4.3.1 [20, T. 2.6.h, p. 67],[43], [5] *If* $L_y : A^*_\xi \to A^*_\xi$ *is the integral operator defined by*

$$L_y[f(z),\xi] = F(z,\xi) = \frac{\gamma+1}{z^\gamma}\int_0^z f(z,\xi)t^{\gamma-1}dt$$

and $\operatorname{Re}\gamma \geq 0$*, then*

(i) $L_\gamma[S^*] \subset S^*$

(ii) $L_\gamma[K^*] \subset K^*$.

Definition 4.3.1 [45, p. 157], [20, p. 4] The function $L : U \times \overline{U} \times [0,\infty) \to \mathbb{C}$ is a strong subordination (or a Löewner) chain if $L(z,\xi;t)$ is analytic and univalent in U for $\xi \in \overline{U}$, $t \geq 0$, $L(z,\xi;t)$ is continuously differentiable function of t on $[0,\infty)$ for all $z \in U$, $\xi \in \overline{U}$ and $L(z,\xi;s) \prec\prec L(z,\xi,t)$ where $0 \leq s \leq t$.

The following lemma provides a sufficient condition for $L(z,\xi;t)$ be a strong subordination chain.

Lemma 4.3.2 [45, p. 159], [20, p. 4] *The function*

$$L(z,\xi;t) = a_1(\xi,t)z + a_2(\xi,t)z^2 + \dots$$

with $a_1(\xi,t) \neq 0$ *for* $\xi \in \overline{U}$*,* $t \geq 0$ *and* $\lim_{t\to\infty}|a_1(\xi,t)| = \infty$ *is a strong subordination chain if*

$$\operatorname{Re} z \cdot \frac{\partial L(z,\xi;t)/\partial z}{\partial L(z,\xi;t)/\partial t} > 0, \quad z \in U,\ \xi \in \overline{U},\ t \geq 0.$$

Lemma 4.3.3 [35, Th. 2] *Let* $h(\cdot,\xi)$ *be analytic in* $U \times \overline{U}$*,* $q(\cdot,\xi) \in H^*[a,n,\xi]$ *,* $\varphi : \mathbb{C}^2 \times U \times \overline{U} \to \mathbb{C}$ *and suppose that*

$$\varphi(q(z,\xi), tzq'(z,\xi);\zeta,\xi) \in h(U \times \overline{U}), \tag{4.3.1}$$

for $z \in U$, $\zeta \in \partial U$, $\xi \in \overline{U}$ *and* $0 < t \leq \dfrac{1}{n} \leq 1$. *If* $p(\cdot, \xi) \in Q(a)$ *and* $\varphi(p(z,\xi), zp(z,\xi); z, \xi)$ *is univalent in* U, *for all* $\xi \in \overline{U}$ *then*

$$h(z,\xi) \prec\prec \varphi(p(z,\xi), zp'(z,\xi); z, \xi)$$

implies

$$q(z,\xi) \prec\prec p(z,\xi), \quad z \in U,\ \xi \in \overline{U}.$$

Furthermore, if $\varphi(p(z,\xi), zp'(z,\xi); z,\xi) = h(z,\xi)$, $\xi \in \overline{U}$ *has a univalent solution* $q(\cdot,\xi) \in Q(a)$, *then* $q(\cdot,\xi)$ *is the best subordinant.*

Theorem 4.3.1 [36] *Let* $h_1(z,\xi)$ *convex function in* U, *for all* $\xi \in \overline{U}$ *with* $h_1(0,\xi) = a$, $\gamma \neq 0$ *with* $\operatorname{Re}\gamma > 0$ *and* $p \in H^*[a,1,\xi] \cap Q$. *If* $p(z,\xi) + \dfrac{zp'(z,\xi)}{\gamma}$ *is univalent in* U, *for all* $\xi \in \overline{U}$,

$$h_1(z,\xi) \prec\prec p(z,\xi) + \frac{zp'(z,\xi)}{\gamma} \tag{4.3.2}$$

and

$$q_1(z,\xi) = \frac{\gamma}{z^\gamma}\int_0^z h_1(t,\xi)t^{\gamma-1}dt, \tag{4.3.3}$$

then

$$q_1(z,\xi) \prec\prec p(z,\xi), \quad z \in U,\ \xi \in \overline{U}.$$

Function $q_1(z,\xi)$ *is convex and is the best subordinant.*

Theorem 4.3.2 [36] *Let* $q(z,\xi)$ *convex function in* U, *for all* $\xi \in \overline{U}$ *and let* $h(z,\xi)$ *be defined by*

