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Fixed Point Theory in Reflexive Metric Spaces

Ph.D. Thesis Summary

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Introduction

Fixed point theory is a mathematical domain that has enjoyed a prosperous development in the last fifty years. This theory was extended in various directions, classical instruments were generalized, new notions and results have been given and constantly improved. Fixed point theorems are used in many branches of mathematics such as analysis, geometry or dynamical systems. In addition, they are important tools applied in other domains with no immediate connection with mathematics at first glance. Many stability and equilibrium problems can be modeled using fixed points. Such examples can be found in economics, game theory, compiler theory and many others.

Metric fixed point theory was born with the well-known Banach-Caccioppoli Contraction Principle that was initially published in 1922. This result states that every self-contraction defined on a complete metric space has a unique fixed point and any sequence of Picard iterates converges to the unique fixed point. A speed of convergence of the Picard iterates to the unique fixed point is also established. Since then, this principle was constantly improved and extended in many directions.

A new milestone in the development of metric fixed point theory was achieved by the publication of a fixed point result for nonexpansive mappings. This result is due to Kirk [42] and states that, in a Banach space, every nonexpansive self-mapping defined on a nonempty, weakly compact and convex set with normal structure has a fixed point. This theorem is usually known in a more particular form as the Browder-Göhde-Kirk Theorem since Browder [6, 7] and Göhde [30] also independently proved the result in Hilbert and uniformly convex Banach spaces. These results called the attention of a large number of researchers who began to study in more detail the interplay between the geometrical properties of the working space and the existence of fixed points, as well as the convergence of iterates for certain iterative schemes or contractive conditions.

Not surprisingly, in the general context of metric spaces, nonexpansive mappings need not possess fixed points. A fixed point theorem for nonexpansive mappings was proved by Kirk [43] in the framework of bounded metric spaces for which there exists a convexity structure that is countably compact and normal.

Metric spaces are important tools used in the modeling of day-to-day life problems. Of course, the structure of a metric space is sometimes far too general in order to apply existing theories used in the study of such processes. In order to assure a certain regularity, specific restrictions were considered on the metric space. Some of these properties provide sufficient information which allows the development and extension of mathematical theories that play an essential role in solving such problems. The existence of distance-preserving curves between any two points of the space is one of the most important properties that can be imposed in a metric space since it endows the space with a structure that resembles in some way the linear structure of a normed space. Such spaces are called geodesic metric spaces. More precisely, having a metric space (X, d), a geodesic path from $x \in X$ to $y \in X$ is a distance-preserving mapping $c : [0, l] \subseteq \mathbb{R} \to X$ such that $c(0) = x$ and $c(l) = y$. The image $c\left([0, l]\right)$ of c forms a geodesic segment which joins x and y and is not necessarily unique. A metric space is geodesic if every two points of it can be joined by a geodesic path. A comprehensive treatment of geodesic metric spaces can be found, for instance, in [5, 8, 64].

In this work we focus on geodesic spaces with additional properties that allow us to study fixed point problems, as well as minimization and maximization problems between two sets. One important property that has a significant impact on the study of such problems is the reflexivity of a metric space. Having a metric space for which a certain type of convexity is defined, we say that the space is reflexive if the intersection of every descending sequence of nonempty, bounded, closed and convex subsets is nonempty. A simple example of a reflexive metric space is a reflexive Banach space with the usual convexity.

In the case of geodesic metric spaces, a set is convex if it contains every geodesic segment that joins any two points of the set. An important class of geodesic metric spaces are those of bounded curvature introduced by Alexandrov [2]. Later, Gromov [33] contributed to a better understanding of such spaces and named them CAT(κ) spaces after Cartan, Alexandrov and Topogonov, each of whom considered similar conditions for spaces. Gromov's work led to a significant development in theoretical physics and attracted a high amount of interest from researchers. The basic idea behind the concept of CAT(κ) spaces is that geodesic triangles are in some way "thin". Complete CAT(0) spaces are reflexive and, for $\kappa > 0$, complete CAT(κ) spaces are somehow reflexive. Other examples of reflexive spaces include hyperconvex spaces and complete uniformly convex metric spaces with a monotone or lower semi-continuous from the right modulus of uniform convexity. We introduce here another class of reflexive metric spaces, namely complete geodesic Ptolemy spaces with a uniformly continuous midpoint map. More details about all these spaces can be found in the ensuing chapters.

More recently, metric fixed point theory for nonexpansive mappings started to be investigated in reflexive metric spaces. Baillon showed in [4] that every nonexpansive mapping defined on a bounded hyperconvex space with values into itself has fixed points. Kirk proved in [45, 46] the counterpart of the Browder-Göhde-Kirk Theorem for spaces of curvature bounded above. Namely, it was proved that every nonexpansive self-mapping has a fixed point when defined on a subset of a complete CAT(κ) space which is nonempty, bounded, closed, convex and of diameter bounded above by $\pi/(2\sqrt{\kappa})$ for $\kappa > 0$. An analogue of the Browder-Göhde-Kirk Theorem in the setting of uniformly convex metric spaces was given in [15]. More precisely, it was shown that any nonexpansive mapping defined on a nonempty, bounded, closed and convex subset of a complete uniformly convex geodesic metric space with a monotone or lower semi-continuous from the right modulus of uniform convexity has a fixed point. The same result is proved here to hold in the framework of complete geodesic Ptolemy spaces with a uniformly continuous midpoint map.

The purpose of this thesis is to prove in the setting of reflexive metric spaces fixed point and convergence results for classes of single and multivalued mappings that satisfy certain contraction and nonexpansive-like conditions. We study the regularity of geodesic Ptolemy spaces and describe some of their geometric properties which are key tools for proving fixed point results in such settings. We also give generic results in different geodesic spaces on the well-posedness of minimization and maximization problems between two sets. This work is divided into five chapters which are organized as follows. More details about the results we prove here can be found at the beginning of each chapter.

Chapter 1 contains some preliminary notions and results that are used in the sequel.

In Chapter 2 we use condition (C) introduced in [74] which is a generalized nonexpansivity condition in order to give fixed point and convergence results in the setting of different reflexive metric spaces. For the multivalued case we assume condition (C) as in [66]. We also include extensions of condition (C) and apply our findings to obtain results on the existence of common fixed points for commuting mappings.

In Chapter 3 we give fixed point and well-posedness results, as well as demi-closed principles for classes of singlevalued mappings that satisfy assumptions milder than contraction or nonexpansive conditions using the concept of orbits. We focus on generalized contractions in the sense of Walter [77] and on different types of mappings that extend pointwise contractions, asymptotic contractions, asymptotic pointwise contractions, and nonexpansive and asymptotic pointwise nonexpansive mappings. In the multivalued case we extend uniformly Lipschitz and asymptotically nonexpansive mappings.

Chapter 4 recalls the Ptolemy inequality and studies the regularity of geodesic Ptolemy metric spaces and their relation to CAT(0) spaces. We prove that if these spaces are complete and satisfy a certain convexity-like condition, then they are reflexive. We also prove other important properties that allow us to give a series of fixed point results. Other versions of the Ptolemy inequality which hold in CAT(κ) spaces are also studied here.

Chapter 5 deals with the well-posedness of the minimization and maximization problem between two sets in geodesic spaces under different conditions for the sets. A variant of the Drop Theorem in Busemann convex geodesic spaces is given and applied to obtain an optimization result for convex functions.

Most of the original results proved in this work are part of the following publications:

- R. Espínola, P. Lorenzo, A. Nicolae, Fixed points, selections and common fixed points for nonexpansive-type mappings, J. Math. Anal. Appl., 382 (2011), 503-515.

- R. Espínola, A. Nicolae, Geodesic Ptolemy spaces and fixed points, Nonlinear Anal., 74 (2011), 27-34.

- R. Espínola, A. Nicolae, Mutually nearest and farthest points of sets and the Drop Theorem in geodesic spaces, Monatsh. Math., doi 10.1007/s00605-010-0266-0 (in press).

- A. Nicolae, On some generalized contraction type mappings, Appl. Math. Lett., 23 (2010), 133-136.

- A. Nicolae, Generalized asymptotic pointwise contractions and nonexpansive mappings involving orbits, Fixed Point Theory Appl., 2010 (2010), Article ID 458265, 19 pages.

- A. Nicolae, Fixed point theorems for multi-valued mappings of Feng-Liu type, Fixed Point Theory, 12 (2011), 145-154.

- A. Nicolae, Fixed points of uniformly Lipschitz type and asymptotically nonexpansive multivalued mappings (submitted for publication).

- A. Nicolae, D. O'Regan, A. Petruşel, Fixed point theorems for single and multivalued generalized contractions in metric spaces endowed with a graph, Georgian Math. J., 18 (2011), 307-327.

Chapter 1

Preliminaries

In this chapter we give some preliminary notions and results that are needed in the sequel. Each of the following sections introduces one relevant framework we work in, with emphasis on the geometrical aspects of these spaces that play an important role in metric fixed point theory. Some of the sections also contain fixed point results that hold in the described setting and are significant in this work.

1.1 Metric spaces

Section 1.1 presents the notion of metric spaces and some basic definitions, notations and properties related to these spaces. Notice that all the definitions given here carry over to normed spaces by replacing the metric with the norm. We define reflexive metric spaces which constitute the main general setting we work in throughout this thesis. We also include some well-known single and multivalued fixed point results that can be stated in the broad setting of metric spaces. Most of the notions and results we recall here can be found, for instance, in [29, 32, 34, 39, 51, 69].

1.2 Banach spaces

Although this work mainly focuses on the metric setting, we include in Section 1.2 some basic concepts of Banach space geometry that are needed later. We also recall some classical fixed point results for nonexpansive mappings in Banach spaces. For a more detailed discussion on properties of Banach spaces that are significant in fixed point theory see [29, 51].

1.3 Geodesic spaces

In Section 1.3 we define length and geodesic spaces, convex sets and various concepts of convexity in geodesic spaces. We introduce the notions of comparison triangles and angles in the Euclidean plane and define the Alexandrov angle between two geodesic paths. For a comprehensive treatment of length and geodesic metric spaces the reader may check [5, 8, 64].

1.4 The model spaces M_κ^n

Section 1.4 introduces the classical model spaces M_κ^n with emphasis on the spherical and hyperbolic space and on the model space M_κ^2. We define κ-comparison triangles and angles in M_κ^2 and state Alexandrov's Lemma which is a very important geometric property of M_κ^2. More about n-spheres, hyperbolic n-spaces and related topics can be found in [5, 33].

1.4.1 The spherical n-space

1.4.2 The hyperbolic n-space

1.4.3 The model spaces M_κ^2

1.5 CAT(κ) spaces

Section 1.5 contains the notion of CAT(κ) spaces and some characterizations of these spaces. We define spaces of curvature bounded above and below and focus afterwards on the properties of CAT(0), CAT(1) spaces and $\mathbb{R}$-trees including some important fixed point results. More details about CAT(κ) spaces can be found in [5, 8, 45, 46].

1.5.1 CAT(0) spaces

1.5.2 CAT(1) spaces

1.5.3 $\mathbb{R}$-trees

1.6 Hyperconvex spaces

Section 1.6 defines hyperconvexity giving examples and other notions in connection to this concept such as metric convexity, binary intersection property or external and weakly external hyperconvexity. We characterize hyperconvexity in terms of the existence of nonexpansive retractions and recall some well-known fixed point and selection results. For a more detailed discussion on fixed point theory in hyperconvex spaces see, for example, [4, 40, 51, 72].

