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- D. Dumitrescu, Rodica Ioana Lung, **Noémi Gaskó**, Réka Nagy, Job Scheduling and Bin Packing from a Game Theoretical Perspective. An Evolutionary Approach, 12th International Symposium on Symbolic and Numeric Algorithms for Scientific Computing (SYNASC 2010), ISBN: 978-1-4244-9816-1, pp. 209-214, DOI 10.1109/SYNASC.2010.55, 2010.
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Introduction

Problem statement

A strategic game can model the interactions of decision-makers. Real-world phenomena can be modelled with strategic games.

The strategic game can be defined as a set of players (decision-makers), a set of actions for each player, and for each player preferences over the set of action profiles (which in some cases can be considered as the payoff functions).

The most important question in the case of the strategic games is how can a player decide which action to choose form the set of the available actions. A chosen action profile wherewith all players are satisfied is called an equilibrium of the game.

The most important equilibrium concept in non-cooperative Game Theory is the Nash equilibrium [Nash, 1951]. Computing Nash equilibrium is a complex problem.

Some games can have several Nash equilibria, which make the players indecisive. Several refinements are developed in order to solve this selection problem, but there are no effective computational methods to detect these equilibria. This thesis proposes an evolutionary method to detect the Nash refinements.

Nash equilibrium and its refinements detection can be a powerful concept if all players are thinking rationally (a robotics example in [Istenes et al., 2011]). Behavioral game theory proves that players can be affected by emotions, irrationality, etc. therefore the classical equilibrium concepts can not be always a good choice.

Berge-Zhukovskii equilibrium is an alternative solution concept for non-cooperative games. For the best of our known there is no other computational method to detect this equilibrium.

Returning to the players behavior, we developed new equilibria (called joint), which model better real-world situations. We combine in different ways the standard non-cooperative equilibrium concepts and we receive so new types of equilibria. We present an evolutionary method to detect these new equilibria.

Thesis structure

The thesis is organized in seven chapters and a bibliography. Chapter 2 presents some related work in the field of Evolutionary Computation, Multi-Objective Optimization and Game Theory. The third Chapter describes different equilibria types, the Nash refinements, the Berge-Zhukovskii equilibrium, and a new concept: the joint equilibrium, which allows players to play in different biases. Chapter 4 contains the evolutionary detection of the Nash equilibrium refinements based on the generative relations. Some case studies with real-world applications are presented. Chapter 5 describes the evolutionary detection method of the Berge-Zhukovskii equilibrium, and some numerical experiments. In Chapter 6 the detection method for the new joint equilibrium is presented. Generative relations for Nash–Berge-Zhukovskii, Nash-Aumann, Pareto-Berge-Zhukovskii, and Pareto-Aumann equilibria. The new joint equilibrium allows us to define heterogenous players. Chapter 7 presents some conclusions and further work.

Contributions

The main contributions of the thesis include:

1. generative relations of some refinements of Nash equilibrium;
2. evolutionary detection of Nash equilibrium refinements using proposed generative relations:
   (a) Aumann (strong Nash) equilibrium;
   (b) coalition proof Nash equilibrium;
   (c) modified strong Nash equilibrium;
   (d) strong Berge equilibrium;
   (e) strong Berge Pareto equilibrium;
3. generative relation of Berge-Zhukovskii equilibrium (an alternative solution for non-cooperative games);

   evolutionary detection of Berge-Zhukovskii equilibrium;
4. new equilibria types based on the different rationality of the players:
   (a) Nash–Berge-Zhukovskii equilibrium;
   (b) Nash-Aumann equilibrium;
   (c) Pareto-Berge-Zhukovskii equilibrium;
   (d) Pareto-Aumann equilibrium;

   generative relations for the new equilibria types;

   evolutionary detection of these new equilibria.

Game equilibria

Introduction

The actions and payoffs of all agents are common knowledge and agents are supposed to behave in a rational manner. The most important solution concept in non-cooperative game theory is the *game equilibrium*. The most used equilibrium concept is Nash equilibrium [Nash, 1951] which describes a steady-state situation in the game. The concept of Nash equilibrium is based on the idea of stability against unilateral deviations.

When a game has several Nash equilibria it can appear a selection problem. Therefore several refinements and generalizations of Nash equilibrium have been proposed [Osborne, 2004].

Aumann (1959) proposed the concept of strong Nash equilibrium. A strong Nash equilibrium is a game strategy from which no subset of players has a joint deviation that strictly benefits all of them.

