BABEŞ-BOLYAI UNIVERSITY OF CLUJ-NAPOCA
Faculty of Mathematics and Computer Science

LUMINIŢA–IOANA COTÎRLĂ

SPECIAL CLASSES OF UNIVALENT FUNCTIONS

ABSTRACT OF THE PH. D. THESIS

Scientific supervisor:
PROF. UNIV. DR. GRIGORE ŞTEFAN SĂLĂGEAN

2010

Cuprins

2000 Mathematical Subject Classification: 30C45, 30C50, 31A05, 32H02

Keywords and Phrases: univalent function, harmonic function, subordination, superordination, dominant, integral operator, differential operator, Loewner chain, generalized almost starlike mapping.

Introduction

The complex analysis is a field of research in which romanian school of mathematics has had an important contribution and it is also a part of mathematics with multiple applications in other domains of science and technique.

The geometric theory of the functions of one complex variables is an special branch for complex analysis. The basis of this theory was put on in the early twenty century once with the papers of P. Koebe, T. H. Gromwall and L. Bieberbach. In 1916 L. Bieberbach was spoken the notorious conjecture which was prove in 1984 by Louis de Branges.

G. Călugăreanu is the creator of the romanian school of univalent functions theory and P. T. Mocanu was introduced the class of $\alpha-$ convex functions, he has approached the injectivity problem of non-analytic functions and with S. S. Miller created the method of admissible functions, the method of differential subordinations and the theory of differential superordinations.

We remember papers dedicated the domains of theory of univalent functions by: P. Duren, A. W. Goodman, S.S. Miller and P. T. Mocanu, P. Montel, C. Pommerenke.

The paper has 5 chapters, an introduction and a bibliography with 124 titles, 12 of which are signed by the author (10 as unique author and 2 in collaboration).

I would like to use this opportunity to thank my scientific supervisor, Ph. D. Professor G. St. Sălăgean for his guidance and constant support and help.

I would also want to thank to all the mathematicians that are part of Cluj-Napoca school of Geometric function theory.

1 Concepts and preliminary results

This chapter contains two paragraphs and it present preliminary results from the geometric theory of univalent functions.

Definition 1.1.1.[51] Let be D an open set in the complex plane $\mathbb{C}$. A complex function f is holomorphic on D if f is derivative in all point z_0 from D. The set of holomorphic functions on D is denoted by $\mathcal{H}(D)$.

Definition 1.1.2.[51] A complex function f is holomorphic on a some set $A \subset \mathbb{C}$, if exists an open set D which include A such that f is holomorphic on D.

Definition 1.1.3.[51] The function f is holomorphic in the point z_0 if exists a neighborhood $V \in V(z_0)$ such that f is derivable in this neighborhood.

An holomorphic function on $\mathbb{C}$ is an integer function.

Definition 1.1.4.[51] An holomorphic function (or meromorphic) and injective on the domain D from $\mathbb{C}$ is univalent on D. We denote by $\mathcal{H}_u(D)$ the set of univalent functions on the domain D.

Definition 1.1.5.[51] An holomorphic function (or meromorphic) on the domain D is $p-$ valent in this domain, if some values is compute at most p single points from D and exists at least a value compute in p single points.

Definition 1.1.6.[51] Let be $f : D \to \mathbb{C}$, $z_0 \in D$. We say that the function f is analytic in the point z_0 if exists an disc $U(z_0, R) \subset D$ such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad z \in U(z_0, R).$$

We say that the function f is analytic on D if it is analytic in all point of D.

The notion of analytic function is matter of great concern in geometric theory of analytic functions. The first paper is due to P. Koebe [61] and it was publicated in 1907. Now exists many papers dedicated to univalent functions to rank among Montel [89], Z. Nehari [94], L.V. Ahlfors [2], Ch. Pommerenke [100], A.W. Goodman [38], P.L. Duren [30], D.J. Hallenbeck, T. H. MacGregor [48], S.S. Miller and P.T. Mocanu [79], I. Graham and G. Kohr [39].

Definition 1.1.7. Let be D and Δ domains from $\mathbb{C}$. An univalent function f from D such that $f(D) = \Delta$ is worthy representation of D on Δ. The domains D and Δ are worthy equivalently if exists an worthy representation of D on Δ.

We use the notations:

$U = \{z \in \mathbb{C} : |z| < 1\}$ (the unit disc in the complex plane);

$U_r = \{z \in \mathbb{C} : |z| < r\}$ for $r \in (0, 1)$ (the interior of the unit disc from the complex plane);

$U^- = \{z \in \mathbb{C} : |z| > 1\}$ (the exterior of the unit disc from the complex plane).

Let for $a \in \mathbb{C}$ and $n \in \mathbb{N}^*$, the set

$$\mathcal{H}[a,n] = \{f \in \mathcal{H}(U) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + ...\},$$

$$\mathcal{A}_n = \{f \in \mathcal{H}(U) : f(z) = z + a_{n+1} z^{n+1} + ...\}$$

şi

$$\mathcal{A} = \mathcal{A}_1.$$

In the second paragraph are presented some special classes of univalent functions.

A. The classes S and Σ

We denote by $S = \{f \in A : f \in \mathcal{H}_u(U)\}$ the class of univalent functions in the unit disc and normalized , by the condition $f(0) = 0$ and $f'(0) = 1$, for which

$$f(z) = z + a_2 z^2 + ..., |z| < 1. \tag{1.1}$$

The study of the meromorphic and univalent functions could be done parallel with the class S, considering the class Σ and the class Σ_0

$$\Sigma = \big\{\varphi \in \mathcal{H}_u(U^-) : \varphi(\zeta) = \zeta + \alpha_0 + \frac{\alpha_1}{z} + ... + \frac{\alpha_n}{z^n} + ..., |\zeta| > 1\big\}$$

and

$$\Sigma_0 = \{\varphi \in \Sigma : \varphi(\zeta) \neq 0, \zeta \in U^-\}.$$

Remark 1.2.1. Between the classes S and Σ_0 there is a bijection, therefore the class Σ is larger than the class S.

B. The class of starlike functions

Definition 1.2.2.[87] Let be the function $f \in \mathcal{H}(U)$ with $f(0) = 0$. We say that the function f is starlike if f is univalent in U and $f(U)$ is a starlike domain with respect to origin.

The notion of starlike function was introduced by J.Alexander [5] in 1915.

Definition 1.2.3.[87] We denote by S^* the class of functions $f \in \mathcal{A}$ which are starlike in the unit disc,

$$S^* = \big\{f \in \mathcal{A} : \mathrm{Re}\frac{zf'(z)}{f(z)} > 0\big\}.$$

We have that $S^* \subset S$.

Definition 1.2.4.[87] We define the class of starlike functions of order α, $\alpha < 1$, by

$$S^*(\alpha) = \big\{f \in \mathcal{A} : \mathrm{Re}\frac{zf'(z)}{f(z)} > \alpha, z \in U\big\}.$$

C. The class of convex functions

The notion of convex function was introduced by E. Study [118] in 1913.

Definition 1.2.5. The function $f \in \mathcal{H}(U)$ is convex in U (or convex) if f is univalent in U and $f(U)$ is an convex domain.

Theorem 1.2.6.[87]*(Analytic characterization of convexity theorem)* Let $f \in \mathcal{H}(U)$. Then f is convex if and only if $f'(0) \neq 0$ and

$$\mathrm{Re}\frac{zf''(z)}{f'(z)} + 1 > 0, \quad z \in U. \tag{1.2}$$

Theorem 1.2.7.[5]*(Alexander duality theorem)* The function f is convex in U if and only if the function $F(z) = zf'(z)$ is starlike in U.

Definition 1.2.8. We denote by K the class of functions $f \in \mathcal{A}$ which are convex and normalized in the unit disc U,

$$K = \big\{f \in \mathcal{A} : \mathrm{Re}\frac{zf''(z)}{f'(z)} + 1 > 0, \quad z \in U\big\}.$$

Definition 1.2.9.[87] We define the class of convex function of order $\alpha, \alpha < 1$, by

$$K(\alpha) = \big\{f \in \mathcal{A} : \mathrm{Re}\frac{zf''(z)}{f'(z)} + 1 > \alpha, z \in U\big\}.$$

We have: $K(\alpha) \subset K$.

D. The class of the functions S_m

Definition 1.2.10.[111] The differential Sălăgean operator $D^m : \mathcal{A} \to \mathcal{A}$, is defined as

$$D^0 f(z) = f(z)$$

$$D^1 f(z) = zf'(z),$$

$$D^m f(z) = D^1(D^{m-1} f(z)), m \in \mathbb{N}^*.$$

Definition 1.2.11.[111] The integral Sălăgean operator $I^m : \mathcal{A} \to \mathcal{A}$, is defined as

$$I^0 f(z) = f(z);$$

$$I^1 f(z) = If(z) = \int_0^z f(t)t^{-1}dt;$$

$$I^m f(z) = I(I^{m-1} f(z)), f \in \mathcal{A}, m \in \mathbb{N}^*.$$

Remark 1.2.12. If $f \in \mathcal{A}, f(z) = z + \sum_{n=2}^{\infty} a_n z^n, z \in U$, then

$$D^m f(z) = z + \sum_{n=2}^{\infty} n^m a_n z^n, z \in U.$$

Definition 1.2.13.[111] We say that the function $f \in \mathcal{A}$ is $m-$ starlike, $m \in \mathbb{N}$, if

$$\mathrm{Re}\frac{D^{m+1} f(z)}{D^m f(z)} \geq 0, \quad z \in U.$$

We denote by S_m the class of this functions.

Remark 1.2.14. We can see that $S_0 = S^*$ and $S_1 = K$.

The class of $m-$ starlike functions was introduced by G.Ş. Sălăgean in [111]. G. Ş. Sălăgean proof that $S_{m+1} \subset S_m \subseteq S_0, m \in \mathbb{N}$, which follows that $S_m \subset S$, where $m \in \mathbb{N}$.

E. The class of K_m functions

Definition 1.2.15. If we have $f, g \in \mathcal{A}$, of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, the product of convolution (or the product Hadamard) of f and g is defined as

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

Definition 1.2.16.[109] We defined the operator $\mathcal{D}^m : \mathcal{A} \to \mathcal{A}, \quad m \in \mathbb{N}$, as

$$\mathcal{D}^m f(z) = \frac{z}{(1-z)^{m+1}} * f(z) = \frac{z(z^{m-1} f(z))^{(m)}}{m!}, \quad z \in U.$$

Definition 1.2.17.[109] We say that the function $f \in \mathcal{A}$ is in the class K_m if

$$\mathrm{Re} \frac{\mathcal{D}^{m+1} f(z)}{\mathcal{D}^m f(z)} > \frac{1}{2}, \quad z \in U.$$

Remark 1.2.18. We see that $K_0 = S^*(1/2)$ and $K_1 = K$.

The class $K_m \subset \mathcal{A}$ was studied by S. Ruscheweyh in [109], which prove in this paper that $K_{m+1} \subset K_m \subseteq K_0, \quad m \in \mathbb{N}$. It follows that $K_m \subset S, \forall m \in \mathbb{N}$.

F. The class of spiralike functions

The class of spiralike functions was introduced by L. Spacek in 1932.

Definition 1.2.19.[87] The region $D \subset \mathbb{C}$, was contains the origin, is an region spirallike of type γ, with $|\gamma| < \pi/2$, if for all point $w_0 \in D \setminus \{0\}$ the arc of the $\gamma-$ scroll associate the point w_0 with the origin is including in D.

