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TRAPEZOIDAL APPROXIMATION OF FUZZY NUMBERS

Ph.D. Thesis Summary

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KEY-WORDS

Fuzzy sets, fuzzy numbers, triangular fuzzy number, trapezoidal fuzzy number, ambiguity, value, expectancy interval, expectancy value, medium value, the entropy of fuzzy numbers, trapezoidal approximation, weighted approximation.

INTRODUCTION

Fuzzy Mathematics became known in the second half of the 20 th century, after Lotfi A. Zadeh published the first article on fuzzy Mathematics [82], 1965. The article generated lots of research and implicitly lead to the founding of new branches in Mathematics. Fuzzy Mathematics may be regarded as a parallel to the classic Mathematics or as a natural continuation of the latter, but, in most cases, it is considered a new Mathematics, very useful in solving problems expressed through a vague language.

The promoter of the fuzzy sets theory, professor L. Zadeh, states that fuzzy Mathematics is a useful instrument in moulding problems which are either ungradable or too complex to be adequately modeled through traditional methods. If Zadeh is the international promoter of the fuzzy Mathematics, the first concepts of fuzzy Mathematics in Romania belong to G. Moisil. A detailed presentation of Zadeh's bibliography and works is given in [23].

The study of the fuzzy Mathematics is organized on several subdomains. Among these, we can mention: the fuzzy logic, the fuzzy arithmetic, the fuzzy control theory and the fuzzy nodulations. Compared to other research domains, the fuzzy arithmetic gained great interest only during the last years. Fuzzy numbers and sets are tools that are successfully used in scientific areas, such as: decision problems, social sciences, control theory, etc. The potential of the fuzzy numbers is accepted and illustrated in exercises presented in [44] or [54]. Results obtained by using fuzzy numbers are practically much better than those obtained by classic methods. The study is oriented on: fuzzy sets and fuzzy logic ([70], [82], [83], [84]) fuzzy numbers ([21], [45], [54]) operations on fuzzy numbers ([12], [18], [19], [42], [54]) ordering of the fuzzy numbers [22], defuzzification measures [18], approximation operators of the fuzzy numbers ([9], [51], [75]), distance between two fuzzy numbers ([74], [39]), fuzzy equations [73], linguistic variables [34] of fuzzy application [20].

Trapezoidal approximation operators was studied in many papers ([1], [2], [3], [9], [15], [37], [49], [48], [51], [52], [68], [78], [80], [81], [85]).

The paper has 4 chapters. In chapter I, „Preliminaries", we introduced elementary theoretical concepts that we are going to use during the whole paper. Starting from the concept of fuzzy numbers, we made a short presentation of operations, arriving at the central element of this paper: the fuzzy number. We present a classification of fuzzy numbers and elementary operations introduced over the fuzzy numbers set. Further on, in „Real numbers and intervals attached to fuzzy numbers", we discuss a set of measures which, applied to fuzzy numbers, defuzzify it, or turn it into an interval of real numbers (e.g. expected interval) or into a real number (entropy, medium value, ambiguity or value).

Starting with the second chapter, we present methods of trapezoidal and weighted approximation of fuzzy numbers. Fuzzy numbers approximation is a method required to simplify the processing of data that contain fuzzy numbers and can be seen from different perspectives. Since there are many approximation methods and many algorithms that do these approximations, we consider that it is not important to find the best approximation operator, but to estimate the features and characteristics of operators which preserve fuzzy measures or level sets and find interesting use for these operators. The most brutal form of approximation is defuzzification and it is a procedure that attaches a real number to a fuzzy number [43], [22]. This manner of approximation leads to great losses of important information and we can call this method, according to [51], „ type I approximation". Another version of approximation is the attaching of a real numbers interval to a fuzzy number (e.g. in [41]), so a „ type II approximation". A more generous approximation of the fuzzy number is with a triangular fuzzy number, so a „ type III approximation" (e.g. in [11]), and the approximation with a trapezoidal will be called „type IV approximation" (e.g. in [8], [9], [50], [51], [52], [53]).

Trapezoidal operators of fuzzy numbers approximation have been studied by many authors, because fuzzy numbers have lots of uses in the efficient adjustment of imprecise information. It is obvious that there is an infinite number of methods for the approximation of fuzzy numbers with trapezoidal numbers. One of the pioneers of trapezoidal approximation, Delgado, in [37], suggests that an approximation must preserve at least some of the initial parameters of the fuzzy number. Any method of trapezoidal approximation is easy to suggest, because there is a great variety of characteristics that can be preserved from the beginning and, under these conditions, we can reach a new approximation. In [51], authors propose a trapezoidal operator of approximation and a list of criteria that an operator should fulfill in order to be „good".

The characteristics an approximation operator must fulfill, according to [51], are the following:

1. $\alpha-$cut invariance;
2. Translation invariance;
3. Scalar invariance;
4. Monotony;
5. Identity;
6. Nearest criterion;
7. Expected value invariance;
8. Continuity;
9. Compatibility with extension principle;
10. Order invariance;
11. Correlation invariance;
12. Uncertain invariance.

In many works that introduce approximation operators, problems of minimum have been required, in order to identify these operators, because, for a better approximation, authors used minimization of distance between the fuzzy number and its approximate. This problem was solved using Lagrange's multipliers method [51] or the Karush-Kuhn-Tuker theorem [9], [53]. Another manner of approach is given in [85], where trapezoidal approximation is studied from the angle of weighted distance. In [53], they discuss the weighted trapezoidal approximation of fuzzy numbers that preserve expected interval and new algorithms of trapezoidal approximation of fuzzy numbers are introduced, keeping a distance based on weighted bisymmetrical functions.

In chapter II, we introduced and studied the trapezoidal approximation operator which preserves the expected interval and the fuzzy number core. In identifying this operator, the four conditions at the beginning for identifying the parameters of the approximation operator lead to a determined, one solution system. Although the shape of this operator is very simple and easy to obtain, we deal with a linear operator, a rare feature among trapezoidal approximation operators, therefore a valuable feature. Beside that, the operator is invariant in translations, it respects the criteria of identity, it preserves the order induced into the set of fuzzy numbers by the expected value, is invariant in relations based on expected value or expected interval, so it meets many of the criteria mentioned in [51]. However, the operator does not preserve the ambiguity of the fuzzy number, nor its value. Another studied characteristic is continuity, in connection to two metrics. In order to reflect the utility of this operator, we gave a calculation formula for fuzzy numbers, introduced by Bodjanova in [22] and a set of other exercises and examples: solving the fuzzy equation systems, estimation of the multiplicity order for trapezoidal numbers.

The second paragraph of this chapter introduces the trapezoidal operator of approximation, which preserves the expected interval and the support of the fuzzy number [26]. If, in the case of the first operator, all fuzzy numbers could be approximated to a trapezoidal number, in this operator's case, only a family of fuzzy numbers can be approximated with trapezoidal numbers, preserving the requested characteristics: preservation of the zero level set and the expected interval. The study of this operator is made from the perspective of important characteristics, such as: translation invariance, linearity, identity criteria, preservation of measures based on the preservation of the expected interval, continuity in connection with different metrics, the study of the multiplicity defect.

We close the chapter with an approximation operator, called the trapezoidal operator that preserves expected value and core of the fuzzy number [28]. In the cases of the first two operators, problems were reduced to a system of four equations and four unknowns, in the case of this operator from the condition of minimization of the distance between the fuzzy number and the trapezoidal one, the preservation of the expected value and of the core, the solving is done using a minimization theory, and we have chosen the Karush-Kuhn-Tucker theory. Beside generating this operator, we also studied their characteristics and generated examples.

Chapter III introduces approximation operators that preserve value, ambiguity and a level set. The first is the operator that preserves the ambiguity and the value and it is the nearest trapezoidal approximation of a fuzzy number, it was introduced in [13]. Even though the operator is introduced by minimizing the distance between the fuzzy number and the trapezoidal that approximates it, we do not use minimization theories, but an extended trapezoidal. We study the properties of this operator, introduce calculation algorithms, demonstrate continuity and give examples. For the two operators that preserve value, ambiguity and a level set, introduced in [24], [29], we studied properties and presented a set of examples.

Chapter IV studies approximation weighted operators. First, we present the weighted operator that preserves the fuzzy number support, introduced in [25]. The operator is generated by Lagrange's multipliers method, then we present its particular cases, its properties and a set of examples. The weighted operator that preserves the expected interval is introduced in [30]. As a method of solving the minimization problem, we have chosen the Karush-Kuhn-Tucker theorem, presenting particular cases, properties and examples that activate this operator.