1.7 Uniformly convex metric spaces

In Section 1.7 we define uniformly convex geodesic metric spaces and the modulus of uniform convexity for a geodesic space. We analyze the modulus of convexity in CAT(0) and CAT(1) spaces and give some fixed point results for uniformly convex metric spaces with a modulus of convexity that is monotone or lower semi-continuous from the right (called here UC spaces). More details about uniformly convex metric spaces can be found in [15, 53].

Chapter 2

A class of nonexpansive-type mappings in geodesic, hyperconvex and Banach spaces

The purpose of this chapter is to give fixed point and convergence results in the setting of reflexive metric spaces for some classes of mappings that satisfy a generalized nonexpansivity condition recently introduced in [74]. We also include extensions of this condition for both the single and multivalued case. Most of the results of this chapter are part of [18].

In Section 2.1 we recall condition (C) which is an extension of the concept of nonexpansivity for singlevalued mappings. This condition was introduced by Suzuki in [74]. We give some basic properties, convergence and fixed point results for this class of mappings in Banach spaces, uniformly convex and hyperconvex metric spaces, as well as in $\mathbb{R}$-trees. This section is divided into four subsections. The first one focuses on some generalizations of condition (C) considered by García-Falset, Llorens-Fuster and Suzuki in [24]. In the second subsection we adapt some of the results previously stated in Banach or CAT(0) spaces (see [66, 74]) to the framework of uniformly convex metric spaces (Lemma 2.1.1, Theorems 2.1.3, 2.1.4). Other contributions are included in the next subsection where we study the question whether mappings with condition (C) also have fixed points when defined on bounded hyperconvex spaces. In the compact case we provide an affirmative answer (Theorem 2.1.5). For the more general case we need to introduce a new condition on the mappings under consideration (Definition 2.1.5). In particular, we show that a 2-Lipschitz self-mapping with condition (C) defined on a bounded hyperconvex space has a fixed point (Theorem 2.1.6, Corollary 2.1.3). This result is significant among the class of known results for mappings with condition (C) since it is the first one without compactness conditions for which neither the uniqueness of asymptotic centers nor anything related to the Opial property is required. In the last subsection we study mappings with condition (C) in complete geodesically bounded $\mathbb{R}$-trees. We prove that a mapping with condition (C) defined on complete geodesically bounded $\mathbb{R}$-trees has a fixed point.

Section 2.2 contains the main results of this chapter. We study condition (C) for multivalued mappings in the context of geodesic metric spaces (with special attention to the case of $\mathbb{R}$-trees) and Banach spaces. We assume condition (C) for multivalued mappings as in [66] where different results in this direction were obtained for CAT(0) spaces. Here, our contributions are structured into two subsections. The first one focuses on geodesic spaces where we prove a technical lemma (Lemma 2.2.1) which is a multivalued version of the key fact that is behind the main results in [24, 74]. Our results (Proposition 2.2.1, Theorems 2.2.1, 2.2.2) are first obtained for complete uniformly convex geodesic spaces with convex metric and then particularized for more precise geometries. Since CAT(0) spaces are a particular class of uniformly convex geodesic spaces with convex metric, we obtain more general results than those from [66]. We introduce condition (C') which is a new condition for multivalued mappings in the spirit of condition (C) (Definition 2.2.3). We give examples showing that this condition is actually weaker than condition (C) (Proposition 2.2.2) and prove a selection theorem in $\mathbb{R}$-trees for mappings satisfying this newly introduced condition (Theorem 2.2.4), from where a stronger fixed point result for multivalued mappings follows (Corollary 2.2.1). This selection result resembles a very important one, see for instance [40, 72], for hyperconvex spaces (notice, see [44], that complete $\mathbb{R}$-trees are hyperconvex) although the approach here is completely different as the proof relies on very particular properties of $\mathbb{R}$-trees rather than on hyperconvexity. We close this subsection by introducing a generalized version of condition (C') for multivalued mappings (Definition 2.2.4) and giving a selection result in the context of $\mathbb{R}$-trees for mappings which fulfill this condition (Theorem 2.2.5). In the second subsection we revisit the classical theory of nonexpansive multivalued mappings in Banach spaces to study it under condition (C). We show the existence of fixed points for such a mapping in a Banach space with the Opial property (Theorem 2.2.6). The method of asymptotic centers allows us to establish the same result in UCED Banach spaces (Theorem 2.2.7). Moreover, if we also assume the continuity of the mapping we can prove the existence of fixed points in a Banach space for which the asymptotic center of a bounded sequence with respect to a bounded, closed and convex subset is nonempty and compact, that is, a counterpart of the Kirk-Massa Theorem (Theorem 2.2.8). We finish this subsection with another extension of condition (C) in the multivalued case.

In Section 2.3 we apply some of the fixed point theorems stated in the previous sections to obtain results on the existence of common fixed points for commuting mappings. More precisely, we focus on the commutativity between single and multivalued mappings. We extend a result of [66] in the setting of uniformly convex metric spaces with convex metric (Theorem 2.3.1). Likewise, we prove a similar result in the framework of $\mathbb{R}$-trees (Theorem 2.3.2).

2.1 Fixed points for singlevalued mappings

In this section we introduce a more general condition than the nonexpansivity condition and give convergence and fixed point results for this class of mappings in Banach spaces, complete UC spaces, hyperconvex spaces and $\mathbb{R}$-trees.

Suzuki extended in [74] the concept of singlevalued nonexpansive mappings as follows.

Definition 2.1.1 (Suzuki [74]). *Let X be a Banach space and $K \in \mathcal{P}(X)$. A mapping $T : K \to X$ is said to satisfy condition (C) if for every $x, y \in K$,*

$$\frac{1}{2}\|x - T(x)\| \leq \|x - y\| \quad \Longrightarrow \quad \|T(x) - T(y)\| \leq \|x - y\|.$$

Suzuki [74] proved the following convergence theorem for mappings with condition (C).

Theorem 2.1.1 (Suzuki [74]). *Let X be a Banach space and $K \in \mathcal{P}_{cp,cv}(X)$. Assume that the mapping $T : K \to K$ satisfies condition (C). Let $\alpha \in [1/2, 1)$. Define a sequence*

$(x_n) \subseteq K$ by taking $x_1 \in K$ and for $n \in \mathbb{N}$,

$$x_{n+1} = (1-\alpha)x_n + \alpha T(x_n).$$

Then (x_n) converges strongly to a fixed point of T.

Suzuki [74] also gave a variant of the Browder-Göhde-Kirk Theorem for mappings satisfying condition (C).

Theorem 2.1.2 (Suzuki [74]). *Let X be a UCED Banach space and $K \in \mathcal{P}_{cv}(X)$ be weakly compact. Assume $T : K \to K$ satisfies condition (C). Then $\mathrm{Fix}(T) \neq \emptyset$.*

2.1.1 Generalizations of condition (C) in Banach spaces

Motivated by the results of [74], García-Falset, Llorens-Fuster and Suzuki considered in [24] two generalizations of condition (C) giving examples and establishing fixed point results. The first studied condition is the following.

Definition 2.1.2 (García-Falset, Llorens-Fuster, Suzuki [24]). *Let X be a Banach space, $K \in \mathcal{P}(X)$, $T : K \to X$ and $\mu \geq 1$. The mapping T satisfies condition (E_μ) if for all $x, y \in K$,*

$$\|x - T(y)\| \leq \mu\|T(x) - x\| + \|x - y\|.$$

T is said to satisfy *condition* (E) if it satisfies (E_μ) for some $\mu \geq 1$. Condition (C) implies (E_3), but the reversed implication does not hold.

Another natural extension of condition (C) studied in [24] is given in the following.

Definition 2.1.3 (García-Falset, Llorens-Fuster, Suzuki [24]). *Let X be a Banach space, $K \in \mathcal{P}(X)$, $T : K \to X$ and $\lambda \in (0,1)$. The mapping T satisfies condition (C_λ) if for all $x, y \in K$,*

$$\lambda\|x - T(x)\| \leq \|x - y\| \implies \|T(x) - T(y)\| \leq \|x - y\|.$$

For more details about conditions (E) and (C_λ), as well as fixed point results for mappings satisfying these conditions one may consult [24].

2.1.2 Condition (C) in UC spaces

In the remaining of this section we focus on condition (C) in the metric setting. We start by formulating another special case of [28, Proposition 2] in the context of geodesic metric spaces with convex metric.

Lemma 2.1.1 (Goebel, Kirk [28]). *Let X be a geodesic metric space with convex metric, $\alpha \in (0,1)$ and (x_n),(y_n) bounded sequences in X such that for every $n \in \mathbb{N}$,*

$$x_{n+1} = (1-\alpha)x_n + \alpha y_n \quad \text{and} \quad d(y_{n+1}, y_n) \leq d(x_{n+1}, x_n).$$

Then $\lim_{n\to\infty} d(x_n, y_n) = 0$.

Theorem 2.1.3. *Let X be a uniquely geodesic metric space and $K \in \mathcal{P}_{cl,cv}(X)$. Suppose $T : K \to K$ satisfies condition (C) and $\mathrm{Fix}(T) \neq \emptyset$. Then $\mathrm{Fix}(T)$ is closed and convex.*

Theorem 2.1.4. *Let X be a complete UC space with convex metric and suppose $K \in \mathcal{P}_{b,cl,cv}(X)$. If $T : K \to K$ satisfies condition (C), then $\mathrm{Fix}(T)$ is nonempty, closed and convex.*

2.1.3 Condition (C) in hyperconvex spaces

Hyperconvex spaces provide a very specific and interesting class of metric spaces with a large literature on fixed point results for nonexpansive mappings (see [51, Chapter 13] or [40, 72] and the references therein). It is well-known that nonexpansive self-mappings defined on nonempty and bounded hyperconvex spaces have fixed points. Therefore, it is natural to wonder whether mappings with condition (C) also have fixed points when defined from a bounded hyperconvex space into itself. The goal of this subsection is to study this question. As a result, we provide partial positive answers.

Although a mapping with condition (C) need not be continuous, it is shown in Theorem 2.1.1 that, if T is a self-mapping with condition (C) defined on a nonempty, compact and convex subset of a Banach space, then it has a fixed point. In order to obtain the same result for hyperconvex metric spaces, we first need to give a meaning to convex combinations of two points in such spaces.

Let H be a hyperconvex space. The space H embeds isometrically into $\ell^\infty(H)$ and there exists a nonexpansive retraction R from $\ell^\infty(H)$ into H (see [51, Chapter 13] for details).

Definition 2.1.4. *Let H be a hyperconvex metric space and R as above. Then, for $x, y \in H$ and $\alpha \in [0,1]$, define*

$$(1-\alpha)x \oplus \alpha y = R((1-\alpha)x + \alpha y),$$

where $(1-\alpha)x + \alpha y$ stands for the usual convex combination in $\ell^\infty(H)$.

Notice that this definition provides a structure of segments (also called bicombing in the literature) which makes the metric convex as it is required in Lemma 2.1.1. In consequence, the adaptation of this lemma to this new setting (see also [28, Proposition 2]) is straightforward.

Lemma 2.1.2. *Let H be a hyperconvex metric space and consider the bicombing given by R as above. Let $\alpha \in (0,1)$ and (x_n), (y_n) be two bounded sequences in H such that for every $n \in \mathbb{N}$,*

$$x_{n+1} = (1-\alpha)x_n \oplus \alpha y_n \quad \text{and} \quad d(y_{n+1}, y_n) \leq d(x_{n+1}, x_n).$$

Then $\lim_{n\to\infty} d(x_n, y_n) = 0$.

