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NEW CLASSES OF UNIVALENT FUNCTIONS

PhD thesis-abstract

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CONTENTS

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PREFACE

Geometrical theory of univalent functions, is one of the areas of complex analysis the analytical rigor of reasoning is closely intertwined with the geometric intuition and the first concepts were introduced in the early twentieth century when they appeared first major works such as written by P. Koebe, T. H. Gronwall, I. W. Alexander, L. Bieberbach.

Is noteworthy that in developing this area of mathematics, Romanian mathematicians had distinguished merit. Creator of Romanian school univalent functions theory is G. Călugareanu who obtained necessary and sufficient conditions for univalention. Continuer it is Academician P.T. Mocanu who has imposed worldwide with outstanding results and we mention a few, has introduced new classes of univalent functions, alpha-convex functions or functions Mocanu, has addressed the issue of nonanalytic function injectivity and has created together with S.S. Miller a new method of study of specific classes of univalent functions and namely "admissible functions method" or "method of differential subordination". Later, Academician P.T. Mocanu and S.S. Miller introduced the dual concept of differential subordination called “differential superordontion”. Under the leadership of Academician Mr P.T. Mocanu has been formed a strong school of geometric functions theory in Cluj. Among his numerous collaborators at the national level we mention: N.N. Pascu, G.Ş. Sălăgean, T. Bulboacă, G. Kohr, P. Curt, Gheorghe Oros, M. Acu and others, and internationally, S.S. Miller, M.O. Reade, S. Ruscheweyh, S. Owa, R. Fourier, M.K. Aouf and others.

In this PhD thesis have been obtained new results regarding the differential subordinations and superordinations and some subclasses of univalent functions.

The paper contains four chapters, an introduction and a bibliography, containing 138 titles, among which 21 are signed by the author.

In the first chapter, entitled "General on the concept of univalent function" and structured in four paragraphs, are presented general problems concerning the theory of univalent functions, their special family, analytic functions with positive real part and the concept of subordination.

The second chapter entitled "Special classes of univalent functions" is divided into nine sections. Here are the important results on the class of starlike functions, convex, close-to-convex, alpha-convex, alpha-convex p-symmetric type, starlike of α type, spirallike, stellate and convex of order α and analytical with negative coefficients, being exposed definitions, lemmas and fundamental theorems. These notions and results are necessary to confirm the original results contained in Chapter IV.

The next two chapters contain original results, already published or under publication. "Differential subordination and superordination" is the third chapter and is divided into ten paragraphs. In the first two paragraphs are definitions, lemmas and fundamental theorems for differential subordination and Briot-Bouque differential subordination. These notions and results are necessary to confirm the original results contained in the following paragraphs of this chapter. In paragraphs III.3 and III.4 were determined applications of differential subordination and Briot-

Bouque differential subordination and are contained in [57], [61], [64]. In the following two paragraphs are the basic definitions and theorems for superordination differential and Briot-Bouque differential superordination, necessary to confirm the original results contained in the following paragraphs of this chapter. Some of these results are contained in [56]. In paragraph III.7 were determined applications of Briot-Bouque differential superordination obtained by using integral operators, and [54], [55], [58] containing the results. III.8 paragraph contains applications of differential subordination and superordination and theorems sandwich. These results are contained in [63], [65], [66]. Using the Ruscheweyh operator in paragraph III.9 were determined and subordination and differential superordination for analytical functions. The results of this paragraph are contained in [67]. In the last paragraph of this chapter are presented using subordination and superordination differential operator $I_k(r,\lambda)f(z)$ are contained in [60], [68].

The last chapter entitled "Subclasses of univalent functions" is structured in six sections. All results of this chapter are original. In paragraph IV.1 are shown subclasses of univalent functions defined by convolution, denoted, S_g^*, K_g, C_g, and are contained in [62]. Paragraph IV.2 entitled "Subclasses of normalized starlike and convex functions" show the subclasses denoted $S^*(\zeta)$, $K(\zeta)$, contained in [69]. The following paragraph is an intermingling of the previous paragraphs, within the meaning that we determined subclasses of normalized univalent functions defined by convolution, denoted, $S_g^*(\zeta)$, $K_g(\zeta)$, $C_g(\zeta)$, $M_{\alpha,g}(\zeta)$, $\hat{S}_{\gamma,g}(\zeta)$. These results are contained in [70]. Subclasses of normalized starlike and convex functions of order α are presented in paragraph IV.4, are denoted $S^*(\alpha;\zeta)$, $K(\alpha;\zeta)$, and are contained in [71]. Results of paragraph IV.5, subclasses of normalized alpha-convex functions, denoted $M_\alpha(\zeta)$, are contained in [72]. The last paragraph of this chapter is entitled "Subclasses of univalent functions with negative coefficients" presents a new class of functions denoted $TS^*_{n,\lambda}(\alpha)$ and these results are contained in [73].

On this way I want to bring sincere thanks and to express my feelings of esteem and respect for scientific leader of the work, academician Mr. Petru T. Mocanu for the way he direct the development of this work, for the support and confidence that inspired me and permanent encouraging . Also, my thanks goes to prof. dr. Grigore Şt. Sălăgean whose results in field of functions with negative coefficients have been very helpful, Mrs. prof. dr. Gabriela Kohr, and other professor of the Department of Theory of Functions. And last but not least I would like to thanks my children, parents and husband for support, understanding, encouragement and support.

In what follows, I have selected the most important results of each chapter.

I. GENERAL ON THE CONCEPT OF UNIVALENT FUNCTION

In this chapter are presented the basic concepts and results on theory univalent functions, their special family, analytic functions with positive real part and the notion of subordination.

I.1. General issues concerning the theory of univalent functions

Definition I.1.1 [38]: *Let the domain* $D \subseteq \mathbb{C}$ *and let the function* $f : D \to \mathbb{C}$. *We say that function f is* ***univalent function****, if* $f \in \mathcal{H}(D)$ *and f is injective on D.*

Definition I.1.2 [38]: *We denote with*: $\mathcal{H}_u(D) = \{ f \in \mathcal{H}(D)$*: f is univalent on D* $\}$. $\mathcal{H}_u(D)$ *denote the* ***classes of univalent functions****.*

Theorem I.1.1 [20,38]: *If the function* $f \in \mathcal{H}_u(D)$, *then* $f'(z) \neq 0$, *for any* $z \in D$.

Corollary I.1.2 [38]: *Let D be a convex domain in the plane* $\mathbb{C}$, $f \in \mathcal{H}(D)$ *such that* $\operatorname{Re} f'(z) > 0$, *for any* $z \in D$, *then* $f \in \mathcal{H}_u(D)$.

I.2. Special families of univalent functions in *U*

We denote the open unit disc in complex plan: $U = \{ z \in \mathbb{C} : |z| < 1 \}$.

The set of functions $f : U \to \mathbb{C}$ holomorphic in the unit disc is denoted by $\mathcal{H}(U)$.

For $a \in \mathbb{C}$ and $n \in \mathbb{N}^*$, we denote

$$\mathcal{H}[a,n] = \{ f \in \mathcal{H}(U) : \ f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots \},$$

$$\mathcal{A}_n = \{ f \in \mathcal{H}(U) : \ f(z) = z + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \ldots \},$$

and for $n = 1$, $\mathcal{A}_1 = \mathcal{A}$.

An important place they occupy in the theory of univalent functions of class *S* of functions of the form:

$$f(z) = z + a_2 z^2 + a_3 z^3 + \ldots + a_n z^n + \ldots, \ z \in U,$$

holomorphic and univalent in the unit disc *U*.

We denote: $S = \{ f \in \mathcal{H}_u(D)$: $f(0) = f'(0) - 1 = 0 \}$.

Theorem I.2.4 [7]: *If* $f \in S$, $f(z) = z + a_2 z^2 + a_3 z^3 + \ldots + a_n z^n + \ldots$, $z \in U$, *then* a_2 *satisfie the relation* $|a_2| \leq 2$. *Equality holds* $|a_2| = 2$ *if and only if f is a Koebe function, if* $f(z) = K_\tau(z) = \dfrac{z}{\left(1 + e^{i\tau} z\right)^2}$, $z \in U$, $\tau \in \mathbb{R}$.

Conjectura I.2.1 [9]: *If* $f \in S$, $f(z) = z + a_2 z^2 + a_3 z^3 + \ldots + a_n z^n + \ldots$, $z \in U$, *then* $|a_n| \leq n$ *for any* $n \in \mathbb{N}$, $n \geq 2$.

Corollary I.2.3 [95]: *Class S is compact.*

I.3. Analytic functions with positive real part

Definition I.3.1 [95]: *We introduce a class of functions* $\mathcal{P} = \{ p \in \mathcal{H}(U) : p(0) = 1, \ \operatorname{Re} p(z) > 0, \ z \in U \}$ *called* ***class of Caratheodory functions***.

Definition I.3.2 [95]: *We introduce a class of functions* $\mathcal{B} = \{ \varphi \in \mathcal{H}(U) : \varphi(0) = 0, \ |\varphi(z)| < 1, \ z \in U \}$ *called* ***class of Schwarz functions***.

Theorem I.3.4 [18]: *If* $p \in \mathcal{P}$, $p(z) = 1 + p_1 z + p_2 z^2 + \ldots$, $z \in U$, *then* $|p_n| \le 2$ *for any* $n \in \mathbb{N}^*$. *Equality holds if and only if:*

$$p(z) = \frac{1+\lambda z}{1-\lambda z}, \ z \in U, \ \lambda \in \mathbb{C}, \ |\lambda| = 1.$$

I.4. Subordination

Definition I.4.1 [95]: *Let the functions f, F* $\in \mathcal{H}(U)$. *We say that* ***function f is subordinated to the function F*** *and we note* $f \prec F$ *or* $f(z) \prec F(z), z \in U$ *if there is a function* $w \in \mathcal{H}(U)$, *with* $w(0) = 0$ *and* $|w(z)| < 1$, $z \in U$ *such that* $f(z) = F[w(z)]$, $z \in U$.

Theorem I.4.1 [95]: *Let the functions* $f, F \in \mathcal{H}(U)$ *and suppose that F is univalent in U. Then* $f(z) \prec F(z), z \in U$ *if and only if* $f(0) = F(0)$ *and* $f(U) \subseteq F(U)$.

Corollary I.4.1 [49]: *Let the functions* $f, F \in \mathcal{H}(U)$ *such that F is univalent in U.*

a) *If* $f(0) = F(0)$ *and* $f(U) \subseteq F(U)$, *then* $f(\overline{U_r}) \subseteq F(\overline{U_r}), 0 < r < 1$.

b) *The equality* $f(\overline{U_r}) = F(\overline{U_r})$ *there is a* $r < 1$ *if and only if* $f(U) = F(U)$ *or* $f(z) = F(\lambda z)$, $|\lambda| = 1$.

II. SPECIAL CLASSES OF UNIVALENT FUNCTIONS

In this chapter are presented the important results on starlike functions, convex functions, close-to-convex functions, alpha-convex functions, p-fold symmetric alpha-convex functions, starlike functions type α, spirallike functions, starlike and convex functions of order α, analytic functions with negative coefficients, being exposed to definitions, lemmas and fundamental theorems.

II.1. Starlike functions

The starlike functions are first introduced in 1920 by J. W. Alexander.

Theorem II.1.1 [95]: *Let the functin* $f \in \mathcal{H}(U), f(0) = 0$. *Then function f is starlike in U if and only if* $f'(0) \neq 0$ *and*

$$\mathrm{Re}\frac{z f'(z)}{f(z)}>0,\ z\in U.$$

Definition II.1.6 [95]: *Let* $S^* = \{ f\in\mathcal{A}:\ \mathrm{Re}\frac{z f'(z)}{f(z)}>0,\ z\in U\}$. S^* *denote the* ***class of starlike function.***

Theorem II.1.3 [95]: *Class* S^* *is compact.*

Theorem II.1.5 [50,101]: *If* $f\in S^*$, *where* $f(z)=z+a_2z^2+a_3z^3+\ldots+a_nz^n+\ldots$, $z\in U$, *then* $|a_n|\le n$ *for any* $n\in\mathbb{N}$, $n\ge 2$. *Equality holds if and only if* $f(z)=K_\tau(z)$, $z\in U$, $\tau\in\mathbb{R}$.

II.2. Convex functions

The convex functions were introduced in 1913 by E. Study [138], and their study was continued T. H. Gronwall [35] and K. Lowner [50].

Lemma II.2.2 [123]: *Let the function* $p\in\mathcal{H}(U)$, *such that* $\mathrm{Re}\,p(0)>0$ *and let* $\alpha\in\mathbb{R}$. *Then:*

$$\mathrm{Re}\left[p(z)+\alpha\frac{z p'(z)}{p(z)}\right]>0,\ z\in U\ \Rightarrow\ \mathrm{Re}\,p(z)>0,\ z\in U.$$

Theorem II.2.1 [95]: *Let the function* $f\in\mathcal{H}(U)$. *Then function* f *is convex in U if and only if* $f'(0)\neq 0$ *and*

$$\mathrm{Re}\frac{z f''(z)}{f'(z)}+1>0,\ z\in U.$$

Definition II.1.4 [95]: *Let* $K = \{ f\in\mathcal{A}:\ \mathrm{Re}\frac{z f''(z)}{f'(z)}+1>0,\ z\in U\}$, *K denote the* ***class of convex function.***

Theorem II.2.2 [95]: *The function* $f\in K$ *if and only if* $g\in S^*$, *where* $g(z)=z f'(z)$, $z\in U$ *or* $f\in K\ \Leftrightarrow\ z f'(z)\in S^*$.

Theorem II.2.5 [95]: *Class K is compact.*

Theorem II.2.6 [50]: *If* $f\in K$, $f(z)=z+a_2z^2+\ldots+a_nz^n+\ldots$, $z\in U$, *then* $|a_n|\le 1$ *for any* $n\in\mathbb{N}$, $n\ge 2$. *Equality holds if and only if:* $f(z)=\frac{z}{1+e^{i\tau}z}$, $z\in U$, $\tau\in\mathbb{R}$.

G. S. Sălăgean and S. Ruscheweyh introduce two differential operators which allow, in certain situations, study the stars and convex functions simultaneously and of their subclasses.

Definition II.2.8 [125]: *Let* D^n *be the* ***Sălăgean differential operator,*** $D^n:\mathcal{A}\to\mathcal{A}$, $n\in\mathbb{N}$, *defined as:*

$$\begin{aligned}
&D^0 f(z)=f(z),\\
&D^1 f(z)=Df(z)=z f'(z),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&D^n f(z)=D\left(D^{n-1}f(z)\right).
\end{aligned}$$

Observation II.2.10: If function $f \in \mathcal{A}$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, $z \in U$, then:

$$D^n f(z) = z + \sum_{j=2}^{\infty} j^n a_j z^j, \ z \in U.$$

Definition II.2.9 [125]: We say that function $f \in \mathcal{A}$ *is* ***n-starlike****,* $n \in \mathbb{N}$*, if verify inequality:*

$$\operatorname{Re} \frac{D^{n+1} f(z)}{D^n f(z)} > 0, \ z \in U.$$

We note S_n^* *class of these functions.*

Theorem II.2.10 [5] *Let* $\phi \in \mathcal{A}$ *is convex,* $g \in S^*$ *and* $F \in \mathcal{H}(U)$ *such that* $\operatorname{Re} F(z) > 0$*,* $z \in U$*. Then* $\dfrac{\phi * Fg}{\phi * g}$ *is convex.*

Definition II.2.11 [121]: *Let* R^n *be the* ***Ruscheweyh differential operator****,* $R^n : \mathcal{A} \to \mathcal{A}$*,* $n \in \mathbb{N}$*, defined as:*

$$R^n f(z) = \frac{z}{(1-z)^{n+1}} * f(z) = \frac{z\left(z^{n-1} f(z)\right)^{(n)}}{n!}, \ z \in U.$$

Observația II.2.12 [121]: If the function $f \in \mathcal{A}$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, $z \in U$, then

$$R^n f(z) = z + \sum_{j=2}^{\infty} C_{n+j-1}^{n} a_j z^j, \ z \in U.$$

II.3. Close-to-convex functions

Definition II.3.1 [95]: *A function* $f : U \to \mathbb{C}$*,* $f \in \mathcal{H}(U)$ *is called close-to-convex if there a convex function* φ *in U such that:*

$$\operatorname{Re} \frac{f'(z)}{\varphi'(z)} > 0, \ z \in U.$$

We say that function f is ***close-to-convex against with the function*** φ.

Definition II.3.2 [95]: *We denote* $C = \{ f \in \mathcal{A} : (\exists)\, \varphi \in K, \ \operatorname{Re} \frac{f'(z)}{\varphi'(z)} > 0, z \in U \}$*, C denote the* ***class of close-to-convex function****.*

Theorem II.3.1 [43,106]: *Let the domain* $D \subset \mathbb{C}$ *and let the function* $f \in \mathcal{H}(D)$*. Suppose that there a function* $\varphi \in \mathcal{H}_u(D)$ *such that* $\varphi(D) = \Delta$ *is a convex domain. Then* $\operatorname{Re} \frac{f'(z)}{\varphi'(z)} > 0$*,* $z \in D$ *(ie function* f *is close-to-convex against with* φ*) involving the function* f *is univalent in D.*

Theorem II.3.3 [116,117]: *If* $f \in C$*,* $f(z) = z + a_2 z^2 + \ldots + a_n z^n + \ldots$*,* $z \in U$*, then* $|a_n| \le n$ *for any* $n \ge 2$*. Equality holds if and only if* $f(z) = K_\tau(z) = \dfrac{z}{\left(1 + e^{i\tau} z\right)^2}$*,* $z \in U$*,* $\tau \in \mathbb{R}$*.*

II.4. Alpha-convex functions (Mocanu functions)

In order to establish a link between the notions of convexity and the star, in 1969, P. T. Mocanu [92] introduces the concept of alpha-convexity. Later their various properties were obtained by P. T. Mocanu, S. S. Miller and M. O. Reade [88,89].

Theorem II.4.1 [92]: *Let the function f alpha-convex on the circle* $\{z \in \mathbb{C} : |z| = r\}$ *if and only if* $\operatorname{Re} J(\alpha, f; z) > 0$ *for* $|z| = r$, *where:*

$$J(\alpha, f; z) = (1-\alpha)\frac{z f'(z)}{f(z)} + \alpha\left(1 + \frac{z f''(z)}{f'(z)}\right).$$

Definition II.4.3 [92]: *We denote* $M_\alpha = \{ f \in \mathcal{H}(U) : f(0) = f'(0) - 1 = 0,\ \frac{f(z) f'(z)}{z} \neq 0, \operatorname{Re} J(\alpha, f; z) > 0, z \in U \}$, M_α *denote the* ***class of alpha-convex functions or Mocanu functions***.

II.5. p-fold symmetric alpha-convex functions

Definition II.5.1 [26]: *Let* $\alpha \in \mathbb{R}$ *and* $p \in \mathbb{N}$, $p \geq 1$. *We denote* $M_{\alpha,p} = \left\{ f \in M_\alpha : f(z) = z + a_{p+1} z^{p+1} + a_{2p+1} z^{2p+1} + \ldots, z \in U \right\}$, $M_{\alpha,p}$ *denote the* ***class of p-fold symmetric alpha-convex functions***.

Theorem II.5.2 [26]: *If the function* $f \in M_{\alpha,p}$, *where* $\alpha > 0$ *and* $z \in U$ *is a fixed point, then:*

$$\left[-M(-r^p, \alpha p)\right]^{\frac{1}{p}} \leq |f(z)| \leq \left[M(r^p, \alpha p)\right]^{\frac{1}{p}},$$

where $M(r, \alpha) = \left[\frac{1}{\alpha}\int_0^r \frac{\rho^{\frac{1}{\alpha}-1}}{(1-\rho)^{\frac{2}{\alpha}}} d\rho\right]^\alpha$. *Equality is achieved* (*both sides*) *if the function* f *is form* $f(z) = \left[K_\tau(z^p, \alpha p)\right]^{\frac{1}{p}}$.

II.6. Starlike functions type α

Definition II.6.1 : *Let the function* $f \in S^*$. *We say that function f is* ***starlike type*** α, *and* ***note*** $f \in S^*[\alpha]$, *If* $\alpha = \alpha(f) = \sup\{\beta : f \in M_\beta\}$.

Definition II.6.2: *Let* $\alpha \in \mathbb{R}$ *and* $p \in \mathbb{N}$, $p \geq 1$. *We denote* $S_p^*[\alpha] = \left\{ f \in S^*[\alpha] : f(z) = z + a_{p+1} z^{p+1} + a_{2p+1} z^{2p+1} + \ldots, z \in U \right\}$, $S_p^*[\alpha]$ *denote the* ***class of starlike function type*** α ***p- symmetric***.

Theorem II.6.1 [26]: *The function* $f \in S_p^*[\alpha]$ *if and only if* $g \in S^*[\alpha p]$, $\alpha \geq 0$, *where* $f(z) = \left[g(z^p)\right]^{\frac{1}{p}}$, $p = 2, 3, \ldots$.

II.7. Spirallike functions

In 1932, L. Spacek [136] presents a generalization of specific functions and class spirallike functions.