Our original contribution may be found in chapters 2,3,4, and a large part of paragraph 1.3.2.

1 Preliminaries

1.1 Fuzzy sets

Definition 1 *Let X be a set. A **fuzzy set** is characterized by a function called membership function and defined as the [82]:*

$$f_X : X \to [0,1]$$

associating each element of X to a real number in the range [0.1].

1.2 Fuzzy numbers

Definition 2 *A fuzzy number A is fuzzy subset of a real line $\mathbb{R}$ with the membership function μ_A which is (see [41]) normal, fuzzy convex, upper semicontinous, supp A is bound, where the support of A, denoted by supp A, is the closure of the set* $\{x \in X : \mu_A(x) > 0\}$.

Any fuzzy number can be expressed (see [**?**]) in the form

$$\mu_A(x) = \begin{cases} g(x), & \text{when } x \in [a,b) \\ 1, & \text{when } x \in [b,c] \\ h(x), & \text{when } x \in (c,d] \\ 0, & \text{otherwise,} \end{cases}$$

where $a, b, c, d \in \mathbb{R}$ such that $a < b \le c < d$, g is a real valued function that is increasing and upper semicontinuous and h is a real valued function that is decreasing and lower semicontinuous.

Let us denote $F(\mathbb{R})$ the set of fuzzy numbers.

Starting at the general definition of fuzzy numbers in [54] are presented the main sets of fuzzy numbers: $L-R$ fuzzy numbers, trapezoidal fuzzy numbers, triangular fuzzy numbers, gaussian fuzzy numbers, quasi-gaussian fuzzy numbers, quasi-quadric fuzzy numbers, exponential fuzzy numbers, quasi-exponential fuzzy numbers and single-ton fuzzy numbers.

Definition 3 *A particular case of $L-R$ fuzzy numbers introduced by Dubois and Prade in 1981 [43] is*

$$A(x)=\begin{cases}\left(\frac{x-a}{b-a}\right)^n, & \text{if } x\in[a,b]\\ 1, & \text{if } x\in[b,c]\\ \left(\frac{d-x}{d-c}\right)^n, & \text{if } x\in[c,d]\\ 0, & \text{else.}\end{cases} \tag{1}$$

Let us use the notation $A(x)=(a,b,c,d)_n$.

For the fuzzy number given by (1) has following $\alpha-$cut

$$A_\alpha=\left[a+\sqrt[n]{\alpha}\,(b-a)\,,d-\sqrt[n]{\alpha}\,(d-c)\right];$$

The algebraic structure for fuzzy mathematics was introduced by Mizumoto and Tanaka in [66] and [67], by Dubois and Prade in [43], [41] and [40], by Ma, Friderman and Kandel in [64]. The algebraic operations can be derived from the so-called extended principles [65] or using the $\alpha-$ cuts ([60], [76]). In this paper we use the last method.

1.3 Real numbers and intervals attached to fuzzy numbers

In applications, sometimes for practical reasons, a fuzzy number is assigned a real value or an interval of real numbers. This process captures relevant information of a fuzzy number and leads to simplify the representation and handling fuzzy numbers.

Next, we present some entropy measures and parameters attached to fuzzy numbers and used in fuzzy numbers theory.

1.3.1 Ambiguity, value and expected interval

fuzzy number A is fuzzy subset of a real line $\mathbb{R}$ with the membership function μ_A which is (see [41]): normal (i.e. there exists an element x_0 such that $\mu_A(x_0)=1$), fuzzy convex (i.e. $\mu_A(\lambda x_1+(1-\lambda)x_2)\geq\min(\mu_A(x_1),\mu_A(x_2))$, $\forall x_1,x_2\in\mathbb{R}$, $\lambda\in[0,1]$), upper semicontinuous and supp A is bounded, where supp $A=cl\{x\in X:\mu_A(x)>0\}$ is the support of A and cl is the closure operator.

The α-cut of A , $\alpha\in]0,1]$, of a fuzzy number A is a crisp set defined as:

$$A_\alpha=\{x\in X:\mu_A(x)\geq\alpha\}.$$

Every α-cut of a fuzzy number A is a closed interval $A_\alpha = [A_L(\alpha), A_U(\alpha)]$, where

$$\begin{aligned} A_L(\alpha) &= \inf\{x \in X : \mu_A(x) \geq \alpha\} \\ A_U(\alpha) &= \sup\{x \in X : \mu_A(x) \geq \alpha\}. \end{aligned}$$

The support of a fuzzy number A is denoted by (see [5]):

$$supp(A) = [A_L(0), A_U(0)].$$

The pair of functions (A_L, A_U) gives a parametric representation of a fuzzy number A.

For two arbitrary fuzzy numbers A and B with $\alpha-$cut sets $[A_L(\alpha), A_U(\alpha)]$ and $[B_L(\alpha), B_U(\alpha)]$, $\alpha \in [0,1]$ the metrics

$$D(A,B) = \sqrt{\int_0^1 [A_L(\alpha) - B_L(\alpha)]^2 \, d\alpha + \int_0^1 [A_U(\alpha) - B_U(\alpha)]^2 \, d\alpha},$$

and

$$d(A,B) = \sup_{0 \leq \alpha \leq 1} \{\max\{|A_L(\alpha) - B_L(\alpha)|, |A_U(\alpha) - B_U(\alpha)|\}\}.$$

are presented in [48].

The ambiguity and the value of a fuzzy number denoted with $Amb(A)$ and $Val(A)$ was introduced in [37] as follows

$$Amb(A) = \int_0^1 \alpha(A_U(\alpha) - A_L(\alpha))d\alpha, \tag{2}$$

$$Val(A) = \int_0^1 \alpha(A_U(\alpha) + A_L(\alpha))d\alpha. \tag{3}$$

The expected interval of a fuzzy number A, denoted with $EI(A)$ was introduced in ([40], [55]) as follows:

$$EI(A) = [E_*(A), E^*(A)] = \left[\int_0^1 A_L(\alpha)\, d\alpha, \int_0^1 A_U(\alpha)\, d\alpha\right]$$

The middle of the expected interval is named the expected value, $EV(A)$ was given in [55], as follows:

$$EV(A) = \frac{E_*(A) + E^*(A)}{2}.$$

1.3.2 The entropy of fuzzy numbers

Until now the fuzzy entropy calculus has been studied from different angles, which led to several methods for calculating the entropy. In the first stage the entropy was defined based on the distance. Kaufmann in [59] proposed a way of measuring the degree of fuzzification of a fuzzy set using fuzzy distance between the nearest non-fuzzy set and the fuzzy set. Another way of measuring the entropy was proposed by Yager [77], and this is to measure the distance between the fuzzy set and its complementary. Another approach is to use the entropy function to calculate the entropy.

It should be noted that entropy is a measure that characterizes the elements of fuzzy (sets or numbers) and interest for this concept is increasingly grown in recent years. This can be claimed on the basis of many papers on the theme of entropy [18], [61], [62], [76].

Let $h : [0,1] \to [0,1]$ be a function of calculating the entropy, which satisfies the following properties: it is increasing on the interval $\left[0, \frac{1}{2}\right]$ and descressing on the interval $\left[\frac{1}{2}, 1\right]$,with $h(0) = h(1) = 0, h\left(\frac{1}{2}\right) = 1$ and $h(u) = h(1-u)$.

Let H be

$$H(A) = \int_X h(A(x))dx$$

in [76] is called entropy function.

Some entropy functions h are defined in ([69] and [76]):

$$\begin{aligned}
h_1(u) &= \begin{cases} 2u, & \text{if } u \in \left[0, \frac{1}{2}\right] \\ 2(1-u), & \text{if } u \in \left[\frac{1}{2}, 1\right], \end{cases} , \\
h_2(u) &= 4u(1-u), \\
h_3(u) &= -u \ln u - (1-u)\ln(1-u), \text{ and } 0 \ln 0 := 0,
\end{aligned}$$

h_3 was known as Shannon's function.

We will denote with H_i fuzzy entropy which respects the entropy function h_i.

Using h_i functions is easy to obtain fuzzy entropy of multiplication for two trapezoidal fuzzy numbers.