Theorem 2.1.1 can now be easily adapted to our setting.

Theorem 2.1.5. *Let H be a compact hyperconvex metric space. Suppose $T : H \to H$ satisfies condition (C) and consider any bicombing as above on H. Let $\alpha \in [1/2, 1)$. Define a sequence $(x_n) \subseteq H$ by taking $x_1 \in H$ and for $n \in \mathbb{N}$,*

$$x_{n+1} = (1-\alpha)x_n \oplus \alpha T(x_n).$$

Then (x_n) converges to a fixed point of T.

Compactness in the previous theorem is only used to obtain the fixed point once it is known that $\lim_{n\to\infty} d(x_n, T(x_n)) = 0$. Therefore, the following corollary follows.

Corollary 2.1.1. *If T and (x_n) are as above, and H is a hyperconvex metric space, not necessarily compact, then (x_n) is an approximate fixed point sequence for T.*

The next corollary follows from the fact that mappings with condition (C) are quasi-nonexpansive.

Corollary 2.1.2. *In the conditions of Theorem 2.1.5,* $\mathrm{Fix}(T)$ *is hyperconvex.*

In order to approach the noncompact case we consider a new condition.

Definition 2.1.5. *Let X be a metric space, $K \subseteq X$ and $T : K \to X$. We say that T satisfies condition (D) if for all $x, y \in K$,*

$$\frac{1}{2}d(x, T(x)) \geq d(x, y) \quad \Longrightarrow \quad d(T(x), T(y)) \leq d(x, T(x)).$$

In the conjunction of conditions (C) and (D) we can adapt the classical proof of Baillon (see [4, Theorem 5]) for the existence of fixed points for nonexpansive mappings in hyperconvex spaces.

Theorem 2.1.6. *Let X be a nonempty and bounded hyperconvex space. Suppose $T \colon X \to X$ satisfies conditions (C) and (D). Then* $\mathrm{Fix}(T)$ *is nonempty and hyperconvex.*

Corollary 2.1.3. *Let X be a nonempty and bounded hyperconvex space. Suppose $T \colon X \to X$ is a 2-Lipschitz mapping satisfying condition (C). Then* $\mathrm{Fix}(T)$ *is nonempty and hyperconvex.*

2.1.4 Condition (C) in $\mathbb{R}$-trees

$\mathbb{R}$-trees form a very particular, but still very wide and important in applications, class of geodesic and hyperconvex spaces. Their particular geometry has made it possible to prove, for instance, that continuous mappings defined on complete geodesically bounded $\mathbb{R}$-trees (and so not necessarily bounded) have fixed points. In the following we give a similar result for mappings satisfying condition (C). Recall that there is no direct relation between continuity and condition (C).

Theorem 2.1.7. *Let X be a complete geodesically bounded $\mathbb{R}$-tree and $T : X \to X$ a mapping satisfying condition (C). Then* $\mathrm{Fix}(T) \neq \emptyset$.

Remark 2.1.1. In the above result the hypothesis that X is geodesically bounded cannot be removed.

2.2 Fixed points for multivalued mappings

In this section we study condition (C) for multivalued mappings in the context of geodesic metric spaces (with special attention to the case of $\mathbb{R}$-trees) and Banach spaces. We work with the definition of condition (C) for multivalued mappings that was given in [66] where different results in this direction were obtained for CAT(0) spaces.

Definition 2.2.1 (Razani, Salahifard [66])**.** *Let X be a metric space and $K \in \mathcal{P}(X)$. A mapping $T : K \to \mathcal{P}(X)$ is said to satisfy condition (C) if for each $x, y \in K$ and $u_x \in T(x)$ such that*

$$\frac{1}{2}d(x, u_x) \leq d(x, y),$$

there exists $u_y \in T(y)$ with

$$d(u_x, u_y) \leq d(x, y).$$

In the rest of this chapter we use condition (C) for both single and multivalued mappings with the context distinguishing between the two cases. The same also holds for other conditions we make use of.

2.2.1 Condition (C) in geodesic spaces

Following the singlevalued case, we introduce the next condition and prove that, for $\mu = 3$, it is a generalization of condition (C).

Definition 2.2.2. *Let X be a metric space, $K \in \mathcal{P}(X)$, $T : K \to \mathcal{P}(X)$ and $\mu \geq 1$. The mapping T satisfies condition (E_μ) if for each $x, y \in K$ and $u_x \in T(x)$ there exists $u_y \in T(y)$ such that*

$$d(x, u_y) \leq \mu d(x, u_x) + d(x, y).$$

Lemma 2.2.1. *Let X be a metric space, $K \in \mathcal{P}(X)$ and let $T : K \to \mathcal{P}(K)$ satisfy condition (C). Then T satisfies condition (E_3).*

The next result provides an approximate fixed point sequence for a multivalued mapping satisfying condition (C).

Proposition 2.2.1. *Let X be geodesic metric space with convex metric, $K \in \mathcal{P}_{b,cv}(X)$ and $T : K \to \mathcal{P}(K)$. If T satisfies condition (C), then T has an approximate fixed point sequence.*

Our first fixed point result for multivalued mappings is given for self-mappings defined on a compact and convex set.

Theorem 2.2.1. *Let X be a geodesic space with convex metric and $K \in \mathcal{P}_{cp,cv}(X)$. Suppose $T : K \to \mathcal{P}_{cl}(K)$ satisfies condition (C). Then* $\mathrm{Fix}(T) \neq \emptyset$.

In the sequel we focus on results where the compactness condition is moved from the domain to the image of the mapping.

Theorem 2.2.2. *Let X be a complete UC space with convex metric and $K \in \mathcal{P}_{b,cl,cv}(X)$. Suppose $T : K \to \mathcal{P}_{cp}(K)$ satisfies condition (C). Then* $\mathrm{Fix}(T) \neq \emptyset$.

In the above result we can drop the convexity of the metric and assume instead that the mapping admits an approximate fixed point sequence.

Theorem 2.2.3. *Let X be a complete UC space and $K \in \mathcal{P}_{b,cl,cv}(X)$. Suppose $T : K \to \mathcal{P}_{cp}(K)$ satisfies condition (C) and admits an approximate fixed point sequence. Then* $\mathrm{Fix}(T) \neq \emptyset$.

In the next result we consider the following new condition for multivalued mappings which will be shown to be weaker than condition (C).

Definition 2.2.3. *Let X be a metric space, $K \in \mathcal{P}(X)$ and $T : K \to \mathcal{P}(X)$. We say that the mapping T satisfies condition (C') if for each $x, y \in K$ and $u_x \in T(x)$ with*

$$d(x, u_x) = \mathrm{dist}(x, T(x)) \quad \text{and} \quad \frac{1}{2} d(x, u_x) \leq d(x, y),$$

there exists $u_y \in T(y)$ such that

$$d(u_x, u_y) \leq d(x, y).$$

We prove next a selection theorem in $\mathbb{R}$-trees for multivalued mappings satisfying condition (C') and analyze afterwards the relation of (C') to (C) and (E_3) respectively.

Theorem 2.2.4. *Let X be an $\mathbb{R}$-tree, $K \in \mathcal{P}(X)$ and $T : K \to \mathcal{P}_{cl,cv}(X)$ a mapping which satisfies (C'). Then the mapping $f : K \to X$ defined by $f(x) = P_{T(x)}(x)$ for each $x \in K$ is a selection of T that satisfies condition (C).*

Corollary 2.2.1. *Let X be a bounded complete $\mathbb{R}$-tree. Suppose $T : X \to \mathcal{P}_{cl,cv}(X)$ satisfies condition (C'). Then* $\operatorname{Fix}(T)$ *is a nonempty complete $\mathbb{R}$-tree.*

Proposition 2.2.2. *Let K be a bounded, closed and convex subset of a complete $\mathbb{R}$-tree and $T : K \to \mathcal{P}_{cl,cv}(K)$. The following hold:*

(i) *if T satisfies (C), then it also satisfies (C'), but the converse does not hold;*

(ii) *if T satisfies (C'), then it also satisfies (E_3), but the converse is false.*

Following the definition of condition (C_λ) in the singlevalued case, we introduce the next generalized version of condition (C') for multivalued mappings.

Definition 2.2.4. *Let X be a metric space, $K \in \mathcal{P}(X)$, $T : K \to \mathcal{P}(X)$ and $\lambda \in (0, 1)$. The mapping T satisfies condition (C'_λ) if for each $x, y \in K$ and $u_x \in T(x)$ with*

$$d(x, u_x) = \operatorname{dist}(x, T(x)) \quad \text{and} \quad \lambda d(x, u_x) \le d(x, y),$$

there exists $u_y \in T(y)$ such that

$$d(u_x, u_y) \le d(x, y).$$

Theorem 2.2.5. *Let X be an $\mathbb{R}$-tree, $K \in \mathcal{P}(X)$ and $T : K \to \mathcal{P}_{cl,cv}(X)$ a mapping which satisfies (C'_λ). Then the mapping $f : K \to X$ defined by $f(x) = P_{T(x)}(x)$ for each $x \in K$ is a selection of T that satisfies condition (C_λ).*

2.2.2 Condition (C) in Banach spaces

In this subsection we revisit the classical theory of nonexpansive multivalued mappings in Banach spaces to study it under condition (C). We start by proving the existence of fixed points for mappings that satisfy condition (C) in a Banach space with the Opial property.

Theorem 2.2.6. *Let X be a Banach space which has the Opial property with respect to τ. Suppose $K \in \mathcal{P}_{b,cv}(X)$ is τ-sequentially compact and $T : K \to \mathcal{P}_{cp}(K)$ is a mapping satisfying condition (C). Then* $\operatorname{Fix}(T) \neq \emptyset$.

The next result is a multivalued analogue of Theorem 2.1.2.

Theorem 2.2.7. *Let X be a UCED Banach space and $K \in \mathcal{P}_{cv}(X)$ be weakly compact. Suppose $T : K \to \mathcal{P}_{cp}(K)$ satisfies condition (C). Then* $\operatorname{Fix}(T) \neq \emptyset$.

We prove next an analogous result to the Kirk-Massa Theorem [49] for mappings satisfying condition (C).

Theorem 2.2.8. *Let X be a Banach space, $K \in \mathcal{P}_{b,cl,cv}(X)$ and $T : K \to \mathcal{P}_{cp,cv}(K)$ a continuous mapping with respect to the Pompeiu-Hausdorff distance satisfying condition (C). Suppose that each sequence in K has a nonempty and compact asymptotic center relative to K. Then* $\operatorname{Fix}(T) \neq \emptyset$.

It is worth pointing out that another natural extension of condition (C) for a multivalued mapping $T : K \to \mathcal{P}_{b,cl}(X)$ is the following: for all $x, y \in K$

$$\frac{1}{2}\mathrm{dist}(x, T(x)) \leq \|x - y\| \quad \Longrightarrow \quad H(T(x), T(y)) \leq \|x - y\|.$$

We refer to this condition as *condition* (C'').

Obviously, a nonexpansive mapping meets condition (C''). However, it is not clear if a mapping satisfying the above condition also satisfies (C). Still, if T takes compact values, it is easy to see that this new condition implies condition (C). Since in our theorems T is assumed to be compact valued, such results generalize classical fixed point theorems for multivalued mappings (see [49, 54, 57]).