Definition 1.2.20.[87] We say that the function $f \in \mathcal{H}(U)$, with $f(0) = 0$, is a spirallike function of type γ in the unit disc U if f is univalent in U and the domain $f(U)$ is spirallike of type γ.

Definition 1.2.21.[87] We say that the function $f \in \mathcal{H}(U)$,with $f(0) = 0$, is spirallike if exist an number γ, with $|\gamma| < \pi/2$, such that f is spirallike of type γ.

Definition 1.2.22.[87] 1. For $\gamma \in (-\pi/2, \pi/2)$, we denote by $\widehat{S}_\gamma$ the class of spirallike functions of type γ and normalized in the unit disc:

$$\widehat{S}_\gamma = \big\{ f \in \mathcal{A} : \mathrm{Re}\big[e^{i\gamma} \frac{z f'(z)}{f(z)} \big] > 0, \quad z \in U \big\}.$$

2. We denote by $\widehat{S}$ the class of spirallike functions and normalized in the unit disc,

$$\widehat{S} = \bigcup_{\gamma \in (-\pi/2, \pi/2)} \widehat{S}_\gamma.$$

Theorem 1.2.23.[87]*(The formula of build for the class $\widehat{S_\gamma}$)* The function $f \in \widehat{S_\gamma}, \gamma \in \big(-\frac{\pi}{2}, \frac{\pi}{2}\big)$ if and only if exists a function $\mu \in M[0, 2\pi]$ such that

$$f(z) = z \exp\big\{-2\cos\gamma e^{-i\gamma} \int_0^{2\pi} \log(1 - ze^{-it}) d\mu(t)\big\}, \quad z \in U, \tag{1.3}$$

where $\log 1 = 0$.

Definition 1.2.24.[87] Let $f \in \mathcal{A}$ and $n \in \mathbb{N}$. We say that f is a $n-$spirallike function of type $\gamma \in (-\pi/2, \pi/2)$ if $D^n f(z) \neq 0, z \in U$ and

$$\mathrm{Re}\big[e^{i\gamma} \frac{D^{n+1} f(z)}{D^n f(z)}\big] > 0, z \in U,$$

where D^n is the differential Sălăgean operator.

We denote this class by $S_{\gamma,n}$.

2 Harmonic functions

This chapter has three paragraphs.

In first paragraph are given results well-known about harmonic functions.

Definition 2.1.1.[31] Let $u : G \to \mathbb{R}$ a function of C^2 class on G. The function u is harmonic on G if $\Delta u \equiv 0$ where

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}. \tag{2.1}$$

Δu is the laplacean of the function u, and the equation $\Delta u = 0$ is the equation of Laplace.

Theorem 2.1.2.[31]*(The principle of the extreme for harmonic functions)* Let $u : G \to \mathbb{R}$ a harmonic function. If $z_0 \in G$ is an point of maxim (or minim) for the function u on G, then u is constant on the conex component of G contained the point z_0.

Theorem 2.1.3.[31]*(The Poison formula)* Let $r > 0$ and $u : \overline{U}(0, r) \to \mathbb{R}$ an harmonic function on $U(0; R)$ and continuous on $\overline{U}(0; r)$. Then

$$u(\rho e^{i\varphi}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{r^2 - \rho^2}{r^2 - 2\rho r \cos(\theta - \varphi) + \rho^2} u(re^{i\Theta}) d\Theta, \tag{2.2}$$

for all $\rho \in [0, r)$ and $\varphi \in \mathbb{R}$.

Corollary 2.1.4.[31] Let be $u : \overline{U}(z_0; R) \to \mathbb{R}$ an harmonic function on $U(z_0; R)$ and continuous on $\overline{U}(z_0; R)$. Then

$$u(z_0 + re^{i\varphi}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr \cos(\Theta - \varphi) + r^2} u(z_0 + Re^{i\Theta}) d\Theta, \tag{2.3}$$

for all $r \in [0, R)$ and $\varphi \in \mathbb{R}$.

In the second paragraph are introduced new classes of harmonic functions defined by the integral operator of Sălăgean.

Let H the class of functions $f = h + \overline{g}$ which are harmonic univalent and sense-preserving in the unit disc $U = \{z : |z| < 1\}$, for which $f = h + \overline{g}$ is normed-space and $f(0) = h(0) = f'_z(0) - 1 = 0$.

Ahuja and Jahangiri defined the class $H_p(n)$ $(p, n \in \mathbb{N})$, which contained the harmonic p-valent functions $f = h + \overline{g}$, sense-preserving in U and h and g are of the form

$$h(z) = z^p + \sum_{k=2}^{\infty} a_{k+p-1} z^{k+p-1}, \qquad g(z) = \sum_{k=1}^{\infty} b_{k+p-1} z^{k+p-1}, \quad |b_p| < 1. \tag{2.4}$$

For $f = h + \overline{g}$ of the form (2.4), the integral operator of Sălăgean, is defined as:

$$I^n f(z) = I^n h(z) + (-1)^n \overline{I^n g(z)}; \quad p > n, \quad z \in U, \tag{2.5}$$

where

$$I^n h(z) = z^p + \sum_{k=2}^{\infty} \big(\frac{p}{k+p-1}\big)^n a_{k+p-1} z^{k+p-1}$$

and

$$I^n g(z)=\sum_{k=1}^{\infty}\big(\frac{p}{k+p-1}\big)^n b_{k+p-1}z^{k+p-1}.$$

Definition 2.2.1.[25] For fixed positive integers n,p, and for $0\leq\alpha<1,\quad \beta\geq 0$ we let $H_p(n+1,n,\alpha,\beta)$ denote the class of multivalent harmonic functions of the form (2.4) that satisfy the condition

$$\operatorname{Re}\{\frac{I^n f(z)}{I^{n+1}f(z)}\}>\beta\big|\frac{I^n f(z)}{I^{n+1}f(z)}-1\big|+\alpha. \tag{2.6}$$

Definiţia 2.2.2.[25] The subclass $H_p^-(n+1,n,\alpha,\beta)$ consists of functions $f_n=h+\overline{g_n}$ in $H_p(n,\alpha,\beta)$ so that h and g are of the form

$$h(z)=z^p-\sum_{k=2}^{\infty}a_{k+p-1}z^{k+p-1},\quad g_n(z)=(-1)^{n-1}\sum_{k=1}^{\infty}b_{k+p-1}z^{k+p-1},\quad |b_p|<1. \tag{2.7}$$

In the following result we introduce a sufficient condition for the coefficient bounded of harmonic functions from $H_p(n+1,n,\alpha,\beta)$.

Theorem 2.2.3.[25] Let $f=h+\overline{g}$ be given by (2.4). If

$$\sum_{k=1}^{\infty}\{\Psi(n+1,n,p,\alpha,\beta)|a_{k+p-1}|+\Theta(n+1,n,p,\alpha,\beta)|b_{k+p-1}|\}\leq 2, \tag{2.8}$$

where

$$\Psi(n+1,n,p,\alpha,\beta)=\frac{\big(\frac{p}{k+p-1}\big)^n(1+\beta)-(\beta+\alpha)\big(\frac{p}{k+p-1}\big)^{n+1}}{1-\alpha},$$

and

$$\Theta(n+1,n,p,\alpha,\beta)=\frac{\big(\frac{p}{k+p-1}\big)^n(1+\beta)+\big(\frac{p}{k+p-1}\big)^{n+1}(\beta+\alpha)}{1-\alpha},$$

$a_p=1,\quad 0\leq\alpha<1,\quad \beta\geq 0,\quad n\in\mathbb{N}$, then $f\in H_p(n+1,n,\alpha,\beta)$.

The next theorem prove that the condition (2.8) is necessary for the function $f_n=h+\overline{g}_n$, where h and g_n are of the form (2.7).

Theorem 2.2.4.[25] Let $f_n=h+\overline{g}_n$ be given by (2.7). Then $f_n\in H_p^-(n+1,n,\alpha,\beta)$ if and only if

$$\sum_{k=1}^{\infty}[\Psi(n+1,n,p,\alpha,\beta)a_{k+p-1}+\Theta(n+1,n,p,\alpha,\beta)b_{k+p-1}]\leq 2, \tag{2.9}$$

$a_p=1, 0\leq\alpha<1, n\in\mathbb{N}$.

The following theorem gives the distortion bounds for functions in $H_p^-(n+1,n,\alpha,\beta)$ which yields a covering results for this class.

Theorem 2.2.5.[25] Let $f_n\in H_p^-(n+1,n,\alpha,\beta)$. For $|z|=r<1$ we have

$$|f_n(z)|\leq(1+b_p)r^p+[\Phi(n+1,n,p,\alpha,\beta)-\Omega(n+1,n,p,\alpha,\beta)b_p]r^{n+1+p}$$

and

$$|f_n(z)|\geq(1-b_p)r^p-\{\Phi(n+1,n,p,\alpha,\beta)-\Omega(n+1,n,p,\alpha,\beta)b_p\}r^{n+p+1}$$

where,

$$\Phi(n+1,n,p,\alpha,\beta) = \frac{1-\alpha}{\left(\frac{p}{p+1}\right)^n(1+\beta) - \left(\frac{p}{p+1}\right)^{n+1}(\beta+\alpha)},$$

$$\Omega(n+1,n,p,\alpha,\beta) = \frac{(1+\beta)+(\alpha+\beta)}{\left(\frac{p}{p+1}\right)^n(1+\beta) - \left(\frac{p}{p+1}\right)^{n+1}(\beta+\alpha)}.$$

The following covering result follows from the left hand inequality in Theorem 2.2.5.

Corollary 2.2.6. Let $f_n \in H_p^-(n+1,n,\alpha,\beta)$, then for $|z| = r < 1$ we have

$$\{w : |w < 1 - b_p - [\Phi(n+1,n,p,\alpha,\beta) - \Omega(n+1,n,p,\alpha,\beta)b_p] \subset f_n(U)\}.$$

Definition 2.2.7.[21] For $0 \le \alpha < 1$, $n \in \mathbb{N}$, $z \in U$, we denoted by $H_p(n,\alpha)$ the family of harmonic function f of the form(2.4) for which

$$\mathrm{Re}\left(\frac{I^n f(z)}{I^{n+1} f(z)}\right) > \alpha. \tag{2.10}$$

Definition 2.2.8.[21] We denote by $H_p^-(n,\alpha)$ the subclass of harmonic functions $f_n = h + \overline{g}_n$ from $H_p(n,\alpha)$ for which h and g_n are of the form

$$h(z) = z^p - \sum_{k=2}^{\infty} a_{k+p-1} z^{k+p-1} \quad \text{şi} \quad g_n(z) = (-1)^{n-1} \sum_{k=1}^{\infty} b_{k+p-1} z^{k+p-1} \tag{2.11}$$

where $a_{k+p-1}, b_{k+p-1} \ge 0$, $|b_p| < 1$.

Definition 2.2.9.[23] For $0 \le \alpha < 1$, $n \in \mathbb{N}$, $z \in U$, $H(n,\alpha)$ is the family of harmonic function f of the form (2.4), with $p = 1$ for which

$$\mathrm{Re}\left\{\frac{I^n f(z)}{I^{n+1} f(z)}\right\} > \alpha. \tag{2.12}$$

Definition 2.2.10.[23] Let we denote $H^-(n,\alpha)$ the subclass of harmonic function $f_n = h + \overline{g}_n$ from $H(n,\alpha)$ for which h and g_n are of the form

$$h(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad g_n(z) = (-1)^{n-1} \sum_{k=1}^{\infty} b_k z^k, \tag{2.13}$$

where $a_k, b_k \ge 0$, $|b_1| < 1$.