Theorem 4 *(Theorem 4.1, [18]) If A, B are fuzzy numbers then*

$$\begin{aligned}
(i)\ H_1(A \cdot B) &= -\frac{1}{2}\left(a_1b_1 - a_2b_2 + a_3b_3 - a_4b_4\right); \\
(ii)\ H_2(A \cdot B) &= -\frac{2}{3}\left(a_1b_1 - a_2b_2 + a_3b_3 - a_4b_4\right); \\
(iii)\ H_3(A \cdot B) &= -\frac{1}{2}\left(a_1b_1 - a_2b_2 + a_3b_3 - a_4b_4\right).
\end{aligned}$$

Theorem 5 *(Theorem 5.1, [18]) If A, B are fuzzy numbers then*
(i)

$$\begin{aligned}
H_1(A/B) &= \frac{2\left(a_2b_4 - a_1b_3\right)}{\left(b_4 - b_3\right)^2} \ln \frac{(b_3 + b_4)^2}{4b_3b_4} \\
&\quad + \frac{2(a_4b_2 - a_3b_1)}{\left(b_2 - b_1\right)^2} \ln \frac{(b_2 + b_1)^2}{4b_2b_1}.
\end{aligned}$$

(ii)

$$
\begin{aligned}
H_2(A/B) \quad = \quad & \frac{4(a_2b_4 - a_1b_3)}{(b_4 - b_3)^2}\left(\frac{b_4 + b_3}{b_4 - b_3}\ln\frac{b_4}{b_3} - 2\right) \\
& + \frac{4(a_4b_2 - a_3b_1)}{(b_2 - b_1)^2}\left(\frac{2b_2 + b_1}{b_2 - b_1}\ln\frac{b_1}{b_1} - 2\right).
\end{aligned}
$$

(iii) Moreover, $2b_3 > b_4$ and $b_2 < b_1$ then

$$
\begin{aligned}
H_3(A/B) = & \tfrac{a_2b_4 - a_1b_3}{(b_4-b_3)b_4}\sum_{k=1}^{\infty}\frac{k}{(k+1)^2}\left(\frac{b_4 - b_3}{b_4}\right)^k + \\
& + \tfrac{a_2b_4 - a_1b_3}{(b_4-b_3)b_3}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}k}{(k+1)^2}\left(\frac{b_4 - b_3}{b_3}\right)^k + \\
& + \tfrac{a_4b_2 - a_3b_1}{(b_2-b_1)b_1}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}k}{(k+1)^2}\left(\frac{b_2 - b_1}{b_1}\right)^k + \\
& + \tfrac{a_4b_2 - a_3b_1}{(b_2-b_1)b_2}\sum_{k=1}^{\infty}\frac{k}{(k+1)^2}\left(\frac{b_2 - b_1}{b_2}\right)^k.
\end{aligned}
$$

Proposition 6 *If*

$$A \in \mathbb{T}, B \in \mathbb{T}\ .$$

Then

$$
\begin{aligned}
H_1(A \odot B) &= \frac{1}{2}(2a_2b_2 - a_2b_1 - a_1b_2 - 2a_3b_3 + a_3b_4 + a_4b_3) \\
H_2(A \odot B) &= \frac{2}{3}(2a_2b_2 - a_2b_1 - a_1b_2 - 2a_3b_3 + a_3b_4 + a_4b_3) \\
H_3(A \odot B) &= \frac{1}{2}(2a_2b_2 - a_2b_1 - a_1b_2 - 2a_3b_3 + a_3b_4 + a_4b_3).
\end{aligned}
$$

Proposition 7 *If*

$$A \in \mathbb{T}, B \in \mathbb{T}$$

then

$$H_i(A \odot B) = b_2H_i(A) + a_2H_i(B) - k\left[(b_4 - b_3)(a_2 - a_3) + (a_4 - a_3)(b_2 - b_3)\right]$$

where

$$k = \begin{cases} \frac{1}{2}, & \text{if } i \in \{1, 3\} \\ \frac{2}{3}, & \text{if } i = 2. \end{cases}$$

1.4 Approximation criteria

Throughout this paper we propose to approximate the fuzzy numbers with trapezoidal fuzzy numbers. Thus, we introduce some operators defined on the set of trapezoidal fuzzy numbers with trapezoidal fuzzy numbers in the set values. Since the trapezoidal approximation can be studied from various angles, looking for operators who perform a large part of the property as outlined in the criteria list from [51]. These criteria are:

Translation invariance

$$T\left(A+z\right)=T\left(A\right)+z \tag{4}$$

for every $A\in F(\mathbb{R})$;

Linearity,

$$\begin{gathered} T\left(\lambda A\right)=\lambda T\left(A\right)\\ T\left(A+B\right)=T\left(A\right)+T\left(B\right) \end{gathered} \tag{5}$$

for every A, $B\in F(\mathbb{R})$ and $\lambda\in\mathbb{R}^{*}$;

α_0 invariance

$$\left(T\left(A\right)\right)_{\alpha_0}=A_{\alpha_0}, \tag{6}$$

for example: support invariance or core invariance.

Identity

$$T\left(A\right)=A, \tag{7}$$

for every $A\in F^{T}\left(\mathbb{R}\right)$;

Expected interval invariance

$$EI\left(T\left(A\right)\right)=EI\left(A\right), \tag{8}$$

for every $A\in F(\mathbb{R})$;

Expected value invariance

$$EV\left(T\left(A\right)\right)=EV\left(A\right), \tag{9}$$

for every $A\in F(\mathbb{R})$;

Order invariance with respect to ordering $\succ$ defined in [77]

$$A\succ B\Leftrightarrow EV\left(A\right)\geqslant EV\left(B\right) \tag{10}$$

so

$$A\succ B\Leftrightarrow T\left(A\right)\succ T\left(B\right) \tag{11}$$

for every A, $B\in F(\mathbb{R})$;

Order invariance with respect to ordering M defined in [58] as follows:

$$M\left(A,B\right)=\left\{\begin{array}{c} 0,\text{if }\ E^{*}(A)-E_{*}(B)<0,\\ \frac{E^{*}(A)-E_{*}(B),}{E^{*}(A)-E_{*}(B)-(E_{*}(A)-E^{*}(B))},\ \text{if } 0\in\left[E_{*}(A)-E^{*}(B),E^{*}(A)-E_{*}(B)\right],\\ 1,\text{if } E_{*}(A)-E^{*}(B)>0, \end{array}\right. \tag{12}$$

so

$$M\left(T\left(A\right),T\left(B\right)\right)=M\left(A,B\right), \tag{13}$$

for every $A,B\in F(\mathbb{R})$;

Nonspecificity invariance defined in [33], as follows:

$$w(A) = \int_{-\infty}^{\infty} \mu_A(x)dx, \tag{14}$$

so

$$w(A) = w(T(A)), \tag{15}$$

for every $A \in F(\mathbb{R})$;

Correlation invariance

$$\rho(T(A), T(B)) = \rho(A, B) \tag{16}$$

for every $A, B \in F(\mathbb{R})$, where $\rho(A, B)$ denotes a correlation of two fuzzy numbers A and B, defined in [57] as follows:

$$\rho(A, B) = \frac{E_*(A)E_*(B) + E^*(A)E^*(B)}{\sqrt{(E_*(A))^2 + (E^*(A))^2}\sqrt{(E_*(B))^2 + (E^*(B))^2}}. \tag{17}$$

Monotony

$$\text{if } A \subset B \text{ then } T(A) \subset T(B). \tag{18}$$

Nearness criterion

$$d(A, T(A)) \leq d(A, B), \tag{19}$$

where d is a metric and $A, B \in F(\mathbb{R})$;

Continuity

$$\forall \varepsilon > 0, \exists\, \delta > 0 \ d(A, B) < \delta \Rightarrow d(T(A), T(B)) < \varepsilon, \tag{20}$$

where d is a metric and $A, B \in F(\mathbb{R})$;

The below version of well-known Karush-Kuhn-Tucker theorem is useful in the solving of the proposed problem.

Theorem 8 *(see Rockafellar [71], pp. 281-283) Let $f, g_1, g_2, ..., g_m : \mathbb{R}^n \to \mathbb{R}$ be convex and differentiable functions. Then $\bar{x}$ solves the convex programming problem*

$$\min f(x)$$

subject to $g_i(x) \leq b_i$, $i \in \{1, 2, 3, ..., m\}$ if and only if there exist μ_i, where $i \in \{1, 2, 3, ..., m\}$, such that

(i) $\nabla f\left(\bar{x}\right) + \sum_{i=1}^{m} \mu_i \nabla g_i\left(\bar{x}\right) = 0;$

(ii) $g_i\left(\bar{x}\right) - b_i \leq 0;$

(iii) $\mu_i \geq 0;$

(iv) $\mu_i\left(b_i - g_i\left(\bar{x}\right)\right) = 0.$

2 Approximation operators that preserve expectancy and level sets

In this chapter we introduce three operators that preserve approximation Expectancy and level set, as follows: trapezoidal approximation operator which preserves the expected interval and the core, the approximation operator which preserves and support and the expected interval in [26] and the trapezoidal approximation which preserves the expected value and the core, in paper [28]. For those operators we study properties like: translation invariance, continuity, preserving the expected value or expected interval, identity criterion.