2.3 Common fixed points for commuting mappings

In this section we apply some of the fixed point theorems stated in the previous sections to obtain results on the existence of common fixed points. We focus on the commutativity between single and multivalued mappings. Recall that, if X is a metric space, $K \in \mathcal{P}(X)$, $f : K \to K$ and $T : K \to \mathcal{P}(K)$, then f and T are *commuting mappings* if $f(y) \in T(f(x))$ for all $x \in K$ and $y \in T(x)$.

The lemma below constitutes a main tool in proving our results.

Lemma 2.3.1. *Let X be a metric space, $K \in \mathcal{P}(X)$, $f : K \to K$ satisfying condition (C) and with $\mathrm{Fix}(f) \neq \emptyset$. Suppose $T : K \to \mathcal{P}(K)$ is such that for every $x, y \in \mathrm{Fix}(f)$, the set $P_{T(y)}(x)$ is a singleton. If f and T commute, then $P_{T(y)}(x) \in \mathrm{Fix}(f)$ for all $x, y \in \mathrm{Fix}(f)$.*

Theorem 2.3.1. *Let X be a complete UC space with convex metric and $K \in \mathcal{P}_{b,cl,cv}(X)$. Suppose $f : K \to K$ and $T : K \to \mathcal{P}_{cp,cv}(K)$ satisfy condition (C). If f and T commute, then there exists $z \in K$ such that $z = f(z) \in T(z)$.*

Theorem 2.3.2. *Let X be a bounded complete $\mathbb{R}$-tree. Suppose $f : X \to X$ and $T : X \to \mathcal{P}_{cl,cv}(X)$ satisfy conditions (C) and (C') respectively. If f and T commute, then there exists $z \in K$ such that $z = f(z) \in T(z)$.*

Chapter 3

Generalizations of contractions and nonexpansive mappings involving orbits

In this chapter we give fixed point and well-posedness results, as well as demi-closed principles for classes of singlevalued mappings that satisfy assumptions milder than contraction or nonexpansive conditions using the concept of orbits. Likewise, we generalize multivalued uniformly Lipschitz and asymptotically nonexpansive mappings. We work in the context of metric, Banach, CAT(0) and uniformly convex geodesic spaces. Most of the results proved here are included in [59, 60, 62].

In Section 3.1 we give fixed point results for singlevalued mappings that satisfy certain contraction-type conditions. We also focus on generalizations of nonexpansive mappings. This section is divided into three subsections. The first one presents some local variants (Theorems 3.1.3, 3.1.4) of a fixed point theorem proved by Walter [77] which uses a contraction condition defined in terms of contractive gauge functions. We also give a negative answer to an open question in connection with this theorem raised by Kirk and Saliga [50] (Example 3.1.1). At the same time, we formulate additional conditions that provide an affirmative answer to this problem. More precisely, we prove that, if the contractive gauge function is additionally assumed to be subadditive or if the space is compact, then one can answer this question in the positive (Remark 3.1.1, Theorem 3.1.5). The next subsection includes fixed point and well-posedness results for mappings that generalize pointwise contractions (Theorem 3.1.6), asymptotic pointwise contractions (Theorem 3.1.7), asymptotic pointwise nonexpansive mappings (Theorem 3.1.8) or strongly asymptotic pointwise contractions (Theorems 3.1.10, 3.1.11). We also prove a demi-closed principle (Theorem 3.1.9) and an asymptotic version (Theorem 3.1.12) of a result given by Walter [77]. These generalizations are obtained by considering the radius or the diameter of orbits in the definition of the mappings. In the last subsection we introduce two extensions of nonexpansive mappings in CAT(0) spaces and prove fixed point results for these classes of mappings (Theorems 3.1.13, 3.1.14). We give examples showing that these conditions are not only different to each other, but also to nonexpansivity. Besides, we include a demi-closed principle using a condition that generalizes these two conditions (Theorem 3.1.15).

Section 3.2 recalls the concept of uniformly k-Lipschitz mappings which was introduced in [27]. In the same paper it was shown that in uniformly convex Banach spaces every uniformly k-Lipschitz mapping has a fixed point when k satisfies a relation depending on the modulus of convexity. Many other fixed point results for uniformly Lipschitz mappings were proved in different contexts. One famous fixed point theorem was given by Lifšic [56] in the general setting of metric spaces. Another important approach was pointed out by Lim and Xu [58]. In the recent paper [41], Khamsi and Kirk gave the definition of multivalued uniformly k-Lipschitz mappings and used this concept to extend Lifšic's Theorem to the multivalued case. In this section we further study and generalize the notion of multivalued uniformly Lipschitz mappings in the context of Banach, metric and CAT(0) spaces. Besides, we investigate an extension of multivalued asymptotically nonexpansive mappings in uniformly convex geodesic metric spaces. In the first subsection we introduce a generalization of multivalued uniformly k-Lipschitz mappings, called k-GL mappings (Definition 3.2.2), and use it to prove Theorem 3.2.1, a variant of the first fixed point result for uniformly Lipschitz mappings given in [27]. In addition, we show that the multivalued version of Lifšic's Theorem proved in [41] also holds for k-GL mappings (Theorem 3.2.2). In the second subsection we prove a fixed point result in CAT(0) spaces for another class of multivalued mappings that extend uniformly Lipschitz mappings using strongly ergodic matrices (Theorem 3.2.3). This result generalizes a fixed point theorem in Hilbert spaces for mappings with Lipschitz iterates due to Górnicki [31]. The last subsection contains a multivalued variant of a well-known fixed point theorem for asymptotically nonexpansive mappings (Theorem 3.2.4).

3.1 Fixed points for singlevalued mappings

In this section we prove fixed point results for mappings that satisfy various generalized contraction and nonexpansive-like conditions. We also study the well-posedness of some of these fixed point problems and prove demi-closed principles. Before giving more details, we define the notion of orbits for singlevalued mappings. To this end, let (X, d) be a metric space and $T : X \to X$. For $x \in X$, we define the *orbit starting at* x by

$$O_T(x) = \{x, T(x), \ldots, T^n(x), \ldots\},$$

where $T^{n+1}(x) = T(T^n(x))$ for $n \geq 0$ and $T^0(x) = x$. The *orbit starting at* x *and* y is defined as $O_T(x, y) = O_T(x) \cup O_T(y)$. However, the orbit starting at x can also be defined as the sequence $(T^n(x))$ itself and not the set of elements of the sequence. In this section the first definition is more convenient.

3.1.1 Generalized contraction-type mappings

We begin by recalling that a function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a *contractive gauge function* if it is continuous, increasing and $\varphi(r) < r$ for every $r > 0$ (see [77]).

Theorem 3.1.1 (Walter [77]). *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping with bounded orbits. If there exists a contractive gauge function φ such that*

$$d(T(x), T(y)) \leq \varphi(\operatorname{diam}(O_T(x, y))) \quad \text{for every } x, y \in X,$$

then T is a Picard operator.

Denote, for $\epsilon \geq 0$, $L_\epsilon = \{x \in X : d(x, T(x)) \leq \epsilon\}$. Kirk and Saliga [50] proved the following related result.

Theorem 3.1.2 (Kirk, Saliga [50])**.** *Let (X,d) be a complete metric space and $T : X \to X$ be a mapping with bounded orbits. If there exists $\alpha < 1$ such that*

$$d\left(T(x),T(y)\right) \leq \alpha \operatorname{diam}\left(O_T(x,y)\right) \quad \text{for every } x,y \in X,$$

then T is a Picard operator and $\lim_{\epsilon \searrow 0} \operatorname{diam}\left(L_\epsilon\right) = 0$. Moreover, $(x_n) \subseteq X$ is an approximate fixed point sequence if and only if (x_n) converges to the unique fixed point of T.

Kirk and Saliga [50] also raised the question whether the conclusions of the above theorem still stand in the weaker setting of Theorem 3.1.1. Addressing this question, Akkouchi showed in [1] that the answer is affirmative for the particular class Φ (see [1]) consisting of continuous and increasing functions $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that the mapping $r \longmapsto r - \varphi(r)$ from $\mathbb{R}_+$ to $\mathbb{R}_+$ is strictly increasing. Akkouchi [1] also remarked that a function $\varphi \in \Phi$ is a contractive gauge function. Still, the class of contractive gauge functions is larger than Φ (see [1, Example 2.2]).

We give below two local variants of Theorem 3.1.1.

Theorem 3.1.3. *Let (X,d) be a complete metric space, $x_0 \in X$ and $r > 0$. Suppose $T : \widetilde{B}(x_0,r) \to X$ is a mapping with $\operatorname{diam}\left(O_T(x_0)\right) \leq r$ and there exists a contractive gauge function φ satisfying*

$$d\left(T(x),T(y)\right) \leq \varphi\left(\operatorname{diam}\left(O_T(x,y)\right)\right) \quad \text{for all } x,y \in O_T(x_0). \tag{3.1}$$

If T has closed graph or the function $x \longmapsto d\left(x,T(x)\right)$, for $x \in \widetilde{B}(x_0,r)$, is T-orbitally lower semi-continuous, then T has a fixed point.

Theorem 3.1.4. *Let (X,d) be a complete metric space, $x_0 \in X, r > 0$ and let $T : \widetilde{B}(x_0,r) \to X$ be a mapping with $\operatorname{diam}\left(O_T(x_0)\right) \leq r$. Suppose there exists a contractive gauge function φ such that for all $x,y \in \widetilde{B}(x_0,r)$ with $O_T(x) \subseteq \widetilde{B}(x_0,r)$ and $O_T(y) \subseteq \widetilde{B}(x_0,r)$,*

$$d\left(T(x),T(y)\right) \leq \varphi\left(\operatorname{diam}\left(O_T(x,y)\right)\right).$$

If T has closed graph or the function $x \longmapsto d\left(x,T(x)\right)$, for $x \in \widetilde{B}(x_0,r)$, is T-orbitally lower semi-continuous, then T has a unique fixed point z and for every $x \in X$ with $O_T(x) \subseteq \widetilde{B}(x_0,r)$, $\lim_{n\to\infty} T^n(x) = z$.

Now we move our attention to the problem raised by Kirk and Saliga in [50] which was mentioned above.

Remark 3.1.1. Let φ be a contractive gauge function which is subadditive. Then $\varphi \in \Phi$.

Thus, we know that for contractive gauge functions, which are subadditive, the answer to the question of Kirk and Saliga [50] is positive. However, in the general case, without additional conditions the answer is negative. The following example illustrates this.

Example 3.1.1. *Let $X = [0,\infty)$ with the usual metric and $T, \varphi : X \to X$,*

$$T(x) = \varphi(x) = \begin{cases} x/2 & \text{if } x \leq \sqrt{2}, \\ x - 1/x & \text{if } x > \sqrt{2}. \end{cases}$$

Then T and φ satisfy the hypotheses of Theorem 3.1.1, but there exists $(x_n) \subseteq X$ such that $\lim_{n\to\infty} |x_n - T(x_n)| = 0$ and (x_n) does not converge 0, the unique fixed point of T.

Still, if the space X is assumed compact, the question formulated by Kirk and Saliga in [50] regarding the possibility of weakening the assumptions in Theorem 3.1.2 in the sense of Theorem 3.1.1 has an affirmative answer.

Theorem 3.1.5. *Let (X,d) be a compact metric space and $T : X \to X$. Suppose there exists a contractive gauge function φ such that*

$$d\left(T(x),T(y)\right) \le \varphi\left(\operatorname{diam}\left(O_T(x,y)\right)\right) \quad \text{for all } x,y \in X.$$

Then

(a) *T is a Picard operator;*

(b) *a sequence $(x_n) \subseteq X$ is an approximate fixed point sequence if and only if it converges to the unique fixed point of T;*

(c) $\lim_{\epsilon \searrow 0} \operatorname{diam}\left(L_\epsilon\right) = 0$.