The Aumann (strong Nash) equilibrium represents a transition between players pursuing only their own interests and cooperative games. This equilibrium emphasizes coalitions of players and therefore could be a better approximation of the real world decision making.

The strong Berge equilibrium, proposed by Berge, is an other refinement of the Nash equilibrium. This equilibrium is stable against deviations.

Other important refinements are the coalition proof equilibrium and modified strong Nash equilibria.

Berge equilibrium generalizes the concept of Nash equilibrium.

An important solution concept is the Berge-Zhukovskii equilibrium. This equilibrium can be an alternative solution for games which don't have Nash equilibria, or for games, where Nash ensures not the highest payoff for players.

The all described equilibria types are based on players rationality. In real-wold situations the decision can be affected by emotions, irrationality. New combined, joint equilibria are introduced to eliminate these factors.

Non-cooperative games: basic notions

A non-cooperative game can be described as a system of players, actions and payoffs. Each player has some available actions, and for each action a corresponding payoff.

Mathematically, a finite strategic non-cooperative game is a system

$$G = (N, (S_i, u_i), i = 1, ..., n),$$

where:

- N represents a set of players, and n is the number of players;
- for each player $i \in N$, S_i is the set of actions available,

$$S = S_1 \times S_2 \times ... \times S_n$$

is the set of all possible situations of the game.

Each $s \in S$ is a strategy (or strategy profile) of the game;

- for each player $i \in N$, $u_i : S \to R$ represents the payoff function of i.

We can describe the Nash equilibrium [Nash, 1951] as a state, such that no player can change unilaterally her strategy to increase the payoff.

Let us denote by (s_i, s^*_{-i}) the strategy profile obtained from s^* by replacing the strategy of player i with s_i :

$$(s_i, s^*_{-i}) = (s^*_1, ..., s_i, ..., s^*_n).$$

Definition A strategy profile $s^* \in S$ is a Nash equilibrium if the inequality

$$u_i(s_i, s^*_{-i}) \geq u_i(s^*),$$

holds $\forall i = 1, ..., n, \forall s_i \in S_i$.

Refinements of Nash equilibrium

Several refinements of Nash equilibrium are considered.

Aumann (strong Nash) equilibrium

The Aumann (or strong Nash) equilibrium [Aumann, 1959] is a game strategy for which no coalition of players has a joint deviation that improve the payoff of each member of the coalition. More formally we have the next definition.

Let (s_I, s^*_{-I}) denotes the strategy profile in which $i \in I$ chooses the individual strategy s_i, and each $j \in N - I$ chooses s^*_j.

Definition The strategy s^* is an Aumann equilibrium if for each coalition $I \subseteq N, I \neq \phi$ the inequality

$$u_i(s_I, s^*_{-I}) \leq u_i(s^*), \forall i \in I$$

holds.

Let us denote by $SE(G)$ the set of Aumann (strong) equilibria of the game G and by $NE(G)$ the set of Nash equilibria in the G game. Thus we have:

$$SE(G) \subseteq NE(G).$$

Remark If each deviating coalition is composed from a unique player the strong Nash equilibrium reduces to the Nash equilibrium.

Remark SE does not always exists for all non-cooperative games.

k-Aumann equilibrium

The k-Aumann equilibrium is a state in which no coalition of players with size at most k has a joint deviation that improves the payoff of the members of the coalition.

Let us denote the k-Aumann equilibrium of the game G by k-$SE(G)$, $1 \le k \le n$.

Remark 1-Aumann equilibrium is equivalent to the Nash equilibrium, i.e.

$$1 - SE(G) = NE(G).$$

Remark It is easy to establish the claim of inclusions:

$$n - SE(G) \subseteq (n-1) - SE(G) \subseteq ... \subseteq 1 - SE(G) = NE(G).$$

Modified strong Nash equilibrium

In some cases the Aumann equilibrium concept can be to strong, therefore this can be modified using a weaker condition. The modified strong Nash equilibrium is introduced by Ray [Ray, 1989] and Greenberg [Greenberg, 1987].

Let us consider a finite strategic game and the following notations: $S_I = \prod_{i \in I} S_i$ and $s_I = (s_i)_{i \in I}$.

The following definitions are necessary to introduce the modified strong Nash equilibrium:

Definition For $I \in 2^N - \{\emptyset\}$, $s^* \in S_N, s_I \in S_I$ we say that s_I is blocked by $T \subset I$ given s^* if there exists a vector $z_T \in S_T$ such that:

$$u_T(z_T, s_{I-T}, s^*_{N-I}) \ge u_T(s_I, s^*_{N-I}).$$

Definition I is credible given s^* if there is a $s_I \in S_I, s_I \ne s^*_I$, that is not blocked by any credible $T \subset I$ given s^*.