2.1 Trapezoidal approximation which preserves the expected interval and the core

2.2 Trapezoidal approximation which preserves the expected interval and the support

Let us denote

$$F_{ES}(\mathbb{R})=\left\{A\in F(\mathbb{R})|\ 2\int_0^1\left[A_U(\alpha)-A_L(\alpha)\right]d\alpha\geq A_U(0)-A_L(0)\right\}$$

Theorem 9 *If $A\in F_{ES}(\mathbb{R})$ then*

$$T(A)=\left(A_L(0),2\int_0^1 A_L(\alpha)\,d\alpha-A_L(0),2\int_0^1 A_U(\alpha)\,d\alpha-A_U(0),A_U(0)\right)$$

is the trapezoidal approximation which preserves the expected interval and the support.

Remark 10 *If $A\notin F_{ES}(\mathbb{R})$ then it doesn't exist a trapezoidal fuzzy number which preserves the expected interval and the support of the fuzzy number A.*

Example 11 *Let A be a fuzzy number*

$$\begin{aligned} A_L(\alpha) &= 1+\alpha^2,\\ A_U(\alpha) &= 3-\alpha^2,\alpha\in[0,1] \end{aligned}$$

then the trapezoidal approximation which preserves the expected interval and the support is

$$T(A)=\left(1,\frac{5}{3},\frac{7}{3},3\right).$$

Example 12 *Let us consider a fuzzy number A given by*

$$\begin{aligned} &A_L(\alpha)=1+\sqrt{\alpha},\\ &A_U(\alpha)=45-35\sqrt{\alpha},\alpha\in[0,1]. \end{aligned}$$

Because

$$2\int_0^1 \left[A_U(\alpha) - A_L(\alpha)\right] d\alpha = 40 < 44 = A_U(0) - A_L(0)$$

we get $A \notin F_{ES}(\mathbb{R})$ *and not exists a trapezoidal fuzzy number which preserves the expected interval and the support of* A*.*

Remark 13 *If* $A = (a, b, c, d)_n$*, is a fuzzy number,* $n \in \mathbb{R}_+^*$ *and*

$$2n(b-c) \leq (n-1)(a-d)$$

then

$$T(A) = \left(a, \frac{2bn - an + a}{n+1}, \frac{2cn - dn + d}{n+1}, d\right)$$

is the trapezoidal fuzzy number which preserves the expected interval and the support of A*.*

Example 14 *Let* $B = (5, 8, 12, 14)_{\frac{1}{3}}$ *be a fuzzy number then the trapezoidal approximation which preserves the expected interval and the support is:*

$$T(B) = \left(5, \frac{13}{2}, 13, 14\right).$$

The main properties of the trapezoidal approximation operator which preserves the expected interval and the support are introduced in the following theorem.

Theorem 15 *The trapezoidal approximation operator preserving the expected interval and the support* $T : F(\mathbb{R}) \longrightarrow F^T(\mathbb{R})$*:*

(i) is invariant to translation, that is

$$T(A + z) = T(A) + z$$

for every $A \in F(\mathbb{R})$*;*

(ii) is a linear operator, that is

$$\begin{aligned} T(\lambda A) &= \lambda T(A) \\ T(A+B) &= T(A) + T(B) \end{aligned}$$

for every $A, B \in F(\mathbb{R})$ *and* $\lambda \in \mathbb{R}^*$;

(iii) fulfills the identity criterion, that is

$$T(A) = A,$$

for every $A \in F^T(\mathbb{R})$*;*

(iv)is expected interval invariant, that is

$$EI(T(A)) = EI(A),$$

for every $A \in F(\mathbb{R})$;

(v) is order invariant with respect to the preference relation $\succ$ *defined by (see [77])*

$$A \succ B \Leftrightarrow EV(A) \geqslant EV(B)$$

that is

$$A \succ B \Leftrightarrow T(A) \succ T(B)$$

for every $A,\ B \in F(\mathbb{R})$;

(vi) is order invariant with respect to the preference relation M *defined by (see [58])*

$$M(A,B) = \begin{cases} 0, & \text{if } E^*(A) - E_*(B) < 0 \\ \frac{E^*(A)-E_*(B),}{E^*(A)-E_*(B)-(E_*(A)-E^*(B))}, & \text{if } 0 \in [E_*(A) - E^*(B), E^*(A) - E_*(B)] \\ 1, & \text{if } E_*(A) - E^*(B) > 0 \end{cases}$$

that is

$$M(T(A), T(B)) = M(A,B)$$

for every $A, B \in F(\mathbb{R})$;

(vii) is uncertainty invariant with respect to the nonspecificity measure defined by (see [**?**] *)*

$$w(A) = \int_{-\infty}^{\infty} \mu_A(x)dx$$

that is

$$w(A) = w(T(A))$$

for every $A \in F(\mathbb{R})$;

(viii) is correlation invariant, that is

$$\rho(T(A), T(B)) = \rho(A,B)$$

for every $A, B \in F(\mathbb{R})$, *where* $\rho(A,B)$ *denotes the correlation coefficient between A and B, defined as (see [57])*

$$\rho(A,B) = \frac{E_*(A)E_*(B) + E^*(A)E^*(B)}{\sqrt{(E_*(A))^2 + (E^*(A))^2}\sqrt{(E_*(B))^2 + (E^*(B))^2}}$$

2.3 Trapezoidal approximation which preserves expected value and core

Theorem 16 *If A is a fuzzy number then* $T(A) = (t_1, t_2, t_3, t_4)$ *is the nearest trapezoidal approximation of A, with respect the metric D, preserving the core and the expected value.*

(i) If

$$\int_0^1 [(2-6\alpha) A_U(\alpha) + (6\alpha - 10) A_L(\alpha)]\, d\alpha + 7A_L(1) + A_U(1) < 0 \tag{21}$$

and

$$\int_0^1 [A_L(\alpha) + A_U(\alpha)]\, d\alpha \geq A_L(1) + A_U(1), \tag{22}$$

then

$$\begin{aligned} t_1 &= t_2 = A_L(1) \\ t_3 &= A_U(1) \\ t_4 &= 2\int_0^1 A_L(\alpha)\, d\alpha + 2\int_0^1 A_U(\alpha)\, d\alpha - 2A_L(1) - A_U(1). \end{aligned}$$

(ii) If

$$\int_0^1 [(2-6\alpha) A_L(\alpha) + (6\alpha - 10) A_U(\alpha)]\, d\alpha + A_L(1) + 7A_U(1) > 0 \tag{23}$$

and

$$\int_0^1 [A_L(\alpha) + A_U(\alpha)]\, d\alpha \leq A_L(1) + A_U(1), \tag{24}$$

then

$$\begin{aligned} t_1 &= 2\int_0^1 A_L(\alpha)\, d\alpha + 2\int_0^1 A_U(\alpha)\, d\alpha - A_L(1) - 2A_U(1) \\ t_2 &= A_L(1) \\ t_3 &= A_U(1) \\ t_4 &= A_U(1). \end{aligned}$$

(iii) If

$$\int_0^1 [(2-6\alpha) A_U(\alpha) + (6\alpha - 10) A_L(\alpha)]\, d\alpha + 7A_L(1) + A_U(1) \geq 0 \tag{25}$$

şi

$$\int_0^1 [(2-6\alpha) A_L(\alpha) + (6\alpha - 10) A_U(\alpha)]\, d\alpha + A_L(1) + 7A_U(1) \leq 0 \tag{26}$$

then

$$t_1 = -\frac{3}{2}\int_0^1 \alpha\left(A_L(\alpha) - A_U(\alpha)\right) d\alpha - \frac{3}{4}A_L(1) + \frac{1}{2}\int_0^1 \left(5A_L(\alpha) - A_U(\alpha)\right) d\alpha - \frac{1}{4}A_U(1)$$

$$\begin{aligned} t_2 &= A_L(1) \\ t_3 &= A_U(1) \end{aligned}$$

$$t_4 = \frac{3}{2}\int_0^1 \alpha\left(A_L(\alpha) - A_U(\alpha)\right) d\alpha - \frac{1}{4}A_L(1) - \frac{1}{2}\int_0^1 \left(A_L(\alpha) - 5A_U(\alpha)\right) d\alpha - \frac{3}{4}A_U(1).$$