3.1.2 Generalized pointwise, asymptotic pointwise and strongly asymptotic pointwise contractions

Four recent papers [15, 16, 35, 52] present simple and elegant proofs of fixed point results for pointwise contractions, asymptotic pointwise contractions and even asymptotic pointwise nonexpansive mappings. Kirk and Xu [52] studied these mappings in the context of weakly compact and convex subsets of Banach spaces and in uniformly convex Banach spaces respectively. Hussain and Khamsi [35] considered these problems in the framework of metric and CAT(0) spaces. In [16], Espínola and Hussain proved coincidence results for asymptotic pointwise nonexpansive mappings. Espínola, Fernández-León and Piątek [15] examined the existence of fixed points and the convergence of iterates for asymptotic pointwise contractions in uniformly convex metric spaces. In this subsection we formulate less restrictive conditions than the ones which appear in the classical definitions of these mappings and show that the conclusions of the fixed point theorems still stand. We also give well-posedness results.

A mapping $T : X \to X$ is called a *pointwise contraction* if there exists a function $\alpha : X \to [0,1)$ such that

$$d\left(T(x),T(y)\right) \le \alpha(x)d(x,y) \quad \text{for every } x,y \in X.$$

Let $T : X \to X$ and for $n \in \mathbb{N}$ let $\alpha_n : X \to \mathbb{R}_+$ such that

$$d\left(T^n(x),T^n(y)\right) \le \alpha_n(x)d(x,y) \quad \text{for every } x,y \in X.$$

If the sequence (α_n) converges pointwise to the function $\alpha : X \to [0,1)$, then T is called an *asymptotic pointwise contraction.*
If for every $x \in X$, $\limsup_{n\to\infty} \alpha_n(x) \le 1$, then T is called an *asymptotic pointwise nonexpansive mapping.*
If there exists $0 < k < 1$ such that for every $x \in X$, $\limsup_{n\to\infty} \alpha_n(x) \le k$, then T is called a *strongly asymptotic pointwise contraction.*

In the sequel we extend the results obtained by Hussain and Khamsi in [35] using the radius of the orbit. We start by introducing a property for a mapping $T : X \to X$, where X is a metric space. Namely, we say that T satisfies property (S) if

(S) for every approximate fixed point sequence (x_n) and for every $m \in \mathbb{N}$, the sequence $(d(x_n, T^m(x_n)))$ converges to 0 as $n \to \infty$ uniformly with respect to m.

Theorem 3.1.6. *Let X be a bounded metric space such that $\mathcal{A}(X)$ is nested compact. Also let $T : X \to X$ for which there exists $\alpha : X \to [0, 1)$ such that*

$$d(T(x), T(y)) \leq \alpha(x) r_x(O_T(y)) \quad \text{for every } x, y \in X. \tag{3.2}$$

Then T is a Picard operator. Moreover, if additionally T satisfies (S), then the fixed point problem is well-posed.

Theorem 3.1.7. *Let X be a bounded metric space, $T : X \to X$ and suppose there exists a convexity structure $\mathcal{F}$ which is nested compact and T-stable. Assume*

$$d(T^n(x), T^n(y)) \leq \alpha_n(x) r_x(O_T(y)) \quad \text{for every } x, y \in X,$$

where for each $n \in \mathbb{N}, \alpha_n : X \to \mathbb{R}_+$ and the sequence (α_n) converges pointwise to a function $\alpha : X \to [0, 1)$. Then T is a Picard operator. Moreover, if additionally T satisfies (S), then the fixed point problem is well-posed.

Theorem 3.1.8. *Let X be a complete CAT(0) space, $K \in \mathcal{P}_{b,cl,cv}(X)$, $T : K \to K$ and for $n \in \mathbb{N}$, let $\alpha_n : K \to \mathbb{R}_+$ be such that $\limsup_{n\to\infty} \alpha_n(x) \leq 1$ for all $x \in K$. If for all $n \in \mathbb{N}$,*

$$d(T^n(x), T^n(y)) \leq \alpha_n(x) r_x(O_T(y)) \quad \text{for every } x, y \in K,$$

then $\mathrm{Fix}(T)$ is nonempty, closed and convex.

We prove below a demi-closed principle similarly to [35, Proposition 1]. To this end, for $K \in \mathcal{P}_{cl,cv}(X)$, $(x_n) \subseteq K$ a bounded sequence and $\varphi : K \to \mathbb{R}_+, \varphi(x) = \limsup_{n\to\infty} d(x, x_n)$, as in [35], we introduce the following notation

$$x_n \overset{\varphi}{\rightharpoonup} \omega \quad \text{if and only if} \quad \varphi(\omega) = \inf_{x\in K} \varphi(x).$$

Theorem 3.1.9. *Let X be a CAT(0) space and $K \in \mathcal{P}_{b,cl,cv}(X)$. Suppose $T : K \to K$ satisfies (S) and for $n \in \mathbb{N}$, let $\alpha_n : K \to \mathbb{R}_+$ be such that $\limsup_{n\to\infty} \alpha_n(x) \leq 1$ for all $x \in K$. Suppose that for $n \in \mathbb{N}$,*

$$d(T^n(x), T^n(y)) \leq \alpha_n(x) r_x(O_T(y)) \quad \text{for every } x, y \in K.$$

If $(x_n) \subseteq K$ is an approximate fixed point sequence such that $x_n \overset{\varphi}{\rightharpoonup} \omega$, then $\omega \in \mathrm{Fix}(T)$.

In the sequel we generalize the strongly asymptotic pointwise contraction condition by using the diameter of the orbit.

Theorem 3.1.10. *Let X be a complete metric space and $T : X \to X$ a mapping with bounded orbits that is orbitally continuous. Also, for $n \in \mathbb{N}$, let $\alpha_n : X \to \mathbb{R}_+$ for which there exists $0 < k < 1$ such that for every $x \in X$, $\limsup_{n\to\infty} \alpha_n(x) \leq k$. If for each $n \in \mathbb{N}$,*

$$d(T^n(x), T^n(y)) \leq \alpha_n(x) \mathrm{diam}(O_T(x, y)) \quad \text{for every } x, y \in X,$$

then T is a Picard operator. Moreover, if additionally T satisfies (S), then the fixed point problem is well-posed.

Theorem 3.1.11. *Let X be a bounded metric space such that $\mathcal{A}(X)$ is nested compact and let $T : X \to X$ be an orbitally continuous mapping. Also, for $n \in \mathbb{N}$, let $\alpha_n : X \to \mathbb{R}_+$ for which there exists $0 < k < 1$ such that for every $x \in X$, $\limsup_{n\to\infty} \alpha_n(x) \leq k$. If for each $n \in \mathbb{N}$,*

$$d(T^n(x), T^n(y)) \leq \alpha_n(x) \mathrm{diam}(O_T(x, y)) \quad \text{for every } x, y \in X,$$

then T is a Picard operator. Moreover, if additionally T satisfies (S), then the fixed point problem is well-posed.

We conclude this subsection by proving an asymptotic version of Theorem 3.1.1 given in the previous subsection.

Theorem 3.1.12. *Let (X, d) be a complete metric space and $T : X \to X$ an orbitally continuous mapping with bounded orbits. Suppose there exist a continuous function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\varphi(t) < t$ for all $t > 0$ and the functions $\varphi_n : \mathbb{R}_+ \to \mathbb{R}_+$ such that the sequence (φ_n) converges pointwise to φ and for each $n \in \mathbb{N}$,*

$$d(T^n(x), T^n(y)) \leq \varphi_n\left(\operatorname{diam}(O_T(x, y))\right) \quad \text{for all } x, y \in X.$$

Then T is a Picard operator. Moreover, if additionally T satisfies (S) and φ_n is continuous for each $n \in \mathbb{N}$, then the fixed point problem is well-posed.

3.1.3 Generalized nonexpansive mappings

In this subsection we give fixed point results in CAT(0) spaces for two classes of mappings which are more general than nonexpansive ones.

Theorem 3.1.13. *Let X be a bounded complete CAT(0) space and $T : X \to X$ such that for every $x, y \in X$,*

$$d\left(T(x), T(y)\right) \leq r_x\left(O_T(y)\right). \tag{3.3}$$

Then $\operatorname{Fix}(T)$ *is nonempty, closed and convex.*

Theorem 3.1.14. *Let X be a bounded complete CAT(0) space and $T : X \to X$ such that for every $x, y \in X$,*

$$d(T(x), T(y)) \leq \operatorname{diam}\left(\{x\} \cup O_T(y)\right), \tag{3.4}$$

and

$$d(T(x), T(y)) \leq r_x(O_T(y)) + \sup_{k,p\in\mathbb{N}} \{\operatorname{diam}(\{T^k(x)\} \cup O_T(T^{k+p}(y))) - \operatorname{diam}(O_T(T^{k+p}(y)))\}. \tag{3.5}$$

Then $\operatorname{Fix}(T)$ *is nonempty, closed and convex.*

We prove next a demi-closed principle.

Theorem 3.1.15. *Let X be a CAT(0) space and $K \in \mathcal{P}_{b,cl,cv}(X)$. Let $T : K \to K$ be a mapping that satisfies (S) and (3.4) for each $x, y \in K$ and let $(x_n) \subseteq K$ be an approximate fixed point sequence such that $x_n \overset{\varphi}{\rightharpoonup} \omega$. Then $\omega \in \operatorname{Fix}(T)$.*

3.2 Fixed points for multivalued mappings

Let (X, d) be a metric space. A mapping $T : X \to X$ is called *uniformly Lipschitz* if there exists $k \geq 0$ such that for all $x, y \in X$, $d(T^n(x), T^n(y)) \leq kd(x, y)$ for any $n \geq 1$. The mapping T is also called *uniformly k-Lipschitz*. These mappings were introduced in [27].

In [41], Khamsi and Kirk considered the concept of uniformly Lipschitz multivalued mappings using orbits. For a multivalued mapping $T : X \to \mathcal{P}(X)$, an *orbit starting at* x is a sequence $(x_n) \subseteq X$ with $x_0 = x$ and $x_{n+1} \in T(x_n)$ for any $n \geq 0$. The set of all orbits of T starting at x is denoted by $O_T(x)$.

Definition 3.2.1 (Khamsi, Kirk [41])**.** *Let (X,d) be a metric space. The multivalued mapping $T : X \to \mathcal{P}(X)$ is uniformly k-Lipschitz (with $k \geq 0$) if for every $x, y \in X$ and every $(x_n) \in O_T(x)$ there exists $(y_n) \in O_T(y)$ such that*

$$d(x_{n+h}, y_n) \leq kd(x_h, y) \quad \textit{for any } n \geq 1, h \geq 0.$$

The main result of [41] extends Lifšic's Theorem to the multivalued case.

3.2.1 K-GL mappings in geodesic and Banach spaces

We begin this subsection by introducing a notion which generalizes the concept of uniformly Lipschitz multivalued mappings.