Definition A strategy profile $s^* \in S_N$ is a modified strong Nash equilibrium if it is not blocked by any credible coalition (given s^*).

Coalition proof Nash equilibrium

Bernheim [Bernheim et al., 1987] introduced the coalition proof Nash equilibrium. A coalition-proof equilibrium is a correlated strategy from which no coalition has an improving and self-enforcing deviation.

Definition Let $s^* \in S$ and let P be the set of the subsets. An internally consistent improvement (ICI) of P upon s^* is defined by induction on $card(P)$ [Keiding, Peleg, 2002]:

- if $card(P) = 1$, then $P = \{i\}$, then s_i is an ICI upon s^*, if

$$u_i(s_i, s^*_{N-i}) > u_i(s^*);$$

- if $card(P) > 1$, then $s_P \in S_P$ is an ICI of P upon s^*

  $($i) s_P is an improvement of P upon s^* :

$$u_i(s_P, s^*_{N-P}) > u_i(s^*),$$

  and

  $($ii) if $T \subset P$ and $card(T) < card(S)$ then T has no ICI upon (s_P, s^*_{N-S}).

Definition A strategy profile $s \in S$ is a coalition proof Nash equilibrium, if no P subcoalition has an ICI upon s^*.

Remark The Aumann equilibrium is a subset of the coalition proof Nash equilibrium (CNE):

$$SE \subseteq CNE.$$

Remark For two person games the strategy $s^* \in S$ is a coalition proof Nash equilibrium, if [Keiding, Peleg, 2002]:

- s^* is a NE;
- there is no $s \in S$ which is a NE such that $u_i(s) > u_i(s^*)$ for $i = 1,2$;

Strong Berge equilibrium

Berge introduced the concept of the strong Berge equilibrium [Berge, 1957]. The strong Berge equilibrium is stable against deviation of all the players except one of them. If a player chooses her strategy in a strong Berge equilibrium, then she obliges all the other player to do so.

Definition A strategy profile $s^* \in S$ is a strong Berge equilibrium (SBE) of the game, if

$$u_j(s^*_i, s_{-i}) \le u_j(s^*), \forall i \in N, \forall j \in N - i, \forall s_{-i} \in S_{-i}.$$

Remark If the number of players is equal to 2, the strong Berge equilibrium and the Nash equilibrium coincide.

Strong Berge Pareto equilibrium

Strong Berge Pareto equilibrium [Nessah et al., 2008] is a refinement of the strong Berge equilibrium.

Definition A strategy profile $s^* \in S$ is Pareto efficient, when it does not exist a strategy $s \in S$, such that

$$u_i(s) \ge u_i(s^*), i \in N,$$

with at least one strict inequality.

Definition A strategy profile $s^* \in S$ is a strong Berge equilibrium (SBE) of the game, if

$$u_j(s_i^*, s_{-i}) \leq u_j(s^*), \forall i \in N, \forall j \in N - i, \forall s_{-i} \in S_{-i}.$$

Definition A strategy profile $s^* \in S$ is a strong Berge and Pareto equilibrium of the game, if s^* is a strong Berge equilibrium, and it is also Pareto efficient.

Berge equilibrium

The Berge equilibrium is a more general equilibrium concept than the Nash equilibrium.

Abalo and Kostreva [Abalo and Kostreva, 2005] gave a general definition to the Berge equilibrium.

Definition Let M be a finite set of indices. Denote by $P = \{P_t\}, t \in M$ a partition of N and $R = \{R_t\}, t \in M$ be a set of subsets of N. A strategy profile $s^* \in S$ is an equilibrium strategy for the partition P with respect to the set R, or simply a Berge equilibrium strategy, if and only if the condition

$$u_{p_m}(s^*) \geq u_{p_m}(s_{R_m}, s^*_{N-R_m})$$

holds for each given $m \in M$, any $p_m \in P_m$ and $s_{R_m} \in S_{R_m}$.

The payoff of each player p_m from the class P_m does not decreases if one or more players from the class R_m deviate from the equilibrium strategy s^*.

Berge-Zhukovskii equilibrium

The Berge-Zhukovskii equilibrium [Zhukovskii, 1994] can be a solution for games, which do not have Nash equilibrium, or for games which have more than one Nash equilibrium.