Theorem II.7.1 [8]: *Let the function* $f \in \mathcal{H}(U)$, *with* $f(0) = 0$, $f'(0) \neq 0$ *and* $f(z) \neq 0$, $z \in U$ *and let* $\gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. *Then function* f *is spirallike type* γ *if and only if:*

$$\operatorname{Re}\left[e^{-i\gamma} \frac{z f'(z)}{f(z)}\right] > 0, \; z \in U.$$

Definition II.7.7 [8]: *We denote* $\hat{S}_\gamma = \left\{ f \in \mathcal{A} : \operatorname{Re}\left[e^{-i\gamma} \frac{z f'(z)}{f(z)}\right] > 0, z \in U \right\}$, *where* $\gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $\hat{S}_\gamma$ *denote the* ***class of type*** γ ***spirallike functions***.

II.8. Starlike and convex functions of order α

Definition II.8.1 [95]: *Let* $0 \leq \alpha < 1$. *We denote* $S^*(\alpha) = \left\{ f \in \mathcal{A} : \operatorname{Re} \frac{z f'(z)}{f(z)} > \alpha, z \in U \right\}$, $S^*(\alpha)$ *denote the* ***class starlike functions of order*** α.

Definition II.8.2 [95]: *Let* $0 \leq \alpha < 1$. *We denote:* $K(\alpha) = \left\{ f \in \mathcal{A} : \operatorname{Re} \frac{z f''(z)}{f'(z)} + 1 > \alpha, z \in U \right\}$, $K(\alpha)$ *denote the* ***class convex functions of order*** α.

II.9. Analytic functions with negative coefficients

Definition II.9.1: *We denote:* $T = \left\{ f \in S : f(z) = z - \sum_{n=2}^{\infty} a_n z^n, a_n \geq 0, n \in \mathbb{N} \setminus \{0,1\}, z \in U \right\}$ *and* $T^* = T \cap S^*$, T^* *denote* ***the class starlike functions with negative coefficients***, $T^*(\alpha) = T \cap S^*(\alpha)$, $T^*(\alpha)$ *denote the class* ***starlike functions of order*** α ***with negative coefficients***, $T^c = T \cap K$, T^c *denote the class* ***convex functions with negative coefficients*** *and* $T^c(\alpha) = T \cap K(\alpha)$, *with* $0 \leq \alpha < 1$ *denote the class* ***convex functions of order*** α ***with negative coefficients***.

Definition II.9.2: *We denote:* $T_d^* = \left\{ f \in T : \left| \frac{z f'(z)}{f(z)} - 1 \right| < 1, z \in U \right\}$.

Theorem II.9.2 [134]: *Let the function* f *defined by the relation* (II.9.1). *Then* $f \in T^{*}(\alpha)$ *if and only if:* $\sum_{n=2}^{\infty} \frac{n-\alpha}{1-\alpha} a_n \le 1$ *and* $f \in T^{c}(\alpha)$ *if and only if:* $\sum_{n=2}^{\infty} \frac{n(n-\alpha)}{1-\alpha} a_n \le 1$.

III. DIFFERENTIAL SUBORDINATIONS AND SUPERORDINATIONS

In this are presented chapter the definitions, lemmas and the fundamental theorem on differential subordinations and superordinations, Briot-Bouque differential subordinations and superordinations, and their applications and using various functions and linear operator.

III.1. Differential subordinations

The method of differential subordination, known as the method admissible functions is one of the latest methods used in the geometric theory of analytic functions and was introduced by S. S. Miller and P. T. Mocanu in [76,77] and then developed in many other work.

Definition III.1.1: *Let* $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ *and let the univalent function h in unit disc U. If the function* $p \in \mathcal{H}[a,n]$ *satiesfies the differential subordination:*

$$\psi\left(p(z), z\,p'(z), z^2\,p''(z); z\right) \prec h(z), \quad z \in U\ , \tag{III.1.4}$$

then function p is called an ***(a, n) solution of the differential subordination*** (III.1.4).

Definition III.1.2: *Subordination of the relation* (III.1.4) *is called* ***a second-order differential subordination,*** *and the univalent function q in U, is called* ***(a, n) dominant solution of the differential subordination*** (III.1.4).

Definition III.1.3: *A dominant* $\tilde{q}$ *that satisfies* $\tilde{q}(z) \prec q(z)$ *for all dominants q of relația* (III.1.4) *is said to be* ***the best (a, n) dominant***.

Definition III.1.4: *Let* $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ *and let the univalent function h in unit disc U. If p is a function analytic in unit disc U and satiesfies the differential subordination of relation* (III. 1.4), *then function p is called* ***solution of the differential subordination***.

Definition III.1.5: *The univalent function* q *is called* ***a dominant the differential subordination****n of relation* (III.1.4)*, If* $p \prec q$ *, for any* p *care satisfies the relation* (III.1.4).

Lemma III.1.1 [76]: *Let* $z_0 = r_0\, e^{i\theta_0}$*, with* $0 < r_0 < 1$ *and let* $f(z) = a_n z^n + a_{n+1} z^{n+1} + \ldots$ *a function be continuous on U (0; r₀) and analytic on* $U(0;r_0) \cup \{z_0\}$ *with* $f(z) \not\equiv 0$ *and* $n \ge 1$. *If* $|f(z_0)| = \max\{|f(z)| : z \in \overline{U}(0;r_0)\}$*, then there a real number m,* $m \ge n$*, such that:*

a) $\frac{z_0\, f'(z_0)}{f(z_0)} = m$; b) $\mathrm{Re}\frac{z_0\, f''(z_0)}{f'(z_0)} + 1 \ge m$.

Definition III.1.6: *We denote by* $\mathcal{Q}$ *the set of functions q that are analytic and injective on* $\overline{U} \setminus E(q)$*, where*

$$E(q) = \left\{\zeta \in \partial U : \lim_{z \to \zeta} q(z) = \infty\right\},$$

and are such that $q'(\zeta)\neq 0$ *for* $\zeta\in\partial U\setminus E(q)$. *Let* $E(q)$ *is called exception set. Denote by* $\mathcal{Q}(a)$ *the subclass* $\mathcal{Q}$ *for which* $q(0)=a$.

Definition III.1.6 [76, 77]: *Let the set of* $\Omega\subset\mathbb{C}$, *let the function* $q\in\mathcal{Q}$ *and* $n\in\mathbb{N}, n\geq 1$. *We note with* $\Psi_n[\Omega,q]$ *class functions* $\psi:\mathbb{C}^3\times U\to\mathbb{C}$ *that satisfy the condition* $\psi(r,s,t;z)\notin\Omega$, *whenever:*

$$r=q(\zeta),\ s=m\zeta\, q'(\zeta),\ \operatorname{Re}\left[\frac{t}{s}+1\right]\geq\operatorname{Re}\left[\frac{\zeta\, q''(\zeta)}{q'(\zeta)}+1\right], \tag{III.1.5}$$

where $z\in U$, $\zeta\in\partial U\setminus E(q)$ *and* $m\geq n$. *The set of* $\Psi_n[\Omega,q]$ *is called* ***class of admissibile functions****, and the condition* $\psi(r,s,t;z)\notin\Omega$ *is called* ***the admissibility condition****.*

Theorem III.1.3 [82]: *Let* $\psi\in\Psi_n[h,q]$, *where* $q(0)=a$ *and* $\psi(a,0,0;0)=h(0)$. *If the function* $p(z)=a+p_n z^n+...,\ p\in\mathcal{H}[a,n]$, *and the function* $\psi\left(p(z),z\,p'(z),z^2\,p''(z);z\right)\in\mathcal{H}(U)$, *then we have:*

$$\psi\left(p(z),z\,p'(z),z^2p''(z);z\right)\prec h(z)\ \Rightarrow\ p(z)\prec q(z).$$

Theorem III.1.4 [76,77]: *Let the univalent functions* $h,q\in\mathcal{H}_u(U)$, *with* $q(0)=a$ *and we note* $h_\rho(z)=h(\rho z)$, $q_\rho(z)=q(\rho z)$. *Let the function* $\psi:\mathbb{C}^3\times U\to\mathbb{C}$, *with* $\psi(a,0,0;0)=h(0)$ *satisfy one of the following conditions:*

a) $\psi\in\Psi_n[\Omega,q_\rho]$, *for some* $\rho\in(0,1)$, *or*

b) *there exists* $\rho_0\in(0,1)$ *such that* $\psi\in\Psi_n[h_\rho,q_\rho]$, *for any* $\rho\in(\rho_0,1)$.

If the function $p(z)=a+p_n z^n+...,$ $p\in\mathcal{H}[a,n]$, *and the function* $\psi\left(p(z),z\,p'(z),z^2p''(z);z\right)\in\mathcal{H}(U)$, *then we have:*

$$\psi\left(p(z),z\,p'(z),z^2p''(z);z\right)\prec h(z)\ \Rightarrow\ p(z)\prec q(z).$$

Theorem III.1.7 [53]: *Let the function* $p(z)=a+p_n z^n+...,\ p\in\mathcal{H}[a,n]$.

a) *If* $\psi\in\Psi_n[\Omega,a]$, *then we have:*

$$\psi\left(p(z),z\,p'(z),z^2p''(z);z\right)\in\Omega,\ z\in U\Rightarrow |p(z)|<1,\ z\in U.$$

b) *If* $\psi\in\Psi_n[a]$, *then we have:*

$$\psi\left(p(z),z\,p'(z),z^2p''(z);z\right)<1,\ z\in U\Rightarrow |p(z)|<1,\ z\in U.$$

Theorem III.1.9 [95]: *Let the function* $p(z)=a+p_n z^n+...,\ p\in\mathcal{H}[a,n]$.

a) *If* $\psi\in\Psi_n\{\Omega,a\}$, *then we have:*

$$\psi\left(p(z),z\,p'(z),z^2p''(z);z\right)\in\Omega,\ z\in U\Rightarrow \operatorname{Re}p(z)>0,\ z\in U.$$

b) *If* $\psi\in\Psi_n\{a\}$, *then we have:*

$$\psi\left(p(z),z\,p'(z),z^2p''(z);z\right)>0,\ z\in U\Rightarrow \operatorname{Re}p(z)>0,\ z\in U.$$

Theorem III.1.11 [53]: *Let the function* $p(z)=a+p_n z^n+...,\ p\in\mathcal{H}[a,n]$, *where* $|a|<1$ *and let the function* $P:U\to\mathbb{C}$, *with* $|P(z)|<1,\ z\in U$. *Then we have:*

$$|p(z)+P(z)\,z\,p'(z)|<1,\ \ z\in U\Rightarrow |p(z)|<1,\ z\in U.$$

Theorem III.1.12 [53]: *Let the function* $p(z)=a+p_n z^n+\ldots$, $p\in\mathcal{H}[a,n]$, *where* $|a|<M$, $M>0$ *and let the function* $P:U\to\mathbb{C}$, *with* $|P(z)|<M$, $z\in U$. *Then we have:*

$$|p(z)+P(z)\,z\,p'(z)|<M,\ \ z\in U \Rightarrow |p(z)|<M,\ z\in U.$$

Definition III.1.7: *Let* $c\in\mathbb{C}$, *with* $\operatorname{Re} c>0$, *let* $n\in\mathbb{N}^*$ *and let*

$$C_n=C_n(c)=\frac{n}{\operatorname{Re} c}\left[|c|\sqrt{1+\frac{2\operatorname{Re} c}{n}}+\operatorname{Im} c\right].$$

If the univalent function R is defined in U by $R(z)=\dfrac{2C_n z}{1-z^2}$, *then we note with* $R_{c,n}$ ***the „Open Door"function*** *defined by the relation:*

$$R_{c,n}(z)=R\left(\frac{z+b}{1+\bar{b}\,z}\right)=2C_n\frac{(z+b)(1+\bar{b}\,z)}{(1+\bar{b}\,z)^2-(z+b)^2},\ \text{where } b=R^{-1}(c).$$

Lemma III.1.7 (lema „Open Door") [85]: *Let* $c\in\mathbb{C}$, *with* $\operatorname{Re} c>0$, *let* $n\in\mathbb{N}^*$ *and* $R_{c,n}$ *the„Open Door" function and let the function* $P\in\mathcal{H}[c,n]$ *satisfy the differential subordination* $P(z)\prec R_{c,n}(z)$. *If the function* $p\in\mathcal{H}\left[\frac{1}{c},n\right]$ *satisfies the differential eqution* $z\,p'(z)+P(z)\,p(z)=1$, *then* $\operatorname{Re} p(z)>0$, $z\in U$.

Theorem III.1.14 [83,84]: *Let the functions* $\phi,\varphi\in\mathcal{H}[1,n]$, *with* $\phi(z)\varphi(z)\neq 0$, $z\in U$. *Let* $\alpha,\beta,\gamma,\delta\in\mathbb{C}$, *with* $\beta\neq 0$, $\alpha+\delta=\beta+\gamma$ *and* $\operatorname{Re}(\alpha+\delta)>0$. *Let the function* $f\in\mathcal{A}_n$ *and suppose that*

$$P(z)\equiv\alpha\frac{z\,f'(z)}{f(z)}+\frac{z\,\varphi'(z)}{\varphi(z)}+\delta\prec R_{\alpha+\delta}(z),$$

where $R_{c,n}$ *is the „Open Door" function. If* $F=I_{\beta,\gamma}(f)$ *is defined by*

$$F(z)=\left[\frac{\beta+\gamma}{z^{\gamma}\phi(z)}\int_0^z f^{\alpha}(t)\,t^{\delta-1}\,\varphi(t)\,dt\right]^{\frac{1}{\beta}}=z+A_{n+1}z^{n+1}+\ldots,$$

then $F\in\mathcal{A}_n$, $\dfrac{F(z)}{z}\neq 0$, $z\in U$, *and*

$$\operatorname{Re}\left[\beta\frac{z\,F'(z)}{F(z)}+\frac{z\,\phi'(z)}{\phi(z)}+\gamma\right]>0,\ z\in U.$$

III.2. Briot-Bouquet differential subordinations

Definition III.2.1: *By* ***Briot–Bouquet differential operator*** *means an operator of the form:*

$$\Phi\left(p(z),z\,p'(z)\right),\ \text{where } \Phi(r,s)=r+\frac{s}{\beta r+\gamma}.$$

Definition III.2.2: *Let* $\beta,\gamma\in\mathbb{C}$, *let the function* $h\in\mathcal{H}(U)$ *and let the function* $p\in\mathcal{H}(U)$, $p(z)=h(0)+p_1 z+\ldots$, *with* $p(0)=h(0)$. *By* ***Briot–Bouquet differential subordination*** understand form*:*

$$p(z)+\frac{z\,p'(z)}{\beta\,p(z)+\gamma}\prec h(z).$$

Lemma III.2.1 [95]: *The function* $L(z,t)=a_1(t)\,z+a_2(t)\,z^2+a_3(t)\,z^3+\ldots$, *with* $a_1(t)\neq 0$ *for* $t\geq 0$ *and* $\lim_{t\to\infty}|a_1(t)|=\infty$, *is a subordination chain if and only if there exist constants* $r\in(0,1]$ *and* $M>0$ *such that:*

a) *L(z,t) is analytic in* $|z|<r$ *for each* $t\geq 0$, *is measurable in* $[0,\infty)$ *for each* $|z|<r$ *, and satisfies* $|L(z,t)|\leq M|a_1(t)|$ *for* $|z|<r$ *and* $t\geq 0$.

b) *There a function p(z,t) analytic in U for any* $t\in[0,\infty)$ *and measurable in* $[0,\infty)$ *for each* $z\in U$, *such that* $\operatorname{Re}p(z,t)>0$, $z\in U$, $t\geq 0$ *and* $\frac{\partial L(z,t)}{\partial t}=z\frac{\partial L(z,t)}{\partial z}p(z,t)$ *for* $|z|<r$ *and for almost all* $t\in[0,\infty)$.

Theorem III.2.1 [78,79]: *Let* $\beta,\gamma\in\mathbb{C}$, *with* $\beta\neq 0$ *and let be the convex function h which satisfies:*

$$\operatorname{Re}\left[\beta\,h(z)+\gamma\right]>0,\ z\in U.$$

If the function $p\in\mathcal{H}\ [h(0),n]$, *then we have:*

$$p(z)+\frac{z\,p'(z)}{\beta\,p(z)+\gamma}\prec h(z)\quad\Rightarrow\quad p(z)\prec h(z).$$

Theorem III.2.4 [85]: *Let* $\beta,\gamma\in\mathbb{C}$, *with* $\beta\neq 0$ *and let be the univalent function* $q\in\mathcal{H}_u(U)$, *with* $q(0)=a$, *such that* $\beta\,q(z)+\gamma\neq 0$, $z\in U$ *and* $\operatorname{Re}\left[\beta\,q(0)+\gamma\right]>0$. *We denote:*

$$Q(z)=\frac{z\,q'(z)}{\beta\,q(z)+\gamma}\ \text{ and }\ h(z)=q(z)+n\,Q(z)=q(z)+\frac{n\,z\,q'(z)}{\beta\,q(z)+\gamma}.$$

Suppose that:

a) $\operatorname{Re}\frac{z\,h'(z)}{Q(z)}=\operatorname{Re}\left[\beta\,q(z)+\gamma+\frac{n\,z\,Q'(z)}{\beta\,Q(z)}\right]>0,\ z\in U$

and

b) *h is convex or*

b') $\log[\beta\,q+\gamma]$ *is convex* (*or Q is starlike*).

If $p\in\mathcal{H}\ [a,n]$ *satiesfies the Briot – Bouquet differential subordination*

$$p(z)+\frac{z\,p'(z)}{\beta\,p(z)+\gamma}\prec h(z),\tag{III.2.10}$$

then $p(z)\prec q(z)$ *and the function q is* (*a, n*) *dominant solution of the differential subordination* (III.2.10). *The extremal function is* $p(z)=q(z^n)$, *and the function q is solution of Briot – Bouquet differential equation.*

Theorem III.2.13 [78]: *Let be the univalent function* $q\in\mathcal{H}_u(U)$ *and let* $\theta,\phi\in\mathcal{H}(D)$, *where* $D\supset q(U)$, *such that* $\phi(w)\neq 0$, $w\in q(U)$.

*Să **note*** $Q(z)=z\,q'(z)\,\phi\big(q(z)\big)$, $h(z)=\theta\big(q(z)\big)+Q(z)$ *and suppose that:*

a) *h is convex or* Q *is starlike inU,*

b) $\operatorname{Re}\dfrac{z\,h'(z)}{Q(z)} = \operatorname{Re}\left[\dfrac{\theta'(q(z))}{\phi(q(z))} + \dfrac{z\,Q'(z)}{Q(z)}\right] > 0,\ z \in U$.

If function $p \in \mathcal{H}(U)$, *with* $p(0) = q(0)$ *and* $p(U) \subset D$, *then we have:*

$$\theta(p(z)) + z\,p'(z)\phi(p(z)) \prec \theta(q(z)) + z\,q'(z)\phi(q(z)) = h(z) \qquad \text{(III.2.31)}$$

implică $p(z) \prec q(z)$, *and the function q is the best dominant of the subordination* (III.2.31).

III.3. Applications of differential subordinations

In paragraph were determined applications of differential subordinations, using function $q(z) = \dfrac{1+A\,z}{1-A\,z}$. The results are original and are contained in [57], [61].

Theorem III.3.1 [61]**:** *Let* $q \in \mathcal{H}_u(U)$, $q(z) = \dfrac{1+A\,z}{1-A\,z}$, *with* $A \in (-1,0) \cup (0,1)$ *and let* $\alpha \in (0,1]$, *such that* $\dfrac{1-\alpha}{\alpha} + \dfrac{1+A}{1-A} > 0$. *If* $p \in \mathcal{H}(U)$, *with* $p(0) = q(0) = 1$ *and*

$$(1-\alpha)\,p(z) + \alpha\,z\,p'(z) \prec (1-\alpha)\frac{1+A\,z}{1-A\,z} + \alpha\frac{2A\,z}{(1-A\,z)^2},\ \textit{then}\ p(z) \prec q(z).$$

Theorem III.3.2 [61]**:** *Let* $A \in (-1,0) \cup (0,1)$ *and let* $\alpha \in (0,1]$, *such that* $\dfrac{1-\alpha}{\alpha} + \dfrac{1+A}{1-A} > 0$. *If* $p \in \mathcal{H}(U)$, *with* $p(0) = 1$ *and*

$$(1-\alpha)\,p(z) + \alpha\,z\,p'(z) \prec (1-\alpha)\frac{1+A\,z}{1-A\,z} + \alpha\frac{2A\,z}{(1-A\,z)^2}\ ,\ \textit{then}\ \operatorname{Re} p(z) > 0.$$

Theorem III.3.3 [61]**:** *Let* $q \in \mathcal{H}_u(U)$, $q(z) = \dfrac{1+A\,z}{1-A\,z}$, *with* $A \in (-1,0) \cup (0,1)$ *and let* $\alpha, \beta > 0$, $\gamma \in (0,1]$, *such that*

$$\frac{2\alpha}{\gamma}\frac{1+A}{1-A} + \frac{\beta}{\gamma} + \frac{1+A}{1-A} > 0. \qquad \text{(III.3.9)}$$

If $p \in \mathcal{H}(U)$, *with* $p(0) = q(0) = 1$ *and*

$$\alpha\,p^2(z) + \beta\,p(z) + \gamma\,z\,p'(z) \prec \alpha\left(\frac{1+A\,z}{1-A\,z}\right)^2 + \beta\frac{1+A\,z}{1-A\,z} + \gamma\frac{2A\,z}{(1-A\,z)^2},\ \textit{then}\ p(z) \prec q(z).$$

Theorem III.3.4 [61]**:** *Let* $A \in (-1,0) \cup (0,1)$ *and let* $\alpha, \beta > 0$, $\gamma \in (0,1]$, *suppose that satisfies the relation* (III.3.9). *If* $p \in \mathcal{H}(U)$, *with* $p(0) = 1$ *and*

$$\alpha\,p^2(z) + \beta\,p(z) + \gamma\,z\,p'(z) \prec \alpha\left(\frac{1+A\,z}{1-A\,z}\right)^2 + \beta\frac{1+A\,z}{1-A\,z} + \gamma\frac{2A\,z}{(1-A\,z)^2},\ \textit{then}\ \operatorname{Re} p(z) > 0.$$

Theorem III.3.5 [61]**:** *Let* $q \in \mathcal{H}_u(U)$, $q(z) = \dfrac{1+A\,z}{1-A\,z}$, *with* $A \in (-1,0) \cup (0,1)$ *and let* $\alpha > 0$ *and* $\beta \in (0,1]$, *such that:*

$$\frac{1-A^2}{1-2A+A^2}+(\beta-1)\frac{2A-2A^3t}{1-2A^2+A^4}>0, \tag{III.3.16}$$

$$\frac{(1-\alpha)\beta}{\alpha}+\frac{1+A}{1-A}\left(\beta+1+\frac{1}{\alpha}\right)+(\beta-1)\frac{2A}{1-A^2}>0. \tag{III.3.17}$$

If $p\in \mathcal{H}(U)$, *with* $p(0) = q(0) = 1$ *and*

$$\left(p(z)\right)^{\beta}\left[(1-\alpha)+\alpha p(z)\right]+\alpha z p'(z)\left(p(z)\right)^{\beta-1}\prec\left(\frac{1+Az}{1-Az}\right)^{\beta}\left[(1-\alpha)+\alpha\frac{1+Az}{1-Az}\right]+\alpha\frac{2Az}{(1-Az)^2}\left(\frac{1+Az}{1-Az}\right)^{\beta-1}$$

then $p(z)\prec q(z)$.