Definition 3.2.2. *Let (X,d) be a metric space. A multivalued mapping $T : X \to \mathcal{P}(X)$ is called k-GL (with $k \geq 0$) if for $n \in \mathbb{N}$ there exists $\alpha_n : X \to \mathbb{R}_+$ such that*

$$\limsup_{n\to\infty} \alpha_n(x) \leq k \quad \textit{for any } x \in X,$$

and for every $x, y \in X$ and every $(x_n) \in O_T(x)$ there exists $(y_n) \in O_T(y)$ with

$$d(x_{n+h}, y_n) \leq \alpha_n(y) \sup_{i \geq h} d(x_i, y) \quad \textit{for all } n \geq 1, h \geq 0.$$

Using k-GL mappings, we give next a generalized multivalued variant of the first fixed point result for uniformly Lipschitz mappings given in [27].

Theorem 3.2.1. *Let X be a uniformly convex Banach space with modulus of convexity δ_X. Suppose $C \in \mathcal{P}_{b,cl,cv}(X)$ and $T : C \to \mathcal{P}_{cl}(C)$ is a k-GL mapping where*

$$k\left(1 - \delta_X\left(\frac{1}{k}\right)\right) < 1 \quad \textit{for } k \geq 1.$$

Then $\operatorname{Fix}(T) \neq \emptyset$.

In the sequel we generalize the multivalued version of Lifšic's Theorem proved in [41].

Theorem 3.2.2. *Let (X,d) be a bounded complete metric space and $T : X \to \mathcal{P}_{cl}(X)$ a k-GL mapping where $k < \kappa(X)$. Then* $\operatorname{Fix}(T) \neq \emptyset$.

Corollary 3.2.1. *Let (X,d) be a bounded complete $CAT(0)$ space, $T : X \to X$ and for $n \in \mathbb{N}$, let $\alpha_n : X \to \mathbb{R}_+$ be such that there exists $k < \sqrt{2}$ with $\limsup_{n\to\infty} \alpha_n(x) \leq k$ for all $x \in X$. If for every $x, y \in X$,*

$$d\left(T^n(x), T^n(y)\right) \leq \alpha_n(x) \sup_{i \geq 0} d(T^i(y), x) \quad \textit{for each } n \geq 1,$$

then $\operatorname{Fix}(T) \neq \emptyset$.

3.2.2 Generalized uniformly Lipschitz mappings using strongly ergodic matrices

In this subsection we study another class of mappings that generalize uniformly Lipschitz multimaps using strongly ergodic matrices. A matrix of positive real numbers $[a_{n,k}]_{n,k\geq 1}$ is called *strongly ergodic* if

(i) $\lim_{n\to\infty} a_{n,k} = 0$ for every $k \geq 0$;

(ii) $\sum_{k\geq 1} a_{n,k} = 1$ for every $n \geq 1$;

(iii) $\lim_{n\to\infty} \sum_{k\geq 1} |a_{n,k+1} - a_{n,k}| = 0$.

The idea of this extension of uniformly Lipschitz mappings has its roots in [31], where Górnicki proved a fixed point result in Hilbert spaces for mappings with Lipschitz iterates.

Theorem 3.2.3. *Let (X, d) be a complete CAT(0) space, $C \in \mathcal{P}_{b,cl,cv}(X)$ and $T : C \to \mathcal{P}_{cl}(C)$ such that for every $x, y \in C$ and every $(x_n) \in O_T(x)$ there exists $(y_n) \in O_T(y)$ with*

$$d(x_{n+h}, y_n) \leq \alpha(n) d(x_h, y) \quad \text{for all } n \geq 1, h \geq 0,$$

where $\alpha : \mathbb{N} \to \mathbb{R}_+$. Suppose also $[a_{n,k}]_{n,k\geq 1}$ is a strongly ergodic matrix and

$$\liminf_{n\to\infty} \inf_{m\geq 0} \sum_{k\geq 1} a_{n,k}\alpha(k+m)^2 < 2.$$

Then $\mathrm{Fix}(T) \neq \emptyset$.

3.2.3 Asymptotically nonexpansive mappings

In [26], Goebel and Kirk defined asymptotically nonexpansive mappings as a natural extension of the notion of nonexpansive mappings. Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be *asymptotically nonexpansive* if there exists $(k_n) \subseteq [0, \infty)$ with $\lim_{n\to\infty} k_n = 0$ such that

$$d(T^n(x), T^n(y)) \leq (1 + k_n) d(x, y) \quad \text{for every } x, y \in X \text{ and for any } n \geq 1.$$

We prove a fixed point result in UC spaces for a multivalued generalization of this notion.

Theorem 3.2.4. *Let (X, d) be a complete UC space and denote by δ_X a modulus of uniform convexity which is monotone or lower semi-continuous from the right. Suppose $C \in \mathcal{P}_{b,cl,cv}(X)$ and $T : C \to \mathcal{P}(C)$ is such that for every $x, y \in C$ and every $(x_n) \in O_T(x)$ there exists $(y_n) \in O_T(y)$ with*

$$d(x_{n+h}, y_n) \leq (1 + k_n) \sup_{i\geq h} d(x_i, y) \quad \text{for any } n \geq 1, h \geq 0,$$

where $(k_n) \subseteq [0, \infty)$ with $\lim_{n\to\infty} k_n = 0$. Then $\mathrm{Fix}(T) \neq \emptyset$.

Chapter 4

Geometric aspects of geodesic Ptolemy spaces and fixed points

In this chapter we study the regularity of geodesic Ptolemy spaces and apply our findings to metric fixed point theory. It is an open question whether such spaces with a continuous midpoint map are CAT(0). We prove that if a certain uniform continuity is imposed on such a midpoint map then these spaces, if complete, are reflexive and bounded sequences have unique asymptotic centers. We also show that these spaces are in fact uniformly convex. Moreover, if the space is bounded and we assume a stronger variant of the uniform continuity of a midpoint map, then the modulus of convexity does not depend on the radius of the balls. The regularity properties proved here are applied to yield a series of fixed point results specific to CAT(0) spaces. We also study forms of the Ptolemy inequality and the Busemann convexity for CAT(κ) spaces and raise the problem of characterizing CAT(κ) spaces in terms of these notions. Some of the results of this chapter are part of [19].

A metric space (X, d) is called a *Ptolemy space* if

$$d(x,z)d(y,p) \leq d(x,y)d(z,p) + d(x,p)d(y,z) \quad \text{for every } x,y,z,p \in X.$$

The above relation is known as the *Ptolemy inequality*.

The classical theorem of Ptolemy states that, in the Euclidean plane, the Ptolemy inequality holds with equality if and only if x, y, z, p lie in this order on a circle. In the case of normed spaces, the Ptolemy inequality has a great impact on the regularity of the space. Namely, it was proved in [71] that a normed space is an inner product space if and only if it is a Ptolemy space.

The Ptolemy inequality was used by Foertsch and Schroeder in [22] to study the boundary at infinity of CAT(-1) spaces. It was shown that the boundary of a CAT(-1) space endowed with a Bourdon or a Hamenstädt metric is a Ptolemy space. At the same time, a condition for equality to hold in the Ptolemy inequality was also provided. This property led to the study of the relation between Gromov hyperbolic spaces and CAT(-1) spaces (see [23] for details). Motivated by these results, Ptolemy metric spaces were further investigated in [21] where a characterization of CAT(0) spaces in terms of the Ptolemy inequality is given.

CAT(0) spaces are Ptolemy spaces, but a geodesic Ptolemy space is not necessarily uniquely geodesic (see [21]) and thus cannot satisfy the CAT(0) condition. However, it was shown in the same paper that a proper geodesic Ptolemy space is uniquely geodesic, where the properness assumption may be replaced by the existence of a continuous midpoint map. Naturally, the authors raised then the still open question of whether a proper geodesic Ptolemy space (or a geodesic Ptolemy space with a continuous midpoint map) is CAT(0).

A direct consequence of the CAT(0) condition is the fact that CAT(0) spaces are Busemann convex, but being Busemann convex is a weaker property than being CAT(0). However, it was proved in [21] that the Busemann convexity implies the CAT(0) condition in the setting of Ptolemy spaces.

Furthermore, Foertsch and Schroeder showed in [22] that proper geodesic Ptolemy spaces are, in particular, strictly convex. As consequence of this result, the authors proved that the metric projection onto a closed and convex subset is a singlevalued and continuous mapping. It is still open whether this projection is nonexpansive as it is the case in CAT(0) spaces.

In Section 4.1 we study the regularity of geodesic Ptolemy spaces. Our contributions are structured into three subsections. In the first one we ask a bit more than continuity of the midpoint map and show that certain properties which so far have been proved to hold in UC spaces also hold in geodesic Ptolemy spaces. More precisely, we prove that a complete geodesic Ptolemy space with a uniformly continuous midpoint map is reflexive (Theorem 4.1.1). As a consequence of this result, we prove that bounded sequences have a unique asymptotic center (Theorem 4.1.2). We also include an example which shows that the uniform continuity property we impose on a midpoint map is weaker than the Busemann convexity (Example 4.1.1). In the second subsection we prove that every geodesic Ptolemy space which admits a uniformly continuous midpoint map is uniformly convex (Theorem 4.1.3). As a consequence, we show that, in a bounded complete Ptolemy space with a uniformly continuous midpoint map, the metric projection onto a closed and convex subset is a singlevalued and uniformly continuous mapping (Proposition 4.1.1). We introduce a strengthened version of the uniform continuity of a midpoint map, namely the strong uniform continuity of a midpoint map (Definition 4.1.2), and show that this notion is in general still weaker than the Busemann convexity. We conclude that every bounded geodesic Ptolemy space admitting such a midpoint map is uniformly convex and the modulus of uniform convexity does not depend on the radius of the balls (Theorem 4.1.4). The last subsection introduces the κ-Ptolemy inequality which is an analogue of the Ptolemy inequality in M_κ^2 spaces. In this way one can define κ-Ptolemy metric spaces. We show first that the κ-Ptolemy inequality becomes the classical Ptolemy inequality when κ tends to zero (Proposition 4.1.2). We prove that the κ-Ptolemy inequality is satisfied in CAT(κ) spaces (Theorem 4.1.5) and focus on some properties of κ-Ptolemy spaces (Propositions 4.1.3, 4.1.4, Remark 4.1.4). We also show that geodesic κ-Ptolemy spaces (with $\kappa < 0$) which admit a continuous midpoint map are uniquely geodesic (Theorem 4.1.6). We introduce the κ-Busemann convexity which is a version of the Busemann convexity in CAT(κ) spaces and notice that the κ-Busemann convexity becomes the Busemann convexity when κ tends to zero (Proposition 4.1.5). We show that a κ-Busemann convex space (with $\kappa < 0$) is Busemann convex (Remark 4.1.5) and raise the question whether one could make use of these notions to give a characterization of CAT(κ) spaces (Remark 4.1.6).

In Section 4.2 we apply the results obtained in the previous section to prove fixed point theorems in geodesic Ptolemy spaces. We find that many known fixed point results for CAT(0) spaces can be stated in the context of geodesic Ptolemy spaces which admit a uniformly continuous midpoint map. We start from Kirk's Theorem (Theorem 4.2.1), continue with generalized pointwise contractions, asymptotic pointwise contractions as well as nonexpansive mappings, and end with a fixed point result for multivalued mappings.

4.1 Regularity of geodesic Ptolemy spaces

In this section we prove some results on the regularity of geodesic Ptolemy spaces. We start by noticing that the metric of a geodesic Ptolemy space is convex.