In the following result we introduce a sufficient condition for the coefficient bounded of harmonic functions from $H_p(n,\alpha)$.

Theorem 2.2.11.[21] Let $f = h + \overline{g}$ given by (2.4). If

$$\sum_{k=1}^{\infty} \{\psi(n,p,k,\alpha)|a_{k+p-1}| + \theta(n,p,k,\alpha)|b_{k+p-1}|\} \le 2 \tag{2.14}$$

where

$$\psi(n,p,k,\alpha) = \frac{\left(\frac{p}{k+p-1}\right)^n - \alpha\left(\frac{p}{k+p-1}\right)^{n+1}}{1-\alpha}$$

$$\theta(n,p,k,\alpha)=\frac{\left(\frac{p}{k+p-1}\right)^n+\alpha\left(\frac{p}{k+p-1}\right)^{n+1}}{1-\alpha},$$

$$a_p=1,\quad 0\le\alpha<1,\quad n\in\mathbb{N},$$

then f is sense-preserving in U and $f\in H_p(n,\alpha)$.

For $p=1$ in Theorem 2.2.11, we obtain:

Corollary 2.2.12.[23] Let $f=h+\overline{g}$ given by (2.4) with $p=1$. If

$$\sum_{k=1}^{\infty}\{\psi(n,k,\alpha)|a_k|+\theta(n,k,\alpha)|b_k|\}\le 2, \tag{2.15}$$

where

$$\psi(n,k,\alpha)=\frac{(k)^{-n}-\alpha(k)^{-(n+1)}}{1-\alpha}\ \text{şi}\ \theta(n,k,\alpha)=\frac{(k)^{-n}+\alpha(k)^{-(n+1)}}{1-\alpha},$$

$a_1=1$, $0\le\alpha<1$, $n\in\mathbb{N}$. Then f is sense-preserving in U and $f\in H(n,\alpha)$.

The next theorem prove that the condition (2.14) is necessary for the function $f_n=h+\overline{g}_n$, where h and g_n are of the form (2.11).

Theorem 2.2.13.[21] Let $f_n=h+\overline{g}_n$ be given by (2.11). Then $f_n\in H_p^-(n,\alpha)$ if and only if

$$\sum_{k=1}^{\infty}\{\psi(n,p,k,\alpha)a_{k+p-1}+\theta(n,p,k,\alpha)b_{k+p-1}\}\le 2, \tag{2.16}$$

where $a_p=1$, $0\le\alpha<1$, $n\in\mathbb{N}$.

For $p=1$ in Theorem 2.2.13, we obtain:

Corollary 2.2.14.[23] Let $f_n=h+\overline{g}_n$ be given by (2.13). Then $f_n\in H^-(n,\alpha)$ if and only if

$$\sum_{k=1}^{\infty}\{\psi(n,k,\alpha)a_k+\theta(n,k,\alpha)b_k\}\le 2 \tag{2.17}$$

where $a_1=1$, $0\le\alpha<1$, $n\in\mathbb{N}$.

In the following theorem we give the extreme points for the convex bounded hull from $H_p^-(n,\alpha)$, by clco$H_p^-(n,\alpha)$.

Theorem 2.2.15.[21] Let f_n given by (2.11). Then $f_n\in H_p^-(n,\alpha)$ if and only if

$$f_n(z)=\sum_{k=1}^{\infty}[x_{k+p-1}h_{k+p-1}(z)+y_{k+p-1}g_{n_{k+p-1}}(z)],$$

where

$$h_p(z)=z^p,\quad h_{k+p-1}(z)=z^p-\frac{1}{\psi(n,p,k,\alpha)}z^{k+p-1},\quad k=2,3,\dots$$

and

$$g_{n_{k+p-1}}(z)=z^p+(-1)^{n-1}\cdot\frac{1}{\theta(n,p,k,\alpha)}\overline{z}^{k+p-1},\quad k=1,2,3,\dots$$

$$x_{k+p-1}\ge 0,\quad y_{k+p-1}\ge 0,\quad x_p=1-\sum_{k=2}^{\infty}x_{k+p-1}-\sum_{k=1}^{\infty}y_{k+p-1}.$$

In particular, the extreme points for $H_p^-(n,\alpha)$ are $\{h_{k+p-1}\}$ and $\{g_{n_{k+p-1}}\}$.

For $p=1$ in theorem 2.2.15, we obtain

Corollary 2.2.16.[23] Let f_n be given by (2.13). Then $f_n \in H^-(n,\alpha)$ if and only if

$$f_n(z)=\sum_{k=1}^{\infty}[x_k h_k(z)+y_k g_{n_k}(z)],$$

where

$$h(z)=z, \quad h_k(z)=z-\frac{1}{\psi(n,k,\alpha)}z^k, \quad (k=2,3,\dots)$$

and

$$g_{n_k}(z)=z+(-1)^{n-1}\frac{1}{\theta(n,k,\alpha)}\overline{z}^k \quad (k=1,2,3,\dots)$$

$$x_k\geq 0, \quad y_k\geq 0, \quad x_p=1-\sum_{k=2}^{\infty}x_k-\sum_{k=1}^{\infty}y_k.$$

In particular, the extreme points of $H^-(n,\alpha)$ are $\{h_k\}$ and $\{g_{n_k}\}$.

The following theorem gives the distortion bounds for functions in $H_p^-(n,\alpha)$, which yields a covering results for this class.

Theorem 2.2.17.[21] Let $f_n \in H_p^-(n,\alpha)$. Then, for $|z|=r<1$ we have

$$|f_n(z)|\leq (1+b_p)r^p+\{\phi(n,p,k,\alpha)-\Omega(n,p,k,\alpha)b_p\}r^{p+1}$$

and

$$|f_n(z)|\geq (1-b_p)r^p-\{\phi(n,p,k,\alpha)-\Omega(n,p,k,\alpha)b_p\}r^{p+1},$$

where

$$\phi(n,p,k,\alpha)=\frac{1-\alpha}{\left(\dfrac{p}{p+1}\right)^n-\alpha\left(\dfrac{p}{p+1}\right)^{n+1}},$$

$$\Omega(n,p,k,\alpha)=\frac{1+\alpha}{\left(\dfrac{p}{p+1}\right)^n-\alpha\left(\dfrac{p}{p+1}\right)^{n+1}}.$$

For $p=1$ in the theorem 2.2.17, we obtain

Corollary 2.2.18.[23] Let $f_n \in H^-(n,\alpha)$. Then, for $|z|=r<1$ we have

$$|f_n(z)|\leq (1+b_1)r+\{\phi(n,k,\alpha)-\Omega(n,k,\alpha)b_1\}r^{n+1}$$

and

$$|f_n(z)|\geq (1-b_1)r-\{\phi(n,k,\alpha)-\Omega(n,k,\alpha)b_1\}r^{n+1},$$

where

$$\phi(n,k,\alpha)=\frac{1-\alpha}{\left(\dfrac{1}{2}\right)^n-\alpha\left(\dfrac{1}{2}\right)^{n+1}}$$

and

$$\Omega(n,k,\alpha) = \frac{1+\alpha}{\left(\frac{1}{2}\right)^n - \alpha\left(\frac{1}{2}\right)^{n+1}}.$$

The following covering result follows from the left hand inequality in Theorem 2.2.17.

Corollary 2.2.19.[23] Let $f_n \in H_p^-(n,\alpha)$, for $|z| = r < 1$ we have

$$\{w : \; |w| < 1 - b_p - [\phi(n,p,k,\alpha) - \Omega(n,p,k,\alpha)b_p]\} \subset f_b(U).$$

In the second paragraph are introduced new classes of harmonic function defined by generalized Sălăgean operator.

The differential Sălăgean operator was generalized by F. M. Al-Oboudi in [6]. It is defined as: let $f \in \mathcal{A}$ and $m \in \mathbb{N}$, then we consider

$$D_\lambda^0 f(z) = f(z);$$

$$D_\lambda^1 f(z) = (1-\lambda)f(z) + \lambda z f'(z), \quad \lambda > 0;$$

$$D_\lambda^m f(z) = D_\lambda^1(D_\lambda^{m-1} f(z)).$$

Definition 2.3.1.[58] For $0 \le \alpha < 1, k \in \mathbb{N}, \lambda \ge 0$ and $z \in U$, let $H(k,\alpha)$ the family of harmonic functions f for which

$$\mathrm{Re}\left(\frac{D_\lambda^k f(z)}{D_\lambda^{k+1} f(z)}\right) > \alpha. \tag{2.18}$$

Definition 2.3.2[58] We denote by $H^-(k,\alpha)$ the subclass of harmonic functions $f_k = h + \overline{g_k}$ in $H^-(k,\alpha)$ for which h and g_k are of the form

$$h(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad , \quad g_k(z) = (-1)^{k-1}\sum_{n=2}^{\infty} b_n z^n,$$

where $a_n, b_n \ge 0, |b_n| < 1$.

Theorem 2.3.3.[58] Let $f = h + \overline{g}$. If

$$\sum_{n=1}^{\infty}\{\Psi(k,n,\alpha)|a_n| + \Theta(k,n,\alpha)|b_n|\} \le 2, \tag{2.19}$$

where

$$\Psi(k,n,\alpha) = \frac{(1+(n-1)\lambda)^k - \alpha(1+(n-1)\lambda)^{k+1}}{1-\alpha},$$

$$\Theta(k,n,\alpha) = \frac{(1+(n-1)\lambda)^k + \alpha(1+(n-1)\lambda)^{k+1}}{1-\alpha},$$

$$a_1 = 1, \quad 0 \le \alpha \le 1, \quad k \in \mathbb{N},$$

then f is sense-preserving in U and $f \in H(k,\alpha)$.

Theorem 2.3.4.[58] Let $f_n = h + \overline{g_n}$ given by (2.18). Then $f_n \in H^-(k, \alpha)$ if and only if

$$\sum_{n=1}^{\infty} \{\Psi(k, n, \alpha)a_n + \Theta(k, n, \alpha)b_n\} \leq 2, \tag{2.20}$$

where $a_1 = 1, 0 \leq \alpha < 1, k \in \mathbb{N}$.

We give in the following theorem an result of distortion for the functions from the class $H^-(k, \alpha)$.

Theorem 2.3.5.[58] Let $f_n \in H^-(k, \alpha)$. Then, for $|z| = r < 1$ we have

$$|f_k(z)| \leq (1 + b_1)r + [\varphi(k, n, \alpha) - \Omega(k, n, \alpha)b_1]r^2$$

and

$$|f_k(z)| \geq (1 - b_1)r - [\varphi(k, n, \alpha) - \Omega(k, n, \alpha)b_1]r^2$$

where

$$\varphi(k, n, \alpha) = \frac{1 - \alpha}{(1 + \lambda)^k - \alpha(1 + \lambda)^{k+1}}$$

and

$$\Omega(k, n, \alpha) = \frac{1 + \alpha}{(1 + \lambda)^k - \alpha(1 + \lambda)^{k+1}}.$$

The result is sharp for the functions

$$f_k(z) = z + b_1\overline{z} + [\varphi(k, n, \alpha) - \Omega(k, n, \alpha)b_1]\overline{z}^2, \quad 0 \leq b_1 < \frac{1 - \alpha}{1 + \alpha}, \quad z = r$$

$f_k(z) = z - b_1\overline{z} - [\varphi(k, n, \alpha) - \Omega(k, n, \alpha)b_1]\overline{z}^2, \quad \frac{1-\alpha}{1+\alpha} < b_1 < 1, \quad z = r.$

3 Differential subordination and superordinations

This chapter has two paragraphs. First paragraph present the elementary notions about differential subordinations and differential superordinations Briot-Bouquet.