Example 17 *Let A be a fuzzy number*

$$A_\alpha = \left[1 + 99\sqrt{\alpha}, 200 - 95\sqrt{\alpha}\right]$$

then $T(A) = (t_1, t_2, t_3, t_4)$ is

$$\begin{aligned} t_1 &= -\frac{3}{2}\int_0^1 \alpha\left(A_L(\alpha) - A_U(\alpha)\right) d\alpha - \frac{3}{4}A_L(1) + \\ &\quad + \frac{1}{2}\int_0^1 \left(5A_L(\alpha) - A_U(\alpha)\right) d\alpha - \frac{1}{4}A_U(1) \\ &= \frac{923}{30} \end{aligned}$$

$$\begin{aligned} t_2 &= A_L(1) = 100 \\ t_3 &= A_U(1) = 105 \\ t_4 &= A_U(1) = 105 \end{aligned}$$

Theorem 18 *If $A = (a, b, c, d)_n$ is a fuzzy number then $T(A) = (t_1, t_2, t_3, t_4)$ is the nearest trapezoidal fuzzy number preserving the core and the expected value*

(i) If

$$(n-1)(d-c) + (17n+7)(b-a) < 0$$

and

$$a - b - c + d \geq 0,$$

then

$$\begin{aligned} t_1 &= t_2 = b \\ t_3 &= c \\ t_4 &= \frac{2a - 2b - c + 2d + cn}{n+1}. \end{aligned}$$

(ii) If

$$(b-a)(1-n) - (17n+7)(d-c) > 0$$

and

$$a - b - c + d \leq 0,$$

then

$$\begin{aligned} t_1 &= \frac{2a - b - 2c + 2d + bn}{n+1} \\ t_2 &= b \\ t_3 &= c \\ t_4 &= c. \end{aligned}$$

(iii) If

$$(n-1)(d-c) + (17n+7)(b-a) \geq 0$$

and
$$(b-a)(1-n)-(17n+7)(d-c)\leq 0$$
then
$$t_1=\frac{8bn^2+17an-5bn+cn-dn+7a-3b-c+d}{4(n+1)(2n+1)}$$

$$\begin{aligned} t_2 &= b \\ t_3 &= c \end{aligned}$$

$$t_4=\frac{8cn^2-an+bn-5cn+17dn+a-b-3c+7d}{4(n+1)(2n+1)}.$$

3 Approximation operators which preserve ambiguity and value

3.1 Trapezoidal approximation which preserves the ambiguity and the value

Sometimes (see [80]) it is useful to denote a trapezoidal fuzzy number by

$$T=[l,u,x,y]$$

with $l,u,x,y\in\mathbb{R}$ such that $x,y\geq 0, x+y\leq 2(u-l)$,

$$T_L(\alpha)=l+x(\alpha-\frac{1}{2}),$$
$$T_U(\alpha)=u-y(\alpha-\frac{1}{2}),$$

for every $\alpha\in[0,1]$.

It is immediate that

$$l=\frac{t_1+t_2}{2}, \tag{27}$$
$$u=\frac{t_3+t_4}{2}, \tag{28}$$
$$x=t_2-t_1, \tag{29}$$
$$y=t_4-t_3 \tag{30}$$

and $T\in F^S(\mathbb{R})$ if and only if $x=y$. Also, by direct calculation we get

$$Amb(T)=\frac{-6l+6u-x-y}{12}, \tag{31}$$
$$Val(T)=\frac{6l+6u+x-y}{12}. \tag{32}$$

The distance between $T, T' \in F^T(\mathbb{R}), T = [l, u, x, y]$ and $T' = [l', u', x', y']$ becomes ([79])

$$D^2(T, T') = (l - l')^2 + (u - u')^2 + \frac{1}{12}(x - x')^2 + \frac{1}{12}(y - y')^2. \tag{33}$$

An extended trapezoidal fuzzy number ([79]) is an order pair of polynomial functions of degree less than or equal to 1. An extended trapezoidal fuzzy number may not be a fuzzy number, but the distance between two extended trapezoidal fuzzy numbers is similarly defined as in (**??**) or (33). In addition, we define the value and the ambiguity of an extended trapezoidal fuzzy number in the same way as in the case of a trapezoidal fuzzy number. We denote by $F_e^T(\mathbb{R})$ the set of all extended trapezoidal fuzzy numbers, that is

$$F_e^T(\mathbb{R}) = \left\{ T = [l, u, x, y] : T_L(\alpha) = l + x\left(\alpha - \tfrac{1}{2}\right), \right.$$
$$\left. T_U(\alpha) = u - y\left(\alpha - \tfrac{1}{2}\right), \alpha \in [0, 1], l, u, x, y \in \mathbb{R} \right\},$$

where T_L and T_U have the same meaning as above.

The extended trapezoidal approximation $T_e(A) = [l_e, u_e, x_e, y_e]$ of a fuzzy number A is the extended trapezoidal fuzzy number which minimizes the distance $d(A, X)$ where $X \in F_e^T(\mathbb{R})$. In the paper [5] the authors proved that $T_e(A)$ is not always a fuzzy number. The extended trapezoidal approximation $T_e(A) = [l_e, u_e, x_e, y_e]$ of a fuzzy number A is determined ([79]) by the following equalities

$$l_e = \int_0^1 A_L(\alpha) d\alpha, \tag{34}$$

$$u_e = \int_0^1 A_U(\alpha) d\alpha, \tag{35}$$

$$x_e = 12 \int_0^1 (\alpha - \frac{1}{2}) A_L(\alpha) d\alpha, \tag{36}$$

$$y_e = -12 \int_0^1 (\alpha - \frac{1}{2}) A_U(\alpha) d\alpha. \tag{37}$$

The real numbers x_e and y_e are non-negative (see [79]) and from the definition of a fuzzy number we have $l_e \leq u_e$.

In the paper [78] the author proved two very important distance properties for the extended trapezoidal approximation operator, as follows.

Proposition 19 *([78], Proposition 4.2) Let A be a fuzzy number. Then*

$$D^2(A, B) = D^2(A, T_e(A)) + D^2(T_e(A), B) \tag{38}$$

for any trapezoidal fuzzy number B.

Proposition 20 *([78], Proposition 4.4.) $D(T_e(A), T_e(B)) \leq D(A, B)$ for all fuzzy numbers A, B.*

Remark 21 *Let $A, B \in F(\mathbb{R})$ and $T_e(A) = [l_e, u_e, x_e, y_e]$, $T_e(B) = [l'_e, u'_e, x'_e, y'_e]$ the extended trapezoidal approximations of A and B. Proposition 20 and (33) imply*

$$(l_e - l'_e)^2 + (u_e - u'_e)^2 \leq D^2(A, B)$$

and

$$(x_e - x'_e)^2 + (y_e - y'_e)^2 \leq 12D^2(A, B).$$

Therefore $T(A) = [l_T, u_T, x_T, y_T]$ if and only if $(l_T, u_T, x_T, y_T) \in \mathbb{R}^4$ is a solution of the problem

$$\min\left((l - l_e)^2 + (u - u_e)^2 + \frac{1}{12}(x - x_e)^2 + \frac{1}{12}(y - y_e)^2\right) \tag{39}$$

under the conditions

$$x \geqslant 0, \tag{40}$$
$$y \geqslant 0, \tag{41}$$
$$x + y \leqslant 2u - 2l, \tag{42}$$
$$-6l + 6u - x - y = -6l_e + 6u_e - x_e - y_e, \tag{43}$$
$$6l + 6u + x - y = 6l_e + 6u_e + x_e - y_e, \tag{44}$$

where l_e, u_e, x_e, y_e are given by (34)-(37). We immediately obtain that (39)-(44) is equivalent to

$$\min\left((x - x_e)^2 + (y - y_e)^2\right) \tag{45}$$

under the conditions

$$x \geqslant 0, \tag{46}$$
$$y \geqslant 0, \tag{47}$$
$$x + y \leqslant 3u_e - 3l_e - \frac{1}{2}x_e - \frac{1}{2}y_e. \tag{48}$$

In addition,

$$l = -\frac{1}{6}(x - x_e) + l_e \tag{49}$$

and

$$u = \frac{1}{6}(y - y_e) + u_e. \tag{50}$$

Let us consider the set

$$M_A = \left\{(x, y) \in \mathbb{R}^2 : x \geqslant 0, y \geqslant 0, x + y \leqslant 3u_e - 3l_e - \frac{1}{2}x_e - \frac{1}{2}y_e\right\},$$

d_E the Euclidean metric on $\mathbb{R}^2$ and let us denote by $P_M(Z)$ the orthogonal projection of $Z \in \mathbb{R}^2$ on non-empty set $M \subset \mathbb{R}^2$, with respect to d_E.