In contrast to the Nash equilibrium, where the players are selfregarding, the Berge-Zhukovskii equilibrium allows us to reach cooperative issues and therefore it is possible to determine cooperation in a non-cooperative framework.

The strategy s^* is a Berge-Zhukovskii equilibrium, if at least one of the players of the coalition $N - \{i\}$ deviates from her equilibrium strategy, the payoff of the player i in the resulting strategy profile would be at most equal to her payoff $u_i(s^*)$ in the equilibrium strategy.

Formally we can write:

Definition A strategy profile $s^* \in S$ is Berge-Zhukovskii equilibrium if the inequality

$$u_i(s^*) \geq u_i(s_i^*, s_{N-i})$$

holds for each player $i = 1, ..., n$, and $s_{N-i} \in S_{N-i}$.

Remark The Berge-Zhukovskii is a particular case of the Berge equilibrium. Consider each class P_i of the partition P consists from a player i and each set of R_i is the set N of players except i. Therefore $M = N$, $P_i = \{i\}$ and $R_i = N - i, \forall i \in N$. In this case Definition becomes Definition .

k-Berge-Zhukovskii equilibrium

k-Berge-Zhukovskii equilibrium is a subset of the Berge-Zhukovskii equilibrium. In this case the size of the coalition equal to k.

Definition A strategy profile $s^* \in S$ is a k-Berge-Zhukovskii equilibrium if the inequality

$$u_i(s^*) \geq u_i(s^*_{N-K}, s_K)$$

holds for each player $i \in N$, $s_K \in S_K$, $K \subseteq N$, $card(K) = k$.

ϵ-Berge equilibrium

ϵ-Nash equilibrium detection is described in [Barlo and Dalkiran, 2009].

ϵ-Berge-Zhukovskii can be defined in the following way:

Definition A strategy profile $s^* \in S$ is an ϵ-Berge-Zhukovskii equilibrium if the inequality

$$u_i(s^*) + \epsilon \geq u_i(s^*_i, s_{N-i})$$

holds for each player $i = 1, ..., n$, and $s_{N-i} \in S_{N-i}$.

The ϵ can be interpreted in several ways [Dumitrescu et al., 2009b]:

- penalization of players;
- noise of the game;
- payoff function approximation;

Detecting refinements of Nash equilibrium

Generative relations

Generative relations for different equilibria types are presented.

In order to compute certain equilibria we would like to characterize these with a relation on the strategy set.

This relation we can call the *generative relation* of the equilibrium.

Generative relations are an algebraic tool in order to detect several equilibria.

For the generative relations first we need a quality measure:

$$Q : S \times S \to \mathbb{N},$$

where S is the set of the strategy profiles.

Let s and s^* be two strategy profiles, $s, s^* \in S$.

In this case $Q(s, s^*)$ measures the quality of strategy s with respect to the strategy s^*.

The quality Q is used to define a relation $\prec_Q$:

$$s \leq_Q s^*, \text{ if and only if } Q(s, s^*) \leq Q(s^*, s).$$

In [Lung, Dumitrescu, 2008] was introduced the first generative relation for the Nash equilibrium.

The quality measures for the Nash refinements are the following:

- for the Aumann equilibrium:

$$a(s^*, s) = card[i \in I, \phi \neq I \subseteq N, u_i(s_I, s^*_{-I}) \geq u_i(s^*), s_i \neq s^*_{-i}];$$

- for the modified strong Nash equilibrium:

$$ma(s^*, s) = card[t \in T, T \neq \phi, T \subset I, \phi \neq I \subseteq N, u_t(z_t, s_{I-T}, s^*_{N-I}) \geq u_t(s_I, s^*_{N-I}),$$

$$s_I \neq s^*_I, z_t \in S_T];$$

- for strong Berge equilibrium:

$$sb(s^*, s) = card[j \in -i, i \in N, u_j(s^*_i, s_{-i}) \geq u_j(s^*), \forall s_{-i} \in S_{-i}];$$

An evolutionary technique for equilibrium detection

Generative relations allow an evolutionary technique to be applied for equilibria detection. Selection methods are based on generative relations.

Our goal is to find different equilibria types with evolutionary multi-objective optimization algorithms based on non-domination.

The used method can be described as follows:

A population of game strategies is evolved. Every individual is encoded as a n-dimensional vector, representing a strategy $s \in S$.

The initial population is generated randomly. At each step the actual population can be considered as the approximation of the equilibrium.