Theorem III.3.6 [61]**:** *Let* $A\in(-1,0)\cup(0,1)$, *let* $\alpha>0$ *and* $\beta\in(0,1]$, *suppose that satisfies the relations* (III.3.16) *and* (III.3.17). *If* $p\in \mathcal{H}(U)$, *with* $p(0) = 1$ *and*

$$\left(p(z)\right)^{\beta}\left[(1-\alpha)+\alpha p(z)\right]+\alpha z p'(z)\left(p(z)\right)^{\beta-1}\prec\left(\frac{1+Az}{1-Az}\right)^{\beta}\left[(1-\alpha)+\alpha\frac{1+Az}{1-Az}\right]+\alpha\frac{2Az}{(1-Az)^2}\left(\frac{1+Az}{1-Az}\right)^{\beta-1}$$

then $\operatorname{Re} p(z)>0$.

Theorem III.3.7 [57]: *If the function* $f\in\mathcal{A}$, *then:*

$$\frac{zf''(z)}{f'(z)}+1\prec\frac{1+z}{1-z}\overset{(a)}{\Rightarrow}\frac{zf'(z)}{f(z)}\prec\frac{1}{1-z}\overset{(b)}{\Rightarrow}\frac{f(z)}{z}\prec\frac{1}{1-z}.$$

Theorem III.3.8 [57]: *If the function* $f\in\mathcal{A}$ *and* $0<A\le 1$, *then:*

$$\frac{zf''(z)}{f'(z)}+1\prec\frac{1+Az}{1-Az}\overset{(a)}{\Rightarrow}\frac{zf'(z)}{f(z)}\prec\frac{1}{1-Az}\overset{(b)}{\Rightarrow}\frac{f(z)}{z}\prec\frac{1}{1-Az}.$$

Theorem III.3.9 [57]: *If the function* $f\in\mathcal{A}$, *then:*

$$\frac{zf''(z)}{f'(z)}+1\prec\frac{1+z}{1-z}\overset{(a)}{\Rightarrow}f'(z)\prec\frac{1}{(1-z)^2}\overset{(b)}{\Rightarrow}\frac{f(z)}{z}\prec\frac{1}{1-z}.$$

Theorem III.3.10 [57]: *If the function* $f\in\mathcal{A}$ *and* $0<A\le 1$, *then:*

$$\frac{zf''(z)}{f'(z)}+1\prec\frac{1+Az}{1-Az}\overset{(a)}{\Rightarrow}f'(z)\prec\frac{1}{(1-Az)^2}\overset{(b)}{\Rightarrow}\frac{f(z)}{z}\prec\frac{1}{1-Az}.$$

III.4. Applications of Briot-Bouquet differential subordinations

In paragraph were determined applications of Briot-Bouquet differential subordination, using the function $q(z)=\frac{1+Az}{1-Az}$. The results are original and are contained in [64].

Theorem III.4.1 [64]**:** *Let* $q\in \mathcal{H}_u(U)$, $q(z)=\frac{1+Az}{1-Az}$, *where* $A\in(-1,0)\cup(0,1)$. *If* $p\in \mathcal{H}(U)$, *with* $p(0) = q(0) = 1$ *and*

$$p(z)+\frac{zp'(z)}{p(z)}\prec\frac{1+Az}{1-Az}+\frac{2Az}{1-A^2z^2},\ \textit{then}\ p(z)\prec q(z).$$

Theorem III.4.2 [64]**:** *Let* $q \in \mathcal{H}_u(U)$, $q(z)=\dfrac{1+Az}{1-Az}$, *where* $A\in(-1,0)\cup(0,1)$. *If* $p \in \mathcal{H}(U)$, *with* $p(0) = q(0) = 1$ *and*

$$p(z)+\frac{z\,p'(z)}{p(z)} \prec \frac{1+Az}{1-Az}+\frac{2Az}{1-A^2z^2}, \text{ then } \operatorname{Re} p(z)>0.$$

Theorem III.4.3 [64]**:** *Let* $A\in(-1,0)\cup(0,1)$ *and let be the convex function* h *in U, with* $h(0)=1$. *Suppose that satisfies differential eqution:*

$$h(z)=q(z)+\frac{z\,q'(z)}{q(z)},\ z\in U. \tag{III.4.9}$$

has the univalent solution $q(z)=\dfrac{1+Az}{1-Az}$ *satisfies* $q(0)=1$ *and* $h(z)\prec q(z)$.

If $f\in\mathcal{A}$ *and* $\dfrac{z\,f'(z)}{f(z)}$ *is univalent,* $\dfrac{z\,F'(z)}{F(z)}\in\mathcal{H}[1,1]\cap\mathcal{Q}$ *and*

$$\frac{z\,f'(z)}{f(z)}\prec h(z),\ z\in U, \text{ then } \frac{z\,F'(z)}{F(z)}\prec q(z),\ z\in U,$$

where

$$F(z)=\int_0^z \frac{f(t)}{t}\,dt. \tag{III.4.12}$$

Theorem III.4.4 [64]**:** *Let* $A\in(-1,0)\cup(0,1)$ *and let the function h defined by* (III.4.9). *If* $f\in\mathcal{A}$ *and* $\dfrac{z\,f'(z)}{f(z)}$ *is univalent,* $\dfrac{z\,F'(z)}{F(z)}\in\mathcal{H}\,[1,1]\cap\mathcal{Q}$ *and*

$$\frac{z\,f'(z)}{f(z)}\prec h(z),\ z\in U, \text{ then } \operatorname{Re}\frac{z\,F'(z)}{F(z)}>0,\ z\in U,$$

where the function F is defined by (III.4.12).

Theorem III.4.5 [64]: *Let* $q\in\mathcal{H}_u(U)$, $q(z)=\dfrac{1+Az}{1-Az}$, *with* $A\in(-1,0)\cup(0,1)$, *such that:*

$$1+\frac{A}{1-A}-\frac{A(1-\gamma)}{1+\gamma+A-A\gamma}>0, \tag{III.4.19}$$

$$\frac{1}{(1-A)(1+\gamma+A-A\gamma)}>0, \tag{III.4.20}$$

$$(1+\gamma)^2-2A\gamma(1+\gamma)+2A^3\gamma(\gamma-1)-A^4(\gamma-1)^2>0. \tag{III.4.21}$$

If $p\in\mathcal{H}(U)$, *with* $p(0) = q(0) = 1$ *and*

$$p(z)+\frac{z\,p'(z)}{p(z)+\gamma}\prec\frac{1+Az}{1-Az}+\frac{2Az}{(1-Az)(1+\gamma+Az-A\gamma z)}, \text{ then } p(z)\prec q(z).$$

Theorem III.4.6 [64]**:** *Let* $q\in\mathcal{H}_u(U)$, $q(z)=\dfrac{1+Az}{1-Az}$, *with* $A\in(-1,0)\cup(0,1)$ *and suppose that satisfies the relations* (III.4.19), (III.4.20) *and* (III.4.21). *If* $p\in\mathcal{H}(U)$, *with* $p(0) = q(0) = 1$ *and*

$$p(z)+\frac{z\,p'(z)}{p(z)+1}\prec\frac{1+Az}{1-Az}+\frac{2Az}{(1-Az)(1+\gamma+Az-A\gamma z)}, \text{ then } \operatorname{Re} p(z)>0.$$

Theorem III.4.7 [64]: *Let* $A \in (-1,0) \cup (0,1)$ *and let be the convex function in U, with* $h(0) = 1$. *Suppose that satisfies differential eqution:*

$$h(z) = q(z) + \frac{z\,q'(z)}{q(z)+\gamma}, \; z \in U. \tag{III.4.29}$$

has the univalent solution $q(z) = \dfrac{1+Az}{1-Az}$ *satisfies* $q(0) = 1$ *and* $h(z) \prec q(z)$. *If* $f \in \mathcal{A}$ *and* $\dfrac{z\,f'(z)}{f(z)}$ *is univalent,* $\dfrac{z\,F'(z)}{F(z)} \in \mathcal{H}\,[1,1] \cap \mathcal{Q}$ *and*

$$\frac{z\,f'(z)}{f(z)} \prec h(z), \; z \in U, \textit{ then } \frac{z\,F'(z)}{F(z)} \prec q(z), \; z \in U,$$

where

$$F(z) = \frac{\gamma+1}{z^{\gamma}} \int_0^z f(t)\,t^{\gamma-1}\,dt. \tag{III.4.32}$$

Theorem III.4.8 [64]**:** *Let* $A \in (-1,0) \cup (0,1)$ *and let the function h defined by* (III.4.29). *If* $f \in \mathcal{A}$ *and* $\dfrac{z\,f'(z)}{f(z)}$ *is univalent,* $\dfrac{z\,F'(z)}{F(z)} \in \mathcal{H}\,[1,1] \cap \mathcal{Q}$ *and*

$$\frac{z\,f'(z)}{f(z)} \prec h(z), \; z \in U, \textit{ then } \; \mathrm{Re}\,\frac{z\,F'(z)}{F(z)} > 0, \; z \in U,$$

Where the function F *is defined by* (III.4.32).

III.5. Differential superordination

Differential superordination method was introduced by S. S. Miller and P. T. Mocanu in article „Subordinants of Differential Superordinations" [86]. Using these methods allowed to obtain new results in the geometric theory of analytic functions.

Definition III.5.1: *Let* f *and* F *be members of* $\mathcal{H}(U)$. *The function f is said to be* ***subordinate to*** *F, or* F *is said to be* ***superordinate*** *to f, if there exists a function w analytic in U, with* $w(0) = 0$ *and* $|w(z)| < 1$, *such that* $f(z) = F\left(w(z)\right)$. *In such a case we write* $f \prec F$ *or* $f(z) \prec F(z)$. *If the function* F *is univalent, then* $f \prec F$ *if and only if* $f(0) = F(0)$ *and* $f(U) \subset F(U)$.

Definition III.5.2: *Let* $\varphi : \mathbb{C}^3 \times U \to \mathbb{C}$ *and let be the univalent function h in unit disc U. If the function* $p \in \mathcal{H}\left[a,n\right]$ *satiesfies the differential subordination:*

$$h(z) \prec \varphi\Big(p(z), z\,p'(z), z^2\,p''(z); z\Big), \quad z \in U, \tag{III.5.4}$$

then function p is called ***(a, n) solution of the differential superordination*** (III.5.4).

Definition III.5.3: *The superordination* (III.5.4) *is called the* ***second-order differential superordination,*** *and the function q univalent in U, is called* ***(a, n) subordinant solution of the differential superordination*** (III.5.4).

Definition III.5.4: *Let* $\varphi : \mathbb{C}^3 \times U \to \mathbb{C}$ *and let be the univalent function h in unit disc U. If p is a function analytic in unit disc U and satisfy the differential superordination* (III.5.4), *then function p is called* ***solution of the differential superordination***.

Definition III.5.5: *The univalent function q is called* ***a subordinant of the differential superordination*** (III.5.4)*, If $q \prec p$, for any p which satisfies the relation* (III.5.4).

Definition III.5.6: *A subordinant $\tilde{q}$ such that $q(z) \prec \tilde{q}(z)$ for all subordinants q of* (III.5.4) *is said to be* ***the best subordinant***.

Definition III.5.7 [77, 81]: *Let the set of $\Omega \subset \mathbb{C}$, let the function $q \in \mathcal{H}[a,n]$ and $n \in \mathbb{N}, n \geq 1$. Denote with $\Phi_n[\Omega, q]$ the class functions $\varphi : \mathbb{C}^3 \times U \to \mathbb{C}$ that satisfy the condition $\varphi(r,s,t;\zeta) \in \Omega$, whenever:*

$$r = q(z),\ s = \frac{z\,q'(z)}{m},\ \ \mathrm{Re}\frac{t}{s} + 1 \leq \frac{1}{m}\mathrm{Re}\left[\frac{z\,q''(z)}{q'(z)} + 1\right],$$

where $z \in U$, $\zeta \in \partial U$ and $m \geq n$. The set of $\Phi_n[\Omega, q]$ is called ***class of admissible functions****, iar the condition $\varphi(r,s,t;z) \in \Omega$ is called* ***the admissible condition*** .

Theorem III.5.1 [85]: *Let $\Omega \subset \mathbb{C}, q \in \mathcal{H}[a,n]$ and let $\varphi \in \Phi_n[\Omega, q]$, where $q(0) = a$. If the function $p \in \mathcal{Q}(a)$ and $\varphi\left(p(z), z\,p'(z), z^2\,p''(z); z\right)$ is is univalent function in unit disc U, then*

$$\Omega \subset \left\{\varphi\left(p(z), z\,p'(z), z^2\,p''(z); z\right), z \in U\right\} \Rightarrow q(z) \prec p(z).$$

Theorem III.5.2 [56]: *Let $\Omega \subset \mathbb{C}$, let $q \in \mathcal{H}[a,n]$, with $q(0) = a$ and let $\varphi \in \Phi_n\left[\Omega, q_\rho\right]$, for some $\rho \geq 1$, where $q_\rho(z) = q(\rho z)$. If the function $p \in \mathcal{Q}(a)$ and $\varphi\left(p(z), z\,p'(z), z^2\,p''(z); z\right)$ is is univalent function in unit disc U, then*

$$\Omega \subset \left\{\varphi\left(p(z), z\,p'(z), z^2 p''(z); z\right); z \in U\right\} \Rightarrow q(z) \prec p(z)\ .$$

Theorem III.5.3 [86]: *Let $q \in \mathcal{H}[a,n]$ and let be h analytic in U and let $\varphi \in \Phi_n[h, q]$. If the function $p \in \mathcal{Q}(a)$ and the function $\varphi\left(p(z), z\,p'(z), z^2\,p''(z); z\right)$ is is univalent function in unit disc U, then*

$$h(z) \prec \varphi\left(p(z), z\,p'(z), z^2 p''(z); z\right) \Rightarrow q(z) \prec p(z)\ .$$

Theorem III.5.4 [56]: *Let $\Omega \subset \mathbb{C}$, let the functions $h, q \in \mathcal{H}[a,n]$, with $q(0) = a$ and note $h_\rho(z) = h(\rho z)$, $q_\rho(z) = q(\rho z)$. Let the function $\varphi : \mathbb{C}^3 \times U \to \mathbb{C}$, with $\varphi(a,0,0;0) = h(0)$ satisfy one of the following conditions:*

a) $\varphi \in \Phi_n\left[\Omega, q_\rho\right]$, *for some $\rho \geq 1$,*

or

b) *there un $\rho_0 \geq 1$ such that $\varphi \in \Phi_n\left[h_\rho, q_\rho\right]$ for any $\rho \in (1, \rho_0)$.*

If the function $p \in \mathcal{Q}(a)$ and the function $\varphi\left(p(z), z\,p'(z), z^2\,p''(z); z\right)$ is univalent in unit disc U, then

$$h(z) \prec \varphi\left(p(z), z\,p'(z), z^2 p''(z); z\right) \Rightarrow q(z) \prec p(z)\ .$$

Theorem III.5.5 [86]: *Let be h a analytic function in U and let $\varphi : \mathbb{C}^3 \times U \to \mathbb{C}$. Suppose that the differential equation:*

$$\varphi\left(q(z), z\,q'(z), z^2 q''(z); z\right) = h(z)$$

has solution $q \in \mathcal{Q}(a)$. *If* $\varphi \in \Phi[h,q]$, $p \in \mathcal{Q}(a)$ *and* $\varphi\left(p(z), z\,p'(z), z^2 p''(z); z\right)$ *is univalent in U, then:*

$$h(z) \prec \varphi\left(p(z), z\,p'(z), z^2 p''(z); z\right) \Rightarrow q(z) \prec p(z) \ .$$

and the function q is the best subordinant .

Theorem III.5.6 [56]: *Let be the univalent function* $h \in \mathcal{H}_u(U)$ *and let* $\varphi : \mathbb{C}^3 \times U \to \mathbb{C}$. *Suppose that differential equation:*

$$\varphi\left(q(z),\ z\,q'(z), z^2 q''(z); z\right) = h(z)$$

has solution q, with q(0) = *a, and one of the following conditions is verified:*

a) $q \in \mathcal{Q}$ *and* $\varphi \in \Phi[h,q]$,

b) *q is univalent in U and* $\varphi \in \Phi\left[h, q_\rho\right]$,*for some* $\rho \geq 1$,

c) *q is univalent in U and there un* $\rho_0 \geq 1$, *such that* $\varphi \in \Phi\left[h_\rho, q_\rho\right]$ *for any* $\rho \in (1, \rho_0)$.

If the function $p \in \mathcal{Q}(a)$ *and the function* $\varphi\left(p(z), z\,p'(z), z^2\ p''(z); z\right)$ *is univalent in unit disc U, then:*

$$h(z) \prec \varphi\left(p(z), z\,p'(z), z^2 p''(z); z\right) \Rightarrow q(z) \prec p(z) \ .$$

and the function q is the best subordinant.

Theorem III.5.7 [56]: *Let be the univalent function* $h \in \mathcal{H}_u(U)$ *and let* $\varphi : \mathbb{C}^3 \times U \to \mathbb{C}$. *Suppose that differential equation:*

$$\varphi\left(q(z), n\,z\,q'(z), n(n-1)z\,q'(z) + n^2 z^2 q''(z); z\right) = h(z)$$

has solution q, with q(0) = *a, and one of the following conditions is verified:*

a) $q \in \mathcal{Q}$ *and* $\varphi \in \Phi_n[h,q]$,

b) *q is univalent in U and* $\varphi \in \Phi_n\left[h, q_\rho\right]$, *for some* $\rho \geq 1$,

c) *q is univalent in U and there* $\rho_0 \geq 1$, *such that* $\varphi \in \Phi_n\left[h_\rho, q_\rho\right]$ *for any* $\rho \in (1, \rho_0)$.

If the function $p \in \mathcal{Q}(a)$ *and the function* $\varphi\left(p(z), z\,p'(z), z^2\ p''(z); z\right)$ *is univalent in the unit disc U, then*

$$h(z) \prec \varphi\left(p(z), z\,p'(z), z^2 p''(z); z\right) \Rightarrow q(z) \prec p(z)$$

and the function q is the best subordinant.

Theorem III.5.15 [13,15]: *Let be q a convex (univalent) function in unit disc U, let be the functions* ϑ *and* φ *analytic in a domain* $D \supset q(U)$ *and let* $\mu \in \mathcal{H}(\mathbb{C})$. *Suppose that:*

$$\operatorname{Re}\frac{\vartheta'(q(z)) + \varphi'(q(z))\,\mu(t\,z\,q'(z))}{\varphi(q(z))\,\mu'(t\,z\,q'(z))} > 0,\ z \in U,\ t \geq 0.$$

If $p \in \mathcal{H}[q(0),1] \cap \mathcal{Q}$, *with p*(0) = *q*(0), $p(U) \subset D$ *and* $\vartheta(p(z)) + \mu(z\,p'(z))\varphi(p(z))$ *is univalent in U, and*

$$\vartheta(q(z)) + \mu(t\,z\,q'(z))\varphi(q(z)) \prec \vartheta(p(z)) + \mu(z\,p'(z))\varphi(p(z))$$

then $q(z) \prec p(z)$. *The function* q *is the best subordinant.*

Corolarul III.5.6 [13]: *Let be q a convex (univalent) function in unit disc U, let be the function* φ *analytic in a domain* $D \supset q(U)$ *and let* $\varphi \in \mathcal{H}(\mathbb{C})$. *Suppose that:*

a) $\xi(z) = z\,q'(z)\varphi\big(q(z)\big)$ *is starlike in* U,

b) $\mathrm{Re}\dfrac{\vartheta'\big(q(z)\big)}{\varphi\big(q(z)\big)} > 0,\ z \in U$.