$$q(z,\xi) + \frac{zq'(z,\xi)}{\gamma} = h(z,\xi), \quad z \in U,\ \xi \in \overline{U} \tag{4.3.4}$$

with $\operatorname{Re}\gamma > 0$. *If* $p(z,\xi) \in H^*[a,1,\xi] \cap Q$, $p(z,\xi) + \dfrac{zp'(z,\xi)}{\gamma}$ *is univalent in* U, *for all* $\xi \in \overline{U}$ *and and satisfy*

$$h(z,\xi) \prec\prec p(z,\xi) + \frac{zp'(z,\xi)}{\gamma}, \quad z \in U,\ \xi \in \overline{U} \tag{4.3.5}$$

then

$$q(z,\xi) \prec\prec p(z,\xi), \quad z \in U,\ \xi \in \overline{U},$$

where

$$q(z,\xi) = \frac{\gamma}{z^\gamma}\int_0^z h(t,\xi)t^{\gamma-1}dt, \quad z \in U,\ \xi \in \overline{U}.$$

Function q is the best subordinant.

Theorem 4.3.3 [36] *Let $h(z,\xi)$ be starlike function in U, for all $\xi \in \overline{U}$, with $h(0,\xi) = 0$. If $p(z,\xi) \in H^*[0,1;\xi] \cap Q$ and $zp'(z,\xi)$ is univalent in U, for all $\xi \in \overline{U}$, then*

$$h(z,\xi) \prec\prec zp'(z,\xi) \tag{4.3.6}$$

implies

$$q(z,\xi) \prec\prec p(z,\xi), \quad z \in U,\ \xi \in \overline{U},$$

where

$$q(z,\xi) = \int_0^z h(t,\xi)t^{\gamma-1}dt. \tag{4.3.7}$$

Function q is convex and is the best subordinant.

Chapter 5

Order of convolution consistence

5.1 Analytic functions with negative coefficients

In this section we list some results already known about the univalent functions with negative coefficients. We denote

$$\mathcal{N} = \left\{ f \in A: \ f(z) = z - \sum_{j=2}^{\infty} a_j z^j, \ a_j \geq 0, \ j \geq 2 \right\}.$$

Remark 5.1.1 [54] (i)Denoting by T subset of S be the class of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0,$$

where $T = S \cap \mathcal{N}$.

(ii)We denote $T^* = T \cap S^*$ and T_1^* the families consisting of functions in T (respectively starlike functions) and satisfy

$$|(zf'/f) - 1| \leq 1, \quad z \in U.$$

Theorem 5.1.1 [54] *For* $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$, $a_n \geq 0$, *the following are equivalent:*

(i) $\sum_{n=2}^{\infty} n a_n \leq 1$;

(ii) $f \in T$;

(iii) $f \in T^*$;

(iv) $f \in T_1^*$;

(v) $f' \neq 0,\ z \in U$;

(vi) $\mathrm{Re}\, f' > 0,\ z \in U$.

We defined the classes $T_n(\alpha)$, $\alpha < 1$, $n \in \mathbb{N}$, by

$$T_n(\alpha) = \left\{ f \in \mathcal{N} :\ \mathrm{Re}\, \frac{S^{n+1} f(z)}{S^n f(z)} > \alpha,\ z \in U \right\}.$$

About functions from these classes we have next theorem.

Theorem 5.1.2 [52], [13] *Let* f *a function from* $\mathcal{N}$,

$$f(z) = z - \sum_{j=2}^{\infty} a_j z^j.$$

A function $f \in T_n(\alpha)$, $n \in \mathbb{N}$, $\alpha < 1$ *if and only if*

$$\sum_{j=2}^{\infty} \frac{j^n (j - \alpha)}{1 - \alpha} \leq 1.$$

In the particular case, $T_0(0) = T^*$ *is the class of starlike functions with negative coefficients, and* $T_1(0)$ *is the class of convex functions with negative coefficients.*

We study $h(z) = f(z) * g(z)$, where $f(z)$ şi $g(z)$ is members from the class $T_n(\alpha)$, $n \in \mathbb{N}$, $\alpha < 1$.