The non-dominated sorting segregates the population into layers by first finding the non-dominated solutions in the population, using the generative relations, and labels these points as the first front. These points are removed and the non-dominated solutions in the remaining population are then identified and removed.

This process continues until the whole population is classified into layers. The algorithm updates the current archive by identifying all the non-dominated solutions in the union of the old archive and current population. The layers are taken in turn until the maximum size of the archive is reached (often the population size).

Strategy population at iteration t may be regarded as the current equilibrium approximation. Subsequent application of the such operators (like the simulated binary crossover (SBX) [Deb and Beyer, 1995] and real polynomial mutation [Deb et al., 2000]) is guided by a specific selection operator induced by the generative relation.

Selection for survival can be done by using a procedure based on the same selection operator or another one, also correlated to the generative relation.

Selection, recombination, and mutation is repeated until the maximum number of generation is reached.

This approach can be summarized in a technique called Relational Evolutionary Equilibria Detection (REED) as described in Algorithm 1.

Algorithm 1 REED method

Set $t = 0$;
Randomly initialize a population $P(0)$ of strategies;
while (not termination-condition) **do**
  Binary tournament selection and recombination using the simulated binary crossover (SBX) operator for $P(t) \to Q$;
  Mutation on Q using real polynomial mutation $\to P$;
  Compute the rank of each population member in $P(t) \cup P$ with respect to the generative relation. Order by rank $(P(t) \cup P)$;
  Rank based selection for survival → P (t + 1);
end while

Evolutionary detection of Berge-Zhukovskii equilibrium

Generative relation for Berge-Zhukovskii equilibrium

Consider two strategy profiles s and s^* from S. Denote by $b(s,s^*)$ the number of players who lose by remaining to the initial strategy s, while the other players are allowed to play the corresponding strategies from s^* and at least one player switches from s to s^*.

We may express $b(s,s^*)$ as:

$$b(s,s^*) = card[i \in N, u_i(s) < u_i(s_i, s^*_{N-i})].$$

Definition Let $s, s^* \in S$. We say the strategy s is better than strategy s^* with respect to Berge-Zhukovskii equilibrium, and we write $s \prec_B s^*$, if and only if the inequality

$$b(s,s^*) < b(s^*,s)$$

holds.

Definition The strategy profile $s^* \in S$ is a Berge-Zhukovskii non-dominated strategy, if and only if there is no strategy $s \in S, s \neq s^*$ such that s dominates s^* with respect to $\prec_B$ i.e.

$$s \prec_B s^*.$$

Denote by BNS the set of all non-dominated strategies with respect to the relation $\prec_B$.

We may consider relation $\prec_B$ as a candidate for generative relation of the Berge-Zhukovskii equilibrium . This means the set of the non-dominant strategies with respect to the relation $\prec_B$ equals the set of Berge-Zhukovskii equilibria.

We may consider the set of all Berge-Zhukovskii equilibrium strategies as representing the Berge-Zhukovskii equilibrium (BE) of the game.

Evolutionary detection method

In Differential Evolution [Storn, Price, 1995] an initial population is generated randomly and evaluated. The method enters a loop of generating offspring, evaluates offspring, and selects individuals to create the next generation, until the maximum number of generation is reached.

Let us denote:

- $U(0,k)$ - is a uniformly distributed number between 0 and k;
- pc - the crossover probability;
- F - the scaling factor;
- dim - the number of problem parameters;

The procedure for creating offspring is depicted in Algorithm

Algorithm 2 Procedure create offspring $O[i]$ from parent $P[i]$

$O[i] = P[i]$
randomly select parents $P[i_1]$, $P[i_2]$, $P[i_3]$, where $i_1 \neq i_2 \neq i_3 \neq i$
$n = U(0, dim)$;
for $j = 0; j < dim$ and $U(0,1) < pc; j = j+1$ **do**
  $O[i][n] = P[i_1][n] + F * (P[i_2][n] - P[i_3][n])$
  $n = (n+1) \bmod dim$
end for

Differential Evolution technique is presented in Algorithm .

Algorithm 3 Procedure Differential Evolution

initialize population with random individuals
evaluate individuals
while (not termination-condition) **do**
  for $i = 0; i < popsize; i = i+1$ **do**
    create offspring $O[i]$ from parent $P[i]$
    evaluate offspring $O[i]$
    if offspring $O[i]$ is better than parent $P[i]$ **then**
      replace parent with offspring
    else
      keep parent in population
    end if
  end for
end while

Crowding Differential Evolution [Thomsen, 2004] extends the Differential Evolution (DE) algorithm with a crowding scheme.