In the following we use the concept of continuous midpoint maps. We say that X admits a *continuous midpoint map* if there exists a map $m : X \times X \to X$ such that

$$d(x, m(x,y)) = d(y, m(x,y)) = \frac{d(x,y)}{2} \quad \text{for all } x, y \in X,$$

and for $x, y, x_n, y_n \in X$ where $n \in \mathbb{N}$ such that $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(y_n, y) = 0$ we have that $\lim_{n\to\infty} d(m(x_n, y_n), m(x,y)) = 0$.

In [21], the authors raised the still open question of whether a proper geodesic Ptolemy space (or, more generally, whether a geodesic Ptolemy space with a continuous midpoint map) is CAT(0). Using the Busemann convexity, they proved that a metric space is CAT(0) if and only if it is Ptolemy and Busemann convex. Notice that the Busemann convexity clearly implies that the space is uniquely geodesic and the only midpoint map that one can define is continuous.

4.1.1 Reflexivity and asymptotic centers

In this subsection we obtain new results on the regularity of geodesic Ptolemy spaces when a stronger condition than continuity is considered on the midpoint map. We begin with a result stating that geodesic Ptolemy spaces with a uniformly continuous midpoint map are reflexive. First, we recall the following notion.

Definition 4.1.1. *Let X be a geodesic space. We say that X admits a uniformly continuous midpoint map if there exists a map $m : X \times X \to X$ such that*

$$d(x, m(x,y)) = d(y, m(x,y)) = \frac{d(x,y)}{2} \quad \textit{for all } x, y \in X,$$

and for $n \in \mathbb{N}$ and $x_n, x_n', y_n, y_n' \in X$ with

$$\lim_{n\to\infty} d(x_n, x_n') = 0 \quad \textit{and} \quad \lim_{n\to\infty} d(y_n, y_n') = 0$$

we have that

$$\lim_{n\to\infty} d(m(x_n, y_n), m(x_n', y_n')) = 0.$$

Clearly, every Busemann convex geodesic space admits a uniformly continuous midpoint map. The following example shows however that there exist spaces with a uniformly continuous midpoint map, but without being Busemann convex.

Example 4.1.1. *Let X be the positive octant of the spherical space $(\mathbb{S}^2, d)$. Then X is not Busemann convex, but admits a uniformly continuous midpoint map.*

Since every geodesic Ptolemy space with a continuous midpoint map is uniquely geodesic, it immediately follows that every geodesic Ptolemy space with a uniformly continuous midpoint map is also uniquely geodesic. We prove next the reflexivity of geodesic Ptolemy spaces with a uniformly continuous midpoint map. This result is the key tool in the proof of Theorem 4.1.2.

Theorem 4.1.1. *A complete geodesic Ptolemy space with a uniformly continuous midpoint map is reflexive.*

Remark 4.1.1. So far reflexivity of geodesic metric spaces has only been proved for UC spaces. In the following subsection we study the uniform convexity of geodesic Ptolemy spaces when certain continuity properties of the midpoint map are assumed. Notice that this problem lies somewhere in between what is known and the open question raised in [21] of whether such spaces are CAT(0).

The following theorem lies at the heart of proving fixed point results for geodesic Ptolemy spaces.

Theorem 4.1.2. *In a complete geodesic Ptolemy space with a uniformly continuous midpoint map, the asymptotic center of every bounded sequence with respect to a closed and convex subset is a singleton.*

4.1.2 Uniform convexity

In this subsection we study the uniform convexity of geodesic Ptolemy spaces when assuming additional convexity-like conditions. We begin with the following result for geodesic Ptolemy spaces with a uniformly continuous midpoint map.

Theorem 4.1.3. *A geodesic Ptolemy space with a uniformly continuous midpoint map is uniformly convex.*

Remark 4.1.2. *A proper geodesic Ptolemy space is pointwise uniformly convex.*

Remark 4.1.3. *A compact geodesic Ptolemy space is uniformly convex.*

In order to understand how close geodesic Ptolemy spaces with a uniformly continuous midpoint map fall to CAT(0) spaces, it would be interesting to see how regular a modulus of uniform convexity can get. A first natural question is whether there exists a modulus which does not depend on the radius of the balls.

Corollary 4.1.1. *Let X be a geodesic Ptolemy space with a uniformly continuous midpoint map. Then for every $\epsilon \in (0, 2]$ and every $R > 0$ there exists $\delta(\epsilon) \in (0, 1]$ such that it is a modulus of convexity for any ball of radius r with $\frac{1}{R} \leq r \leq R$.*

We study below the metric projection in the setting of bounded and complete Ptolemy spaces with a uniformly continuous midpoint map.

Proposition 4.1.1. *Let X be a bounded complete geodesic Ptolemy space with a uniformly continuous midpoint map and let $C \in \mathcal{P}_{cl,cv}(X)$. Then the metric projection P_C is a singlevalued and uniformly continuous mapping.*

We have seen in Corollary 4.1.1 that, for balls of radius not converging to zero or to infinity, the modulus of convexity does not depend on the radius. One way to avoid these two extreme situations is to consider the space to be bounded and to satisfy the following property.

Definition 4.1.2. *Let X be a geodesic space. We say that X admits a strong uniformly continuous midpoint map if there exists a map $m : X \times X \to X$ such that*

$$d(x, m(x, y)) = d(y, m(x, y)) = \frac{d(x, y)}{2} \quad \text{for all } x, y \in X,$$

and for $n \in \mathbb{N}$ and $\varphi_n \geq 0$, $x_n, x'_n, y_n, y'_n \in X$ with

$$\lim_{n\to\infty} \varphi_n d(x_n, x'_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} \varphi_n d(y_n, y'_n) = 0$$

we have that

$$\lim_{n\to\infty} \varphi_n d(m(x_n, y_n), m(x'_n, y'_n)) = 0.$$

Obviously, a strong uniformly continuous midpoint map is also uniformly continuous. It is easy to see that a Busemann convex geodesic space admits a strong uniformly continuous midpoint map. However, there exist spaces which have a strong uniformly continuous midpoint map, but are not Busemann convex.

The next result is a consequence of the aforementioned facts.

Theorem 4.1.4. *A bounded geodesic Ptolemy space with a strong uniformly continuous midpoint map is uniformly convex and we can find a modulus of uniform convexity that does not depend on the radius of the balls.*

4.1.3 Spherical and hyperbolic versions of the Ptolemy inequality

In this subsection we present two analogues of the Ptolemy inequality for the spherical and hyperbolic spaces. These inequalities were studied by Valentine in [75, 76]. Here we are interested in variants of these inequalities for M^2_κ spaces. We prove that these Ptolemy-like inequalities are also satisfied in CAT(κ) spaces and introduce generalized versions of the Busemann convexity. We then raise the question whether one could make use of these notions to give a characterization of CAT(κ) spaces similarly to the one for CAT(0) spaces given in [21]. We do not have an answer to this question, but we include some results that study these notions.

We start by noticing that the sphere in $\mathbb{E}^3$ is not a Ptolemy space. However, Valentine [75] showed that for every $x, y, z, p \in \mathbb{S}^2$,

$$\sin\frac{d(x,z)}{2}\sin\frac{d(y,p)}{2} \leq \sin\frac{d(x,y)}{2}\sin\frac{d(z,p)}{2} + \sin\frac{d(x,p)}{2}\sin\frac{d(y,z)}{2}.$$

Based on this, it is immediate that for every $x, y, z, p \in M^2_\kappa$ with $\kappa > 0$,

$$\sin\frac{d(x,z)\sqrt{\kappa}}{2}\sin\frac{d(y,p)\sqrt{\kappa}}{2} \leq \sin\frac{d(x,y)\sqrt{\kappa}}{2}\sin\frac{d(z,p)\sqrt{\kappa}}{2} + \sin\frac{d(x,p)\sqrt{\kappa}}{2}\sin\frac{d(y,z)\sqrt{\kappa}}{2}. \tag{4.1}$$

A similar result was proved in [76] for hyperbolic spaces using the hyperbolic sine instead of the sine function. Namely, for every $x, y, z, p \in \mathbb{H}^2$,

$$\sinh\frac{d(x,z)}{2}\sinh\frac{d(y,p)}{2} \leq \sinh\frac{d(x,y)}{2}\sinh\frac{d(z,p)}{2} + \sinh\frac{d(x,p)}{2}\sinh\frac{d(y,z)}{2}.$$

Hence, for every $x, y, z, p \in M^2_\kappa$ with $\kappa < 0$,

$$\begin{aligned}\sinh\frac{d(x,z)\sqrt{-\kappa}}{2}\sinh\frac{d(y,p)\sqrt{-\kappa}}{2} \leq{}& \sinh\frac{d(x,y)\sqrt{-\kappa}}{2}\sinh\frac{d(z,p)\sqrt{-\kappa}}{2}\\ &+ \sinh\frac{d(x,p)\sqrt{-\kappa}}{2}\sinh\frac{d(y,z)\sqrt{-\kappa}}{2}.\end{aligned} \tag{4.2}$$

In the sequel we refer to inequalities (4.1) and (4.2) as the *κ-Ptolemy inequality* with the value of κ (less or greater than 0) distinguishing between the two cases. Notice that one could define the 0-Ptolemy inequality as the classical Ptolemy inequality. A *κ-Ptolemy space* is a metric space where the κ-Ptolemy inequality is satisfied. We prove next the following continuity property.

Proposition 4.1.2. *The κ-Ptolemy inequality becomes the classical Ptolemy inequality when κ tends to* 0.

We prove next that CAT(κ) spaces are κ-Ptolemy.

Theorem 4.1.5. *Let X be a CAT(κ) space with* $\operatorname{diam}(X) < \pi/(2\sqrt{\kappa})$ *for $\kappa > 0$. Then X is a κ-Ptolemy space.*

In the following we give some properties of geodesic κ-Ptolemy spaces.

Proposition 4.1.3. *Let X be a geodesic κ-Ptolemy space with $\kappa > 0$ and* $\operatorname{diam}(X) < 2\pi/\sqrt{\kappa}$. *Then, for every $x, y, z \in X$ and m a midpoint of a segment joining x and y we have that*

$$\sin \frac{d(z,m)\sqrt{\kappa}}{2} \leq \frac{1}{2} \frac{1}{\cos \frac{d(x,y)\sqrt{\kappa}}{4}} \left(\sin \frac{d(z,x)\sqrt{\kappa}}{2} + \sin \frac{d(z,y)\sqrt{\kappa}}{2} \right).$$

Proposition 4.1.4. *Let X be a geodesic κ-Ptolemy space with $\kappa < 0$. Then, for every $x, y, z \in X$ and m a midpoint of a segment joining x and y we have that*

$$\sinh \frac{d(z,m)\sqrt{-\kappa}}{2} \leq \frac{1}{2} \frac{1}{\cosh \frac{d(x,y)\sqrt{-\kappa}}{4}} \left(\sinh \frac{d(z,x)\sqrt{-\kappa}}{2} + \sinh \frac{d(z,y)\sqrt{-\kappa}}{2} \right).$$

Remark 4.1.4. The metric of a geodesic κ-Ptolemy space with $\kappa < 0$ is convex.

Theorem 4.1.6. *Let X be a geodesic κ-Ptolemy space with $\kappa < 0$ which admits a continuous midpoint map. Then X is uniquely geodesic.*

We define next generalized versions of the Busemann convexity. Consider a geodesic triangle of side lengths a, b, c and let m be the length of a segment joining midpoints of the sides of lengths a and b respectively.