If $p \in \mathcal{H}\,[q(0),1] \cap \mathcal{Q}$, *with* $p(0) = q(0)$, $p(U) \subset D$ *and* $\vartheta\big(p(z)\big) + z\,p'(z)\varphi\big(p(z)\big)$ *is univalent in U, and*

$$\vartheta\big(q(z)\big) + z\,g'(z)\varphi\big(z\,q'(z)\big) \prec \vartheta\big(p(z)\big) + z\,p'(z)\varphi\big(p(z)\big)$$

then $q(z) \prec p(z)$. *The function* q *is the best subordinant.*

Teorema III.5.16: *Let* $q \in \mathcal{H}_u(U)$ *and let be the functions* ϑ *and* φ *analytic in a domain* $D \supset q(U)$, *with* $\varphi(w) \neq 0$, *where* $w \in q(U)$. *Let* $\xi(z) = z\,q'(z)\varphi\big(q(z)\big)$, $l(z) = \vartheta\big(q(z)\big) + \xi(z)$ *and suppose that:*

a) ξ *is starlike,*

b) $\mathrm{Re}\dfrac{z\,l'(z)}{\xi(z)} = \mathrm{Re}\left[\dfrac{\vartheta'\big(q(z)\big)}{\varphi\big(q(z)\big)} + \dfrac{z\,\xi'(z)}{\xi(z)}\right] > 0,\ z \in U$.

If $p \in \mathcal{H}\,[q(0),1] \cap \mathcal{Q}$, *with* $p(0) = q(0)$, $p(U) \subset D$ *and* $\vartheta\big(p(z)\big) + z\,p'(z)\varphi\big(p(z)\big)$ *is univalent in U and*

$$\vartheta\big(q(z)\big) + z\,q'(z)\varphi\big(q(z)\big) \prec \vartheta\big(p(z)\big) + z\,p'(z)\varphi\big(p(z)\big),$$

then $q(z) \prec p(z)$. *The function* q *is is the best subordinant.*

III.6. Briot-Bouquet differential superordination

Definition III.6.1: *Let* $\beta, \gamma \in \mathbb{C}$, *let* $h \in \mathcal{H}(U)$ *and let* $p \in \mathcal{H}(U)$, $p(z) = h(0) + p_1 z + \dots$, *with property* $p(0) = h(0)$. *By* ***Briot–Bouquet differential superordination*** *understand form:*

$$h(z) \prec p(z) + \frac{z\,p'(z)}{\beta\,p(z) + \gamma}.$$

Theorem III.6.1 [87]: *Let be h a convex function in U, with* $h(0) = a$, *and let be the functions* Θ *and* Φ *analytic in a domain D. Let* $p \in \mathcal{H}\,[a,1] \cap \mathcal{Q}$ *and suppose that* $\Theta\big(p(z)\big) + z\,p'(z)\Phi\big(p(z)\big)$ *is univalent in U. If differential equation:*

$$\Theta\big(q(z)\big) + z\,q'(z)\Phi\big(q(z)\big) = h(z)$$

it has univalent solution q, $q(0) = a$, $q(U) \subset D$ *and:*

$$\Theta\big(q(z)\big) \prec h(z),$$

Then:

$$h(z) \prec \Theta\big(p(z)\big) + z\,p'(z)\Phi\big(p(z)\big) \ \Rightarrow\ q(z) \prec p(z).$$

The function q is the best subordinant.

Theorem III.6.3 [87]: *Let be the functions* Θ *and* Φ *analytic in a domain D and let be q, where q is univalent function in U, with* $q(0)=a$, $q(U)\subset D$. *Let* $Q(z)=z\,q'(z)\Phi\left(q(z)\right)$, $h(z)=Q(z)+\Theta\left(q(z)\right)$ *and suppose that:*

a) $Q(z)$ *is starlike,*

b) $\operatorname{Re}\dfrac{\Theta'\left(q(z)\right)}{\Phi\left(q(z)\right)}>0$.

If $p\in\mathcal{H}[a,1]\cap\mathcal{Q}$, $p(U)\subset D$ *and suppose that* $\Theta\left(p(z)\right)+z\,p'(z)\Phi\left(p(z)\right)$ *is univalent in U, then*

$$h(z)\prec\Theta\left(p(z)\right)+z\,p'(z)\Phi\left(p(z)\right)\ \Rightarrow\ q(z)\prec p(z).$$

The function q is the best subordinant.

III.7. Applications of Briot-Bouquet differential superordination using an integral operator

In paragraph were determined applications of Briot-Bouquet differential superordination with the help of integral operators. The results are original and are contained in [54], [55], [58].

Theorem III.7.1 [54]: *Let* $A\in(-1,0)\cup(0,1)$. *The function* $h(z)=\dfrac{1+Az}{1-Az}+\dfrac{Az}{1-Az}$, $z\in U$ *is convex*.

Theorem III.7.2 [55]: *Let* $A\in(-1,0)\cup(0,1)$ *and the function h is convex in U, with* $h(0)=1$. *Suppose that we have differential equation:*

$$h(z)=q(z)+\frac{z\,q'(z)}{q(z)+1},\ z\in U$$

with the univalent solution $q(z)=\dfrac{1+Az}{1-Az}$, $q(0)=1$ *and* $q(z)\prec h(z)$. *If* $f\in\mathcal{A}$ *and* $\dfrac{z\,f'(z)}{f(z)}$ *is univalent,* $\dfrac{z\,F'(z)}{F(z)}\in\mathcal{H}[1,1]\cap\mathcal{Q}$ *and:*

$$h(z)\prec\frac{z\,f'(z)}{f(z)},\ z\in U\text{, then }q(z)\prec\frac{z\,F'(z)}{F(z)},\ z\in U,$$

where:

$$F(z)=\frac{2}{z}\int_0^z f(t)dt\,. \qquad \text{(III.7.10)}$$

Corollary III.7.1 [55]: *Let* $A\in(-1,0)\cup(0,1)$. *If* $f_1,f_2\in\mathcal{A}$, $\dfrac{z\,f_1'(z)}{f_1(z)}$ *and* $\dfrac{z\,f_2'(z)}{f_2(z)}$ *are univalent,* $\dfrac{z\,F_1'(z)}{F_1(z)},\dfrac{z\,F_2'(z)}{F_2(z)}\in\mathcal{H}[a,1]\cap\mathcal{Q}$ *and:*

$$\frac{z\,f_1'(z)}{f_1(z)}\prec\frac{1+Az}{1-Az}+\frac{Az}{1-Az}\prec\frac{z\,f_2'(z)}{f_2(z)},\ z\in U\text{, then }\frac{z\,F_1'(z)}{F_1(z)}\prec\frac{1+Az}{1-Az}\prec\frac{z\,F_2'(z)}{F_2(z)},\ z\in U,$$

Where:

$$F_i(z)=\frac{2}{z}\int_0^z f_i(t)dt\ ,\ i=\overline{1,2}\,. \tag{III.7.11}$$

Theorem III.7.3 [58]**:** *Let* $a\in A_0$, *where* $A_0=(-\infty,-4.5115...)\cup(0.7571...,+\infty)$. *The function:*

$$h(z)=z+a+\frac{z}{z+a+1},\ z\in U$$

is convex.

Theorem III.7.4 [58]: *Let* $a\in A_0$ *and let* h *is convex function in U, with* $h(0)=a$. *Suppose that differential equation:*

$$h(z)=q(z)+\frac{z\,q'(z)}{q(z)+1},\ z\in U$$

has univalent solution $q(z)=z+a$, $q(0)=a$ *and* $q(z)\prec h(z)$. *If* $f\in\mathcal{A}$ *and* $\frac{z\,f'(z)}{f(z)}$ *is univalent,* $\frac{z\,F'(z)}{F(z)}\in\mathcal{H}\,[a,1]\cap\mathcal{Q}$ *and*

$$h(z)\prec\frac{z\,f'(z)}{f(z)},\ z\in U\text{ , then }q(z)\prec\frac{z\,F'(z)}{F(z)},\ z\in U\,,$$

where the function F *is defined by the relationship* (III.5.10).

Corollary III.7.2 [58]: *Let* $a\in A_0$. *If* $f_1,f_2\in\mathcal{A}$, $\frac{z\,f_1'(z)}{f_1(z)}$ *and* $\frac{z\,f_2'(z)}{f_2(z)}$ *are univalent,* $\frac{z\,F_1'(z)}{F_1(z)},\frac{z\,F_2'(z)}{F_2(z)}\in\mathcal{H}\,[a,1]\cap\mathcal{Q}$ *and*

$$\frac{z\,f_1'(z)}{f_1(z)}\prec z+a+\frac{z}{z+a+1}\prec\frac{z\,f_2'(z)}{f_2(z)},\ z\in U\text{ , then }\frac{z\,F_1'(z)}{F_1(z)}\prec z+a\prec\frac{z\,F_2'(z)}{F_2(z)}\ ,\ z\in U\,,$$

where F_i, $i=\overline{1,2}$, *is defined by the relationship* (III.5.11).

III.8. Applications of differential subordinations and superordinations, sandwich theorems

In paragraph were determined applications of differential subordination and superordination. The results are original and are contained in [63], [65], [66].

Theorem III.8.1 [63]: *Let be the convex function q in U and suppose that* $\operatorname{Re}q(z)>\beta$. *Let* $f\in\mathcal{A}$ (k,n) , $k\in\mathbb{N}$ *and* $\alpha>0$. *Suppose that the function* q *satisfies the relation:*

$$\operatorname{Re}\left[\frac{z\,q''(z)}{g'(z)}+q(z)-\beta+1\right]>0\,. \tag{III.8.1}$$

If

$$\frac{1}{2}\left(\frac{f(z)}{z^k}\right)^{2\alpha}-(\alpha k+\beta)\left(\frac{f(z)}{z^k}\right)^{\alpha}+\alpha\frac{f'(z)}{z^{k-1}}\left(\frac{f(z)}{z^k}\right)^{\alpha-1}\prec\frac{q^2(z)}{2}-\beta\,q(z)+z\,q'(z),$$

then

$$\left(\frac{f(z)}{z^k}\right)^{\alpha} \prec q(z).$$

The function q is the best dominant.

Theorem III.8.2 [63]: *Let be the convex function q in U and suppose that* $\operatorname{Re} q(z) > \beta$. *Let* $f \in \mathcal{A}$ (k,n), $k \in \mathbb{N}$, $\left(\frac{f(z)}{z^k}\right)^{\alpha} \in \mathcal{H}\,[q(0),1] \cap \mathcal{Q}$, $\alpha > 0$ *and let*

$$\frac{1}{2}\left(\frac{f(z)}{z^k}\right)^{2\alpha} - (\alpha k + \beta)\left(\frac{f(z)}{z^k}\right)^{\alpha} + \alpha \frac{f'(z)}{z^{k-1}}\left(\frac{f(z)}{z^k}\right)^{\alpha-1}$$

is univalent function in U. Suppose that the function q satisfies the relation:

$$\operatorname{Re}\left[q(z)q'(z) - \beta q'(z)\right] > 0. \tag{III.8.4}$$

If

$$\frac{q^2(z)}{2} - \beta q(z) + z\,q'(z) \prec \frac{1}{2}\left(\frac{f(z)}{z^k}\right)^{2\alpha} - (\alpha k + \beta)\left(\frac{f(z)}{z^k}\right)^{\alpha} + \alpha \frac{f'(z)}{z^{k-1}}\left(\frac{f(z)}{z^k}\right)^{\alpha-1},$$

then

$$q(z) \prec \left(\frac{f(z)}{z^k}\right)^{\alpha}$$

and q is the best subordinant.

Theorem III.8.3 [63]: *Let the functions q_1 is convex and q_2 is univalent in U and suppose that* $\operatorname{Re} q(z) > \beta$. *Let* $f \in \mathcal{A}$ (k,n), $k \in \mathbb{N}$, $\left(\frac{f(z)}{z^k}\right)^{\alpha} \in \mathcal{H}\,[q(0),1] \cap \mathcal{Q}$, $\alpha > 0$ *and let*

$$\frac{1}{2}\left(\frac{f(z)}{z^k}\right)^{2\alpha} - (\alpha k + \beta)\left(\frac{f(z)}{z^k}\right)^{\alpha} + \alpha \frac{f'(z)}{z^{k-1}}\left(\frac{f(z)}{z^k}\right)^{\alpha-1}$$

is univalent in U. Suppose that the function q_1 satisfies the relation (III.8.4) *and the function q_2 satisfies the relation* (III.8.1). *If*

$$\frac{q_1^{\,2}(z)}{2} - \beta q_1(z) + z\,q_1'(z) \prec \frac{1}{2}\left(\frac{f(z)}{z^k}\right)^{2\alpha} - (\alpha k + \beta)\left(\frac{f(z)}{z^k}\right)^{\alpha} + \alpha \frac{f'(z)}{z^{k-1}}\left(\frac{f(z)}{z^k}\right)^{\alpha-1}$$
$$\prec \frac{q_2^{\,2}(z)}{2} - \beta q_2(z) + z\,q_2'(z),$$

then

$$q_1(z) \prec \left(\frac{f(z)}{z^k}\right)^{\alpha} \prec q_2(z)$$

and q_1 is the best subordinant, iar q_2 is the best dominant.

Theorem III.8 [65]**:** *Let* $f \in \mathcal{A}(k,n)$, $k \in \mathbb{N}$, $\gamma > 0$ *and* $\alpha > 0$. *Let be the convex function q in U and suppose that the function q satisfies the relation:*

$$\operatorname{Re}\left[\frac{z\,q''(z)}{g'(z)} + \frac{\alpha}{\gamma} + 1\right] > 0. \tag{III.8.7}$$

If:

$$\gamma\left(\frac{f(z)}{z^k}\right)^{\alpha-1}\frac{f'(z)}{z^{k-1}}+(1-\gamma k)\left(\frac{f(z)}{z^k}\right)^{\alpha}\prec q(z)+\frac{\gamma z q'(z)}{\alpha},$$

then:

$$\left(\frac{f(z)}{z^k}\right)^{\alpha}\prec q(z)$$

and q is the best dominant.

Theorem III.8.5 [65]**:** *Let* $f\in\mathcal{A}(k,n)$, $k\in\mathbb{N}$, *let be the convex function q in U and let* $\left(\frac{f(z)}{z^k}\right)^{\alpha}\in\mathcal{H}[q(0),1]\cap\mathcal{Q}$, $\gamma>0$ *and* $\alpha>0$ *. Let*

$$\gamma\left(\frac{f(z)}{z^k}\right)^{\alpha-1}\frac{f'(z)}{z^{k-1}}+(1-\gamma k)\left(\frac{f(z)}{z^k}\right)^{\alpha}$$

is univalent in U . Suppose that the function q satisfies the relation

$$\operatorname{Re}\left[\frac{\gamma q'(z)}{\alpha}\right]>0. \tag{III.8.10}$$

If :

$$q(z)+\frac{\gamma z q'(z)}{\alpha}\prec\gamma\left(\frac{f(z)}{z^k}\right)^{\alpha-1}\frac{f'(z)}{z^{k-1}}+(1-\gamma k)\left(\frac{f(z)}{z^k}\right)^{\alpha},$$

then:

$$q(z)\prec\left(\frac{f(z)}{z^k}\right)^{\alpha}$$

and q is the best subordinant.

Theorem III.8.6 [65]**:** *Let* $f\in\mathcal{A}(k,n)$, $\left(\frac{f(z)}{z^k}\right)^{\alpha}\in\mathcal{H}[q(0),1]\cap Q$, $k\in\mathbb{N}$ $\gamma>0$ *and* $\alpha>0$. *Let*

$$\gamma\left(\frac{f(z)}{z^k}\right)^{\alpha-1}\frac{f'(z)}{z^{k-1}}+(1-\gamma k)\left(\frac{f(z)}{z^k}\right)^{\alpha}$$

is univalent in U . Let the functions q_1 *is convex and* q_2 *is univalent in U. Suppose that the function* q_1 *satisfies the relation* (III.8.10), *and* q_2 *satisfies the relation* (III.8.7). *If*

$$q_1(z)+\frac{\gamma z q_1'(z)}{\alpha}\prec\gamma\left(\frac{f(z)}{z^k}\right)^{\alpha-1}\frac{f'(z)}{z^{k-1}}+(1-\gamma k)\left(\frac{f(z)}{z^k}\right)^{\alpha}\prec q_2(z)+\frac{\gamma z q_2'(z)}{\alpha},$$

then

$$q_1(z)\prec\left(\frac{f(z)}{z^k}\right)^{\alpha}\prec q_2(z)$$

and q_1 *is the best subordinant, iar* q_2 *is the best dominant.*

Theorem III.8.7 [66]**:** *Let* $f\in\mathcal{A}(k,n)$, $k\in\mathbb{N}$, $\gamma>0$ *and* $\alpha>0$. *Let be the univalent function q in U and suppose that satisfies the relation:*

$$\operatorname{Re}\left[\frac{z\,q''(z)}{g'(z)}-\frac{z\,q'(z)}{q(z)}+1\right]>0. \tag{III.8.13}$$

If :

$$1+\alpha\gamma\frac{z\,f'(z)}{f(z)}-\alpha k\gamma \prec 1+\frac{\gamma z\,q'(z)}{q(z)},$$

then:

$$\left(\frac{f(z)}{z^k}\right)^{\alpha} \prec q(z)$$

and q is the best dominant.

Theorem III.8.8 [66]**:** *Let* $f\in\mathcal{A}(k,n)$ *,* $\left(\frac{f(z)}{z^k}\right)^{\alpha}\in\mathcal{H}[q(0),1]\cap\mathcal{Q},\ k\in\mathbb{N}\ \ \gamma>0$ *and* $\alpha>0$ *. Let*

$$1+\alpha\gamma\frac{z\,f'(z)}{f(z)}-\alpha k\gamma$$

univalent in U*. Let be q convex function in U and suppose that satisfies the relation* (III.7.13). *If*

$$1+\frac{\gamma z\,q'(z)}{q(z)}\prec 1+\alpha\gamma\frac{z\,f'(z)}{f(z)}-\alpha k\gamma,$$

then:

$$q(z)\prec\left(\frac{f(z)}{z^k}\right)^{\alpha}$$

and q is the best subordinant.

Theorem III.8.9 [66]**:** *Let* $f\in\mathcal{A}(k,n)$ *,* $\left(\frac{f(z)}{z^k}\right)^{\alpha}\in\mathcal{H}[q(0),1]\cap\mathcal{Q},\ k\in\mathbb{N},\ \gamma>0$ *and* $\alpha>0$ *. Let* $1+\alpha\gamma\frac{z\,f'(z)}{f(z)}-\alpha k\gamma$ *is univalent in* U*. Let the functions* q_1 *is convex and* q_2 *is univalent in U and suppose that satisfies the relation* (III.8.13). *If*

$$1+\frac{\gamma z\,q_1'(z)}{q_1(z)}\prec 1+\alpha\gamma\frac{z\,f'(z)}{f(z)}-\alpha k\gamma\prec 1+\frac{\gamma z\,q_2'(z)}{q_2(z)},$$

then :

$$q_1(z)\prec\left(\frac{f(z)}{z^k}\right)^{\alpha}\prec q_2(z),$$

and q_1 *is the best subordinant, iar* q_2 *is the best dominant.*

Theorem III.810 [66]**:** *Let* $f\in\mathcal{A}(k,n)$ *,* $k\in\mathbb{N}$ *and* $\alpha>0$*. Let be the univalent function q in U and suppose that satisfies the relations:*

$$\operatorname{Re} q(z)>0 \tag{III.8.18}$$

and

$$\operatorname{Re}\left[\frac{z\,q''(z)}{g'(z)}-\frac{z\,q'(z)}{q(z)}+1\right]>0. \tag{III.8.19}$$

If

$$\left(\frac{f(z)}{z^k}\right)^{\alpha}+\alpha\gamma\frac{z f'(z)}{f(z)}-\alpha k\gamma\prec q(z)+\frac{z q'(z)}{q(z)},$$

then

$$\left(\frac{f(z)}{z^k}\right)^{\alpha}\prec q(z),$$

and q is the best dominant.

Theorem III.8.11 [66]**:** *Let* $f\in\mathcal{A}(k,n)$, $\left(\frac{f(z)}{z^k}\right)^{\alpha}\in\mathcal{H}[q(0),1]\cap\mathcal{Q}$, $k\in\mathbb{N}$ *and* $\alpha>0$ *. Let*

$$\left(\frac{f(z)}{z^k}\right)^{\alpha}+\alpha\gamma\frac{z f'(z)}{f(z)}-\alpha k\gamma$$

is univalent in U *. Let be the convex function q in U. Suppose that the function q satisfies the relations* (III.8.19) *and*

$$\operatorname{Re}\left[q(z)g'(z)\right]>0. \qquad \text{(III.8.22)}$$

If:

$$q(z)+\frac{z q'(z)}{q(z)}\prec\left(\frac{f(z)}{z^k}\right)^{\alpha}+\alpha\frac{z f'(z)}{f(z)}-\alpha k,$$

then:

$$q(z)\prec\left(\frac{f(z)}{z^k}\right)^{\alpha}$$

and q is the best subordinant.