In our case the offspring replaces the most similar individual among the population if it is better than the parents with respect to the Berge-Zhukovskii equilibrium (using the generative relation for the Berge-Zhukovskii equilibrium).

Application

A classical form of the prisoner's dilemma (PD) [Flood, 1958] is the following: two suspects are arrested. The police has insufficient information to condemn them. They decide to separate them and to make for each of them an offer: if one confesses and the other remains silent, the betrayer will be free, and the other receives ten years sentence. If both remain silent, the punishment for them will be two years. If both confess they receive six year prison.

The payoffs can be summarized in Table 1.

The payoffs represented by preferences are described in Table 2. The game has a pure Nash equilibrium *(Defect, Defect)*, which is not the best solution. The paradox is that both agents act

Table 1: The payoff functions of the two players in Prisoner's Dilemma

		Pl. 2	
		Cooperate (Stay silent)	Defect (Confess)
Pl. 1	Cooperate (Stay silent)	(2, 2)	(10, 0)
	Defect (Confess)	(0, 10)	(6, 6)

Table 2: The payoff functions (preferences) of the two players in Prisoner's Dilemma

		Player 2	
		Cooperate	Defect
Player 1	Cooperate	(2, 2)	(0, 3)
	Defect	(3, 0)	(1, 1)

rationally, but producing an apparently irrational result.

The Berge-Zhukovskii equilibrium of the PD game is the *(Cooperate, Cooperate)*. This equilibrium is a better solution, because the payoff for each agent is higher in this case.

Let us consider the n-person version of the PD. The payoff function is expressed as:

$$u_i(s) = \begin{cases} 2\sum_{j\neq i} s_j + 1 & \text{if } s_i = 0; \\ 2\sum_{j\neq i} s_j & \text{if } s_i = 1. \end{cases}$$

Remark For $n = 2$ we get the two person version of the game.

The PD is a classical example that if all players choose a strategy rationally (they play Nash) the result will not be the best for all of them while Berge-Zhukovskii equilibrium can be a better choice.

CrDE is used to compute k-Berge-Zhukovskii and Nash equilibrium for seven instances of the prisoner's dilemma considering 2, 10, 20, 50 and 100 players respectively. Parameters used for CrDE are presented in Table 3. Average and standard deviation of distances to the Berge-Zhukovskii and Nash equilibria respectively over 30 runs are computed. Results are presented in Table 4.

For the second part of the experiments NSGA-II is used for the same PD problem, considering 2, 10, 20, 50 and 100 players respectively. Parameters used for NSGA-II are presented in Table 5. Average distances to the $(n-1)$-Berge-Zhukovskii and Nash equilibria respectively over 30 runs are computed. Results are presented in Table 6. The detection method is based on [Deb et al., 2000] and [Lung, Dumitrescu, 2008].

Table 3: Parameter settings for CrDE used for the PD game

Parameter	2	10	20	50	100
Pop size			50		
Max no evaluations		5×10^5		5×10^6	
CF			50		
F			0.1		
Crossover rate			0.9		

Table 4: Average and standard deviation of distances to $(n-1)$-Berge-Zhukovskii (n is the number of players) and Nash equilibria over 30 runs using CrDE for the PD game

No players	Nash	$(n-1)$-Berge
2	0 ± 0	0 ± 0
10	0 ± 0	0 ± 0
20	0 ± 0	0 ± 0
50	0 ± 0	0 ± 0
100	0 ± 0	0 ± 0

Table 5: Parameter settings for NSGAII used for the PD game

Parameter	2	10	20	50	100
Pop size			50		
Max no evaluations		5×10^5		5×10^6	
prob. of crossover			0.9		
prob. of mutation			0.5		

Table 6: Average and standard deviation distances from the Nash equilibrium strategy and from the $(n-1)$-Berge-Zhukovskii (n is the number of players) equilibrium strategy using NSGA-II in the PD game

No players	Nash equilibrium	$(n-1)$-Berge equilibrium
2	0 ± 0	0 ± 0
10	0 ± 0	0 ± 0
20	0 ± 0	0 ± 0
50	0 ± 0	3.93 ± 0.08
100	1.2 ± 0.29	4.02 ± 0.26

We can do a step further and consider generalized games involving inhomogeneous players. Each player may be biased towards a different equilibrium concept. This idea may be captured by the concept of meta-rationality introduced in [Dumitrescu et al., 2009a].