Theorem 5.1.3 [53] *If* $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$, $a_n \geq 0$, $g(z) = z - \sum_{n=2}^{\infty} b_n z^n$, $b_n \geq 0$ *are elements of* $T_n(\alpha)$, *then*

$$h(z) = f(z) * g(z) = z - \sum_{n=2}^{\infty} a_n b_n z^n$$

is an element of $T_n\left(\dfrac{2 - \alpha^2}{3 - 2\alpha}\right)$. *The result is the best possible.*

5.2 The order of convolution consistence of the analytic functions with negative coefficients

. In this section we present some known results in determining the order of consistency of the univalent functions from A class presented in the [3]. Further we mention original results, which shows the determination of the order of consistency of convolution of the analytical functions with negative coefficients for different subclasses, of the work [51].

Definition 5.2.1 [49] If $\alpha \in [0,1)$ and let $n \in \mathbb{N}$; we define the class $\mathcal{S}_n(\alpha)$ of *n-starlike functions of order* α by

$$\mathcal{S}_n(\alpha) = \left\{ f \in A : \operatorname{Re} \frac{S^{n+1}f(z)}{S^n f(z)} > \alpha,\ z \in U \right\}. \tag{5.2.1}$$

Denote by $\mathcal{S}_n$ the class $\mathcal{S}_n(0)$. We note that $\mathcal{S}_0 = \mathcal{ST}$ is the class of starlike functions and $\mathcal{S}_1 = \mathcal{CV}$ is the class of convex functions.

Definition 5.2.2 [3] If $f, g \in A$, the integral convolution is defined by

$$(f \otimes g)(z) = z + \sum_{j=2}^{\infty} \frac{a_j b_j}{j} z^j.$$

Definition 5.2.3 [3] Let Sălăgean integral operator (see [3], [2], [49]) $I^s : A \to A$, $s \in \mathbb{R}$ such that

$$\mathcal{I}^s f(z) = \mathcal{I}^s \left(z + \sum_{j=2}^{\infty} a_j z^j \right) = z + \sum_{j=2}^{\infty} \frac{a_j}{j^s} z^j. \tag{5.2.2}$$

Definition 5.2.4 [3] Let $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ be subsets of A. We say that the three $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ is S-closed under the convolution if there exists a number $S = S(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ such that

$$\begin{aligned} S(\mathcal{X}, \mathcal{Y}, \mathcal{Z}) &= \min\{s \in \mathbb{R} :\ I^s(f * g) \in \mathcal{Z},\ \text{pentru orice } f \in \mathcal{X} \text{ şi } g \in \mathcal{Y}\} \\ &= \min\{s \in \mathbb{R} :\ I^s(\mathcal{X} * \mathcal{Y}) \subseteq \mathcal{Z}\}, \end{aligned} \tag{5.2.3}$$

where I^s is Sălăgean integral operator . The number $S(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ called the **order of convolution consistence** of the three $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$.

U. Bednarz and J. Sokol in [3] obtained the order of convolution consistence concerning certain classes of univalent functions (starlike, convex, uniform-starlike or uniform-convex functions). For example they proved

Theorem 5.2.1 [3] *We have the following order of convolution consistence:*

(i) $S(S^*, S^*, S^*) = 1$;

(ii) $S(K, K, S^*) = -1$;

(iii) $S(K, S^*, S^*) = 0$;

(iv) $S(S^*, S^*, K) = 2$;

(v) $S(K, K, K) = 0$;

(vi) $S(K, S^*, K) = 1$.

The modified Hadamard product or $\circledast$-convolution of two functions f and g from $\mathcal{N}$ by

$$(5.2.4) \qquad f(z) = z - \sum_{j=2}^{\infty} a_j z^j \text{ and } g(z) = z - \sum_{j=2}^{\infty} b_j z^j, \ \ a_j,\ b_j \geq 0,$$

is the function $(f \circledast g)$ defined as (see [53])

$$(f \circledast g)(z) = z - \sum_{j=2}^{\infty} a_j b_j z^j.$$

As in Definition 5.2.4 we define the **order of $\circledast$-convolution consistence** of the three ($\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$), where $\mathcal{X}$, $\mathcal{Y}$ şi $\mathcal{Z}$ is subsets of $\mathcal{N}$, denoted $S_\circledast$ by

$$(5.2.5) \qquad S_\circledast(\mathcal{X},\ \mathcal{Y},\ \mathcal{Z}) = \min\{s \in \mathbb{R} : \mathcal{I}^s(f \circledast g) \in \mathcal{Z},\ \forall f \in \mathcal{X},\ \forall g \in \mathcal{Y}\}.$$

In this section we obtain similar results as in Theorem 5.2.1 but concerning the class $\mathcal{T}_n$, and for $\circledast$-convoluţion.