Theorem III.8.12 [66]**:** *Let* $f\in\mathcal{A}(k,n)$, $\left(\frac{f(z)}{z^k}\right)^{\alpha}\in\mathcal{H}[q(0),1]\cap\mathcal{Q}$, $k\in\mathbb{N}$ *and* $\alpha>0$ *. Let*

$$\left(\frac{f(z)}{z^k}\right)^{\alpha}+\alpha\frac{z f'(z)}{f(z)}-\alpha k$$

is univalent in U *. Let the functions* q_1 *is convex and* q_2 *is univalent in U. Suppose that the function* q_1 *satisfies the relations* (III.8.19) and (III.8.22), *and the function* q_2 *satisfies the relations* (III.8.18) *and* (III.8.19). *If*

$$q_1(z)+\frac{z q_1'(z)}{q_1(z)}\prec\left(\frac{f(z)}{z^k}\right)^{\alpha}+\alpha\gamma\frac{z f'(z)}{f(z)}-\alpha k\gamma\prec q_2(z)+\frac{z q_2'(z)}{q_2(z)},$$

then

$$q_1(z)\prec\left(\frac{f(z)}{z^k}\right)^{\alpha}\prec q_2(z),$$

and q_1 *is the best subordinant, iar* q_2 *is the best dominant.*

Theorem III.8.13 [65]**:** *Let* $f\in\mathcal{A}(k,n)$, $k\in\mathbb{N}$, $\lambda>0$ *and* $\alpha>0$. *Let be the convex function q in U. If*

$$\alpha\lambda\left(\frac{f(z)}{z^k}\right)^{\alpha-1}\frac{f'(z)}{z^{k-1}}+(1-\lambda-\alpha\lambda k)\left(\frac{f(z)}{z^k}\right)^{\alpha}\prec(1-\lambda)q(z)+\lambda z q'(z),$$

then

$$\left(\frac{f(z)}{z^k}\right)^{\alpha} \prec q(z),$$

and q is the best dominant.

Theorem III.8.14 [65]**:** *Let* $f \in \mathcal{A}\ (k,n)$, $\left(\frac{f(z)}{z^k}\right)^{\alpha} \in \mathcal{H}\,[q(0),1] \cap \mathcal{Q}$, $k \in \mathbb{N}$, $\lambda > 0$ *and* $\alpha > 0$.

Let:

$$\alpha\,\lambda\left(\frac{f(z)}{z^k}\right)^{\alpha-1}\frac{f'(z)}{z^{k-1}} + (1-\lambda-\alpha\,\lambda\,k)\left(\frac{f(z)}{z^k}\right)^{\alpha}$$

is univalent a function in U. Let be the convex function q in U. Suppose that q satisfies the relation:

$$\mathrm{Re}\left[\frac{(1-\lambda)\,q'(z)}{\lambda}\right] > 0. \tag{III.8.27}$$

If :

$$(1-\lambda)q(z) + \lambda\,z\,q'(z) \prec \alpha\,\lambda\left(\frac{f(z)}{z^k}\right)^{\alpha-1}\frac{f'(z)}{z^{k-1}} + (1-\lambda-\alpha\,\lambda\,k)\left(\frac{f(z)}{z^k}\right)^{\alpha},$$

then:

$$q(z) \prec \left(\frac{f(z)}{z^k}\right)^{\alpha}$$

and q is the best subordinant.

Theorem III.8.15 [65]**:** *Let* $f \in \mathcal{A}\ (k,n)$, $\mathcal{H}\,[q(0),1] \cap \mathcal{Q}$, $k \in \mathbb{N}$, $\lambda > 0$ *and* $\alpha > 0$. *Let*

$$\alpha\,\lambda\left(\frac{f(z)}{z^k}\right)^{\alpha-1}\frac{f'(z)}{z^{k-1}} + (1-\lambda-\alpha\,\lambda\,k)\left(\frac{f(z)}{z^k}\right)^{\alpha}$$

is univalent in U . Let the functions q_1 *is convex and* q_2 *is univalent in U and suppose that the function* q_1 *satisfies the relation* (III.8.27). *If*

$$(1-\lambda)q_1(z) + \lambda\,z\,q_1'(z) \prec \alpha\,\lambda\left(\frac{f(z)}{z^k}\right)^{\alpha-1}\frac{f'(z)}{z^{k-1}} + (1-\lambda-\alpha\,\lambda\,k)\left(\frac{f(z)}{z^k}\right)^{\alpha} \prec (1-\lambda)q_2(z) + \lambda\,z\,q_2'(z),$$

then:

$$q_1(z) \prec \left(\frac{f(z)}{z^k}\right)^{\alpha} \prec q_2(z)$$

and q_1 *is the best subordinant, iar* q_2 *is the best dominant.*

III.9. Differential subordinations and superordinations for analytic functions defined by the Ruscheweyh linear operator

In [59] and [67] the author obtained differential subordination and superordination using Ruscheweyh linear operator. These results are original.

We define the operator Ruscheweyh $R^m : \mathcal{A}_n \to \mathcal{A}_n$, $n \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$,

$$R^0 f(z) = f(z)$$
$$R^1 f(z) = z f'(z)$$
$$(m+1)R^{m+1} f(z) = z\left[R^m f(z)\right]' + m R^m f(z), \quad z \in U.$$

If $f \in \mathcal{A}_n$, *then we have*:

$$R^m f(z) = z + \sum_{j=n+1}^{\infty} C_{m+j-1}^{m} a_j z^j .$$

Theorem III.9.1 [59]**:** *Let* $f \in \mathcal{A}_n$, $m \in \mathbb{N} \cup \{0\}$ *and* $\alpha > 0$. *Let* q *is univalent function in* U *and suppose that:*

$$\operatorname{Re} q(z) > 0 \tag{III.9.1}$$

and

$$\operatorname{Re}\left[\frac{z q''(z)}{g'(z)} - \frac{z q'(z)}{q(z)} + 1\right] > 0 . \tag{III.9.2}$$

If :

$$\left(\frac{R^m f(z)}{z}\right)^{\alpha} + \alpha \frac{(m+1)R^{m+1} f(z)}{R^m f(z)} - \alpha\left(m+1\right) \prec q(z) + \frac{z q'(z)}{q(z)} ,$$

then:

$$\left(\frac{R^m f(z)}{z}\right)^{\alpha} \prec q(z)$$

and q is the best dominant.

Theorem III.9.2 [59]**:** *Let* $f \in \mathcal{A}_n$, $\left(\frac{R^m f(z)}{z}\right)^{\alpha} \in \mathcal{H}\,[q(0),1] \cap \mathcal{Q}$, $m \in \mathbb{N} \cup \{0\}$ *and* $\alpha > 0$. *Let* $\left(\frac{R^m f(z)}{z}\right)^{\alpha} + \alpha \frac{(m+1)R^{m+1} f(z)}{R^m f(z)} - \alpha\left(m+1\right)$ *is univalent in U. Let be the convex function q in U and suppose that satisfies the relations* (III.9.2) and

$$\operatorname{Re}\left[q(z) g'(z)\right] > 0 \tag{III.9.5}$$

If:

$$q(z) + \frac{z q'(z)}{q(z)} \prec \left(\frac{R^m f(z)}{z}\right)^{\alpha} + \alpha \frac{(m+1)R^{m+1} f(z)}{R^m f(z)} - \alpha\left(m+1\right) ,$$

then:

$$q(z) \prec \left(\frac{R^m f(z)}{z}\right)^{\alpha}$$

and q is the best subordinant.

Theorem III.9.3 [59]**:** *Let* $f \in \mathcal{A}_n$, $\left(\frac{R^m f(z)}{z}\right)^{\alpha} \in \mathcal{H}[q(0),1] \cap \mathcal{Q}$, $m \in \mathbb{N} \cup \{0\}$ *and* $\alpha > 0$. *Let* $\left(\frac{R^m f(z)}{z}\right)^{\alpha} + \alpha \frac{(m+1)R^{m+1} f(z)}{R^m f(z)} - \alpha(m+1)$ *is univalent in U. Let the functions* q_1 *is convex and* q_2 *is univalent in U. Suppose that the function* q_1 *satisfies the relations* (III.9.2) and (III.9.5), *and the function* q_2 *satisfies the relations* (III.9.1) *and* (III.9.2). *If*

$$q_1(z) + \frac{z q_1'(z)}{q_1(z)} \prec \left(\frac{R^m f(z)}{z}\right)^{\alpha} + \alpha \frac{(m+1)R^{m+1} f(z)}{R^m f(z)} - \alpha(m+1) \prec q_2(z) + \frac{z q_2'(z)}{q_2(z)},$$

then

$$q_1(z) \prec \left(\frac{R^m f(z)}{z}\right)^{\alpha} \prec q_2(z),$$

and q_1 *is the best subordinant, iar* q_2 *is the best dominant.*

Theorem III.9.4 [59]**:** *Let* $f \in \mathcal{A}_n$, $m \in \mathbb{N} \cup \{0\}$ *and* $\alpha > 0$. *Let q is univalent function in U and suppose that satisface relations* (III.9.1) *and* (III.9.2). *If*

$$\frac{R^{m+1} f(z)}{z}\left(\frac{z}{R^m f(z)}\right)^{\alpha} + (m+2)\left[\frac{R^{m+2} f(z)}{R^{m+1} f(z)} - 1\right] + \alpha(m+1)\left[1 - \frac{R^{m+1} f(z)}{R^m f(z)}\right] \prec q(z) + \frac{z q'(z)}{q(z)},$$

then:

$$\frac{R^{m+1} f(z)}{z}\left(\frac{z}{R^m f(z)}\right)^{\alpha} \prec q(z),$$

and q is the best dominant.

Theorem III.9.5 [59]**:** *Let* $f \in \mathcal{A}_n$, $\frac{R^{m+1} f(z)}{z}\left(\frac{z}{R^m f(z)}\right)^{\alpha} \in \mathcal{H}[q(0),1] \cap Q$, $m \in \mathbb{N} \cup \{0\}$ *and* $\alpha > 0$. *Let* $\frac{R^{m+1} f(z)}{z}\left(\frac{z}{R^m f(z)}\right)^{\alpha} + (m+2)\left[\frac{R^{m+2} f(z)}{R^{m+1} f(z)} - 1\right] + \alpha(m+1)\left[1 - \frac{R^{m+1} f(z)}{R^m f(z)}\right]$ *is univalent in U. Let be the convex function q in U and suppose that satisfies the relations* (III.9.2) *and* (III.9.5). *If*

$$q(z) + \frac{z q'(z)}{q(z)} \prec \frac{R^{m+1} f(z)}{z}\left(\frac{z}{R^m f(z)}\right)^{\alpha} + (m+2)\left[\frac{R^{m+2} f(z)}{R^{m+1} f(z)} - 1\right] + \alpha(m+1)\left[1 - \frac{R^{m+1} f(z)}{R^m f(z)}\right]$$

Then:

$$q(z) \prec \frac{R^{m+1} f(z)}{z}\left(\frac{z}{R^m f(z)}\right)^{\alpha}$$

and q is the best subordinant.

Theorem III.9.6 [59]**:** *Let* $f \in \mathcal{A}_n$, $\frac{R^{m+1}f(z)}{z}\left(\frac{z}{R^m f(z)}\right)^{\alpha} \in \mathcal{H}\,[q(0),1] \cap \mathcal{Q}$, $m \in \mathbb{N} \cup \{0\}$ *and* $\alpha > 0$. *Let* $\frac{R^{m+1}f(z)}{z}\left(\frac{z}{R^m f(z)}\right)^{\alpha} + (m+2)\left[\frac{R^{m+2}f(z)}{R^{m+1}f(z)} - 1\right] + \alpha(m+1)\left[1 - \frac{R^{m+1}f(z)}{R^m f(z)}\right]$ *is univalent in U. Let the functions* q_1 *is convex and* q_2 *is univalent in U. Suppose that the function* q_1 *satisfies the relations* (III.9.2) and (III.9.5), *and the function* q_2 *satisfies the relations* (III.9.1) *and* (III.9.2). *If*

$$q_1(z) + \frac{z q_1'(z)}{q_1(z)} \prec \frac{R^{m+1}f(z)}{z}\left(\frac{z}{R^m f(z)}\right)^{\alpha} + (m+2)\left[\frac{R^{m+2}f(z)}{R^{m+1}f(z)} - 1\right] + \\ + \alpha(m+1)\left[1 - \frac{R^{m+1}f(z)}{R^m f(z)}\right] \prec q_2(z) + \frac{z q_2'(z)}{q_2(z)},$$

then

$$q_1(z) \prec \frac{R^{m+1}f(z)}{z}\left(\frac{z}{R^m f(z)}\right)^{\alpha} \prec q_2(z)$$

and q_1 *is the best subordinant, iar* q_2 *is the best dominant.*

Theorem III.9.7 [59]**:** *Let* $f \in \mathcal{A}_n$, $m \in \mathbb{N} \cup \{0\}$ *and* $\alpha > 0$. *Let* q *is univalent function in U and suppose that satisface relation* (III.9.1) *and* (III.9.2). *If*

$$(m+2)\frac{R^{m+2}f(z)}{R^{m+1}f(z)} - m\frac{R^{m+1}f(z)}{R^m f(z)} - 1 \prec q(z) + \frac{z q'(z)}{q(z)},$$

then

$$\frac{R^{m+1}f(z)}{R^m f(z)} \prec q(z)$$

and q is the best dominant.

Theorem III.9.8 [59]**:** *Let* $f \in \mathcal{A}_n$, $\frac{R^{m+1}f(z)}{R^m f(z)} \in \mathcal{H}\,[q(0),1] \cap \mathcal{Q}$, $m \in \mathbb{N} \cup \{0\}$ *and* $\alpha > 0$. *Let* $(m+2)\frac{R^{m+2}f(z)}{R^{m+1}f(z)} - m\frac{R^{m+1}f(z)}{R^m f(z)} - 1$ *is univalent in U. Let be the convex function q in U and suppose that satisfies the relations* (III.9.2) *and* (III.9.5). *If*

$$q(z) + \frac{z q'(z)}{q(z)} \prec (m+2)\frac{R^{m+2}f(z)}{R^{m+1}f(z)} - m\frac{R^{m+1}f(z)}{R^m f(z)} - 1,$$

then

$$q(z) \prec \frac{R^{m+1}f(z)}{R^m f(z)}$$

and q is the best subordinant.

Theorem III.9.9 [59]: *Let* $f \in \mathcal{A}_n$, $\frac{R^{m+1}f(z)}{R^m f(z)} \in \mathcal{H}[q(0),1] \cap \mathcal{Q}$, $m \in \mathbb{N} \cup \{0\}$ *and* $\alpha > 0$. *Let* $(m+2)\frac{R^{m+2}f(z)}{R^{m+1}f(z)} - m\frac{R^{m+1}f(z)}{R^m f(z)} - 1$ *is univalent in U. Let the functions* q_1 *is convex and* q_2 *is univalent in U. Suppose that the function* q_1 *satisfies the relations* (III.9.2) and (III.9.5), *and the function* q_2 *satisfies the relations* (III.9.1) *and* (III.9.2). *If*

$$q_1(z) + \frac{z q_1'(z)}{q_1(z)} \prec (m+2)\frac{R^{m+2}f(z)}{R^{m+1}f(z)} - m\frac{R^{m+1}f(z)}{R^m f(z)} - 1 \prec q_2(z) + \frac{z q_2'(z)}{q_2(z)},$$

then

$$q_1(z) \prec \frac{R^{m+1}f(z)}{R^m f(z)} \prec q_2(z)$$

and q_1 *is the best subordinant, iar* q_2 *is the best dominant.*

Theorem III.9.10 [67]: *Let* $f \in \mathcal{A}(k,n)$, $k,n \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$ *and* $\alpha > 0$. *Let be the univalent function q in U and suppose that satisfies the relations:*

$$\operatorname{Re} q(z) > 0 \tag{III.9.13}$$

and

$$\operatorname{Re}\left[\frac{z q''(z)}{g'(z)} - \frac{z q'(z)}{q(z)} + 1\right] > 0. \tag{III.9.14}$$

If:

$$\left(\frac{R^m f(z)}{z^k}\right)^{\alpha} + \alpha\frac{(m+1)R^{m+1}f(z)}{R^m f(z)} - \alpha\left(m+k\right) \prec q(z) + \frac{z q'(z)}{q(z)},$$

then:

$$\left(\frac{R^m f(z)}{z^k}\right)^{\alpha} \prec q(z)$$

and q is the best dominant.

Theorem III.9.11 [67]: *Let* $f \in \mathcal{A}(k,n)$, $k,n \in \mathbb{N}$, $\left(\frac{R^m f(z)}{z^k}\right)^{\alpha} \in \mathcal{H}[q(0),1] \cap Q$, $m \in \mathbb{N} \cup \{0\}$ *and* $\alpha > 0$. *Let:*

$$\left(\frac{R^m f(z)}{z^k}\right)^{\alpha} + \alpha\frac{(m+1)R^{m+1}f(z)}{R^m f(z)} - \alpha\left(m+k\right)$$

is univalent in U . *Let be the convex function q in U. Suppose that the function q satisfies the relations* (III.9.14) *and*

$$\operatorname{Re}\left[q(z)q'(z)\right] > 0. \tag{III.9.17}$$

If :

$$q(z) + \frac{z q'(z)}{q(z)} \prec \left(\frac{R^m f(z)}{z^k}\right)^{\alpha} + \alpha\frac{(m+1)R^{m+1}f(z)}{R^m f(z)} - \alpha\left(m+k\right),$$

then:

$$q(z) \prec \left(\frac{R^m f(z)}{z^k}\right)^\alpha$$

and q is the best subordinant.

Theorem III.9.12 [67]**:** *Let* $f \in \mathcal{A}(k,n)$, $k,n \in \mathbb{N}$, $\left(\frac{R^m f(z)}{z^k}\right)^\alpha \in \mathcal{H}[q(0),1] \cap Q$, $m \in \mathbb{N} \cup \{0\}$ *and* $\alpha > 0$ *. Let*

$$\left(\frac{R^m f(z)}{z^k}\right)^\alpha + \alpha \frac{(m+1)R^{m+1} f(z)}{R^m f(z)} - \alpha\left(m+k\right)$$

is univalent in U *. Let the functions* q_1 *is convex and* q_2 *is univalent in U. Suppose that* q_1 *satisfies the relations* (III.9.14) *and* (III.9.17), *and* q_2 *satisfies the relations* (III.9.13) *and* (III.9.14). *If*

$$q_1(z) + \frac{z\,q_1'(z)}{q_1(z)} \prec \left(\frac{R^m f(z)}{z^k}\right)^\alpha + \alpha \frac{(m+1)R^{m+1} f(z)}{R^m f(z)} - \alpha\left(m+k\right) \prec q_2(z) + \frac{z\,q_2'(z)}{q_2(z)},$$

then

$$q_1(z) \prec \left(\frac{R^m f(z)}{z^k}\right)^\alpha \prec q_2(z)$$

and q_1 *is the best subordinant, iar* q_2 *is the best dominant.*

Theorem III.9.13 [67]**:** *Let* $f \in \mathcal{A}(k,n)$, $k,n \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$ *and* $\alpha > 0$. *Let be the univalent function q in U and suppose that satisfies the relations* (III.9.13) *and* (III.9.14). *If*

$$\frac{R^{m+1} f(z)}{z^k}\left(\frac{z^k}{R^m f(z)}\right)^\alpha + (m+2)\frac{R^{m+2} f(z)}{R^{m+1} f(z)} - (m+k+1) + \alpha(m+k)$$
$$-\alpha(m+1)\frac{R^{m+1} f(z)}{R^m f(z)} \prec q(z) + \frac{z\,q'(z)}{q(z)},$$

then

$$\frac{R^{m+1} f(z)}{z^k}\left(\frac{z^k}{R^m f(z)}\right)^\alpha \prec q(z)$$

and q is the best dominant.

Theorem III.9.14 [67]**:** *Let* $f \in \mathcal{A}(k,n)$, $k,n \in \mathbb{N}$, $\frac{R^{m+1} f(z)}{z^k}\left(\frac{z^k}{R^m f(z)}\right)^\alpha \in \mathcal{H}[q(0),1] \cap \mathcal{Q}$, $m \in \mathbb{N} \cup \{0\}$ *and* $\alpha > 0$ *. Let*

$$\frac{R^{m+1} f(z)}{z}\left(\frac{z}{R^m f(z)}\right)^\alpha + (m+2)\frac{R^{m+2} f(z)}{R^{m+1} f(z)} - (m+k+1) + \alpha(m+k) - \alpha(m+1)\frac{R^{m+1} f(z)}{R^m f(z)}$$

is univalent in U *. Let be the convex function q in U. Suppose that the function q satisfies the relations* (III.9.14) *and* (III.9.17). *If*

$$q(z)+\frac{zq'(z)}{q(z)}\prec\frac{R^{m+1}f(z)}{z}\left(\frac{z}{R^m f(z)}\right)^{\alpha}+(m+2)\frac{R^{m+2}f(z)}{R^{m+1}f(z)}-(m+k+1)$$
$$+\alpha(m+k)-\alpha(m+1)\frac{R^{m+1}f(z)}{R^m f(z)},$$

then

$$q(z)\prec\frac{R^{m+1}f(z)}{z^k}\left(\frac{z^k}{R^m f(z)}\right)^{\alpha}$$

and q is the best subordinant.