Definition 3.1.1. Let $f, g \in \mathcal{H}(U)$.We say that the function f is subordinated to the function g or g is superordinate to f and we denote

$$f \prec g$$

or

$$f(z) \prec g(z),$$

if exist a function Schwarz $w \in (U)$, with $w(0) = 0$ and $|w(z)| < 1, z \in U$ such that

$$f(z) = g(w(z)), z \in U.$$

Definition 3.1.2. We denote by $\mathcal{Q}$ the class of the functions q which are holomorphic and injective on $\overline{U} \setminus E(q)$, where

$$E(q) = \big\{\zeta \in \partial U : \lim_{z \to \zeta} q(z) = \infty\big\},$$

and$q'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(q)$.

$E(q)$ is called the set of exception.

Lemma 3.1.3.[75] Let the functions $q \in \mathcal{Q}, q(0) = a, p \notin \mathcal{H}[a, n], p(z) \neq a$ and let the number $n \geq 1$. If exists the points $z_0 \in U$ and $\zeta_0 \in \partial U \setminus E(q)$ such that $p(z_0) = q(\zeta_0)$ and $p(U_{r_0}) \subset q(U)$, where $r_0 = |z_0|$, then exists a real number $m, m \geq n$, such that

$$z_0 p'(z_0) = m\zeta_0 q'(\zeta_0)$$

and

$$\mathrm{Re}\frac{z_0 p"(z_0)}{p'(z_0)} + 1 \geq m\mathrm{Re}\big\{\frac{\zeta_0 q"(\zeta_0)}{q'(\zeta_0)} + 1\big\}.$$

We consider the disc $U_M = \{w \in \mathbb{C} : |w| < M\}$ and $q(z) = M \cdot \dfrac{Mz + a}{M + \overline{a}z}$ with $M > 0$ and $|a| < M$, then $q(U) = \Delta, q(0) = a, E(q) = \phi$ and $q \in \mathcal{Q}$.

Definition 3.1.4.[75] Let $\Omega \subset \mathbb{C}$, let the function $q \in \mathcal{Q}$ and $n \in \mathbb{N}, n \geq 1$. We denote by $\Psi_n[\Omega, q]$ the class of the functions $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ which satisfies

$$(1') \quad \psi(r, s, t; z) \notin \Omega$$

when

$$r = q(\zeta), s = m\zeta q'(\zeta), \mathrm{Re}\big[\frac{t}{s} + 1\big] \geq m\mathrm{Re}\big[\frac{\zeta q"(\zeta)}{q'(\zeta)} + 1\big],$$

where $z \in U, \zeta \in \partial U \setminus E(q)$ and $m \geq n.$

The set $\Psi_n[\Omega, q]$ is the class of the admissible functions, and the condition $(1')$ is the admissibility condition.

In the second paragraph are presented differential subordinations and superordinations for analytic functions defined by the integral operator of Sălăgean.
Similar results for differential Sălăgean operator are given in [93], [104].

Theorem 3.2.1.[20] Let q be an univalent function in U with $q(0) = 1$, $\gamma \in \mathbb{C}^*$ such that:

$$\operatorname{Re}\left[1 + \frac{zq''(z)}{q'(z)}\right] > \max\left\{0, -\operatorname{Re}\frac{1}{\gamma}\right\}.$$

If $f \in \mathcal{A}$ and

$$\frac{I^{n+1}f(z)}{I^n f(z)} + \gamma\left\{1 - \frac{I^{n-1}f(z)I^{n+1}f(z)}{[I^n f(z)]^2}\right\} \prec q(z) + \gamma z q'(z), \tag{3.1}$$

then

$$\frac{I^{n+1}f(z)}{I^n f(z)} \prec q(z) \tag{3.2}$$

and q is the best dominant of subordination (3.2).

Theorem 3.2.2.[20] Let q be a convex function in U, with $q(0) = 1$ and $\gamma \in \mathbb{C}$ such that $\operatorname{Re}\gamma > 0$. If $f \in \mathcal{A}$,

$$\frac{I^{n+1}f(z)}{I^n f(z)} \in \mathcal{H}[1,1] \cap Q,$$

$$\frac{I^{n+1}f(z)}{I^n f(z)} + \gamma\left\{1 - \frac{I^{n-1}f(z) \cdot I^{n+1}f(z)}{[I^n f(z)]^2}\right\}$$

is univalent in U and

$$q(z) + \gamma z q'(z) \prec \frac{I^{n+1}f(z)}{I^n f(z)} + \gamma\left\{1 - \frac{I^{n-1}f(z) \cdot I^{n+1}f(z)}{[I^n f(z)]^2}\right\}, \tag{3.3}$$

then

$$q(z) \prec \frac{I^{n+1}f(z)}{I^n f(z)} \tag{3.4}$$

and q is the best subordinant of superordination (3.4).

Now, we give a result of "sandwich" type.

Theorem 3.2.3.[20] Let q_1 and q_2 be convex function in the unit disc U, with $q_1(0) = q_2(0) = 1$, $\gamma \in \mathbb{C}$ such that $\operatorname{Re}\gamma > 0$. If $f \in \mathcal{A}$,

$$\frac{I^{n+1}f(z)}{I^n f(z)} \in \mathcal{H}[1,1] \cap Q, \quad \frac{I^{n+1}f(z)}{I^n f(z)} + \gamma\left\{1 - \frac{I^{n-1}f(z) \cdot I^{n+1}f(z)}{[I^n f(z)]^2}\right\}$$

is univalent in U and

$$q_1(z) + \gamma z q_1'(z) \prec \frac{I^{n+1}f(z)}{I^n f(z)} + \gamma\left\{1 - \frac{I^{n-1}f(z) \cdot I^{n+1}f(z)}{[I^n f(z)]^2}\right\} \prec q_2(z) + \gamma z q_2'(z),$$

then

$$q_1(z) \prec \frac{I^{n+1}f(z)}{I^n f(z)} \prec q_2(z), \tag{3.5}$$

and q_1 and q_2 are the best subordinant and the best dominant respectively of (3.5).

Theorem 3.2.4.[20] Let q be an univalent function in the unit disc U, with $q(0) = 1$, $\gamma \in \mathbb{C}^*$ and suppose that

$$\mathrm{Re}\left[1 + \frac{zq''(z)}{q'(z)}\right] > \max\left\{0, -\mathrm{Re}\frac{1}{\gamma}\right\}.$$

If $f \in \mathcal{A}$ and

$$(1+\gamma)z\frac{I^n f(z)}{[I^{n+1}f(z)]^2} + \gamma z\frac{I^{n-1}f(z)}{[I^{n+1}f(z)]^2} - 2\gamma z\frac{[I^n f(z)]^2}{[I^{n+1}f(z)]^3} \prec q(z) + \gamma z q'(z), \tag{3.6}$$

then

$$z\frac{I^n f(z)}{[I^{n+1}f(z)]^2} \prec q(z) \tag{3.7}$$

and q is the best dominant of subordination (3.7).

Theorem 3.2.5.[20] Let q be a convex function in the unit disc U, $q(0) = 1$, $\gamma \in \mathbb{C}$ such that $\mathrm{Re}\gamma > 0$. If $f \in \mathcal{A}$,

$$z\frac{I^n f(z)}{[I^{n+1}f(z)]^2} \in \mathcal{H}[1,1] \cap Q,$$

$$(1+\gamma)z\frac{I^n f(z)}{[I^{n+1}f(z)]^2} + \gamma z\frac{I^{n-1}f(z)}{[I^{n+1}f(z)]^2} - 2\gamma z\frac{[I^n f(z)]^3}{[I^{n+1}f(z)]^3}$$

is univalent in U and

$$\begin{aligned} q(z) + \gamma z q'(z) \prec (1+\gamma)z&\frac{I^n f(z)}{[I^{n+1}f(z)]^2} \\ +\gamma z\frac{I^{n-1}f(z)}{[I^{n+1}f(z)]^2} &- 2\gamma z\frac{[I^n f(z)]^2}{[I^{n+1}f(z)]^3}, \end{aligned} \tag{3.8}$$

then

$$q(z) \prec z\frac{I^n f(z)}{[I^{n+1}f(z)]^2} \tag{3.9}$$

and q is the best subordinant of superordination (3.9).

Theorem 3.2.6.[20] Let q_1, q_2 be convex function in U, with $q_1(0) = q_2(0) = 1$, $\gamma \in \mathbb{C}$, such that $\mathrm{Re}\gamma > 0$. If $f \in \mathcal{A}$,

$$z\frac{I^n f(z)}{[I^{n+1}f(z)]^2} \in \mathcal{H}[1,1] \cap Q,$$

$$(1+\gamma)z\frac{I^n f(z)}{[I^{n+1}f(z)]^2} + \gamma z\frac{I^{n-1}f(z)}{[I^{n+1}f(z)]^2} - 2\gamma z\frac{[I^n f(z)]^2}{[I^{n+1}f(z)]^3}$$

is univalent in U and

$$\begin{aligned} q_1(z) + \gamma z q_1'(z) \prec (1+\gamma)z\frac{I^n f(z)}{[I^{n+1}f(z)]^2} + \gamma z\frac{I^{n-1}f(z)}{[I^{n+1}f(z)]^2} - 2\gamma z\frac{[I^n f(z)]^2}{[I^{n+1}f(z)]^3} \\ \prec q_2(z) + \gamma z q_2'(z), \end{aligned}$$

then

$$q_1(z) \prec z\frac{I^n f(z)}{[I^{n+1} f(z)]^2} \prec q_2(z) \tag{3.10}$$

and q_1 and q_2 are the best subordinant and the best dominant respectively of (3.10).

Theorem 3.2.7.[22] Let q be an univalent function in U with $q(0) = 1$, $\alpha \in \mathbb{C}^*$, $\delta > 0$ and suppose

$$\mathrm{Re}\left[1 + \frac{zq''(z)}{q'(z)}\right] > \max\left\{0, -\mathrm{Re}\frac{\delta}{\alpha}\right\}.$$

If $f \in \mathcal{A}$ satisfies the subordination

$$(1-\alpha)\left(\frac{I^{n+1} f(z)}{z}\right)^{\delta} + \alpha\left(\frac{I^{n+1} f(z)}{z}\right)^{\delta} \cdot \frac{I^n f(z)}{I^{n+1} f(z)} \prec q(z) + \frac{\alpha}{\delta} zq'(z), \tag{3.11}$$

then

$$\left(\frac{I^{n+1} f(z)}{z}\right)^{\delta} \prec q(z) \tag{3.12}$$

and q is the dominant of (3.12) .

Theorem 3.2.8.[22] Let q be convex in U with $q(0) = 1$, $\alpha \in \mathbb{C}$, Re $\alpha > 0$, $\delta > 0$. If $f \in \mathcal{A}$ such that

$$\left(\frac{I^{n+1} f(z)}{z}\right)^{\delta} \in \mathcal{H}[1,1] \cap Q,$$

$$(1-\alpha)\left(\frac{I^{n+1} f(z)}{z}\right)^{\delta} + \alpha\left(\frac{I^{n+1} f(z)}{z}\right)^{\delta} \cdot \frac{I^n f(z)}{I^{n+1} f(z)}$$

is univalent in U and satisfies the superordination

$$q(z) + \frac{\alpha}{\delta} zq'(z) \prec (1-\alpha)\left(\frac{I^{n+1} f(z)}{z}\right)^{\delta} + \alpha\left(\frac{I^{n+1} f(z)}{z}\right)^{\delta} \cdot \frac{I^n f(z)}{I^{n+1} f(z)}, \tag{3.13}$$

then

$$q(z) \prec \left(\frac{I^{n+1} f(z)}{z}\right)^{\delta} \tag{3.14}$$

and q is the best subordinant of (3.14).