We need the next characterization of the class $\mathcal{T}_n$

Theorem 5.2.2 *Let* $n \in \mathbf{N}$ *şi fie* $f \in \mathcal{N}$ *o function of the form (5.2.4); then* f *belongs to* $\mathcal{T}_n$ *dacă if and only if*

$$\sum_{j=2}^{\infty} j^{n+1} a_j \leq 1\,.$$

The result is sharp and the extremal functions are

$$f_j(z) = z - \frac{1}{j^{n+1}} z^j \ , \ j \in \{2, 3, ...\}. \tag{5.2.6}$$

Theorem 5.2.3 *If* $f \in \mathcal{T}_{n+p}$ *şi* $g \in \mathcal{T}_{n+q}$, *then* $\mathcal{I}^s(f \circledast g) \in T_{n+r}$, *where* $p, \ q, \ r,$ $n \in \mathbb{N}$ *and*

$$s = r - p - q - n - 1. \tag{5.2.7}$$

The result is sharp.

Theorem 5.2.4 *Let* $p, \ q, \ r, \ n \in \mathbb{N}$ *and let* s *be given by (5.2.7); then the order of* $\circledast$*-convolution consistence is*

$$S_{\circledast}(\mathcal{T}_{n+p}, \mathcal{T}_{n+q}, \mathcal{T}_{n+r}) = s = r - p - q - n - 1. \tag{5.2.8}$$

Corolary 5.2.1 *We have the following* $\circledast$*-convolution consistence*

$(a) \quad S_{\circledast}(\mathcal{T}_0, \, \mathcal{T}_0, \, \mathcal{T}_0) = -1,$

$(b) \quad S_{\circledast}(\mathcal{T}_0, \, \mathcal{T}_0, \, \mathcal{T}_1) = 0,$

$(c) \quad S_{\circledast}(\mathcal{T}_1, \, \mathcal{T}_0, \, \mathcal{T}_0) = -2,$

$(d) \quad S_{\circledast}(\mathcal{T}_1, \, \mathcal{T}_1, \, \mathcal{T}_0) = -3,$

$(e) \quad S_{\circledast}(\mathcal{T}_1, \, \mathcal{T}_0, \, \mathcal{T}_1) = -1,$

$(f) \quad S_{\circledast}(\mathcal{T}_1, \, \mathcal{T}_1, \, \mathcal{T}_1) = -2.$

We note that $\mathcal{T}_0 = \mathcal{ST} \bigcap \mathcal{N}$ and $\mathcal{T}_1 = \mathcal{CV} \bigcap \mathcal{N}$ and it is easy to compare the results of first Theorem to those of Corollary 5.2.1.

Bibliography

[1] H. Al-Amiri and P. T. Mocanu, *On certain subclasses of meromorphic close-to-convex functions*, Bull. Math. Soc. Sc. Math. Romanie, Tome 38 (86), Nr. 1-2, 1994, 1-15.

[2] C. M. Bălăeţi, *An integral operator associated with differential superordinations*, An. Stiint. Univ. "Ovidius", Constanţa Ser. Mat. 17(2009),no.3,37-44.

[3] U. Bednarz, J. Sokol, *On order of convoluion consistence of the analytic functions*, Studia Univ. Babeş-Bolyai, Mathematica, 55, 2010, no.3.

[4] L. De Branges, *A proof of the Bieberbach conjecture*, Acta Math. J., **154**(1985), 137-152.

[5] D. V. Breaz, *Operatori integrali pe spaţii de funcţii univalente*, Ed. Academiei Române, Bucureşti,(2004), 120-148.

[6] T. Bulboacă, *Differential subordinations and superordinations. Recent results*, Casa Cărţii de Ştiinţă, Cluj-Napoca, 2005, pg. 97-147.

[7] J. Dziok, H. M. Srivastava, *Classes of analytic functions associated with the generalized hypergeometric function*, Appl. Math. Comput., **103**(1999), 1-13.

[8] J. Dziok, H. M. Srivastava, *Certain subclasses of analytic functions associated with the generalized hypergeometric function*, Integral Transform. Spec. Funct., **14**(2003), 7-18.