Theorem 22 (*Theorem 5, [13]*) *The problem (45)-(48) has an unique solution.*

Theorem 23 (*Theorem 7, [13]*)*Let* $A \in F(\mathbb{R})$, $A_\alpha = [A_L(\alpha), A_U(\alpha)], \alpha \in [0,1]$, *and* $T(A) = [l_T, u_T, x_T, y_T]$ *the nearest trapezoidal fuzzy number to* A *which preserves ambiguity and value.*

(*i*) *If*

$$\int_0^1 (3\alpha - 1) A_L(\alpha) d\alpha - \int_0^1 (3\alpha - 1) A_U(\alpha) d\alpha \leq 0 \tag{51}$$

then

$$\begin{aligned}
x_T &= 6 \int_0^1 (2\alpha - 1) A_L(\alpha) d\alpha, \\
y_T &= -6 \int_0^1 (2\alpha - 1) A_U(\alpha) d\alpha, \\
l_T &= \int_0^1 A_L(\alpha) d\alpha, \\
u_T &= \int_0^1 A_U(\alpha) d\alpha.
\end{aligned}$$

(*ii*) *If*

$$\int_0^1 (3\alpha - 1) A_L(\alpha) d\alpha + \int_0^1 (\alpha - 1) A_U(\alpha) d\alpha > 0 \tag{52}$$

then

$$\begin{aligned}
x_T &= -6 \int_0^1 \alpha A_L(\alpha) d\alpha + 6 \int_0^1 \alpha A_U(\alpha) d\alpha \\
y_T &= 0 \\
l_T &= 3 \int_0^1 \alpha A_L(\alpha) d\alpha - \int_0^1 \alpha A_U(\alpha) d\alpha \\
u_T &= 2 \int_0^1 \alpha A_U(\alpha) d\alpha
\end{aligned}$$

(*iii*) *If*

$$\int_0^1 (\alpha - 1) A_L(\alpha) d\alpha + \int_0^1 (3\alpha - 1) A_U(\alpha) d\alpha < 0 \tag{53}$$

then

$$\begin{aligned}
x_T &= 0 \\
y_T &= -6 \int_0^1 \alpha A_L(\alpha) d\alpha + 6 \int_0^1 \alpha A_U(\alpha) d\alpha \\
l_T &= 2 \int_0^1 \alpha A_L(\alpha) d\alpha \\
u_T &= -\int_0^1 \alpha A_L(\alpha) d\alpha + 3 \int_0^1 \alpha A_U(\alpha) d\alpha
\end{aligned}$$

(iv) If

$$\int_0^1 (3\alpha - 1)\, A_L(\alpha) d\alpha - \int_0^1 (3\alpha - 1)\, A_U(\alpha) d\alpha > 0 \quad (54)$$

$$\int_0^1 (3\alpha - 1)\, A_L(\alpha) d\alpha + \int_0^1 (\alpha - 1)\, A_U(\alpha) d\alpha \leq 0 \quad (55)$$

and

$$\int_0^1 (\alpha - 1)\, A_L(\alpha) d\alpha + \int_0^1 (3\alpha - 1)\, A_U(\alpha) d\alpha \geq 0 \quad (56)$$

then

$$x_T = 3\int_0^1 (\alpha - 1)\, A_L(\alpha) d\alpha + 3\int_0^1 (3\alpha - 1)\, A_U(\alpha) d\alpha$$
$$y_T = -3\int_0^1 (3\alpha - 1)\, A_L(\alpha) d\alpha - 3\int_0^1 (\alpha - 1)\, A_U(\alpha) d\alpha$$
$$l_T = \frac{1}{2}\int_0^1 (3\alpha + 1)\, A_L(\alpha) d\alpha - \frac{1}{2}\int_0^1 (3\alpha - 1)\, A_U(\alpha) d\alpha$$
$$u_T = -\frac{1}{2}\int_0^1 (3\alpha - 1)\, A_L(\alpha) d\alpha + \frac{1}{2}\int_0^1 (3\alpha + 1)\, A_U(\alpha) d\alpha.$$

Corollary 24 *(i) If $A \in \Omega_1$, then*

$$T(A) = (4I - 6L, 6L - 2I, 6U - 2S, 4S - 6U).$$

(ii) If $A \in \Omega_2$, then

$$T(A) = (6L - 4U, 2U, 2U, 2U).$$

(iii)If $A \in \Omega_3$, then

$$T(A) = (2L, 2L, 2L, 6U - 4L).$$

(iv) If $A \in \Omega_4$, then

$$\begin{aligned} T(A) &= (2I - 6U + 2S, 3L - I + 3U - S, \\ &\quad 3L - I + 3U - S, 2I - 6L + 2S). \end{aligned}$$

Example 25 *Let A and B be fuzzy numbers*

$$\begin{aligned} A_L(\alpha) &= 1 + \sqrt{\alpha}, \\ A_U(\alpha) &= 4 - \sqrt{\alpha}, \\ B_L(\alpha) &= 1 + \sqrt{\alpha}, \\ B_U(\alpha) &= 35 - 31\sqrt{\alpha}. \end{aligned}$$

We obtain that $A \in \Omega_1$ and $B \in \Omega_4$

$$\begin{aligned} T(A) &= \left(\frac{19}{15}, \frac{31}{15}, \frac{34}{15}, \frac{76}{15}\right) \\ T(B) &= \left(\frac{29}{15}, 2, 2, \frac{419}{15}\right). \end{aligned}$$

Since

$$\begin{aligned} (A+B)_L(\alpha) &= 2 + 2\sqrt{\alpha} \\ (A+B)_U(\alpha) &= 39 - 32\sqrt{\alpha}, \end{aligned}$$

we obtain that

$$T(A) + T(B) = \left(\frac{48}{15}, \frac{61}{15}, \frac{64}{15}, \frac{495}{15}\right),$$

and

$$T(A+B) = \left(\frac{38}{15}, \frac{62}{15}, \frac{73}{15}, \frac{457}{15}\right),$$

so

$$T(A) + T(B) \neq T(A+B),$$

thus the operator T is not additive.

Theorem 26 *If $A, B \in \Omega_i$, $i \in \{1, 2, 3, 4\}$, then*

$$T(A) + T(B) = T(A+B).$$

3.2 Trapezoidal approximation which preserves the ambiguity, the value and the core

Theorem 27 *If $A_\alpha = [A_L(\alpha), A_U(\alpha)], \alpha \in [0,1]$ is a fuzzy number then one has $T(A) = (t_1, t_2, t_3, t_4)$, where*

$$\begin{aligned} t_1 &= 6\int_0^1 \alpha A_L(\alpha)\, d\alpha - 2A_L(1) \\ t_2 &= A_L(1) \\ t_3 &= A_U(1) \\ t_4 &= 6\int_0^1 \alpha A_U(\alpha)\, d\alpha - 2A_U(1) \end{aligned}$$

is the trapezoidal fuzzy number preserving the core, the ambiguity and the value.

Example 28 *Let A be a fuzzy number, with the parametric representation*

$$\begin{aligned} A_L(\alpha) &= 1 + \sqrt{\alpha} \\ A_U(\alpha) &= 30 - 27\sqrt{\alpha}. \end{aligned}$$

The trapezoidal fuzzy number preserving the core, the ambiguity and the value is

$$T(A)=\left(\frac{7}{5},2,3,\frac{96}{5}\right)$$

Example 29 *Let A be a fuzzy number, with the parametric representation*

$$\begin{aligned} A_L(\alpha) &= 1+27\sqrt{\alpha} \\ A_U(\alpha) &= 30-\sqrt{\alpha}. \end{aligned}$$

The trapezoidal fuzzy number preserving the core, the ambiguity and the value is

$$T(A)=\left(\frac{59}{5},28,29,\frac{148}{5}\right)$$

Theorem 30 *If $A=(a,b,c,d)_n$ is a fuzzy number then the trapezoidal fuzzy number preserving the core, the ambiguity and the value is*

$$T(A)=\left(\frac{2bn+3a-b}{2n+1},b,c,\frac{2cn+3d-c}{2n+1}\right).$$

Theorem 31 *The trapezoidal fuzzy number preserving the core, the ambiguity and the value $T:F(\mathbb{R})\to F^T(\mathbb{R})$ fulfills the following properties:*
(i) is invariant to translations;
(ii) is linear;
(iii) fulfills the identity criterion;

3.3 Trapezoidal approximation which preserves the ambiguity, the value and the support

4 Weighted approximation

4.1 Weighted approximation which preserves the support

The processing of fuzzy numbers is sometimes difficult, therefore several methods have been introduced for approximating the fuzzy numbers with trapezoidal fuzzy numbers. Each of these methods, either trapezoidal approximation [5], [9], [51], [52], [50], [49], [80] or weighted approximation [?], [?] bring some benefits and important properties.