Theorem 3.2.9.[22] Let q_1, q_2 be convex in U with $q_1(0) = q_2(0) = 1$, $\alpha \in \mathbb{C}$, Re $\alpha > 0$, $\delta > 0$. If $f \in \mathcal{A}$ such that

$$\left(\frac{I^{n+1} f(z)}{z}\right)^{\delta} \in \mathcal{H}[1,1] \cap Q$$

$$(1-\alpha)\left(\frac{I^{n+1} f(z)}{z}\right)^{\delta} + \alpha\left(\frac{I^{n+1} f(z)}{z}\right)^{\delta} \cdot \frac{I^n f(z)}{I^{n+1} f(z)}$$

is univalent in U and satisfies

$$q_1(z) + \frac{\alpha}{\delta} zq_1'(z) \prec (1-\alpha)\left(\frac{I^{n+1} f(z)}{z}\right)^{\delta} + \alpha\left(\frac{I^{n+1} f(z)}{z}\right)^{\delta} \cdot \frac{I^n f(z)}{I^{n+1} f(z)}$$

$$\prec q_2(z) + \frac{\alpha}{\delta} z q_2'(z),$$

then

$$q_1(z) \prec \left(\frac{I^{n+1} f(z)}{z} \right)^{\delta} \prec q_2(z) \tag{3.15}$$

and q_1, q_2 are the best subordinant and the best dominant respectively of (3.15).

4 Class of analytic functions defined by operators

This chapter contains two paragraphs.

The first paragraph present properties of analytic functions defined by integral operators Sălăgean.

Let $\mathcal{A}$ the class of functions f normated,

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{4.1}$$

which are analitics in the unit disc U.

Let Ω the class of functions $w(z)$ from U which satisfies the conditions $w(0) = 0$ and $|w(z)| < 1$ for $z \in U$.

Definition 4.1.1.[27] We say that $f(z) \in \mathcal{A}$ is in the class $\mathcal{F}_n(b, M)$ if and only if

$$|\frac{1}{b}(\frac{I^n f(z)}{I^{n+1} f(z)} - 1) + 1 - M| < M, \tag{4.2}$$

where $M > \frac{1}{2}$, $z \in U$ and $b \neq 0$ is a complex number.

We know from [10] that $f(z) \in H_n(b, M)$ if and only if $z \in U$

$$\frac{I^n f(z)}{I^{n+1} f(z)} = \frac{1 + [b(1+m) - m]w(z)}{1 - mw(z)},$$

where $m = 1 - \frac{1}{M}$, $(M > \frac{1}{2})$ şi $w(z) \in \Omega$.

Between the first papers mean to the starlike functions or the convex functions of complex order are: [9], [91], [92].

Theorem 4.1.2.[27] Let be the function $f(z)$ defined by (4.1) . If

$$\sum_{k=2}^{\infty} \{(1 - \frac{1}{k}) + |\frac{b(1+m)}{k} + m(1 - \frac{1}{k})|\} \frac{|a_k|}{k^n} \leq |b(1+m)|, \tag{4.3}$$

then $f(z)$ is in the class $\mathcal{F}_n(b, M)$, where $m = 1 - \frac{1}{M} \quad (M > \frac{1}{2})$.

Theorem 4.1.3.[27] Let be the function $f(z)$ defined by (4.1) in the class $\mathcal{F}_n(b, M)$, $z \in U$.

a).For

$$2m(1 - \frac{1}{k})\mathrm{Re}\{b\} > (1 - \frac{1}{k})^2(1 - m) - |b|^2(1 + m),$$

let

$$N = [\frac{2m(1 - \frac{1}{k})\mathrm{Re}\{b\}}{(1 - \frac{1}{k})^2(1 - m) - |b|^2(1 + m)}], \quad k = 1, 2, 3, ..., j - 1.$$

Then

$$|a_j| \leq \frac{1}{\frac{1}{j^n}(1 - \frac{1}{j})!} \prod_{k=2}^{j} |\frac{b(1+m)}{k} + (\frac{k-2}{k})m|, \tag{4.4}$$

for $j = 2, 3, ..., N + 2$; and

$$|a_j| \leq \frac{1}{\frac{1}{j^n}(1 - \frac{1}{j})(N + 1)!} \prod_{k=2}^{N+3} |\frac{b(1+m)}{k} + (\frac{k-2}{k})m|, \tag{4.5}$$

for $j > N+2$.

b). If

$$2m(1-\frac{1}{k})\text{Re}\{b\} \leq (1-\frac{1}{k})^2(1-m) - |b|^2(1+m),$$

then

$$|a_j| \leq \frac{(1+m)|b|}{\frac{1}{j^n}(1-\frac{1}{j})}, \quad for j \geq 2, \tag{4.6}$$

where $m = 1 - \frac{1}{M} \quad (M > \frac{1}{2})$ and $b \neq 0$ complex number.

Theorem 4.1.4.[27] If a function $f(z)$ defined by (4.1) is in the class $\mathcal{F}_n(b, M)$ and μ is an complex number, then

$$|a_3 - \mu a_2^2| \leq \frac{3^{n+1}}{2}|b(1+m)| max\{1, |d|\} \tag{4.7}$$

where

$$d = \frac{b(1+m)}{2 \cdot 3^{n+1}}[2^{2n+4}\mu - 3^{n+1}] - \frac{m}{2}. \tag{4.8}$$

The result is sharp.

Theorem 4.1.5.[26] If $f \in A$ satisfies

$$\left|\frac{I^n f(z)}{I^{n+1}f(z)} - 1\right|^\alpha \left|z\left(\frac{I^n f(z)}{I^{n+1}f(z)}\right)'\right|^\beta < (1/2)^\beta, \quad (z \in U) \tag{4.9}$$

for all real α and β with $\alpha + 2\beta \geq 0$ şi for all $n \in \mathbb{N}$, then

$$\text{Re}\left(\frac{I^n f(z)}{I^{n+1}f(z)}\right) > 0 \quad (z \in U).$$

Theorem 4.1.6.[26] If $f \in A$ satisfies

$$\left|\frac{I^n f(z)}{I^{n+1}f(z)} - 1\right|^\alpha \left|z\left(\frac{I^n f(z)}{I^{n+1}f(z)}\right)'\right|^\beta < (1/2)^\beta(1-\gamma)^{\alpha+\beta} \quad (z \in U), \tag{4.10}$$

for real α, β, γ and $n \in \mathbb{N}$ with $\alpha + 2\beta \geq 0$ and $0 \leq \gamma < 1$, then

$$\text{Re}\left(\frac{I^n f(z)}{I^{n+1}f(z)}\right) > \gamma \quad (z \in U).$$

Theorem 4.1.7.[26] If $f \in A$ satisfies

$$\left|\frac{I^n f(z)}{I^{n+1}f(z)} - 1\right|^\alpha \left|z\left(\frac{I^n f(z)}{I^{n+1}f(z)}\right)'\right|^\beta < (\gamma/2)^\beta \quad (z \in U) \tag{4.11}$$

for real α, β and $\gamma = \beta/\alpha + \beta$, then

$$\text{Re}\left(\frac{I^n f(z)}{I^{n+1}f(z)}\right)^{1/\gamma} > 0 \quad (z \in U).$$

Definition 4.1.8.[24] We denote by $\mathcal{F}^b_{n+1,n}(A,B)$ the class of functions $f(z)$ in $\mathcal{A}$ which satisfies

$$1+\frac{1}{b}\left(\frac{I^n f(z)}{I^{n+1}f(z)}-1\right)\prec\frac{1+Az}{1+Bz} \quad ,z\in\mathbb{U} \tag{4.12}$$

$b\neq 0$ is an complex number, A and B are real fixed number, $-1\leq B<A\leq 1$, $n\in\mathbb{N}_0$.

We denote by Ω_1 the class of analytic bounded functions $w(z)$ in $\mathbb{U}$ which satisfies the conditions $w(0)=0$ and $|w(z)|<1$, for $z\in\mathbb{U}$.

Theorem 4.1.9.[24] Let be the function $f(z)$ defined by the relation (4.1) in the class $\mathcal{F}^b_{n+1,n}(A,B)$, $z\in\mathbb{U}$ and let be

$$G=\frac{(A-B)^2|b|^2}{(1-\frac{1}{k})\{\frac{2B(A-B)Re\{b\}}{k}+(1-B^2)(1-\frac{1}{k})\}} \quad ,k=2,3,...,n-1$$

$M=[G]$(Gauss symbol), and $[G]$ the integer part of G.

(a) If

$$(A-B)^2|b|^2>(1-\frac{1}{k})\{\frac{2B(A-B)\mathrm{Re}\{b\}}{k}+(1-B^2)(1-\frac{1}{k})\},$$

then

$$|a_j|\leq j^n\frac{\prod_{k=2}^{j}|\frac{(A-B)b}{k}-B[(1-\frac{1}{k})-\frac{1}{k}]|}{\prod_{k=2}^{j}(1-\frac{1}{k})}, for \quad j=2,3,...,M+2 \tag{4.13}$$

and

$$|a_j|\leq\frac{j^n\prod_{k=2}^{M+3}|\frac{(A-B)b}{k}-B[(1-\frac{1}{k})-\frac{1}{k}]|}{(1-\frac{1}{j})\prod_{k=2}^{M+3}(1-\frac{1}{k})}, j>M+2. \tag{4.14}$$

(b) If

$$(A-B)^2|b|^2\leq(1-\frac{1}{k})\{\frac{2B(A-B)\mathrm{Re}\{b\}}{k}+(1-B^2)(1-\frac{1}{k})\},$$

then

$$|a_j|\leq\frac{j^n(A-B)|b|}{(1-\frac{1}{j})}, j\geq 2. \tag{4.15}$$

Theorem 4.1.10.[24] Let be the function $f(z)$ defined by (4.1). If

$$\sum_{k=2}^{\infty}\{(1-\frac{1}{k})+|\frac{(A-B)b}{k}-B(1-\frac{1}{k})|\}\frac{|a_k|}{k^n}\leq(A-B)|b|, \tag{4.16}$$

then $f(z)$ is in the class $\mathcal{F}^b_{n+1,n}(A,B)$.

We present a new subclass of analytic functions with negative coefficients in the unit disc U, used the Ruscheweyh operator.

Let be $T(n)$ the class of functions

$$f(z)=z-\sum_{k=n+1}^{\infty}a_k z^k, \quad (a_k\geq 0, \quad n\in\mathbb{N}) \tag{4.17}$$

which are analytic in the unit disc U.