[9] V. P. Gupta and P. K. Jain, *Certain classes of univalent functions with negative coefficients*, Bull. Austral. Math. Soc. 14 (1976), 409-416.

[10] D. J. Hallenbeck and S. Ruscheweyh, *Subordination by convex functions*, Proc. Amer. Math. Soc., **52**(1975), 191-195.

[11] P. Hamburg, P. Mocanu, N. Negoescu, *Analiză matematică* (*Funcţii complexe*), Editura Didactică şi Pedagogică, Bucureşti, 1982, pg. 143-151.

[12] A. Holhoş, G. S. Sălăgean, *Integral properties of certain classes of analytic functions with negative coefficients*, Pu.M.A. 15 (2004), no. 2-3, (2005), 171-177. MR 2182004.

[13] M. D. Hur and Ge M. Oh, *On certain class of analytic functions with negative coefficients*, Pusan Kyongnam Math. J., **5**(1989), 69-80.

[14] I. S. Jack, *Functions starlike and convex of order* α, J. London Math.Soc., **3**(1971), 469-474.

[15] G. Kohr, P. T. Mocanu, *Capitole speciale de analiză complexă*, Presa Universitară Clujeană, Cluj-Napoca, 2005, pg. 123-159.

[16] K. Loewner, *Undersuchunger über die Verzerrung bei Konformen Abbidungen des Einheitskreises* $|z| < 1$, *die durchh Funktionen mit nichtverschwin dender Ableitung geliefert werden*, S. B. Sächs. Akad. Wiss. Leipzig Berichte, **69**(1917), 89-106.

[17] S. S. Miller, P. T. Mocanu, *On some classes of first-order differential subordinations*, Michigan Math. J., **32**(1985), no. 2, 185-195.

[18] S. S. Miller, P. T. Mocanu, *Differential Subordinations. Theory and Applications*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 225, Marcel Dekker, New York, 2000.

[19] S. S. Miller, P. T. Mocanu, *Differential subordinations and univalent functions*, Michigan Math. J., **28**(1981), 157-171.

[20] S. S. Miller and P. T. Mocanu, *Differential subordinations. Theory and applications*, Pure and Applied Mathematics, Marcel Dekker, Inc., New York, 2000.

[21] S. S. Miller, P. T. Mocanu, *The theory and applications of second-order differential subordinations*, Studia Univ. Babeş-Bolyai, Math., 34, **4**(1989), 3-33.

[22] S. S. Miller, P. T. Mocanu, *Second order differential inequalities in the complex plane*, J. Math. Anal. Appl., **65**(1978), 298-305.

[23] S. S. Miller, P. T. Mocanu, *Subordinants of differential superordinations*, Complex Variables, **48(10)**(2003), pg. 815-826.

[24] S. S. Miller, P. T. Mocanu, *Differential subordinations and inequalities in the complex plane*, J. Diff. Eqn., **67**(1987), pg. 199-211.

[25] P. T. Mocanu, *Une propriété de convexité généralisée dans la théorie de la représentation conforme*, Mathematica. (Cluj), **11(34)**(1969), 127-133.

[26] P. T. Mocanu, T. Bulboacă, Gr. St. Sălăgean, *Teoria geometrică a funcţiilor univalente*, Casa Cărţii de Ştiinţă, Cluj-Napoca, 1999.

[27] R. Nevanlinna, *Über die Konforme Abbildung Sterngebieten*, Översikt av Finska Vet. Soc., Förh. (A), No. 6, **63**(1921).

[28] R. Nevanlina, *Über die Schlichten Abbildungen des Einheitkreises*, Översikt av Finska Vet. Soc., Förh. (A), No. 7, **62**(1920), 1-14.

[29] Gh. Oros, G. I. Oros, *A class of holomorphic function II*, Libertas Mathematics, vol. XXIII (2003), 65-68.

[30] Gh. Oros and G. I. Oros, *A class of holomorphic function II*, Libertas Math., **23**(2003), 65-68.

[31] G. I. Oros, *Strong differential superordination*, Acta Universitatis Apulensis, **19**(2009), 101-106.

[32] Gh. Oros, Adela Olimpia Tăut, *Best subordinants of the strong differential superordination*, Hacettepe Journal of Mathematics and Statistics, **38(3)**(2009), 293-298.

[33] Gh. Oros, Adela Olimpia Tăut, R. Şendruţiu, *On a new best subordinant of the strong differential superordination*, to appear.