Theorem III.9.15 [67]**:** *Let* $f\in\mathcal{A}(k,n)$, $k,n\in\mathbb{N}$, $\frac{R^{m+1}f(z)}{z^k}\left(\frac{z^k}{R^m f(z)}\right)^{\alpha}\in\mathcal{H}[q(0),1]\cap\mathcal{Q}$, $m\in\mathbb{N}\cup\{0\}$ *and* $\alpha>0$ *. Let*

$$\frac{R^{m+1}f(z)}{z}\left(\frac{z}{R^m f(z)}\right)^{\alpha}+(m+2)\frac{R^{m+2}f(z)}{R^{m+1}f(z)}-(m+k+1)+\alpha(m+k)-\alpha(m+1)\frac{R^{m+1}f(z)}{R^m f(z)}$$

is univalent in U . Let the functions q_1 *is convex and* q_2 *is univalent in U. Suppose that* q_1 *satisfies the relations* (III.9.14) *and* (III.9.17), *and* q_2 *satisfies the relations* (III.9.13) *and* (III.9.14). *If :*

$$q_1(z)+\frac{zq_1'(z)}{q_1(z)}\prec\frac{R^{m+1}f(z)}{z}\left(\frac{z}{R^m f(z)}\right)^{\alpha}+(m+2)\frac{R^{m+2}f(z)}{R^{m+1}f(z)}-(m+k+1)$$
$$+\alpha(m+k)-\alpha(m+1)\frac{R^{m+1}f(z)}{R^m f(z)}\prec q_2(z)+\frac{zq_2'(z)}{q_2(z)},$$

then:

$$q_1(z)\prec\frac{R^{m+1}f(z)}{z^k}\left(\frac{z^k}{R^m f(z)}\right)^{\alpha}\prec q_2(z)$$

and q_1 *is the best subordinant, iar* q_2 *is the best dominant.*

III.10. Differential subordinations and superordinations for analytic functions defined by a class of multiplier transformations

The results of this paragraph are original and are contained in [60], [68].

Definition III.10.1: *Let* $f\in\mathcal{A}(k,n)$, $k,n\in\mathbb{N}$. *Let the operator* $I_k(r,\lambda):\mathcal{A}(k,n)\to\mathcal{A}(k,n)$, *defined by*:

$$I_k(r,\lambda)f(z)=z^k+\sum_{j=k+1}^{\infty}\left(\frac{j+\lambda}{k+\lambda}\right)^r a_j z^j\,,\ \lambda\ge 0\,,$$

$$(k+\lambda)I_k(r+1,\lambda)f(z)=z\left[I_k(r,\lambda)f(z)\right]'+\lambda I_k(r,\lambda)f(z)\,.$$

Theorem III.10.4 [60]**:** *Let* $f\in\mathcal{A}(k,n)$, $k,n\in\mathbb{N}$ *and* $\lambda\ge 0$. *Let q is univalent function in U and suppose that satisfies the relations:*

$$\operatorname{Re} q(z) > 0 \tag{III.10.4}$$

and

$$\operatorname{Re}\left[\frac{z\,q''(z)}{g'(z)} - \frac{z\,q'(z)}{q(z)} + 1\right] > 0\,. \tag{III.10.5}$$

If :

$$(k+\lambda)\frac{I_k(r+2,\lambda)f(z)}{I_k(r+1,\lambda)f(z)} - (k+\lambda-1)\frac{I_k(r+1,\lambda)f(z)}{I_k(r,\lambda)f(z)} \prec q(z) + \frac{z\,q'(z)}{q(z)} \tag{III.10.6}$$

then:

$$\frac{I_k(r+1,\lambda)f(z)}{I_k(r,\lambda)f(z)} \prec q(z)$$

and q is the best dominant.

Theorem III.10.5 [60]**:** *Let* $f \in \mathcal{A}(k,n)$, $k,n \in \mathbb{N}$, $\dfrac{I_k(r+1,\lambda)f(z)}{I_k(r,\lambda)f(z)} \in \mathcal{H}\,[q(0),1] \cap \mathcal{Q}$ *and* $\lambda \geq 0$.

Let

$$(k+\lambda)\frac{I_k(r+2,\lambda)f(z)}{I_k(r+1,\lambda)f(z)} - (k+\lambda-1)\frac{I_k(r+1,\lambda)f(z)}{I_k(r,\lambda)f(z)}$$

is univalent in U *. Let be the convex function q in U. Suppose that the function q satisfies the relations* (III.10.5) *and:*

$$\operatorname{Re}\left[q(z)g'(z)\right] > 0 \tag{III.10.8}$$

If :

$$q(z) + \frac{z\,q'(z)}{q(z)} \prec (k+\lambda)\frac{I_k(r+2,\lambda)f(z)}{I_k(r+1,\lambda)f(z)} - (k+\lambda-1)\frac{I_k(r+1,\lambda)f(z)}{I_k(r,\lambda)f(z)} \tag{III.10.9}$$

then:

$$q(z) \prec \frac{I_k(r+1,\lambda)f(z)}{I_k(r,\lambda)f(z)}$$

and q is the best subordinant.

Theorem III.10.6 [60]**:** *Let* $f \in \mathcal{A}(k,n)$, $k,n \in \mathbb{N}$, $\dfrac{I_k(r+1,\lambda)f(z)}{I_k(r,\lambda)f(z)} \in \mathcal{H}\,[q(0),1] \cap \mathcal{Q}$ *and* $\lambda \geq 0$.

Let $(k+\lambda)\dfrac{I_k(r+2,\lambda)f(z)}{I_k(r+1,\lambda)f(z)} - (k+\lambda-1)\dfrac{I_k(r+1,\lambda)f(z)}{I_k(r,\lambda)f(z)}$ *is univalent in U. Let the functions* q_1 *is convex and* q_2 *is univalent in U. Suppose that* q_1 *satisfies the relations* (III.10.5) *and* (III.10.8), *and* q_2 *satisfies the relations* (III.10.4) *and* (III.10.5). *If*

$$q_1(z) + \frac{z\,q_1'(z)}{q_1(z)} \prec (k+\lambda)\frac{I_k(r+2,\lambda)f(z)}{I_k(r+1,\lambda)f(z)} - (k+\lambda-1)\frac{I_k(r+1,\lambda)f(z)}{I_k(r,\lambda)f(z)} \prec q_2(z) + \frac{z\,q_2'(z)}{q_2(z)},$$

then

$$q_1(z) \prec \frac{I_k(r+1,\lambda)f(z)}{I_k(r,\lambda)f(z)} \prec q_2(z)$$

and q_1 *is the best subordinant, iar* q_2 *is the best dominant.*

Theorem III.10.7 [60]: *Let* $f \in \mathcal{A}\ (k,n)$, $k,n \in \mathbb{N}$ *and* $\lambda \geq 0$. *Let q be univalent function in U and suppose that satisfies the relations* (III.10.4) *and* (III.10.5). *If*

$$\left(\frac{I_k(r,\lambda)f(z)}{z^k}\right)^{\alpha} + \alpha(k+\lambda)\frac{I_k(r+1,\lambda)f(z)}{I_k(r,\lambda)f(z)} - \alpha(k+\lambda) \prec q(z) + \frac{zq'(z)}{q(z)},$$

then:

$$\left(\frac{I_k(r,\lambda)f(z)}{z^k}\right)^{\alpha} \prec q(z),$$

and q is the best dominant.

Theorem III.10.8 [60]: *Let* $f \in \mathcal{A}(k,n)$, $k,n \in \mathbb{N}$, $\left(\frac{I_k(r,\lambda)f(z)}{z^k}\right)^{\alpha} \in \mathcal{H}\,[q(0),1] \cap \mathcal{Q}$ *and* $\lambda \geq 0$.

Let $\left(\frac{I_k(r,\lambda)f(z)}{z^k}\right)^{\alpha} + \alpha(k+\lambda)\frac{I_k(r+1,\lambda)f(z)}{I_k(r,\lambda)f(z)} - \alpha(k+\lambda)$ *is univalent in U. Let be the convex function q in U. Suppose that the function q satisfies the relations* (III.10.5) *and* (III.10.8). *If*

$$q(z) + \frac{zq'(z)}{q(z)} \prec \left(\frac{I_k(r,\lambda)f(z)}{z^k}\right)^{\alpha} + \alpha(k+\lambda)\frac{I_k(r+1,\lambda)f(z)}{I_k(r,\lambda)f(z)} - \alpha(k+\lambda),$$

then:

$$q(z) \prec \left(\frac{I_k(r,\lambda)f(z)}{z^k}\right)^{\alpha}$$

and q is the best subordinant.

Theorem III.10.9 [60]: *Let* $f \in \mathcal{A}(k,n)$, $k,n \in \mathbb{N}$, $\frac{I_k(r+1,\lambda)f(z)}{I_k(r,\lambda)f(z)} \in \mathcal{H}\,[q(0),1] \cap \mathcal{Q}$ *and* $\lambda \geq 0$.

Let $\left(\frac{I_k(r,\lambda)f(z)}{z^k}\right)^{\alpha} + \alpha(k+\lambda)\frac{I_k(r+1,\lambda)f(z)}{I_k(r,\lambda)f(z)} - \alpha(k+\lambda)$ *is univalent in* U. *Let the functions* q_1 *be convex and* q_2 *be univalent in U. Suppose that* q_1 *satisfies the relations* (III.10.5) *and* (III.10.8), *and* q_2 *satisfies the relations* (III.10.4) *and* (III.10.5). *If*

$$q_1(z) + \frac{zq_1'(z)}{q_1(z)} \prec \left(\frac{I_k(r,\lambda)f(z)}{z^k}\right)^{\alpha} + \alpha(k+\lambda)\frac{I_k(r+1,\lambda)f(z)}{I_k(r,\lambda)f(z)} - \alpha(k+\lambda) \prec q_2(z) + \frac{zq_2'(z)}{q_2(z)},$$

then:

$$q_1(z) \prec \left(\frac{I_k(r,\lambda)f(z)}{z^k}\right)^{\alpha} \prec q_2(z)$$

and q_1 *is the best subordinant, iar* q_2 *is the best dominant.*

Theorem III.10.10 [68]: *Let* $f \in \mathcal{A}_n$, $n \in \mathbb{N}$ *and* $\lambda \geq 0$. *Let q is univalent function in U and suppose that satisfies the relations* (III.10.4) *and* (III.10.5). *If*

$$(\lambda+1)\frac{I(r+2,\lambda)f(z)}{I(r+1,\lambda)f(z)} - \lambda\frac{I(r+1,\lambda)f(z)}{I(r,\lambda)f(z)} \prec q(z) + \frac{zq'(z)}{q(z)}$$

then:

$$\frac{I(r+1,\lambda)f(z)}{I(r,\lambda)f(z)} \prec q(z)$$

and q is the best dominant.

Theorem III.10.11 [68]**:** *Let* $f \in \mathcal{A}_n$, $n \in \mathbb{N}$, $\frac{I(r+1,\lambda)f(z)}{I(r,\lambda)f(z)} \in \mathcal{H}[q(0),1] \cap \mathcal{Q}$ *and* $\lambda \geq 0$ *. Let*

$$(\lambda+1)\frac{I(r+2,\lambda)f(z)}{I(r+1,\lambda)f(z)} - \lambda\frac{I(r+1,\lambda)f(z)}{I(r,\lambda)f(z)}$$

be univalent in U *. Let be the convex function q in U. Suppose that the function q satisfies the relations* (III.10.5) *and* (III.10.8). *If:*

$$q(z) + \frac{zq'(z)}{q(z)} \prec (\lambda+1)\frac{I(r+2,\lambda)f(z)}{I(r+1,\lambda)f(z)} - \lambda\frac{I(r+1,\lambda)f(z)}{I(r,\lambda)f(z)}$$

then:

$$q(z) \prec \frac{I(r+1,\lambda)f(z)}{I(r,\lambda)f(z)}$$

and q is the best subordinant.

Theorem III.10.12 [68]**:** *Let* $f \in \mathcal{A}_n$, $n \in \mathbb{N}$, $\frac{I(r+1,\lambda)f(z)}{I(r,\lambda)f(z)} \in \mathcal{H}[q(0),1] \cap \mathcal{Q}$ *and* $\lambda \geq 0$. *Let*

$$(\lambda+1)\frac{I(r+2,\lambda)f(z)}{I(r+1,\lambda)f(z)} - \lambda\frac{I(r+1,\lambda)f(z)}{I(r,\lambda)f(z)}$$

be univalent in U *. Let the functions* q_1 *be convex and* q_2 *is univalent in U. Suppose that* q_1 *satisfies the relations* (III.10.5) *and* (III.10.8), *and* q_2 *satisfies the relations* (III.10.4) *and* (III.10.5). *If*

$$q_1(z) + \frac{zq_1'(z)}{q_1(z)} \prec (\lambda+1)\frac{I(r+2,\lambda)f(z)}{I(r+1,\lambda)f(z)} - \lambda\frac{I(r+1,\lambda)f(z)}{I(r,\lambda)f(z)} \prec q_2(z) + \frac{zq_2'(z)}{q_2(z)},$$

then

$$q_1(z) \prec \frac{I(r+1,\lambda)f(z)}{I(r,\lambda)f(z)} \prec q_2(z)$$

and q_1 *is the best subordinant, iar* q_2 *is the best dominant.*

Theorem III.10.13 [68]**:** *Let* $f \in \mathcal{A}_n$ *,* $n \in \mathbb{N}$ *and* $\lambda \geq 0$. *Let* q *is univalent function in U and suppose that satisfies the relations* (III.10.4) *and* (III.10.5). *If*

$$\left(\frac{I(r,\lambda)f(z)}{z}\right)^{\alpha} + \alpha(\lambda+1)\frac{I(r+1,\lambda)f(z)}{I(r,\lambda)f(z)} - \alpha(\lambda+1) \prec q(z) + \frac{zq'(z)}{q(z)} ,$$

then:

$$\left(\frac{I(r,\lambda)f(z)}{z}\right)^{\alpha} \prec q(z)$$

and q is the best dominant.

Theorem III.10.14 [68]**:** *Let* $f \in \mathcal{A}_n$, $n \in \mathbb{N}$, $\left(\frac{I(r,\lambda)f(z)}{z}\right)^{\alpha} \in \mathcal{H}[q(0),1] \cap \mathcal{Q}$ *and* $\lambda \geq 0$. *Let*

$$\left(\frac{I(r,\lambda)f(z)}{z}\right)^{\alpha}+\alpha(\lambda+1)\frac{I(r+1,\lambda)f(z)}{I(r,\lambda)f(z)}-\alpha(\lambda+1)$$

be univalent in U . *Let be the convex function q in U. Suppose that the function q satisfies the relations* (III.10.5) *and* (III.10.8). *If*

$$q(z)+\frac{zq'(z)}{q(z)}\prec\left(\frac{I(r,\lambda)f(z)}{z}\right)^{\alpha}+\alpha(\lambda+1)\frac{I(r+1,\lambda)f(z)}{I(r,\lambda)f(z)}-\alpha(\lambda+1),$$

then

$$q(z)\prec\left(\frac{I(r,\lambda)f(z)}{z}\right)^{\alpha}$$

and q is the best subordinant.

Theorem III.10.15 [68]**:** *Let* $f\in\mathcal{A}_n$, $n\in\mathbb{N}$, $\left(\frac{I(r,\lambda)f(z)}{z}\right)^{\alpha}\in\mathcal{H}\,[q(0),1]\cap\mathcal{Q}$ *and* $\lambda\geq 0$. *Let*

$$\left(\frac{I(r,\lambda)f(z)}{z}\right)^{\alpha}+\alpha(\lambda+1)\frac{I(r+1,\lambda)f(z)}{I(r,\lambda)f(z)}-\alpha(\lambda+1)$$

be univalent in U . *Let the functions* q_1 *be convex and* q_2 *be univalent in U. Suppose that* q_1 *satisfies the relations* (III.10.5) *and* (III.10.8), *and* q_2 *satisfies the relations* (III.10.4) *and* (III.10.5). *If :*

$$q_1(z)+\frac{zq_1'(z)}{q_1(z)}\prec\left(\frac{I(r,\lambda)f(z)}{z}\right)^{\alpha}+\alpha(\lambda+1)\frac{I(r+1,\lambda)f(z)}{I(r,\lambda)f(z)}-\alpha(\lambda+1)\prec q_2(z)+\frac{zq_2'(z)}{q_2(z)},$$

then:

$$q_1(z)\prec\left(\frac{I(r,\lambda)f(z)}{z}\right)^{\alpha}\prec q_2(z)$$

and q_1 *is the best subordinant, iar* q_2 *is the best dominant.*

IV. SUBCLASSES OF UNIVALENT FUNCTIONS

The chapter is presenting subclasses of univalent functions defined with the convolution, subclasses of normalized starlike and convex functions, subclasses of normalized univalent functions defined with the convolution, subclasses of normalized starlike and convex functions of order α and subclass univalent functions with negative coefficients. The results presented in this chapter are original and are contained in [62], [69], [70], [71], [72].

IV.1. Subclasses of univalent functions defined by convolution

Using the definition of convolution have obtained new subclass of univalent functions. The results of this subparagraph are original and are contained in [62].

Definition IV.1.2: *Let f, g* $\in\mathcal{A}$, *defined as* $f(z)=z+\sum_{j=2}^{\infty}a_j z^j$, $z\in U$ *and* $g(z)=z+\sum_{j=2}^{\infty}b_j z^j$, $z\in U$, *with* $(g*f)(0)=0$ *and we denote*:

$$S_g^{\,*} = \left\{ f \in S: \ \operatorname{Re}\frac{z\left(g*f\right)'(z)}{\left(g*f\right)(z)} > 0, z \in U, \ \ \left(g*f\right)'(0) \neq 0 \right\},$$

$$K_g = \left\{ f \in S: \ \operatorname{Re}\frac{z\left(g*f\right)''(z)}{\left(g*f\right)'(z)} + 1 > 0, z \in U, \ \ \left(g*f\right)'(0) \neq 0 \right\}.$$

Theorem IV.1.1 [62]: *If* $f \in S^*$ *and the function g is convex, then* $f \in S_g^{\,*}$.

Theorem IV.1.2 [62]: *If* $f \in S_g^{\,*}$ *and the function* h *is convex, then* $h * f \in S_g^{\,*}$.

Theorem IV.1.3 [62]: *Let be the convex function h, with* $h(0) = h'(0) - 1 = 0$. *Then* $S_g^{\,*} \subseteq S_{h*g}^{\;\;*}$.

Definition IV.1.3: *Let* $0 \le \alpha < 1$. *We denote*:

$$S_g^{\,*}(\alpha) = \left\{ f \in S: \ \operatorname{Re}\frac{z\left(g*f\right)'(z)}{\left(g*f\right)(z)} > \alpha, z \in U, \ \ \left(g*f\right)'(0) \neq 0 \right\},$$

$$K_g(\alpha) = \left\{ f \in S: \ \operatorname{Re}\frac{z\left(g*f\right)''(z)}{\left(g*f\right)'(z)} + 1 > \alpha, z \in U \right\}.$$

Theorem IV.1.4 [62]: *Let be the convex function q. Then* $K_g \subseteq S_g^{\,*}$

Theorem IV.1.5 [62]: *Let be the convex function q. The function* $f \in K_g$ *if and only if* $h \in S_g^{\,*}$, *where* $h(z) = z f'(z)$, $z \in U$ *or* $f \in K_g \Leftrightarrow z f'(z) \in S_g^{\,*}$.

Definition IV.1.6: *Let* $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, $g(z) = z + \sum_{j=2}^{\infty} b_j z^j$, $z \in U$, *with* $(g*f)(0) = 0$ *and* $\varphi \in S_g^{\,*}$. *We denote:* $C_g = \left\{ f \in S: \ \operatorname{Re}\frac{z\left(g*f\right)'(z)}{\left(g*\varphi\right)(z)} > 0, z \in U \right\}$.

Theorem IV.1.6 [62]: *Let be the convex function q. Then* $S_g^{\,*} \subseteq C_g$

Theorem IV.1.7 [62]: *Let be the convex function q. If* $f \in C_g$ *and the function* h *is convex, then* $h * f \in C_g$.

Theorem IV.1.8 [62]: *Let be the convex function* q. *Let* $f \in C_g$ *and let be the convex function h, with* $h(0) = h'(0) - 1 = 0$. *Then* $C_g \subseteq C_{h*g}$.

Definition IV.1.7: *Let* $0 \le \alpha < 1$. *We denote:* $C_g(\alpha) = \left\{ f \in S: \ \operatorname{Re}\frac{z\left(g*f\right)'(z)}{\left(g*\varphi\right)(z)} > \alpha, z \in U \right\}$.

IV.2. Subclasses of normalized starlike and convex functions

In [42] S. Kanas and F. Ronning introduce classes $S^*(\zeta)$ and $K(\zeta)$ starlike and convex functions using the normalization $f(\zeta) = f'(\zeta) - 1 = 0$, where $\zeta \in U$ is a fixed point. In this section we present relations between these classes of functions and are contained in [69].

Definition IV.2.1: *Let fixed point* $\zeta \in U$. *We denote* $\mathcal{P}(\zeta)$ *the class functions*

$$p(z) = 1 + p_n (z-\zeta)^n + p_{n+1}(z-\zeta)^{n+1} + ...$$

holomorphic in U *and satisfy conditions* $p(\zeta) = 1$ *and* $\operatorname{Re} p(z) > 0$. $\mathcal{P}(\zeta)$ *called* ***class of normalized in*** ζ ***Caratheodory functions***.

Definition IV.2.2: *Let fixed point* $\zeta \in U$, *let* $f(z) = (z-\zeta) + a_{n+1}(z-\zeta)^{n+1} + a_{n+2}(z-\zeta)^{n+2} + ...$ *and we denote:* $\mathcal{A}_n(\zeta) = \{ f \in \mathcal{H}(\mathrm{U}) \; : \; f(\zeta) = f'(\zeta) - 1 = 0 \}$,

and for $n = 1$, $\mathcal{A}_1(\zeta) = \mathcal{A}(\zeta) = \{ f \in \mathcal{H}(\mathrm{U}) \; : \; f(\zeta) = f'(\zeta) - 1 = 0 \}$.