Definition 4.2.1. A function $f(z) \in T(1)$ is in the class of starlike functions of order α, $T^*(\alpha)$ if

$$\mathrm{Re}\{\frac{zf'(z)}{f(z)}\} > \alpha, \quad (z \in U, \quad 0 \leq \alpha < 1).$$

Definition 4.2.2.[110] The derived Ruscheweyh of order β denoted by $D^\beta f(z)$ of the function $f(z)$ from $T(n)$ is defined as

$$D^\beta f(z) = \frac{z}{(1-z)^{1+\beta}} \circledast f(z) = z - \sum_{k=n+1}^{\infty} a_k B_k(\beta) z^k,$$

where

$$B_k(\beta) = \frac{(\beta+1)(\beta+2)\cdot ... \cdot(\beta+k-1)}{(k-1)!}.$$

Definition 4.2.3.[59] We say that a function $f \in T(n)$ is in the class $\mathcal{J}_n(\beta, \lambda, \mu; A, B)$ if satisfies

$$\frac{z(D^\beta f(z))' + \lambda z^2 (D^\beta f(z))''}{(1-\mu)f(z) + \mu z(D^\beta f(z))' + (\lambda-\mu)z^2(D^\beta f(z))''} \prec \frac{1+Az}{1+Bz}, \tag{4.18}$$

$$(-1 \leq A < B \leq 1, \quad 0 \leq B \leq 1, \quad 0 \leq \mu \leq 1, \quad \mu \leq \lambda \quad \text{şi} \quad \beta > -1).$$

In particular, $\mathcal{J}_1(0,0,0;-(1-2\alpha),1) \equiv T^*(\alpha)$ and $\mathcal{J}_1(0,1,1;-(1-2\alpha),1) \equiv C(\alpha)$, class which are studied by Silverman in [112]. The class $\mathcal{J}_n(0,\lambda,\lambda;-(1-2\alpha),1)$ was studied by Altintas în [7], and class $\mathcal{J}_1(0,0,0;A,B)$ and $\mathcal{J}_1(0,1,1;A,B)$ are studied by Padmanabhan and Ganesan [95].

Theorem 4.2.4.[59] A function $f(z) \in T(n)$ given by (4.17) is in the class $\mathcal{J}_n(\beta, \lambda, \mu; A, B)$ if and only if

$$\sum_{k=n+1}^{\infty} [(k-1)[k(\mu(1+A)+\lambda(B-A))+(1-\mu)] - A(1-\mu) + k(B-A\mu)]B_k(\beta)a_k \leq B-A, \tag{4.19}$$

$$(-1 \leq A < B \leq 1, \quad 0 \leq B \leq 1, \quad 0 \leq \mu \leq 1, \quad \mu \leq \lambda, \quad \beta > -1).$$

The result is sharp for the function $f(z)$ give by

$$f(z) = z - \frac{B-A}{\{n[(n+1)(\mu(1+A)+\lambda(B-A))+(1-\mu)] - A(1-\mu) + (n+1)(B-A\mu)\}B_k(\beta)} z^{n+1}, \tag{4.20}$$

$n \in \mathbb{N}$.

Corollary 4.2.5.[59] Let be $f(z)$ defined by (4.17) from the class $\mathcal{J}_n(\beta, \lambda, \mu; A, B)$. Then

$$a_k \leq \frac{B-A}{\{(k-1)[k(\mu(1+A)+\lambda(B-A))+(1-\mu)] + k(B-A\mu) - A(1-\mu)\}B_k(\beta)},$$

$(k = n+1, n+2, ..., \quad n \in \mathbb{N})$.

Theorem 4.2.6.[59] If $f \in \mathcal{J}_n(\beta, \lambda, \mu; A, B)$, then

$$r - \frac{(B-A)}{n[(n+1)(\mu(1+A)+\lambda(B-A))+(1-\mu)] + (n+1)(B-A\mu) - A(1-\mu)} r^{n+1} \leq$$

$$\leq |D^\beta f(z)| \leq$$

(4.21) $$r+\frac{B-A}{n[(n+1)(\mu(1+A)+\lambda(B-A))+(1-\mu)]+(n+1)(B-A\mu)-A(1-\mu)}r^{n+1},$$

$(|z|=r<1)$.

Theorem 4.2.7.[59] If $f\in\mathcal{J}_n(\beta,\lambda,\mu;A,B)$, then $f\in T^*(\delta)$, where

$\delta=1-$

$$-\frac{B-A}{(n[(n+1)(\mu(1+A)+\lambda(B-A))+(1-\mu)]+(n+1)(B-A\mu)-A(1-\mu))B_{n+1}(\beta)}.$$

Theorem 4.2.8.[59] Let be the function $f_n(z)=z$ and

$$f_k(z)=$$

$$=z-\frac{B-A}{\{(k-1)[k(\mu(1+A)+\lambda(B-A))+(1-\mu)]+k(B-A\mu)-A(1-\mu)\}B_k(\beta)}z^k,$$

$k\geq n+1,\quad n\in\mathbb{N},\quad -1\leq A<B<1,\quad 0\leq B\leq 1,\quad 0\leq\mu\leq 1,\quad \mu\leq\lambda,\quad \beta>-1$. Then $f(z)\in\mathcal{J}_n(\beta,\lambda,\mu;A,B)$ if and only if it can be write

(4.22) $$f(z)=\sum_{k=n+1}^{\infty}\eta_k f_k(z),$$

where $\eta_k\geq 0, k\geq n$ and $\sum\limits_{k=n}^{\infty}\eta_k=1$.

Corollary 4.2.9.[59] The extreme points for the class of functions $f\in\mathcal{J}_n(\beta,\lambda,\mu;A,B)$ are the functions $f_n(z)=z$ and

$f_k(z)=$

$$z-\frac{B-A}{\{(k-1)[k(\mu(1+A)+\lambda(B-A))+(1-\mu)]+k(B-A\mu)-A(1-\mu)\}B_k(\beta)}z^k,$$

$(k\geq n+1,\quad n\in\mathbb{N})$.

Theorem 4.2.10.[59] For all $i=1,...,m$, let $f_i(z)$ defined by

$$f_i(z)=z-\sum_{k=n+1}^{\infty}a_{k,i}z^k\quad(a_{k,i}\geq 0, i=1,...,m,n\in\mathbb{N})$$

in the class $\mathcal{J}_n(\beta,\lambda,\mu;A,B)$. Then the function $h(z)$ defined by

$$h(z)=\sum_{i=1}^{m}t_if_i(z),\quad(t_i\geq 0,(i=1,...,m);\quad\sum_{i=1}^{m}t_i=1)$$

is in the class $\mathcal{J}_n(\beta,\lambda,\mu;A,B)$.

Theorem 4.2.11.[59] Let be the function $0\leq\mu\leq 1,\mu\leq\lambda,\beta>-1,-1\leq A<B\leq 1,\quad 0\leq B\leq 1$. Then

$\mathcal{J}_n(\beta,\lambda,\mu;A,B)\subseteq\mathcal{J}_n(\beta,0,0;A_1,B_1)$, where $A_1\leq 1-2m, B_1\geq\frac{A_1+m}{1-m}$ and

$m=$

(4.23) $$\frac{n(B-A)}{\{n[(n+1)(\mu(1+A)+\lambda(B-A))+(1-\mu)]+(n+1)(B-A\mu)-A(1-\mu)\}B_{n+1}(\beta)}.$$

Theorem 4.2.12.[59] Let be $0 \leq \mu_1 \leq 1, 0 \leq \mu_2 \leq 1, \mu_1 \leq \mu_2 \leq \lambda_2 \leq \lambda_1, \beta > -1, n \in \mathbb{N}$. Then $\mathcal{J}_n(\beta, \lambda_1, \mu_1; A, B) \subseteq \mathcal{J}_n(\beta, \lambda_2, \mu_2; A, B)$.

Theorem 4.2.13.[59] Let be the function $f(z) \in \mathcal{J}_n(\beta, \lambda, \mu; A, B)$. Then the operator Jung-Kim-Srivastava

$$I^\sigma f(z) = z - \sum_{k=n+1}^{\infty} \big(\frac{2}{n+1}\big)^\sigma a_k z^k, \quad \sigma > 0$$

is in the class $\mathcal{J}_n(\beta, \lambda, \mu; A, B)$.

5 Generalized almost starlike functions

In this chapter, we introduce the notion of generalized almost starlikeness on the unit disc as well as on the unit ball B^n in $\mathbb{C}^n$, and we prove that this notion can be characterized in terms of Loewner chains. Finally, we use the theory of Loewner chains to deduce that certain classes of generalized Roper-Suffridge extension operators preserve generalized almost starlikeness.

In the geometric function theory of one complex variable , Loewner chains and the Loewner differential equation serve as a powerful tool in the study of univalent functions.The Loewner differential equation was first established by Loewner [72] and by Kufarev [64].Pfaltzgraff [96] generalized Loewner chains to higher dimensions.Later contributions permitting generalizations to the unit ball of a complex Banach space by Poreda [101]. Some best-possible results concerning the existence and regularity theory of the Loewner equation in several complex variables were obtained by I. Graham, H. Hamada and G. Kohr [43], I. Graham, G. Kohr and M. Kohr [40], [41],and I. Graham and G. Kohr [39].

Definition 5.1.1. A mapping $f : B^n \times [0, \infty) \to \mathbb{C}^n$ is called a Loewner chain if it satisfies the following conditions:

(i)$f(\cdot, t)$ is holomorphic and univalent on B^n, $f(0, t) = 0$ and $Df(0, t) = e^t I$ for each $t \geq 0$;

(ii)$f(\cdot, s) \prec f(\cdot, t)$ whenever $0 \leq s \leq t < \infty$ and $z \in B^n$.

The subordination condition (ii) implies that there is a unique univalent Schwarz mapping $v = v(z, s, t)$, called the transition mapping associated to $f(z, t)$, such that

$$f(z, s) = f(v(z, s, t), t), 0 \leq s \leq t < \infty, z \in B^n.$$

Further, the normalization of $f(z, t)$ implies the normalization

$$Dv(0, s, t) = e^{s-t} I, 0 \leq s \leq t < \infty,$$

for the transition mapping.

A key role in our discussion is played by the Caratheodory sets:

$$\mathcal{P} = \{p \in H(U) : p(0) = 1, \Re p(z) > 0, z \in U\}$$

$$\mathcal{M} = \{h \in H(B^n) : h(0) = 0, Dh(0) = I, \Re\langle h(z), z\rangle > 0, z \in B^n\}.$$

In the case $n = 1$, f is in the set $\mathcal{M}$ if and only if $\frac{f(z)}{z}$ is in the set $\mathcal{P}$.

We introduce the notion of generalized almost starlikeness, prove a characterization of this notion in terms of Loewner chains, and give an result for the compactness of the class of generalized almost starlikeness mappings.

Definition 5.1.2.[28] Let be $a : [0, \infty) \to \mathbb{C}$ be of class C^∞ with $\eta \leq \Re a(t) \leq 0, t \in [0, \infty), \eta < 0$. A normalized locally biholomorphic mapping $f : B^n \to \mathbb{C}^n$ is said to be *generalized almost starlike* if

$$\Re[(1 - a'(t))e^{-a(t)}\langle [Df(e^{a(t)}z)]^{-1} f(e^{a(t)}z), z\rangle] \geq -\Re a'(t)\|z\|^2, \tag{5.1}$$

$z \in B^n, t \in [0, \infty)$.

It is easy to see that in the case of one variable, the above relation becomes

$$\Re[(1 - a'(t))\frac{f(e^{a(t)}z)}{e^{a(t)}zf'(e^{a(t)}z)}] \geq -\Re a'(t), z \in U, t \geq 0 \tag{5.2}$$

Remark 5.1.3. If $a'(t) = \lambda, t \in [0, \infty)$,(in Definition 5.1.16.) where $\lambda \in \mathbb{C}$, Re $\lambda \leq 0$, one obtains the notion of almost starlikeness of complex order λ. This notion has been recently introduced by M. Balaeti and V. Nechita [14]. On the other hand, if $a'(t) = \alpha/\alpha - 1, t \in [0, \infty)$, where $\alpha \in [0, 1)$, we obtain the notion of almost starlikeness of order α due to Feng [33].

Also, if $a'(t) = -1$ in Definition 5.1.2, we obtain the notion of almost starlikeness of order $1/2$.