Let us consider a game with n players. Each player i has a strategy set S_i, and a certain type of rationality, which is denoted by r_i (for example $r_1 = Nash$, $r_2 = Aumann$, $r_3 = Pareto$, etc.).

A meta-strategy is a system

$$(s_1|r_1, s_2|r_2, ..., s_n|r_n),$$

where

$$(s_1, ..., s_n)$$

is a strategy profile, and

$$(r_1, ..., r_n)$$

describes the agents types of rationality.

The generalized game has a meta-strategy space, denoted by M,

$$M = M_1 \times M_2 \times ... \times M_n,$$

where M_i represents the set of meta-strategies for each player i.

Generative relation of Nash–Berge-Zhukovskii equilibrium

Let us consider two strategies:

$$s = (s_1, s_2, ..., s_n),$$

and

$$y = (s_1^*, s_2^*, ..., s_n^*).$$

The corresponding meta-strategies are denoted by M_1 and M_2,

$$M_1 = (s_1|r_1, s_2|r_2, ..., s_n|r_n),$$

and

$$M_2 = (s_1^*|r_1, s_2^*|r_2, ..., s_n^*|r_n).$$

Let us denote by I_{NA} the set of players which plays Nash (are Nash biased), and by I_{BZ} the set of players which plays Berge-Zhukovskii (are Berge-Zhukovskii biased):

$$I_{NA} = \{i \in \{1, ..., n\} | r_i = Nash\},$$

$$I_{BZ} = \{j \in \{1, ..., n\} | r_j = Berge - Zhukovskii\}.$$

We consider that $I_{NA} \cap I_{BZ} = \emptyset$.

It is possible to introduce an operator P, $P : M \times M \to \mathbb{N}$, defined as

$$P(M_1, M_2) = card\{i \in I_{NA}, u_i(s^*) < u_i(s^*_{-i}, s_i), s_i \neq s^*_i\} +$$

$$+ card[i \in I_{BZ}, u_i(s^*) < u_i(s^*_i, s_{N-i})].$$

$P(M_1, M_2)$ represents a meta-strategy quality measure with respect to a *joint Nash–Berge-Zhukovskii equilibrium*.

Definition Let $M_1, M_2 \in M$. The meta-strategy M_1 is more efficient than meta-strategy M_2 with respect to the joint Nash–Berge-Zhukovskii meta-strategy, and we write $M_1 \prec_{NB} M_2$, if and only if:

$$P(M_1, M_2) < P(M_2, M_1).$$

We consider that Nash–Berge-Zhukovskii efficiency relation induces a new type of equilibrium concept called *joint Nash–Berge-Zhukovskii equilibrium*.

Remark If $I_{NA} = \emptyset$ then $P(x, y) = b(x, y)$.

Remark Consider a two player game with $r_1 = r_2 = Nash$ (both players are Nash biased). In this case $\prec_{NB}$ reduces to the generative relation of the Nash equilibrium.

If $r_1 = r_2 = Berge - Zhukovskii$, meaning that both player are biased towards the strong equilibrium, $\prec_{NB}$ becomes the generative relation of the Berge-Zhukovskii equilibrium.

Nash–Berge-Zhukovskii equilibrium expresses a transition between selfishness and altruism.

Remark Generative relations for Nash-Aumann, Pareto-Aumann,Pareto–Berge-Zhukovskii can be defined similarly.

Numerical experiments

Let us consider the two-person continuous game G_1 [Nessah et al., 2007], having the following payoff functions:

$$u_1(x_1, x_2) = -x_1^2 - x_1 + x_2,$$

$$u_2(x_1, x_2) = 2x_1^2 + 3x_1 - x_2^2 - 3x_2, x_1, x_2 \in [-2, 1].$$

Figure 1 represents the detected Nash, Pareto, Berge-Zhukovskii, Pareto-Berge-Zhukovskii and Berge-Zhukovskii-Pareto equilibria. In the Figure 2 are depicted the Nash, Pareto, Berge-Zhukovskii, Nash–Berge-Zhukovskii and Berge-Zhukovskii-Nash equilibria.

Berge-Zhukovskii-Pareto and Pareto-Berge-Zhukovskii equilibria originate in the Berge–Zhukovskii equilibrium.