Definition IV.2.3: *We denote:* $S(\zeta) = \{ f \in \mathcal{A}(\zeta) : f$ *is univalent in* $U\}$. $S(\zeta)$ *called* ***class of normalized in*** ζ ***univalent functions***.

Definition IV.2.4: *We denote:* $S^*(\zeta) = \left\{ f \in S(\zeta): \; \operatorname{Re}\dfrac{(z-\zeta) f'(z)}{f(z)} > 0, z \in U, \quad f'(\zeta) \neq 0 \right\}$.

$S^*(\zeta)$ *called* ***class of normalized in*** ζ ***starlike functions***.

Definition IV.2.5: *We denote:* $K(\zeta) = \left\{ f \in S(\zeta): \; \operatorname{Re}\dfrac{(z-\zeta) f''(z)}{f'(z)} + 1 > 0, z \in U, \quad f'(\zeta) \neq 0 \right\}$.

$K(\zeta)$ *called* ***class of normalized in*** ζ ***convex functions***.

Lemma IV.2.1 [42]: *Let fixed point* $\zeta \in U$ *and let be the function* $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$, *which satisfies the condition:* $\operatorname{Re}\psi(\rho i, \sigma, \mu + i\nu; z) \leq 0$, *when* $\rho, \sigma, \mu, \nu \in \mathbb{R}$, $\sigma \leq -\dfrac{n}{2}(1+\rho^2)$, $\sigma + \mu \leq 0$, *where* $z \in U$, $n \geq 1$. *If* $p \in \mathcal{H}[1,n]$ *and*

$$\operatorname{Re}\psi\left(p(z), (z-\zeta) p'(z), (z-\zeta)^2 p''(z); z\right) > 0, \; z \in U, \textit{ then } \operatorname{Re} p(z) > 0, \; z \in U.$$

Theorem IV.2.2 [69]: *Let fixed point* $\zeta \in U$. *The function* $f \in K(\zeta)$ *if and only if* $g \in S^*(\zeta)$, *where* $g(z) = (z-\zeta) f'(z)$, $z \in U$ *or* $f \in K(\zeta) \Leftrightarrow (z-\zeta) f'(z) \in S^*(\zeta)$.

Theorem IV.2.3 [69]: *Let fixed point* $\zeta \in U$. *If* $p \in \mathcal{P}(\zeta)$,

$$p(z) = 1 + p_n (z-\zeta)^n + p_{n+1}(z-\zeta)^{n+1} + ...,$$

then we have: $\operatorname{Re}\left[p(z) + \dfrac{(z-\zeta) p'(z)}{\beta p(z) + \gamma} \right] > 0$, $z \in U \;\; \Rightarrow \;\; \operatorname{Re} p(z) > 0$, $z \in U$.

Definition IV.2.7: *Let fixed point* $\zeta \in U$, *let be the function*

$$f(z) = (z-\zeta) + a_2 (z-\zeta)^2 + a_3 (z-\zeta)^3 + ...$$

We define the integrals $L : \mathcal{H}[0,n] \to \mathcal{H}[0,n]$ *defined by* $L(f) = F$, *where*

$$F(z) = \frac{2}{z-\zeta} \int_{\zeta}^{z} f(t)\, dt. \qquad \text{(IV.2.5)}$$

Theorem IV.2.4 [69]: *If* $L : \mathcal{A}(\zeta) \to \mathcal{A}(\zeta)$ *is integral operator defined by* (IV.2.5), *then:*

a) $L[S^*(\zeta)] \subset S^*(\zeta)$, b) $L[K(\zeta)] \subset K(\zeta)$.

Definition IV.2.8 [69]: *Let be fixed point* $\zeta \in U$, *let be the function*

$$f(z) = (z-\zeta) + a_2 (z-\zeta)^2 + a_3 (z-\zeta)^3 + ...$$

We define the integrals $L_\gamma : \mathcal{H}[0,n] \to \mathcal{H}[0,n]$ *defined by* $L_\gamma(f) = F$, *where*

$$F(z) = \frac{\gamma+1}{(z-\zeta)^\gamma}\int_\zeta^z f(t)(t-\zeta)^{\gamma-1}\,dt, \text{ where } \gamma \in \mathbb{N}^*. \qquad \text{(IV.2.7)}$$

Theorem IV.2.5 [69]: *Let be fixed point* $\zeta \in U$. *If* $L_\gamma :\ \mathcal{A}(\zeta) \to \mathcal{A}(\zeta)$ *is integral operator defined by* (IV.2.7) *and* $\operatorname{Re}\gamma \ge 0$, *then:*

a) $L_\gamma[S^*(\zeta)] \subset S^*(\zeta)$, b) $L_\gamma[K(\zeta)] \subset K(\zeta)$.

Definition IV.2.9: *Let be fixed point* $\zeta \in U$ *and let be the function*

$$f(z) = (z-\zeta) + a_2(z-\zeta)^2 + a_3(z-\zeta)^3 + \ldots$$

We define the differential operator $D_\zeta^n :\ \mathcal{A}(\zeta) \to \mathcal{A}(\zeta)$, $n \in \mathbb{N}$, *by:*

$$D_\zeta^0 f(z) = f(z),$$

$$D_\zeta^1 f(z) = D_\zeta f(z) = (z-\zeta)f'(z),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$D_\zeta^n f(z) = D\left(D_\zeta^{n-1} f(z)\right).$$

Observația IV.2.1: *If function* $f \in \mathcal{A}(\zeta)$, $f(z) = (z-\zeta) + \sum_{j=2}^{\infty} a_j(z-\zeta)^j$, $z \in U$, *then:*

$$D_\zeta^n f(z) = (z-\zeta) + \sum_{j=2}^{\infty} j^n a_j (z-\zeta)^j,\ z \in U.$$

Definition IV.2.10: *Let be fixed point* $\zeta \in U$. *We say that function* $f \in \mathcal{A}(\zeta)$ *is* ***n-starlike normalized in*** ζ, $n \in \mathbb{N}$, *if it's check inequality:* $\operatorname{Re}\dfrac{D_\zeta^{n+1} f(z)}{D_\zeta^n f(z)} > 0$, $z \in U$. *We denote with* $S_n^*(\zeta)$ *class of these functions.*

IV.3. Subclasses of normalized univalent functions defined by convolution

The results of this paragraph are original, were obtained using the normalization $f(\zeta) = f'(\zeta) - 1 = 0$, where $\zeta \in U$ is a fixed point and definition of convolution. These results are contained in [70].

Definition IV.3.1 [70]: *Let be fixed point* $\zeta \in U$ *and let f, g* $\in \mathcal{A}(\zeta)$, *defined as:* $f(z) = (z-\zeta) + \sum_{j=2}^{\infty} a_j(z-\zeta)^j$, $z \in U$ *and* $g(z) = (z-\zeta) + \sum_{j=2}^{\infty} b_j(z-\zeta)^j$, $z \in U$. *We denote with* $f * g$ *convolution or Hadamard product of two functions given by the relation:*

$$(f*g)(z) = (z-\zeta) + \sum_{j=2}^{\infty} a_j b_j (z-\zeta)^j,\ z \in U.$$

Definition IV.3.2 [70]: *Let be fixed point* $\zeta \in U$, *let* $f(z)=(z-\zeta)+\sum_{j=2}^{\infty} a_j (z-\zeta)^j$ *and* $g(z)=(z-\zeta)+\sum_{j=2}^{\infty} b_j (z-\zeta)^j$, $z \in U$, *with* $(g*f)(\zeta)=0$ *and we denote:*

$$S_g^*(\zeta) = \left\{ f \in S(\zeta): \quad \operatorname{Re}\frac{(z-\zeta)(g*f)'(z)}{(g*f)(z)} > 0, z \in U, (g*f)'(\zeta) \neq 0 \right\}.$$

$$K_g(\zeta) = \left\{ f \in S(\zeta): \quad \operatorname{Re}\frac{(z-\zeta)(g*f)''(z)}{(g*f)'(z)} + 1 > 0, z \in U, (g*f)'(\zeta) \neq 0 \right\}.$$

Theorem IV.3.1 [70]: *Let be fixed point* $\zeta \in U$. *Then* $K_g(\zeta) \subseteq S_g^*(\zeta)$.

Theorem IV.3.2 [70]: *Let be fixed point* $\zeta \in U$. *If* $f \in S^*(\zeta)$ *and the function g is convex, then* $f \in S_g^*(\zeta)$.

Theorem IV.3.3 [70]: *Let be fixed point* $\zeta \in U$. *The function* $f \in K_g(\zeta)$ *if and only if* $h \in S_g^*(\zeta)$, *where* $h(z)=(z-\zeta)f'(z)$, $z \in U$ *or* $f \in K_g(\zeta) \Leftrightarrow (z-\zeta)f'(z) \in S_g^*(\zeta)$.

Theorem IV.3.4 [70]: *Let be fixed point* $\zeta \in U$. *If* $f \in S_g^*(\zeta)$ *and the function h is convex, then* $h*f \in S_g^*(\zeta)$.

Theorem IV.3.5 [70]: *Let be fixed point* $\zeta \in U$. *Let* $f \in S_g^*(\zeta)$ *and let be the convex function h, with* $h(\zeta)=h'(\zeta)-1=0$. *Then* $S_g^*(\zeta) \subseteq S_{h*g}^*(\zeta)$.

Theorem IV.3.6 [70]: *If* $f \in K_g(\zeta)$ *and the function h is convex, then* $h*f \in K_g(\zeta)$.

Theorem IV.3.7 [70]: *Let* $f \in K_g(\zeta)$ *and let be the convex function h, with* $h(\zeta)=h'(\zeta)-1=0$. *Then* $K_g(\zeta) \subseteq K_{h*g}(\zeta)$.

Definition IV.3.4 [70]: *We denote:*

$$C_g(\zeta) = \left\{ f \in S(\zeta): \quad \operatorname{Re}\frac{(z-\zeta)(g*f)'(z)}{(g*\varphi)(z)} > 0, z \in U, (g*f)'(\zeta) \neq 0 \right\}.$$

Theorem IV.3.8 [70]: *Let be fixed point* $\zeta \in U$. *Then* $S_g^*(\zeta) \subseteq C_g(\zeta)$.

Theorem IV.3.9 [70]: *Let be fixed point* $\zeta \in U$. *If* $f \in C_g(\zeta)$ *and let be the convex function h, then* $h*f \in C_g(\zeta)$.

Theorem IV.3.10 [70]: *Let fixed point* $\zeta \in U$. *Let* $f \in C_g(\zeta)$ *and let be the convex function h, with* $h(\zeta)=h'(\zeta)=1$. *Then* $C_g(\zeta) \subseteq C_{h*g}(\zeta)$.

Definition IV.3.5 [70]: *We denote:*

$$M_{\alpha,g}(\zeta) = \left\{ f \in S(\zeta): \operatorname{Re}(1-\alpha)\frac{(z-\zeta)(g*f)'(z)}{(g*f)(z)} + \alpha\left(1+\frac{(z-\zeta)(g*f)''(z)}{(g*f)'(z)}\right) > 0, z \in U \right\}$$

Theorem IV.3.11 [70]: *Let be fixed point* $\zeta \in U$. *For any* $\alpha \in \mathbb{R}$ *we have* $M_{\alpha,g}(\zeta) \subseteq S_g^*(\zeta)$.

Theorem IV.3.12 [70]: *Let be fixed point* $\zeta \in U$. *Whatever* $\alpha, \beta \in \mathbb{R}$, *with* $0 \le \dfrac{\beta}{\alpha} < 1$, *we have* $M_{\alpha,g}(\zeta) \subset M_{\beta,g}(\zeta)$.

Definition IV.3.6 [70]: *We denote:*

$$\hat{S}_{\gamma,g}(\zeta) = \left\{ f \in S(\zeta) \;:\; \operatorname{Re}\left[e^{i\gamma} \frac{(z-\zeta)(g*f)'(z)}{(g*f)(z)} \right] > 0, z \in U \right\}, \text{ where } \gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2} \right).$$

Definition IV.3.7 [70]: *We denote* $\widehat{S_g}(\zeta) = \bigcup_{\gamma \in \left(-\frac{\gamma}{2}, \frac{\gamma}{2}\right)} \hat{S}_{\gamma,g}(\zeta)$.

Definition IV.3.8 [70]: *We denote:*

$$S_g^{\,*}(\zeta,\alpha) = \left\{ f \in S(\zeta): \quad \operatorname{Re}\frac{(z-\zeta)\left(g*f\right)'(z)}{\left(g*f\right)(z)} > \alpha, z \in U, \left(g*f\right)'(\zeta) \neq 0 \right\}.$$

$$K_g(\zeta,\alpha) = \left\{ f \in S(\zeta): \quad \operatorname{Re}\frac{(z-\zeta)\left(g*f\right)''(z)}{\left(g*f\right)'(z)} + 1 > \alpha, z \in U, \left(g*f\right)'(\zeta) \neq 0 \right\}.$$

$$C_g(\zeta,\alpha) = \left\{ f \in S(\zeta): \quad \operatorname{Re}\frac{(z-\zeta)\left(g*f\right)'(z)}{\left(g*\varphi\right)(z)} > \alpha, z \in U \right\}.$$

$$M_{\alpha,g}(\zeta,\gamma) = \left\{ f \in S(\zeta) : \operatorname{Re}(1-\alpha)\frac{(z-\zeta)(g*f)'(z)}{(g*f)(z)} + \alpha\left(1 + \frac{(z-\zeta)(g*f)''(z)}{(g*f)'(z)} \right) > \gamma, z \in U \right\}.$$

$$\hat{S}_{\gamma,g}(\zeta,\alpha) = \left\{ f \in S(\zeta) \;:\; \operatorname{Re}\left[e^{i\gamma} \frac{(z-\zeta)(g*f)'(z)}{(g*f)(z)} \right] > \alpha, z \in U \right\}, \text{ where } \gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2} \right).$$

Definition IV.3.13 [70]: *Let be fixed point* $\zeta \in U$. *We define operator* $R_\zeta^n : \mathcal{A}(\zeta) \to \mathcal{A}(\zeta)$, $n \in \mathbb{N}$, *by:*

$$R_\zeta^n f(z) = \frac{(z-\zeta)}{(1-(z-\zeta))^{n+1}} * f(z) = \frac{(z-\zeta)\left((z-\zeta)^{n-1} f(z)\right)^{(n)}}{n!}, \; z \in U.$$

Observația IV.3.5: If function $f \in \mathcal{A}(\zeta)$, $f(z) = (z-\zeta) + \sum_{j=2}^{\infty} a_j (z-\zeta)^j$, $z \in U$, then

$$R_\zeta^n f(z) = (z-\zeta) + \sum_{j=2}^{\infty} C_{n+j-1}^n a_j (z-\zeta)^j, \; z \in U.$$

Definition IV.3.14 [70]: *We say that function* $f \in \mathcal{A}(\zeta)$ *is* ***n-convex***, $n \in \mathbb{N}$, *If it's check inequality:*

$\operatorname{Re}\dfrac{R_\zeta^{n+1} f(z)}{R_\zeta^n f(z)} > \dfrac{1}{2}$**,** $z \in U$**.** *We denote with* $K_n(\zeta)$ *class of these functions.*

IV.4. Subclasses of normalized starlike and convex functions of order α

The results of this paragraph are original, were obtained using the normalization $f(\zeta)=f'(\zeta)-1=0$, where $\zeta \in U$ is a fixed point and are contained in [71].

Definition IV.4.1: *Let be fixed point $\zeta \in U$ and let be the function*

$$f(z)=(z-\zeta)+a_2(z-\zeta)^2+a_3(z-\zeta)^3+...$$

Let $0 \le \alpha < 1$. Then we denote:

$$S^*(\alpha;\zeta)=\left\{ f \in \mathcal{A}_n(\zeta) : \operatorname{Re}\frac{(z-\zeta)f'(z)}{f(z)} > \alpha, z \in U, \left(g * f\right)'(\zeta) \neq 0 \right\},$$

$S^*(\alpha;\zeta)$ *called* ***class starlike functions of order*** α ***normalized in*** ζ.

$$K(\alpha;\zeta)=\left\{ f \in \mathcal{A}_n(\zeta) : \operatorname{Re}\frac{(z-\zeta)f''(z)}{f'(z)}+1 > \alpha, z \in U, \left(g * f\right)'(\zeta) \neq 0 \right\}.$$

$K(\alpha;\zeta)$ *called* ***class convex functions of order*** α ***normalized in*** ζ.

Theorem IV.4.1 [71]: *Let be fixed point $\zeta \in U$ and let $0 \le \alpha < 1$. Then $S^*(\alpha,\zeta) \subset S^*(\zeta)$ and $K(\alpha,\zeta) \subset K(\zeta)$.*

Theorem IV.4.2 [71]: *Let be fixed point $\zeta \in U$ and let $0 \le \alpha < 1$. The function $f \in S^*(\alpha,\zeta)$ if and only if $g \in S^*(\zeta)$, where:*

$$g(z)=(z-\zeta)\left[\frac{f(z)}{z-\zeta}\right]^{\frac{1}{1-\alpha}},$$

where $\left[\frac{f(z)}{z-\zeta}\right]^{\frac{1}{1-\alpha}}$ *is holomorphic determination for which* $\left.\left[\frac{f(z)}{z-\zeta}\right]^{\frac{1}{1-\alpha}}\right|_{z=\zeta}=1$.

Corolarul IV.4.1 [71]: *Let be fixed point $\zeta \in U$. If $0 \le \alpha < 1$, then $f \in K(\alpha;\zeta)$ if and only if the function $g \in S^*(\alpha;\zeta)$, where $g(z)=(z-\zeta)\left[f'(z)\right]^{\frac{1}{1-\alpha}}$.*

IV.5. Subclasses of normalized alpha-convex functions

The results of this paragraph are original, were obtained using the normalization $f(\zeta)=f'(\zeta)-1=0$, where $\zeta \in U$ is a fixed point and are contained in [72].

Definition IV.5.1 [72]: *Let be fixed point $\zeta \in U$ and the be function*

$$J(\alpha,f;z,\zeta)=(1-\alpha)\frac{(z-\zeta)f'(z)}{f(z)}+\alpha\left(1+\frac{(z-\zeta)f''(z)}{f'(z)}\right),\ z \in U.$$

We denote $M_\alpha(\zeta)=\{ f \in \mathcal{H}(U) : f(\zeta)=f'(\zeta)-1=0,\ \frac{f(z)f'(z)}{z} \neq 0,\ \operatorname{Re}J(\alpha,f;z,\zeta)>0,\ z \in U \}$. $M_\alpha(\zeta)$ *called* ***class alpha-convex functions of normalized in*** ζ.

Theorem IV.5.1 [72]: *Let be fixed point $\zeta \in U$.*

a) *For any* $\alpha \in \mathbb{R}$ *we have* $M_\alpha(\zeta) \subseteq S^*(\zeta)$,

b) *Whatever* $\alpha, \beta \in \mathbb{R}$ *with* $0 \le \dfrac{\beta}{\alpha} < 1$ *we have* $M_\alpha(\zeta) \subset M_\beta(\zeta)$.

Theorem IV.5.2 [72]: *Let be fixed point* $\zeta \in U$ *and let* $\alpha \ge 0$. *Then* $f \in M_\alpha(\zeta)$ *if and only if* $F \in S^*(\zeta)$, *where* $F(z) = f(z)\left[\dfrac{(z-\zeta)f'(z)}{f(z)}\right]^\alpha$, *with* $\left[\dfrac{(z-\zeta)f'(z)}{f(z)}\right]^\alpha$ *the point* $z = \zeta$ *be* 1.

Definition IV.5.2 [72]: *Let be fixed point* $\zeta \in U$, $\alpha, \beta \in \mathbb{R}$ *and let* $f \in \mathcal{A}_n(\zeta)$ *with*

$$\frac{f(z)f'(z)}{z-\zeta} \neq 0,\ \alpha\frac{(z-\zeta)f'(z)}{f(z)} \neq 0,\ z \in U.$$

We say that function $f \in M_\alpha^n(\zeta)$ *if the function* $F: U \to \mathbb{C}$, *defined by*

$$F(z) = f(z)\left[1 + \alpha\frac{(z-\zeta)f'(z)}{f(z)}\right]^\alpha$$

is a starlike function in U.

Theorem IV.5.3 [72]: *Let fixed point* $\zeta \in U$. *For any* $\alpha < 0$, *we have* $M_\alpha^n(\zeta) \subset S^*(\zeta)$.

Definition IV.5.3 [72]: *Let fixed point* $\zeta \in U$, $\alpha, \beta \in \mathbb{R}$ *and let* $f \in \mathcal{A}_n(\zeta)$ *with*

$$\frac{f(z)f'(z)}{z-\zeta} \neq 0,\ 1 - \alpha + \alpha\frac{(z-\zeta)f'(z)}{f(z)} \neq 0,\ z \in U.$$

We say that function $f \in M_{\alpha,\beta}^n(\zeta)$ *if the function* $F: U \to \mathbb{C}$, *defined by:*

$$F(z) = f(z)\left[\frac{(z-\zeta)f'(z)}{f(z)}\right]^{\alpha(1-\beta)}\left[1 - \alpha + \alpha\frac{(z-\zeta)f'(z)}{f(z)}\right]^\beta$$

is a starlike function in U.

Theorem IV.5.4 [72]: *Let be fixed point* $\zeta \in U$. *For any* $\alpha, \beta \in \mathbb{R}$, $\alpha\beta(1-\alpha) \ge 0$, *we have* $M_{\alpha,\beta}^n(\zeta) \subset S^*(\zeta)$.