The following result provides a necessary and sufficient condition for generalized almost starlike on U in terms of Loewner chains.

Theorem 5.1.4.[28] Let be $f : U \to \mathbb{C}$ be a normalized holomorphic function and let $a : [0, \infty) \to \mathbb{C}$ be a function of class C^∞, such that Re $a(t) \leq 0, t \in [0, \infty)$. Assume that there exists $\mu < 0$ such that Re $a(t) \geq \mu, t \geq 0$. Then f is a generalized almost starlike mapping if and only if

$$g(z, t) = e^{t-a(t)}f(e^{a(t)}z), \quad z \in U, t \geq 0$$

is a Loewner chain. In particular, f is a starlike function (i.e., a(t)=0) if and only if $g(z, t) = e^t f(z)$ is a Loewner chain.

From Theorem 5.1.18 and the well known growth theorem for the class S (see [39], [100]) we obtain the next corollary.

Corollary 5.1.5.[28] Let be $f(z)$ be a generalized almost starlike function . Then

$$\frac{|z|}{(1 + |z|)^2} \leq |e^{-a(t)}f(e^{a(t)}z)| \leq \frac{|z|}{(1 - |z|)^2}, \quad z \in U, t \geq 0.$$

The following result proves the compactness of $S_g^*(B^n)$.

Theorem 5.1.6.[28] The set $S_g^*(B^n)$ is a compact set.

Definition 5.1.7. Let be $f \in H(B^n)$ be a normalized mapping. We say that f has parametric representation if there exists a Loewner chain $f(z, t)$ such that $\{e^{-t}f(\cdot, t)\}_{t \geq 0}$ is a normal family on B^n and $f = f(\cdot, 0)$.

Let $S^0(B^n)$ be the set of mappings which have parametric representation on B^n.

Various properties of the Pfaltzgraff-Suffridge and Roper-Suffridge operators may be found in [40], [97], [122], respectively.

Theorem 5.1.8.[29] Assume that f is a generalized almost starlike mapping. Then $F = \Phi_n(f)$ is also a generalized almost starlike mapping.

Theorem 5.1.9.[29] The set $\Phi_n[S_g^*(B^n)]$ is compact.

Bibliografie

[1] M. Acu, *Some preserving properties of the generalized Alexander operator*, General Mathematics, Vol. 10, No. 3-4(2002), 37-46.

[2] L.V. Ahlfors, *Conformal Invariants. Topics in Geometric Function Theory*, McGraw-Hill, New York, 1973.

[3] O.P. Ahuja, J.M. Jahangiri, *Multivalent harmonic starlike functions*, Ann. Univ. Marie Curie-Sklodowska Sect. A, LV 1(2001), 1-13.

[4] H.S. Al-Amiri, *On Ruscheweyh derivatives*, Ann. Polon. Math. **38**, No 1(1980), 88-94.

[5] J.W. Alexander, *Function which map the interior of the unit circle upon simple regions*, Ann. of Math. **17** (1915), 12-22.

[6] F. M. Al-Oboudi, *On univalent functions defined by generalized Sălăgean operator*, JMMS, **27**(2004), 1429-1436.

[7] O. Altintas, *On a subclasses of certain starlike functions with negative coefficients*, Math. Japan, **36**(1991), 489-495.

[8] O. Altintas, O. Ozkan, H. M. Srivastava, *Majorization by starlike functions of complex order*, Complex Variables Theory Appl., **46**(2001), 207-218.

[9] M. K. Aouf, *A generalization of starlike functions of complex order*, Huston J. of Math., **122**(1986), 155-162.

[10] M. K. Aouf, A. M. Al-Amiri, *On certain fractional operators for certain subclass of prestarlike functions defined by Sălăgean operator*, J. Fract. Calc., **22**(2002), 47-56.

[11] M. K. Aouf, H. E. Darwish, A. A. Attiya, *On a class of certain analytic functions of complex order*, Indian J.Pure Appl. Math., **32**(2001), 1443-1452.

[12] M.K. Aouf, S. Owa, M. Obradovic, *Certain classes of analytic functions of complex order and type beta*, Rend. Mat., **11**(1991), 691-714.

[13] Y. Avci, E. Zlotkiewicz, *On harmonic univalent mappings*, Ann. Univ. Marie Crie-Sklodowska, Sect. A., **44**(1991).

[14] M. Balaeti, V. Nechita, *Loewner chains and almost starlike mappings of complex order λ*, Carpathian Journal of Mathematics, will be appear.

[15] L. Bieberbach, *Über die Koeffizienten derjenigen Potenzreihen, welche eine schlichte Abbildung des Einheitskreises vermitteln*, Preuss. Akad. Wiss. Sitzungsb., (1916),940-955.

[16] L.D. Branges, *A proof of the Bieberbach conjecture*, Acta Math. **154** (1985), 137-152.

[17] T. Bulboacă, *Classes of first order differential superordinations*, Demonstr. Math., **35**(2002), No. 2, 287-292.

[18] T. Bulboacă, *Differential Subordinations and Superordinations*, Casa Cărţii de Ştiinţă, Cluj-Napoca, 2005.

[19] J. Clunie, T. Scheil- Small, *Harmonic univalent functions*, Ann. Acad. Sci. Fenn. Ser. A. I. Math., **9**(1984), 3-25.

[20] **L. I. Cotîrlă**, *Differential subordination and superordination for analytic functions defined by integral operator*, Carpathian Journal of Mathematics, Vol. 25, No. **1**/(2009), pag. 49-54.

[21] **L. I. Cotîrlă**, *Harmonic multivalent functions defined by integral operator*, Studia Univ. Babeş-Bolyai, Mathematica, Vol. 54, No. **1**(2009), pag. 65-74.

[22] **L. I. Cotîrlă**, *A differential sandwich theorem for analytic functions defined by the integral operator*, Studia Univ. Babeş-Bolyai, Mathematica, Vol. 54, No. **2**(2009), pag. 13-22.

[23] **L. I. Cotîrlă**, *Harmonic univalent functions defined by an integral operator*, Acta Universitatis Apulensis, **17**(2009), pag. 95-106, ISSN. 1582-5329.

[24] **L. I. Cotîrlă**, *On a generalization class of bounded starlike functions of complex order*, Analele Universităţii de Vest, Timişoara, will be appear.

[25] **L. I. Cotîrlă**, *A new class of harmonic multivalent functions defined by an integral operator*, Acta Universitatis Apulensis, No. **21**(2010), pag. 55-63, ISSN. 1582-5329.

[26] **L. I. Cotîrlă**, *Properties of analytic functions defined by an integral operator*, Demonstratio Mathematica, will be appear.

[27] **L. I. Cotîrlă**, *Some properties of a new class of certain analytic functions of complex order*, Studia Univ. Babeş-Bolyai, Mathematica, will be appear in Vol. 54, No. **3**(2010).

[28] **L. I. Cotîrlă**, *Loewner chains and generalized almost starlike mappings*, Mathematica, will be appear.

[29] **L. I. Cotîrlă**, *Generalized almost starlikeness asociated with extension operators for biholomorphic mappings*, Mathematica, will be appear.

[30] P.L. Duren, *Univalent Functions*, Springer, New York, 1983.

[31] P.L. Duren, *Harmonic Mappings in the Plane*, Cambridge University Press, 2004.

[32] P.J. Eenigenburg, S.S. Miller, P.T. Mocanu, M.O. Reade, *On a Briot-Bouquet differential subordination*, General Inequalities, 3, International Series of Numerical Mathematics, Vol. **64** Birkhauser Verlag, Basel (1983), 339-348.

[33] S.X. Feng, *Some classes of holomorphic mappings in several complex variables*, University of Science and Technology of China, Doctor thesis, 2004 .

[34] G.M. Goluzin, *On the majorization principle in function theory*, Dokl. Akad., Nauk. SSSR **42** (1935), 647-649.

[35] G.M. Goluzin, *Geometric Theory of Functions of a Complex Variable*, A.M.S. Transl. of Math. Monographs, vol. **26** (1969).

[36] S. Gong, T.S. Liu, *On the Roper-Suffridge extension operator*, J.Anal Math, **88**(2002), 397-404.

[37] S. Gong, T.S. Liu, *The generalized Roper-Suffridge extension operator*, J.Math.Anal.Appl., **284**(2003), 425-434.

[38] A.W. Goodman, *Univalent Functions*, Mariner Publ. Comp., Tampa, Florida, 1983.

[39] I. Graham, G.Kohr, *Geometric function theory in one and higher dimension*, Marcel Dekker, New York , 2003.

[40] I. Graham, G. Kohr, M. Kohr, *Loewner chains and the Roper-Suffridge extension operator*, J.Math.Anal.Appl., **247**(2000), 448-465.

[41] I. Graham, G. Kohr, M. Kohr, *Loewner chains and parametric representation in several complex variables*, J.Math.Anal.Appl., **281**(2003), 425-438.

[42] I. Graham, G. Kohr, J.A. Pfaltzgraff, *Parametric representation and linear functionals associated with extension operators for biholomorphic mappings*, Rev. Roumaine Math. Pure Appl. 52(**1**)(2007), 47-68.

[43] I. Graham, H. Hamada, G. Kohr, *Parametric representation of univalent mappings in several complex variables*, Canad. J.Math., **54**(2002), 324-351.

[44] I. Graham, H. Hamada, G. Kohr, T. Suffridge, *Extension operators for locally univalent mappings*, Michigan Math.J., **50**(2002), 37-55.

[45] I. Graham, G. Kohr, *Univalent mappings associated with the Roper-Suffridge extension operator*, J.Analyse Math., **81**(2000), 331-342.

[46] I. Graham, G. Kohr, *An extension theorem and subclasses of univalent mappings in several complex variables*, Complex Variables, **47**(2002), 59-72.

[47] T. Gronwall, *Some remarks on conformal representation*, Ann. of Math. (2) **16** (1914), 72-76.

[48] D.J. Hallenberck, T.H. MacGregor, *Linear Problems and Convexity Techniques in Geometric Function Theory*, Pitman Adv. Publ. Program, Boston-London-Melbourne, 1984.

[49] D.J. Hallenbeck, St. Ruscheweyh, *Subordination by convex functions*, Proc. Amer. Math. Soc. **52** (1975), 191-195.

[50] H. Hamada, G. Kohr, *Subordination chains and the growth theorem of spirallike mappings*, Mathematica(Cluj), 42(**65**)(2000),153-161.

[51] P. Hamburg, P.T. Mocanu, N. Negoescu, *Analiză Matematică (Funţii complexe)*, Ed. Did. şi Ped., Bucureşti, 1982.

[52] E. Hille, *Ordinary Differential Equations in the Complex Plane*, J. Wiley and Sons, New York, 1976.

[53] I.S. Jack, *Functions starlike and convex of order* α, J. London Math. Soc. **3** (1971), 469-474.

[54] J. M. Jahangiri, *Harmonic functions starlike in the unit disc*, J. Math. Anal. Appl., **235**(1999).

[55] J.M. Jahangiri, O.P. Ahuja, *Multivalent harmonic starlike functions*, Ann. Univ. Marie Curie-Sklodowska, Sect. A., **LVI**(2001), 1-13.

[56] J. M. Jahangiri, G. Murugusundaramoorthy and K. Vijaya, *Sălăgean harmonic univalent functions*, South. J. Pure Appl. Math. , **2**(2002), 77-82.

[57] I. B. Jung, Y. C. Kim and H. M. Srivastava, *The Hardy space of analytic functions associated with certain one-parameter families of integral operators*, J. Math. Anal. Appl., **176**(1993), 138-147.