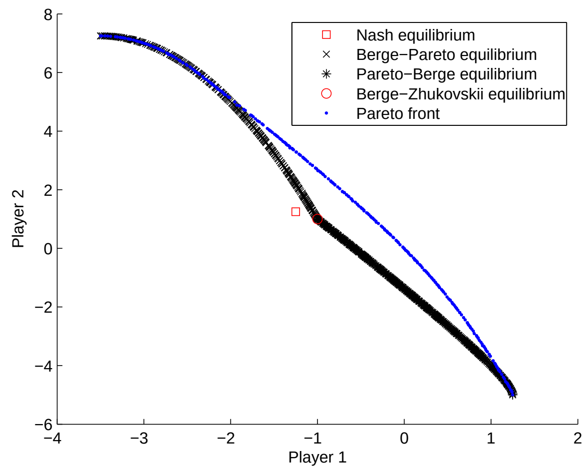

Figure 1: Detected payoffs for Nash, Pareto, Berge-Zhukovskii, Pareto-Berge-Zhukovskii, Berge-Zhukovskii-Pareto joint equilibria of game G_1

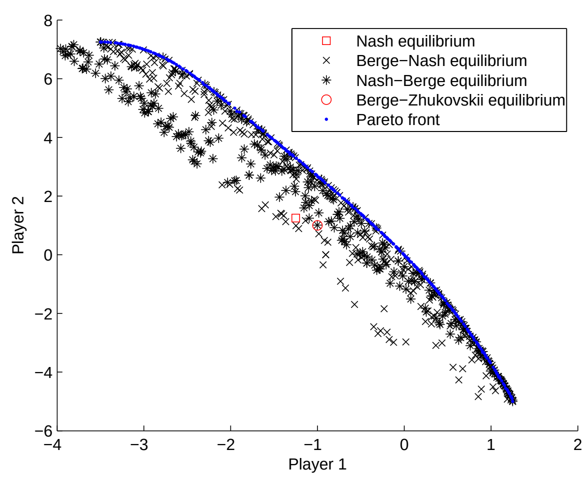

Figure 2: Detected payoffs for Nash, Pareto, Berge-Zhukovskii, Nash–Berge-Zhukovskii, Berge-Zhukovskii-Nash joint equilibria of game G_1

Conclusions

Summary of results

The thesis concerns on Computational Game Theory. Solving a game in standard Game Theory means the equilibrium detection, which can be viewed as the most important task.

The most used equilibrium concept in non-cooperative Game Theory is the Nash equilibrium. Playing in Nash sense means that no player can deviate from the equilibrium strategy in order to increase his/her payoff. Non-cooperative game examples proved that this equilibrium concept is not the best choice in all cases). A game can have several Nash equilibria, which lead the players to a decision problem. Different refinements are introduced to solve this problem (Aumann equilibrium, modified strong Nash equilibrium, coalition proof Nash equilibrium, strong Berge equilibrium, etc.).

Berge-Zhukovskii equilibrium is an alternative solution concept to non-cooperative games. This equilibrium type can be useful in games which have no Nash equilibrium, or have several Nash equilibria, or the Nash equilibrium ensures not the highest payoff.

The literature is not to reach in computational equilibrium detection methods. This thesis focuses on the evolutionary detection method of different equilibria types, based on evolutionary optimization.

In our approach equilibria are characterized by generative relations. Generative relations for Nash equilibrium refinements (Aumann, modified strong Nash, coalition proof Nash, strong Berge, strong Berge Pareto equilibrium) are introduced. Generative relations for Berge-Zhukovskii, k-Berge-Zhukovskii, ϵ-Berge-Zhukovskii equilibrium are described. It is proved that some of these equilibria equal the set of the non-dominated strategies, and some equilibria are subset of the non-dominated strategies.

New joint equilibria types are proposed. These new joint equilibria model games with heterogenous players. For each player is allowed to play in different way. Generative relations for these equilibria are also described.

An evolutionary technique, called Relational Evolutionary Equilibrium Detection (REED) method is used. This method, based on non-domination approximates the certain equilibrium. The main idea is that generative relation induces a specific nondomination concept for each type of equilibrium.

Future work

We would like to apply the introduced joint equilibria in economical applications. Economical models with large number of players are useful examples.

Future work will consider other evolutionary technique for detecting different equilibria types in non-cooperative games. A comparison of the different methods is also necessary.

Future work will also focus on developing new equilibria types, which model the behavior of real-world players. In this thesis we studied only two player games with two different rationalities. A generalization with more than three players and more than two different rationalities is an upcoming work.

Another research direction is the analysis of different equilibria types as a multi-objective optimization tool. The first step of this idea is captured in [Dumitrescu et al. 2011b].

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