Definition IV.5.4 [72]: *Let be fixed point* $\zeta \in U$, $\alpha, \beta \in \mathbb{R}$ *and let* $f \in \mathcal{A}_n(\zeta)$ *with:*

$$\frac{f(z)f'(z)}{z} \neq 0,\ 1 - \alpha + \alpha\frac{(z-\zeta)f'(z)}{f(z)} \neq 0,\ z \in U.$$

We say that function $f \in M_{\alpha,\beta}^n(\zeta)$ *if the function* $F: U \to \mathbb{C}$, *defined by:*

$$F(z) = f(z)\left[\frac{(z-\zeta)f'(z)}{f(z)}\right]^{\alpha(1-\beta)}\left[\alpha\frac{(z-\zeta)f'(z)}{f(z)}\right]^\beta$$

is a starlike function in U.

Theorem IV.5.5 [72]: *For any* $\alpha, \beta \in \mathbb{R}$, *we have* $M_{\alpha,\beta}^n(\zeta) \subset S^*$.

IV. 6. Subclasses of univalent functions with negative coefficients

The results of this paragraph are original and are contained in [73].

H. M. Srivastava and A. A. Attiya in his [137], introduce operator $J_{n,\lambda}: \mathcal{A} \to \mathcal{A}$ defined as:

$$J_{n,\lambda}f(z)=z+\sum_{k=2}^{\infty}\left(\frac{\lambda+1}{\lambda+k}\right)^{n}a_k\,z^k\ ,\ z\in U\ . \qquad \text{(IV.6.3)}$$

$$(1+\lambda)J_{n+1,\lambda}f(z)=z\left(J_{n,\lambda}f(z)\right)'+\lambda J_{n,\lambda}f(z)\,. \qquad \text{(IV.6.4)}$$

We denote $S^{*}_{n,\lambda}(\alpha)$ class of functions $f\in\mathcal{S}$ satisfying the condition:

$$\operatorname{Re}\frac{z\left(J_{n,\lambda}f(z)\right)'}{J_{n,\lambda}f(z)}>\alpha\ ,\ 0\le\alpha<1\ ,\ z\in U\ . \qquad \text{(IV.6.5)}$$

From relations (IV.6.4) and (IV.6.5) that the function $f\in\mathcal{S}$ belongs to class $S^{*}_{n,\lambda}(\alpha)$ if and only if

$$\operatorname{Re}\frac{J_{n+1,\lambda}f(z)}{J_{n,\lambda}f(z)}>\frac{\lambda+\alpha}{\lambda+1}\ ,\ 0\le\alpha<1\ ,\ z\in U\ . \qquad \text{(IV.6.6)}$$

Definition IV.6.1 [73]: *We say that function $f\in\mathcal{T}$ is in the class $TS^{*}_{n,\lambda}(\alpha)$, $n\in\mathbb{C}$, $\lambda\in\mathbb{C}/\mathbb{Z}_0^-$ if the function f satisfy the condition* (IV.6.6).

Theorem IV.6.1 [73]: *Let be $n\in\mathbb{N}^*$, $\lambda\in\mathbb{C}/\mathbb{Z}_0^-$, $0\le\alpha<1$ and let be $f\in\mathcal{T}$. Then function $f(z)$ is in the class $TS^{*}_{n,\lambda}(\alpha)$ if and only if:*

$$\sum_{k=2}^{\infty}\frac{(\lambda+1)^2-(\lambda+k)(\lambda+\alpha)}{\lambda+k}\left(\frac{\lambda+1}{\lambda+k}\right)^{n}a_k<1-\alpha\ . \qquad \text{(IV.6.7)}$$

The result is sharp.

The result (IV.6.7) is sharp for the function:

$$f(z)=z-\frac{(1-\alpha)(\lambda+k)}{(\lambda+1)^2-(\lambda+k)(\lambda+\alpha)}\left(\frac{\lambda+k}{\lambda+1}\right)^{n}z^k\,. \qquad \text{(IV.6.9)}$$

Corollary IV.6.1 [73]: *Let $n\in\mathbb{N}^*$, $\lambda\in\mathbb{C}/\mathbb{Z}_0^-$, $0\le\alpha<1$ and let $f\in\mathcal{T}$. Then function $f(z)$ is in the class $TS^{*}_{n,\lambda}(\alpha)$ If:*

$$a_k\le\frac{(1-\alpha)(\lambda+k)}{(\lambda+1)^2-(\lambda+k)(\lambda+\alpha)}\left(\frac{\lambda+k}{\lambda+1}\right)^{n}. \qquad \text{(IV.6.10)}$$

We have equality in relation (IV.6.10) *if the function $f(z)$ is given by* (IV.6.9).

Theorem IV.6.2 [73]: *Let be $n\in\mathbb{N}^*$, $\lambda\in\mathbb{C}/\mathbb{Z}_0^-$, $0\le\alpha<1$ and let $f\in\mathcal{T}$. Then*

$$TS^{*}_{n,\lambda}(\alpha)\subseteq TS^{*}_{n+1,\lambda}(\alpha)\,. \qquad \text{(IV.6.11)}$$

Theorem IV.6.3 [73]: *Let be $n\in\mathbb{N}^*$, $\lambda\in\mathbb{C}/\mathbb{Z}_0^-$, $0\le\alpha_1\le\alpha_2<1$ and let $f\in\mathcal{T}$ Then*

$$TS^{*}_{n,\lambda}(\alpha_2)\subseteq TS^{*}_{n,\lambda}(\alpha_1)\,. \qquad \text{(IV.6.12)}$$

Theorem IV.6.4 [73]: *Let be $n\in\mathbb{N}^*$, $\lambda\in\mathbb{C}/\mathbb{Z}_0^-$, $0\le\alpha<1$ and let $f\in\mathcal{T}$ If $f\in TS^{*}_{n,\lambda}(\alpha)$ and $|z|=r<1$, then:*

$$r-\frac{(1-\alpha)(\lambda+2)}{(\lambda+1)^2-(\lambda+2)(\lambda+\alpha)}\left(\frac{\lambda+2}{\lambda+1}\right)^{n}r^2\le|f(z)|\le r+\frac{(1-\alpha)(\lambda+2)}{(\lambda+1)^2-(\lambda+2)(\lambda+\alpha)}\left(\frac{\lambda+2}{\lambda+1}\right)^{n}r^2 \quad \text{(IV.6.16)}$$

and

$$1-\left[\frac{2(\lambda+\alpha)}{(\lambda+1)^2-(\lambda+2)(\lambda+\alpha)}-\alpha\right]\frac{\lambda+2}{\lambda+\alpha}\left(\frac{\lambda+2}{\lambda+1}\right)^n r\le|f'(z)|\le 1+\left[\frac{2(\lambda+\alpha)}{(\lambda+1)^2-(\lambda+2)(\lambda+\alpha)}-\alpha\right]\frac{\lambda+2}{\lambda+\alpha}\left(\frac{\lambda+2}{\lambda+1}\right)^n r$$

(IV.6.17)

The bounds (IV.6.16) and (IV.6.17) are attained for the function $f(z)$ given by:

$$f(z)=z-\frac{(1-\alpha)(\lambda+2)}{(\lambda+1)^2-(\lambda+2)(\lambda+\alpha)}\left(\frac{\lambda+2}{\lambda+1}\right)^n z^2,\ (z=\pm r). \tag{IV.6.20}$$

Corollary IV.6.2 [73]: *Let be* $n\in\mathbb{N}^*$, $\lambda\in\mathbb{C}/\mathbb{Z}_0^-$, $0\le\alpha<1$ *and let* $f\in\mathcal{T}$. *If* $f\in TS_{n,\lambda}^*(\alpha)$, *then the unit disc* U *is mapped onto a domain that contains the disc*

$$|w|<1-\frac{(1-\alpha)(\lambda+2)}{(\lambda+1)^2-(\lambda+2)(\lambda+\alpha)}\left(\frac{\lambda+2}{\lambda+1}\right)^n. \tag{IV.6.21}$$

The result is sharp with extremal function $f(z)$ *given by* (IV.6.20).

Let be the functions $f_i(z)$, $i=\overline{1,m}$, defined as:

$$f_i(z)=z-\sum_{k=2}^{\infty}a_{k,i}\,z^k\ ,\ a_{k,i}\ge 0,\ z\in U. \tag{IV.6.22}$$

Theorem IV.6.5 [73]: *Let be* $n\in\mathbb{N}^*$, $\lambda\in\mathbb{C}/\mathbb{Z}_0^-$, $0\le\alpha<1$ *and let be* $f_i(z)\in TS_{n,\lambda}^*(\alpha)$, *where* $i=\overline{1,m}$. *Then function* $h(z)$ *defined by the relationship:*

$$h(z)=\sum_{i=1}^{\infty}c_i\,f_i(z),\ \text{where}\ c_i\ge 0,\sum_{i=1}^{\infty}c_i=1, \tag{IV.6.23}$$

is in the class $TS_{n,\lambda}^*(\alpha)$.

Theorem IV.6.6 [73]: *Let be* $n\in\mathbb{N}^*$, $\lambda\in\mathbb{C}/\mathbb{Z}_0^-$, $0\le\alpha<1$ *and let be* $f_i(z)\in TS_{n,\lambda}^*(\alpha)$, *where* $i=\overline{1,m}$. *Then function* $h(z)$ *defined by the relationship:*

$$h(z)=\gamma f_1(z)+(1-\gamma)f_2(z),\ \text{where}\ 0\le\gamma<1. \tag{IV.6.26}$$

is in the class $TS_{n,\lambda}^*(\alpha)$.

Theorem IV.6.7 [73]: *Let be functions*

$$f_1(z)=z\ \text{and}\ f_k(z)=z-\frac{(1-\alpha)(\lambda+k)}{(\lambda+1)^2-(\lambda+k)(\lambda+\alpha)}\left(\frac{\lambda+k}{\lambda+1}\right)^n z^k,$$

where $k\ge 2$, $n\in\mathbb{N}^*$, $\lambda\in\mathbb{C}/\mathbb{Z}_0^-$, $0\le\alpha<1$. *Then function* $f(z)$ *is in the class* $TS_{n,\lambda}^*(\alpha)$ *if and only if can be expressed in the form:*

$$f(z)=\sum_{k=1}^{\infty}\upsilon_k\,f_k(z),\ \text{where}\ \upsilon_k\ge 0,\sum_{k=1}^{\infty}\upsilon_k=1. \tag{IV.6.29}$$

Definition IV.6.1 [73]: *Let functions* $f_i(z)$, $i=\overline{1,2}$, *be defined* (IV.6.22). *The modified convolution or the modified Hadamard product of the functions* $f_1(z)$ *and* $f_2(z)$ *is defined by:*

$$\left(f_1\otimes f_2\right)(z)=z-\sum_{k=2}^{\infty}a_{k,1}a_{k,2}\,z^k. \tag{IV.6.30}$$

Theorem IV.6.8 [73]: *Let be* $n\in\mathbb{N}^*$, $\lambda\in\mathbb{C}/\mathbb{Z}_0^-$, $0\le\alpha<1$. *Let be* $f_i(z)\in TS_{n,\lambda}^*(\alpha)$, $i=\overline{1,2}$. *Then* $\left(f_1\otimes f_2\right)(z)\in TS_{n,\lambda}^*(\beta)$, *where*

$$\beta = \frac{1-\left(\dfrac{\lambda+2}{\lambda+1}\right)^n \dfrac{(\lambda+2)(\lambda+1)^2(1-\alpha)^2-\lambda(\lambda+2)^2(1-\alpha)^2}{\left[(\lambda+1)^2-(\lambda+2)(\lambda+\alpha)\right]^2}}{1-\left(\dfrac{\lambda+2}{\lambda+1}\right)^n \dfrac{(\lambda+2)^2(1-\alpha)^2}{\left[(\lambda+1)^2-(\lambda+2)(\lambda+\alpha)\right]^2}}. \tag{IV.6.31}$$

The result is sharp for the functions:

$$f_i(z) = z - \frac{(1-\alpha)(\lambda+2)}{(\lambda+1)^2-(\lambda+2)(\lambda+\alpha)}\left(\frac{\lambda+2}{\lambda+1}\right)^n z^2,\ i=\overline{1,2}. \tag{IV.6.32}$$

Corollary IV.6.4 [73]: *Let be* $n\in\mathbb{N}^*$, $\lambda\in\mathbb{C}/\mathbb{Z}_0^-$, $0\le\alpha<1$. *Let* $f_i(z)\in TS^*_{n,\lambda}(\alpha)$, *where* $i=\overline{1,2}$. *Then the function* $h(z)=z-\sum_{k=2}^{\infty}\sqrt{a_{k,1}\,a_{k,2}}\,z^k$ *is in the class* $TS^*_{n,\lambda}(\alpha)$.

Theorem IV.6.9 [73]: *Let be* $n\in\mathbb{N}^*$, $\lambda\in\mathbb{C}/\mathbb{Z}_0^-$, $0\le\alpha<1$, $0\le\beta<1$, $0\le\gamma<1$ *and let* $f_1(z)\in TS^*_{n,\lambda}(\alpha)$ *and* $f_2(z)\in TS^*_{n,\lambda}(\beta)$. *Then* $\left(f_1\otimes f_2\right)(z)\in TS^*_{n,\lambda}(\gamma)$, *where*

$$\gamma = \frac{1-\dfrac{(\lambda+2)(1-\alpha)(1-\beta)\left[(\lambda+1)^2-\lambda(\lambda+2)\right]}{\left[(\lambda+1)^2-(\lambda+2)(\lambda+\alpha)\right]\left[(\lambda+1)^2-(\lambda+2)(\lambda+\beta)\right]}\left(\dfrac{\lambda+2}{\lambda+1}\right)^n}{1-\dfrac{(\lambda+2)^2(1-\alpha)(1-\beta)}{\left[(\lambda+1)^2-(\lambda+2)(\lambda+\alpha)\right]\left[(\lambda+1)^2-(\lambda+2)(\lambda+\beta)\right]}\left(\dfrac{\lambda+2}{\lambda+1}\right)^n}. \tag{IV.6.37}$$

The result is sharp for the functions

$$f_1(z) = z - \frac{(1-\alpha)(\lambda+2)}{(\lambda+1)^2-(\lambda+2)(\lambda+\alpha)}\left(\frac{\lambda+2}{\lambda+1}\right)^n z^2,$$

and:

$$f_2(z) = z - \frac{(1-\beta)(\lambda+2)}{(\lambda+1)^2-(\lambda+2)(\lambda+\beta)}\left(\frac{\lambda+2}{\lambda+1}\right)^n z^2.$$

Theorem IV.6.10 [73]**:** *Let be* $n\in\mathbb{N}^*$, $\lambda\in\mathbb{C}/\mathbb{Z}_0^-$, $0\le\alpha<1$ *and let* $f_i(z)\in TS^*_{n,\lambda}(\alpha)$, $i=\overline{1,2}$. *Then the function* $h(z)$ *defined as:*

$$h(z) = z-\sum_{k=2}^{\infty}\left(a_{k,1}^2+a_{k,2}^2\right)z^k, \tag{IV.6.49}$$

is in the class $TS^*_{n,\lambda}(\upsilon)$, *where:*

$$\upsilon = \frac{1-2\dfrac{(\lambda+2)(1-\alpha)^2\left[(\lambda+1)^2-\lambda(\lambda+2)\right]}{\left[(\lambda+1)^2-(\lambda+2)(\lambda+\alpha)\right]^2}\left(\dfrac{\lambda+2}{\lambda+1}\right)^n}{1-\dfrac{(\lambda+2)^2(1-\alpha)^2}{\left[(\lambda+1)^2-(\lambda+2)(\lambda+\alpha)\right]^2}\left(\dfrac{\lambda+2}{\lambda+1}\right)^n}. \tag{IV.6.50}$$

The result is sharp for the functions $f_i(z)\in TS^*_{n,\lambda}(\alpha)$, $i=\overline{1,2}$ *defined by* (IV.6.32) of *theorem* IV.6.8.

Theorem IV.6.12 [73]: *Let be* $n_1, n_2 \in \mathbb{N}^*$, $\lambda \in \mathbb{C}/\mathbb{Z}_0^-$, $0 \le \alpha < 1$ *and let be* $f_i(z) \in TS^*_{n_i,\lambda}(\alpha)$, $i = \overline{1,2}$. *Then* $\left(f_1 \otimes f_2\right)(z) \in TS^*_{n_1,\lambda}(\alpha) \cap TS^*_{n_2,\lambda}(\alpha)$.

Theorem IV.6.12 [73]: *Let be* $n \in \mathbb{N}^*$, $\lambda \in \mathbb{C}/\mathbb{Z}_0^-$, $0 \le \alpha < 1$ *and let be* $f \in \mathcal{T}$. *If* $f \in TS^*_{n,\lambda}(\alpha)$, *then* $f(z)$ *is close-to-convex of order* γ, *where* $0 \le \gamma < 1$, *in* $|z| < r_1(n,\lambda,\alpha,\gamma)$, *where:*

$$r_1(n,\lambda,\alpha,\gamma) = \inf_k \left[\frac{1-\gamma}{k} \frac{(\lambda+1)^2 - (\lambda+k)(\lambda+\alpha)}{(1-\alpha)(\lambda+k)} \left(\frac{\lambda+1}{\lambda+k}\right)^n \right]^{\frac{1}{k-1}}, \text{ with } k \ge 2. \quad \text{(IV.6.60)}$$

The result is sharp for the extremal function $f(z)$ *defined by* (IV.6.9).

Theorem IV.6.13 [73]: *Let be* $n \in \mathbb{N}^*$, $\lambda \in \mathbb{C}/\mathbb{Z}_0^-$, $0 \le \alpha < 1$ *and let be* $f \in \mathcal{T}$. *If* $f \in TS^*_{n,\lambda}(\alpha)$, *then* $f(z)$ *is starlike of order* γ, *where* $0 \le \gamma < 1$, *in* $|z| < r_2(n,\lambda,\alpha,\gamma)$, *where:*

$$r_2(n,\lambda,\alpha,\gamma) = \inf_k \left[\frac{1-\gamma}{k-\gamma} \frac{(\lambda+1)^2 - (\lambda+k)(\lambda+\alpha)}{(1-\alpha)(\lambda+k)} \left(\frac{\lambda+1}{\lambda+k}\right)^n \right]^{\frac{1}{k-1}}, \text{ with } k \ge 2. \quad \text{(IV.6.63)}$$

The result is sharp for the extremal function $f(z)$ *defined by* (IV.6.9).

Theorem IV.6.14 [73]: *Let be* $n \in \mathbb{N}^*$, $\lambda \in \mathbb{C}/\mathbb{Z}_0^-$, $0 \le \alpha < 1$ *and let* $f \in \mathcal{T}$. *If* $f \in TS^*_{n,\lambda}(\alpha)$, *then* $f(z)$ *is convex of order* γ, *where* $0 \le \gamma < 1$, *in* $|z| < r_3(n,\lambda,\alpha,\gamma)$, *where*

$$r_3(n,\lambda,\alpha,\gamma) = \inf_k \left[\frac{1-\gamma}{k(k-\gamma)} \frac{(\lambda+1)^2 - (\lambda+k)(\lambda+\alpha)}{(1-\alpha)(\lambda+k)} \left(\frac{\lambda+1}{\lambda+k}\right)^n \right]^{\frac{1}{k-1}}, \text{ with } k \ge 2. \quad \text{(IV.6.66)}$$

The result is sharp for the extremal function $f(z)$ *defined by* (IV.6.9).

Theorem IV.6.15 [73]: *Let be* $n \in \mathbb{N}^*$, $\lambda \in \mathbb{C}/\mathbb{Z}_0^-$, $0 \le \alpha < 1$, *let bet* $f \in \mathcal{T}$ *and let the function* $F(z)$ *be defined by:*

$$F(z) = \frac{\gamma+1}{z^\gamma} \int_0^z t^{\gamma-1} f(t)\, dt. \quad \text{(IV.6.69)}$$

If $f \in TS^*_{n,\lambda}(\alpha)$, *then* $F \in TS^*_{n,\lambda}(\alpha)$, *for any* γ, $-1 < \gamma$.

Corollary IV.6.5 [73]: *Let be* $n \in \mathbb{N}^*$, $\lambda \in \mathbb{C}/\mathbb{Z}_0^-$, $0 \le \alpha < 1$, *let be* $f \in \mathcal{T}$ *and let the function* $F(z)$ *be defined by:*

$$F(z) = \int_0^z \frac{f(t)}{t} dt. \quad \text{(IV.6.70)}$$

If $f \in TS^*_{n,\lambda}(\alpha)$, *then* $F \in TS^*_{n,\lambda}(\alpha)$.

Theorem IV.6.16 [73]: *Le bet* $n \in \mathbb{N}^*$, $\lambda \in \mathbb{C}/\mathbb{Z}_0^-$, $0 \le \alpha < 1$, *let* $f \in \mathcal{T}$, *let be* γ *a real number, such that* $-1 < \gamma$ *and let the function* $F(z)$ *defined by* (IV.6.69). *If* $f \in TS^*_{n,\lambda}(\alpha)$, *then* $F(z)$ *is univalent in* $|z| < r$, *where:*

$$r = \inf_k \left[\frac{\gamma+k}{k(\gamma+1)} \frac{(\lambda+1)^2 - (\lambda+k)(\lambda+\alpha)}{(1-\alpha)(\lambda+k)} \left(\frac{\lambda+1}{\lambda+k}\right)^n \right]^{\frac{1}{k-1}}. \quad \text{(IV.6.71)}$$

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