[58] **A.R. Juma, L. I. Cotîrlă**, *On Harmonic univalent function defined by generalized Sălăgean derivatives*, Acta Universitatis Apulensis, will be appear in no. 23/2010.

[59] **A.R. Juma, L. I. Cotîrlă**, *On a subclass of analytic functions defined by Ruscheweyh operator*, submitted for publication.

[60] I. B. Jung, Y. C. Kim and H. M. Srivastava, *The Hardy space of analytic functions associated with certain one-parameter families of integral operators*, J. Math. Anal. Appl., **176**(1993), 138-147.

[61] P. Koebe, *Über die Uniformisierung beliebiger analytischer Kurven*, Nachr. Kgl. Ges. Wiss. Göttingen Math. Phys. (1907), 191-210.

[62] G. Kohr, *On some sufficient conditions of almost starlikeness of order* $1/2$ *in* C^n, Studia Univ.Babes-Bolyai, Mathematica, 41,(**3**)(1996), 51-55.

[63] G. Kohr, P. Liczberski, *Univalent mappings and several complex variables*, Cluj University Press, Cluj Napoca, Romania.

[64] P.P. Kufarev, *A remark on integrals of the Loewner equation*, Dokl.Akad.Nauk SSSR, (**57**)(1947), 655-656.

[65] O. S. Kwon, N. E. Cho, *Generalization of Hadamard products of certain analytic functions*, Math. Sci., Res., Hot-Line, 2(**3**)(1998), 1-8.

[66] E. Lindelöf, *Mémoire sur certaines inégalités dand la théorie des fonctions monogènes et sur quelques propriétés nouvelles de ces fonctions dans le voisinage d'un point singulie essentiel*, Acta Soc. Sc. Fennicae **35** (1909), No. 7.

[67] M.S. Liu, Y.C. Zhu, *On the generalized Roper-Suffridge extension operator in Banach spaces*, Int.J.Math.Sci. (**8**)(2005).

[68] M.S. Liu, Y.C. Zhu, *Loewner chains associated with the generalized Roper-Suffridge extension operator on some domains*, J.Math.Anal.Appl., (**337**)(2008), 946-961.

[69] T.S. Liu, S. Gong, *Family of ϵ starlike mappings (I)*, Chinese Ann. Math. Ser.A, (**23**)(2002), 273-282.

[70] T.S. Liu, S. Gong, *Family of ϵ starlike mappings (II)*, Chinese Ann. Math.Ser.A , (**24**)(2003), 625-638.

[71] T.S. Liu, Q.H. Xu, *Loewner chains associated with the generalized Roper-Suffridge extension operator*, J.Math.Anal.Appl., (**322**)(2006), 107-120.

[72] K. Löewner, *Untersuchungen über schlichte konforme Abbildungen des Einheitskreises*, I.Math.Ann., (**89**)(1923), 103-121.

[73] S.S. Miller, *Distortion properties of alpha-starlike functions*, Proc. Amer. Math. Soc. **38**, No.2 (1973), 311-318.

[74] S.S. Miller, P.T. Mocanu, *Second-order differential inequalities in complex plane*, J.Math.Anal.Appl. **65**(1978), 289-305.

[75] S.S. Miller, P.T. Mocanu, Differential subordinations and univalent functions, Michig. Math. J., **28**(1981), 157-171.

[76] S.S. Miller, P.T. Mocanu, *On some classes of first-order differential subordinations*, Michig. Math. J., **32** (1985), 185-195.

[77] S.S. Miller, P.T. Mocanu, *The Theory and applications of second order differential subordinations*, Studia Univ. Babeş-Bolyai, Math. **34**, no.4 (1989), 3-33.

[78] S.S. Miller, P.T. Mocanu, *Briot-Bouquet differential equations and differential subordinations*, Complex Variables, **33** (1997), 217-237.

[79] S.S. Miller, P.T. Mocanu, *Differential Subordinations. Theory and Applications*, Marcel Dekker, Inc., New York, Basel, 2000.

[80] S.S. Miller, P.T. Mocanu, Subordinants of differential superordinations, Complex Variables, (48) (2003), 815-826.

[81] S.S. Miller, P.T. Mocanu, *Briot-Bouquet differential superordinations and sandwich theorems*, J. Math. Anal. Appl., **329**, No.1 (2007), 327-335.

[82] S.S. Miller, P.T. Mocanu, M.O. Reade, *All α-convex functions are starlike*, Rev. Roum. Math. Pures Appl. **17**, No.((1972), 1395-1397.

[83] S.S. Miller, P.T. Mocanu, M.O. Reade, *All α-convex functions are univalent and starlike*, Proc. Amer. Math. Soc. **37**, No.2(1973), 553-554.

[84] P.T. Mocanu, *Une propriéte de convexité generalisée dans la théorie de la représentation conforme*, Mathematica (Cluj) **11(34)** (1969), 127-133.

[85] P.T. Mocanu, *Some integral operators and starlike functions*, Rev. Roum. Math. Pures Appl. **31**, No. 3(1986), 231-235.

[86] P.T. Mocanu, *On a class of first-order differential subordinations*, Babeş-Bolyai Univ., Fac. of Math. Res. Sem., Seminar on Mathematical Analysis, Preprint 7 (1991), 37-46.

[87] P.T. Mocanu, T. Bulboacă, G.Ş. Sălăgean, *Teoria Geometrică a Funcţiilor Analitice*, Casa Cărţii de Ştiinţă, Cluj-Napoca, 1999.

[88] P. T. Mocanu, *Three-cornered hat harmonic functions*, Complex Variables and Elliptic Equation, **12**(2009), 1079-1084.

[89] P. Montel, *Lecons sur les Fonctions Univalentes ou Multivalentes*, Gauthier-Villars, Paris, 1933.

[90] G. Murugusundaramoorthy, H. M. Srivastava, *Neighorhoods of certain classes of analytic functions of complex order*, J. Pure Appl. Math., 5**2**Article **24**(2004), 1-8(electronic).

[91] M.A. Nasr, M.K. Aouf, *Starlike functions of complex order*, J. Natur. Sci. Math., **25**(1985), 1-12.

[92] M. A. Nasr, M. K. Aouf, *Characterizations for convex functions and starlike functions of complex order in U*, Bull. Fac. Sci. Assiut Univ. A **11**(1982), Nr.2, 117-121.

[93] V. Nechita, *Differential subordinations and superordinations for analytic functions defined by the generalized Sălăgean derivative*, will be appear.

[94] Z. Nehari, *Conformal Mappings*, Mc. Graw-Hill Book Comp., New York, 1952.

[95] K. S. Padmanabhan, M. S. Ganesan, *Convolution of certain classes of univalent functions with negative coefficients*, J. Pure, Appl. Math. 19(**9**), (1988), 880-889.

[96] J.A. Pfaltzgraff, *Subordination chains and univalence of holomorphic mappings in* C^n , Math.Ann., (**210**)(1974), 55-68.

[97] J.A. Pfaltzgraff, T.J. Suffridge, *An extension theorem and linear invariant families generated by starlike maps*, Ann. Univ. Mariae Curie-Sklodowska, Sect. A, (**53**)(1999), 193-207.

[98] G. Polya, G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, Springer-Verlag, Berlin, 1925.

[99] C. Pommerenke, *Über die Subordination analytischer funktionen*, J.Reine Angew.Math., (**218**)(1965), 159-173.

[100] C. Pomerenke, *Univalent functions*, Vandenhoeck Ruprecht, Gottingen, 1975.

[101] T. Poreda, *On the univalent subordination chain of holomorphic mappings in Banach spaces*, Comment. Math., (**28**)(1989), 295-304.

[102] T. Poreda, *On the univalent holomorphic maps of the unit polydisc of* $\mathbb{C}^n$ *which have the parametric representation,II-necessary and sufficient conditions*, Ann. Univ. Mariae Curie-Sklodowska Sect A (**41**)(1987),114-121.

[103] D.V. Prohorov, *Integral transformations in some classes of univalent functions*, Izv. Vyssh. Uchebn. Zaved. Mat. **12**(1980), 45-49.

[104] D. Răducanu, V. Nechita, *A differential sandwich theorem for analytic functions defined by the generalized Sălăgean operator*, will be appear.

[105] M.S. Robertson, *On the theory of univalent functions*, Ann. Math. 37(1936), 347-408.

[106] K. Roper, T.J. Suffridge, *Convexity properties of holomorphic mappings in* $\mathbb{C}^n$, Trans. Amer. Math. Soc. (**351**)(1999), 1803-1833.

[107] K. Roper, T.J. Suffridge, *Convex mappings on the unit ball of* C^n, J.Anal.Math., (**65**)(1995), 333-347.

[108] W. Rudin, *Function Theory in the Unit Ball of* $\mathbb{C}^n$, Springer-Verlag, New-York, 1980.

[109] S. Ruscheweyh, *New criterion for univalent functions*, Proc. Amer. Math. Soc. 49 (1975), 109-115.

[110] $\S$. Ruscheweyh, V. Singh, *On a Briot-Bouquet equation related to univalent functions*, Rev. Roumaine Math. Pures Appl. **24** (1979), 285-290.

[111] Gr. Şt. Sălăgean, *Subclasses of univalent functions*, Complex Analysis, Fifth Romanian-Finnish Seminar, Part 1 (Bucharest, 1981), Lecture Notes in Math., vol. 1013, Springer, Berlin, 1983, 362-372.

[112] H. Silverman, *Univalent functions with negative coefficients*, Proc. Amer. Math. Soc.,**51**(1975), 109-116.

[113] H. Silverman, *Harmonic univalent functions with negative coefficients*, J. Math. Anal. Appl. **220**(1998), 283-289.

[114] H. Silverman, E. M. Silvia, *Subclasses of harmonic univalent functions*, New Zealand J. Math. **28**(1999), 275-284.

[115] N.S. Sohi, L.P. Singh, *A class of bounded starlike functions of complex order*, Indian J. Pure Appl. Math., **33**(1991), 29-35.

[116] H. M. Srivastava, S. Owa and S. K. Chatterjea, *A note on certain classes of starlike functions*, Rend. Sem. Mat. Univ. Padova, **77**(1987), 115-124.

[117] E. Strohhäcker, *Beitrage zur Theorie der schlichten Funktionen*, Math. Z. **37** (1933), 356-380.

[118] E. Study, *Vorlesungen über ausgewählte Gegenstande der Geometrie*, 2. Heft, Teubner, Leipzig und Berlin. 1913.

[119] T. Suffridge, *Some remarks on convex maps of the unit disk*, Duke Math. J. **37** (1970), 775-777.

[120] T.J.Suffridge, *Starlikeness, convexity and other geometric properties of holomorphic maps in higher dimensions*, Lecture Notes in Math., Springer-Verlag, New York, (**599**)(1976), 146-159.

[121] P. Wiatrowski, *The coefficients of a certain family of holomorphyc functions*, Zeszyty Nauk. Univ.Lodz Nauk. Mat-Przyrod, (Ser. 2), **39**(1971), 75-85.

[122] Q-H. Xu, T.S. Liu, *Loewner chains and a subclass of biholomorphic mappings*, J.Math.Anal.Appl.(**334**)(2007), 1096-1105.

[123] S. Yalcin, *A new class of Sălăgean-type harmonic univalent functions*, Applied Mathematics Letters, Vol. 18, **2**(2005), 191-198.

[124] Y.C. Zhu, M.S. Liu, *The generalized Roper-Suffridge extension operator in Banach spaces(II)*, J.Math.Anal.Appl., 303(**2**)(2005), 530-544.