The Karush-Kuhn-Tucker theorem is used in this paper to compute the nearest weighted trapezoidal fuzzy number with respect to weighted distance, such that the support of given fuzzy number is preserved. Important properties such translation invariance, scale invariance, identity, nearest criterion and continuity of the weighted trapezoidal approximation are studied.

Definition 32 *For two arbitrary fuzzy numbers A and B with $\alpha-$cut sets $[A_L(\alpha), A_U(\alpha)]$ and $[B_L(\alpha), B_U(\alpha)]$, $\alpha \in [0,1]$ the quantity*

$$e(A,B) = \left[\int_0^1 f(\alpha)\, g^2(A_\alpha, B_\alpha) d\alpha\right]^{\frac{1}{2}} \tag{57}$$

is the weighted distance between A and B, where

$$g^2(A_\alpha, B_\alpha) = [A_L(\alpha) - B_L(\alpha)]^2 + [A_U(\alpha) - B_U(\alpha)]^2,$$

and f is a non-negative function, increasing on $[0,1]$ such that

$$f(0) = 0 \tag{58}$$

named the weighting function. For a fuzzy number A, $A_\alpha = [A_L(\alpha), A_U(\alpha)], \alpha \in [0,1]$, the problem is to find a trapezoidal fuzzy number $T(A) = (t_1, t_2, t_3, t_4)$, which is the nearest to A with respect to metric e (see 57) and preserves the support of the fuzzy number A.

The fuzzy number has to fulfills the following conditions:

$$\begin{array}{c} \min e(A, T(A)) \\ t_1 = A_L(0) \\ t_4 = A_U(0). \end{array} \tag{59}$$

The problem is to determine four real numbers t_1, t_2, t_3, t_4, which are solutions of system (59), where $t_1 \le t_2 \le t_3 \le t_4$, so we have to minimize the function D, where

$$\begin{aligned} D(t_1, t_2, t_3, t_4) &= \int_0^1 f(\alpha)\left[A_L(\alpha) - (t_1 + (t_2 - t_1)\alpha)\right]^2 d\alpha + \\ &\quad + \int_0^1 f(\alpha)\left[A_U(\alpha) - (t_4 + (t_3 - t_4)\alpha)\right]^2 d\alpha. \end{aligned}$$

We reformulate the problem by (59) as follows

$$\min h(t_2, t_3)$$

where

$$
\begin{aligned}
h\left(t_2,t_3\right) = & \int_0^1 f\left(\alpha\right) A_L^2\left(\alpha\right) d\alpha - 2t_2 \int_0^1 \alpha f\left(\alpha\right) A_L\left(\alpha\right) d\alpha + \\
& + t_2^2 \int_0^1 \alpha^2 f\left(\alpha\right) d\alpha + 2t_2 A_L\left(0\right) \int_0^1 \alpha f\left(\alpha\right)\left(1-\alpha\right) d\alpha \\
& + \int_0^1 f\left(\alpha\right)\left[A_L\left(0\right)\left(1-\alpha\right)\right]^2 d\alpha - 2\int_0^1 f\left(\alpha\right) A_L\left(\alpha\right) A_L\left(0\right)\left(1-\alpha\right) d\alpha \\
& + \int_0^1 f\left(\alpha\right) A_U^2\left(\alpha\right) d\alpha - 2t_3 \int_0^1 \alpha f\left(\alpha\right) A_U\left(\alpha\right) d\alpha \\
& + t_3^2 \int_0^1 \alpha^2 f\left(\alpha\right) d\alpha - 2\int_0^1 f\left(\alpha\right) A_U\left(\alpha\right) A_U\left(0\right)\left(1-\alpha\right) d\alpha \\
& + 2t_3 \int_0^1 \alpha f\left(\alpha\right) A_U\left(0\right)\left(1-\alpha\right) d\alpha + \int_0^1 f\left(\alpha\right)\left[A_U\left(0\right)\left(1-\alpha\right)\right]^2 d\alpha.
\end{aligned}
\tag{60}
$$

subject to

$$
\begin{array}{c}
A_L\left(0\right) \le t_2 \\
t_2 \le t_3 \\
t_3 \le A_U\left(0\right).
\end{array}
\tag{61}
$$

Theorem 33 *Let A, $A_\alpha = \left[A_L\left(\alpha\right), A_U\left(\alpha\right)\right]$ be a fuzzy number and $T(A) = (t_1, t_2, t_3, t_4)$ the nearest (with respect to metric e) trapezoidal fuzzy number to fuzzy number A preserving the support of A.*

(i) If

$$
\begin{array}{c}
\int_0^1 \alpha f\left(\alpha\right)\left[A_L\left(\alpha\right) + A_U\left(\alpha\right)\right] d\alpha - \\
- A_L\left(0\right) \int_0^1 \alpha\left(1-\alpha\right) f\left(\alpha\right) d\alpha - A_U\left(0\right) \int_0^1 \alpha\left(1+\alpha\right) f\left(\alpha\right) d\alpha > 0
\end{array}
\tag{62}
$$

then

$$t_1 = A_L\left(0\right) \tag{63}$$

$$t_2 = A_U\left(0\right) \tag{64}$$

$$t_3 = A_U\left(0\right) \tag{65}$$

$$t_4 = A_U\left(0\right). \tag{66}$$

(ii) If

$$
\begin{array}{c}
- \int_0^1 \alpha f\left(\alpha\right)\left[A_L\left(\alpha\right) + A_U\left(\alpha\right)\right] d\alpha + \\
+ A_L\left(0\right) \int_0^1 \alpha\left(1+\alpha\right) f\left(\alpha\right) d\alpha + A_U\left(0\right) \int_0^1 \alpha\left(1-\alpha\right) f\left(\alpha\right) d\alpha > 0
\end{array}
\tag{67}
$$

then

$$t_1 = A_L\left(0\right) \tag{68}$$

$$t_2 = A_L(0) \tag{69}$$

$$t_3 = A_L(0) \tag{70}$$

$$t_4 = A_U(0). \tag{71}$$

(iii) If

$$\int_0^1 \alpha f(\alpha)\left[A_L(\alpha) - A_U(\alpha)\right] d\alpha \le \left[A_L(0) - A_U(0)\right] \int_0^1 \alpha f(\alpha)(1-\alpha)\, d\alpha \tag{72}$$

then

$$t_1 = A_L(0) \tag{73}$$

$$t_2 = \frac{\int_0^1 \alpha f(\alpha) A_L(\alpha)\, d\alpha - A_L(0) \int_0^1 \alpha f(\alpha)(1-\alpha)\, d\alpha}{\int_0^1 \alpha^2 f(\alpha)\, d\alpha} \tag{74}$$

$$t_3 = \frac{\int_0^1 \alpha f(\alpha) A_U(\alpha)\, d\alpha - A_U(0) \int_0^1 \alpha f(\alpha)(1-\alpha)\, d\alpha}{\int_0^1 \alpha^2 f(\alpha)\, d\alpha} \tag{75}$$

$$t_4 = A_U(0). \tag{76}$$

(iv) If

$$\begin{array}{c} \int_0^1 \alpha f(\alpha)\left[A_L(\alpha) - A_U(\alpha)\right] d\alpha - \\ -\left[A_L(0) - A_U(0)\right] \int_0^1 f(\alpha)\, \alpha (1-\alpha)\, d\alpha > 0 \end{array} \tag{77}$$

$$\begin{array}{c} -\int_0^1 \alpha f(\alpha)\left[A_L(\alpha) + A_U(\alpha)\right] d\alpha + \\ +A_L(0) \int_0^1 \alpha (1+\alpha) f(\alpha)\, d\alpha + A_U(0) \int_0^1 \alpha (1-\alpha) f(\alpha)\, d\alpha \le 0 \end{array} \tag{78}$$

and

$$\begin{array}{c} \int_0^1 \alpha f(\alpha)\left[A_L(\alpha) + A_U(\alpha)\right] d\alpha - \\ -A_L(0) \int_0^1 \alpha (1-\alpha) f(\alpha)\, d\alpha - A_U(0) \int_0^1 \alpha (1+\alpha) f(\alpha)\, d\alpha \le 0, \end{array} \tag{79}$$

then

$$\begin{aligned} t_1 &= A_L(0) \\ t_2 &= t_3 = \frac{\int_0^1 \alpha f(\alpha)\left[A_L(\alpha) + A_U(\alpha)\right] d\alpha}{2\int_0^1 \alpha^2 f(\alpha)\, d\alpha} \\ &\quad - \frac{\left[A_L(0) + A_U(0)\right] \int_0^1 \alpha f(\alpha)(1-\alpha)\, d\alpha}{2\int_0^1 \alpha^2 f(\alpha)\, d\alpha} \\ t_4 &= A_U(0). \end{aligned}$$