[34] G. I. Oros, *On a new strong differential subordination* (to appear).

[35] Gh. Oros, *Briot-Bouquet strong differential superordination and sandwich theorems* Mathematical Reports, Vol. 12(62), No. 3(2010).

[36] Gh. Oros, R. Şendruţiu, Adela Olimpia Tăut, *First-order strong differential superordinations* (to appear).

[37] Georgia Irina Oros, Gheorghe Oros, *Strong differential subordination*, Turkish Journal of Mathematics, **32**(2008).

[38] Georgia Irina Oros, *Sufficient conditions for univalence obtained by using first order nonlinear strong differential subordinations* (to appear).

[39] Georgia Irina Oros, *Sufficient conditions for univalence obtained by using second order linear strong differential subordinations*, Turkish Journal of Mathematics 34 (2010) , pp.13 - 20, doi:10.3906/mat-0810-6, ISSN 1300-0098, Electronic ISSN 1303-6149.

[40] Georgia Irina Oros, Gheorghe Oros, *Second order nonlinear strong differential subordinations*, Bull. Belg. Math. Soc. Simon Stevin, **16**(2009), 171-178.

[41] Georgia Irina Oros, *Strong differential superordination* Acta Universitatis Apulensis, No.19/ 2009 pp.101-106.

[42] Georgia Irina Oros, *On a new strong differential subordination* (to appear).

[43] G. I. Oros, *Utilizarea subordonărilor diferenţiale în studiul unor clase de funcţii univalente*, Casa Cărţii de Stiinţă,Cluj Napoca, (2008).

[44] S. Owa, *On the distortion theorems*, I. Kyungpook Math. J., **18**(1978), 53-59.

[45] Ch. Pommerenke, *Univalent Functions*, Van der Hoeck and Ruprecht, Göttingen, 1975.

[46] M. S. Robertson, *A remark on the odd schlicht functions*, Bull. Amer. Math. Soc., **42**(1936), pg. 366-370.

[47] M. S. Robertson, *Analytic function starlike in one direction*, Amer. J. Math., **58**(1936), 465-472.

[48] S. Ruscheweyh, *New criteria for univalent functions*, Proc. Amer. Math. Soc., **49**(1975), 109-115.

[49] G. S. Sălăgean, *Subclasses of univalent functions*, Complex Analysis, Fifth Romanian Finish Seminar, Part 1 (Bucharest, 1981), 362-372, Lecture Notes in Math., 1013, Springer, Berlin, 1983.

[50] G.S. Sălăgean, *On univalent functions with negative coefficients*, "Babes-Bolyai" Univ., Res. Sem., Prep. 7/1991, 47-54.

[51] G. S. Sălăgean and A.O. Tăut *On the order of convolution consistence of the analytic functions with negative coefficients*, (to appear).

[52] G. S. Sălăgean, *Classes of univalent functions with two fixed points*, Babeş-Bolyai University, Res. Sem., Itin. Sem., Prep. **6**(1984), 181-184.

[53] A. Schild, H. Silverman, *Convolutions of univalent functions with negative coefficients*, Annales Univ. Mariae Curie-Sklodowska, Lublin, Polonia, 1975, 99-106.

[54] H. Silverman, *A survey with open on univalent functions whose coefficients are negative*, Rocky Mountain Journal of Mathematics, 1991, 1099-1120.

[55] H. Silverman, *Univalent functions with negative coefficients*, Proc. Amer. Math. Soc., 51 (1975), 109-116.

[56] H. Silverman, *A survey with open on univalent functions whose coefficients are negative*, Rocky Mountain Journal of Mathematics, 1991, 1099-1120.

[57] Adela Olimpia Tăut, G. I. Oros, R. Şendruţiu, *On a class of univalent functions defined by Sălăgean differential operator*, Banach J. Math. Anal.

[58] Adela Olimpia Tăut, *Differential subordinations obtained using Dziok-Srivastava linear operator*, Acta Universitatis Apulensis, **18**(2009), 79-86.

[59] Adela Olimpia Tăut, *The study of a class of univalent functions defined by Ruscheweyh differential operator*, Journal of Mathematics and Applications, **31**(2009), 107-115.

[60] Adela Olimpia Tăut, *Some strong differential subordinations obtained by Sălăgean differential operator*, Studia Univ. Babeş-Bolyai, Mathematica, Volume LV, Number 3, (2010), 221-228.