Theorem 34 *For a fuzzy number $A = (a,b,c,d)_n$ the nearest trapezoidal fuzzy number preserving the support of A, with $f(\alpha) = \alpha$, is $T(A) = (t_1, t_2, t_3, t_4)$.*

(i) If

$$a - d - an + 4bn + 4cn - 7dn > 0$$

then

$$\begin{aligned} t_1 &= a \\ t_2 &= d \\ t_3 &= d \\ t_4 &= d. \end{aligned}$$

(ii) If

$$a - d + 7an - 4bn - 4cn + dn > 0$$

then

$$\begin{aligned} t_1 &= a \\ t_2 &= a \\ t_3 &= a \\ t_4 &= d. \end{aligned}$$

(iii) If

$$a - d - an + 4bn - 4cn + dn \leq 0$$

then

$$\begin{aligned} t_1 &= a \\ t_2 &= \frac{a - an + 4bn}{3n + 1} \\ t_3 &= \frac{d - dn + 4cn}{3n + 1} \\ t_4 &= d. \end{aligned}$$

(iv) If

$$a - d - an + 4bn - 4cn + dn > 0$$
$$a - d - an + 4bn + 4cn - 7dn \leq 0$$

and

$$a - d + 7an - 4bn - 4cn + dn \leq 0$$

then

$$\begin{aligned} t_1 &= a \\ t_2 &= t_3 = \frac{a + d - an + 4bn + 4cn - dn}{2\,(3n + 1)} \\ t_4 &= d. \end{aligned}$$

Example 35 *The case (i) of Theorem 34 is applicable to the fuzzy number*

$$A = (-100, 2, 10, 12)_2$$

and

$$T(A) = (-100, 12, 12, 12)$$

is the nearest trapezoidal fuzzy number preserving the support of A.

Example 36 *The case (ii) of Theorem 34 is applicable to the fuzzy number*

$$A = (5, 10, 20, 310)_2$$

and

$$T(A) = (5, 5, 5, 310)$$

is the nearest trapezoidal fuzzy number preserving the support of A.

Example 37 *The case (iii) of Theorem 34 is applicable to the fuzzy number*

$$A = (0, 10, 20, 30)_2$$

and

$$T(A) = \left(0, \frac{80}{7}, \frac{130}{7}, 30\right)$$

is the nearest trapezoidal fuzzy number preserving the support of A.

Example 38 *The case (iv) of Theorem 34 is applicable to the fuzzy number*

$$A = (-30, -10, -9, -2)_2$$

and

$$T(A) = \left(-30, \frac{-60}{7}, \frac{-60}{7}, -2\right)$$

is the nearest trapezoidal fuzzy number preserving the support of A.

4.2 Weighted approximation which preserves the expected interval

Let $\lambda_L, \lambda_U : [0,1] \to \mathbb{R}$ be non-negative functions

$$\begin{aligned} \int_0^1 \lambda_L(\alpha)\, d\alpha &> 0 \\ \int_0^1 \lambda_U(\alpha)\, d\alpha &> 0 \end{aligned}$$

named weighting functions and the weighted distance

$$d_\lambda(A, B) = \sqrt{\int_0^1 \lambda_L(\alpha)\left[A_L(\alpha) - B_L(\alpha)\right]^2 d\alpha + \int_0^1 \lambda_U(\alpha)\left[A_U(\alpha) - B_U(\alpha)\right]^2 d\alpha},$$

where A, B are fuzzy numbers with $A_\alpha = [A_L(\alpha), A_U(\alpha)]$ şi $B_\alpha = [B_L(\alpha), B_U(\alpha)]$, $\alpha \in [0,1]$.

Theorem 39 *Let A, $A_\alpha = [A_L(\alpha), A_U(\alpha)]$, $\alpha \in [0,1]$ be a fuzzy number and $T(A) = (l, u, \delta, \sigma)$ the nearest weighted trapezoidal approximation operator preserving the expected interval.*

(i) If

$$\delta^e (1-\omega_L) + \sigma^e (1-\omega_U) - u^e + l^e \leq 0 \tag{80}$$

then

$$\begin{aligned} l &= l^e - \delta^e \left(\omega_L - \frac{1}{2}\right) \\ u &= u^e + \sigma^e \left(\omega_U - \frac{1}{2}\right) \\ \delta &= \delta^e \\ \sigma &= \sigma^e. \end{aligned}$$

(ii) If

$$\delta^e (1-\omega_L) + \sigma^e (1-\omega_U) - u^e + l^e > 0 \tag{81}$$

$$c\delta^e (1-\omega_U)^2 - d\sigma^e (1-\omega_U)(1-\omega_L) + d(u^e - l^e)(1-\omega_L) \geq 0 \tag{82}$$

$$d\sigma^e (1-\omega_L)^2 - c\delta^e (1-\omega_L)(1-\omega_U) + c(u^e - l^e)(1-\omega_U) \geq 0 \tag{83}$$

then

$$\begin{aligned} l &= l^e - \frac{c\delta^e (1-\omega_U)^2 - d\sigma^e (1-\omega_U)(1-\omega_L) + d(u^e - l^e)(1-\omega_L)}{d(1-\omega_L)^2 + c(1-\omega_U)^2} \left(\omega_L - \frac{1}{2}\right) \\ u &= u^e + \frac{d\sigma^e (1-\omega_L)^2 - c\delta^e (1-\omega_L)(1-\omega_U) + c(u^e - l^e)(1-\omega_U)}{d(1-\omega_L)^2 + c(1-\omega_U)^2} \left(\omega_U - \frac{1}{2}\right) \\ \delta &= \frac{c\delta^e (1-\omega_U)^2 - d\sigma^e (1-\omega_U)(1-\omega_L) + d(u^e - l^e)(1-\omega_L)}{d(1-\omega_L)^2 + c(1-\omega_U)^2} \\ \sigma &= \frac{d\sigma^e (1-\omega_L)^2 - c\delta^e (1-\omega_L)(1-\omega_U) + c(u^e - l^e)(1-\omega_U)}{d(1-\omega_L)^2 + c(1-\omega_U)^2}. \end{aligned}$$

(iii) If

$$c\delta^e (1-\omega_U)^2 - d\sigma^e (1-\omega_U)(1-\omega_L) + d(u^e - l^e)(1-\omega_L) < 0 \tag{84}$$

then

$$\begin{aligned} l &= l^e \\ u &= u^e + \frac{u^e - l^e}{1-\omega_U} \left(\omega_U - \frac{1}{2}\right) \\ \delta &= 0 \\ \sigma &= \frac{u^e - l^e}{1-\omega_U}. \end{aligned}$$

(iv) If

$$d\sigma^{e}\left(1-\omega_{L}\right)^{2}-c\delta^{e}\left(1-\omega_{L}\right)\left(1-\omega_{U}\right)+c\left(u^{e}-l^{e}\right)\left(1-\omega_{U}\right)<0 \tag{85}$$

then

$$\begin{aligned}
l &= l^{e}-\frac{u^{e}-l^{e}}{1-\omega_{L}}\left(\omega_{L}-\frac{1}{2}\right) \\
u &= u^{e} \\
\delta &= \frac{u^{e}-l^{e}}{\omega_{L}-1} \\
\sigma &= 0.
\end{aligned}$$

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