Definition 4.1.3. *Let X be a geodesic space with* $\operatorname{diam}(X) < \pi/\sqrt{\kappa}$ *for $\kappa > 0$. We say that X is κ-Busemann convex if*

$$\cos\left(m\sqrt{\kappa}\right) \geq \frac{1 + \cos\left(a\sqrt{\kappa}\right) + \cos\left(b\sqrt{\kappa}\right) + \cos\left(c\sqrt{\kappa}\right)}{4 \cos \frac{a\sqrt{\kappa}}{2} \cos \frac{b\sqrt{\kappa}}{2}} \quad \textit{for } \kappa > 0$$

and

$$\cosh\left(m\sqrt{-\kappa}\right) \leq \frac{1 + \cosh\left(a\sqrt{-\kappa}\right) + \cosh\left(b\sqrt{-\kappa}\right) + \cosh\left(c\sqrt{-\kappa}\right)}{4 \cosh \frac{a\sqrt{-\kappa}}{2} \cosh \frac{b\sqrt{-\kappa}}{2}} \quad \textit{for } \kappa < 0,$$

where a, b, c and m are as above.

Notice that one could define the 0-Busemann convexity as the Busemann convexity in the classical sense. From the definition, it is clear that a CAT(κ) space (of diameter $< \pi/\sqrt{\kappa}$ for $\kappa > 0$) is κ-Busemann convex. Also, a κ-Busemann convex space (of diameter $< \pi/\sqrt{\kappa}$ for $\kappa > 0$) is uniquely geodesic. We prove next the following continuity property.

Proposition 4.1.5. *The κ-Busemann convexity becomes the Busemann convexity when κ tends to* 0.

Remark 4.1.5. Let X be a geodesic space that is κ-Busemann convex for $\kappa < 0$. Then X is Busemann convex.

Remark 4.1.6. We raise the question whether one could characterize CAT(κ) spaces using the κ-Ptolemy inequality and the κ-Busemann convexity. More precisely, is a geodesic κ-Ptolemy space which is κ-Busemann convex a CAT(κ) space?

4.2 Fixed points in geodesic Ptolemy spaces

The properties of geodesic Ptolemy spaces established in Subsection 4.1.1, especially Theorem 4.1.2, allow us to prove a large class of fixed point results in this framework. Here we solely mention Kirk's Theorem in geodesic Ptolemy spaces with the remark that many fixed point results whose proofs rely mainly on the uniqueness of asymptotic centers and the convexity of the metric can be transposed into this setting.

Theorem 4.2.1. *Let X be a complete Ptolemy geodesic space with a uniformly continuous midpoint map and $K \in \mathcal{P}_{b,cl,cv}(X)$. Suppose $T : K \to K$ is a nonexpansive mapping. Then* $\mathrm{Fix}(T)$ *is nonempty, closed and convex.*

Chapter 5

Mutually nearest and farthest points of sets and the Drop Theorem in geodesic spaces

Let A and X be nonempty, bounded and closed subsets of a metric space (E, d). The *minimization* (resp. *maximization*) *problem* denoted by $\min(A, X)$ (resp. $\max(A, X)$) consists in finding $(a_0, x_0) \in A \times X$ such that $d(a_0, x_0) = \inf\{d(a, x) : a \in A, x \in X\}$ (resp. $d(a_0, x_0) = \sup\{d(a, x) : a \in A, x \in X\}$). In this chapter we give generic results on the well-posedness of these problems in different geodesic spaces and under different conditions considering the set A fixed. Besides, we analyze the situations when one set or both sets are compact and prove some specific results for CAT(0) spaces. We also prove a variant of the Drop Theorem in Busemann convex geodesic spaces and apply it to obtain an optimization result for convex functions. Most of the results proved here are included in [20].

For $A \in \mathcal{P}_{cl}(E)$ (resp. $A \in \mathcal{P}_{b,cl}(E)$) and $x \in E \setminus A$, the *nearest point problem* (resp. *farthest point problem*) of x to A consists in finding a point $a_0 \in A$ (the solution of the problem) such that $d(x, a_0) = \operatorname{dist}(x, A)$ (resp. $d(x, a_0) = \operatorname{Dist}(x, A)$). Stečkin [73] was one of the first who realized that in case E is a Banach space, geometric properties like strict convexity, uniform convexity, reflexivity and others play an important role in the study of nearest and farthest point problems. His work triggered a series of results so-called "in the spirit of Stečkin" because the ideas he used were adapted again and again by different authors to various contexts (see, for example, [12, 13]). In [73], Stečkin proved, in particular, that for each nonempty and closed subset A of a uniformly convex Banach space, the complement of the set of all points $x \in E$ for which the nearest point problem of x to A has a unique solution is of first Baire category.

In [13], De Blasi, Myjak and Papini studied more general problems than the ones of nearest and farthest points. Namely, they considered the problem of finding two points which minimize (resp. maximize) the distance between two subsets of a Banach space. They focused on the well-posedness of the problem which consists in showing the uniqueness of the solution and that any approximating sequence of the problem must actually converge to the solution (see Section 5.3 for details). The authors proved that if A is a nonempty, bounded and closed subset of a uniformly convex Banach space E, the family of sets in $\mathcal{P}_{b,cl,cv}(E)$ for which the maximization problem, $\max(A, X)$, is well-posed is a dense G_δ-set in the family $\mathcal{P}_{b,cl,cv}(E)$ endowed with the Pompeiu-Hausdorff distance. For the minimization problem, $\min(A, X)$, a similar result is proved where X belongs to a particular subspace of $\mathcal{P}_{b,cl,cv}(E)$. A nice synthesis of issues concerning nearest and farthest point problems in connection with geometric properties of Banach spaces and some extensions of these problems can be found in [9].

Zamfirescu initiated in [78] the investigation of this kind of problems in the context of geodesic spaces. Later on, researchers have focused on adapting the ideas of Stečkin [73] into the geodesic setting. In particular, Zamfirescu [79] proved that, in a complete geodesic space E without bifurcating geodesics, having a fixed compact set A, the set of points $x \in E$ for which the nearest point problem of x to A has a single solution is a set of second Baire category. Motivated by this result, Kaewcharoen and Kirk [36] showed that if E is a complete CAT(0) space with the geodesic extension property and with curvature bounded below globally, for any fixed closed set A, the set of points $x \in E$ for which the nearest point problem of x to A has a unique solution is a set of second Baire category. A similar result was proved for the farthest point problem. Very recent results in the context of spaces with curvature bounded below globally were obtained in [17] where the authors proved some porosity theorems which are stronger results than the ones in [36].

In this chapter we are also concerned with the geometric result known as the Drop Theorem. The original version of this theorem was proved by Daneš [10] and is a very useful tool in nonlinear analysis. Moreover, it is equivalent to the Ekeland Variational Principle and the Flower Petal Theorem [65]. In [25], generalized versions of the Drop Theorem are proved and afterwards used in the proofs of various minimization problems.

The purpose of this chapter is to study in the context of geodesic metric spaces the problem of minimizing (resp. maximizing) the distance between two sets, originally considered by De Blasi, Myjak and Papini in [13] for uniformly convex Banach spaces.

We begin this chapter with a preliminary section that contains some notions and known results that we use in the ensuing sections.

Section 5.2 recalls some existence and well-posedness results for nearest and farthest point problems for both the Banach and the metric setting.

We start Section 5.3 with well-posedness results for minimization and maximization problems in uniformly convex Banach spaces given in [13]. Our contributions are structured into two subsections. The first one begins with a property of the convex hull of the union of a convex set with a point in Busemann convex spaces (Lemma 5.3.1). Most of the results that we prove in this chapter rely on this lemma. Let A be a nonempty, bounded and closed subset of a Busemann convex geodesic space E with curvature bounded below globally and the geodesic extension property. We show that the family of sets in $\mathcal{P}_{b,cl,cv}(E)$, for which $\max(A, X)$ is well-posed, is a dense G_δ-set in $\mathcal{P}_{b,cl,cv}(E)$ (Theorem 5.3.4). A similar result is given for $\min(A, X)$ with no need of the geodesic extension property (Theorem 5.3.3). These results give natural counterparts to those obtained by De Blasi, Myjak and Papini [13] in the context of uniformly convex Banach spaces. We end this subsection by focusing on the case of CAT(0) spaces, where the rich geometry of these spaces is used to relax certain conditions in relation to the well-posedness problem (Proposition 5.3.3). In the second subsection we start by proving that every reflexive Busemann convex space is complete (Lemma 5.3.3). This property is used in the next results where we show that the boundedness condition on the curvature of the space is no longer needed if we impose compactness conditions on the sets. Both minimization and maximization problems (Theorems 5.3.5, 5.3.6) are discussed in this context where we replace the condition on the curvature by that of not having bifurcating geodesics introduced by Zamfirescu in [79]. We also derive a corollary for the well-posedness of the minimization problem in CAT(0) spaces (Corollary 5.3.1).

In Section 5.4 we prove the Drop Theorem in geodesic Busemann convex spaces

(Theorem 5.4.2). Then we use this theorem to study an optimization problem for convex and continuous real-valued functions defined on geodesic spaces (Theorem 5.4.3). This result is significant since one can use it to prove best approximation results as simple consequences thereof (for example Corollary 5.4.1).

5.1 Preliminaries

In this section we recall some notions and results that are used in this chapter and were not needed until this point.

5.2 Nearest and farthest point problems

We give in this section some existence and well-posedness results for nearest and farthest point problems (see [17, 12, 36, 73, 79]).

5.3 Minimization and maximization problems between two sets

In [13], De Blasi, Myjak and Papini studied the problem of finding two points which minimize (resp. maximize) the distance between two subsets of a Banach space. Although the next notions and the proposition below were originally given in the setting of Banach spaces they can be also introduced in the framework of geodesic metric spaces (or even general metric spaces). In this section, if nothing else is mentioned, E denotes a complete geodesic metric space. Following [13], for $X, Y \in \mathcal{P}_{b,cl}(E)$ and $\sigma > 0$, we set

$$\lambda_{XY} = \inf\{d(x,y) : x \in X, y \in Y\}, \quad \mu_{XY} = \sup\{d(x,y) : x \in X, y \in Y\},$$

$$L_{XY}(\sigma) = \{x \in X : \operatorname{dist}(x,Y) \le \lambda_{XY} + \sigma\},$$
$$M_{XY}(\sigma) = \{x \in X : \operatorname{Dist}(x,Y) \ge \mu_{XY} - \sigma\}.$$

The *minimization* (resp. *maximization*) *problem* denoted by $\min(X,Y)$ (resp. $\max(X,Y)$) consists in finding $(x_0,y_0) \in X \times Y$ (the *solution* of the problem) such that $d(x_0,y_0) = \lambda_{XY}$ (resp. $d(x_0,y_0) = \mu_{XY}$). A sequence (x_n,y_n) in $X \times Y$ such that $d(x_n,y_n) \to \lambda_{XY}$ (resp. $d(x_n,y_n) \to \mu_{XY}$) is called a *minimizing* (resp. *maximizing*) *sequence*. The problem $\min(X,Y)$ (resp. $\max(X,Y)$) is said to be *well-posed* if it has a unique solution $(x_0,y_0) \in X \times Y$ and for every minimizing (resp. maximizing) sequence (x_n,y_n) we have $x_n \to x_0$ and $y_n \to y_0$. In the following we give a characterization of the well-posedness of $\min(X,Y)$ (resp. $\max(X,Y)$).

Proposition 5.3.1 (De Blasi, Myjak, Papini [13]). *Let (E,d) be a complete geodesic metric space and $X, Y \in \mathcal{P}_{b,cl}(E)$. The problem $\min(X,Y)$ (resp. $\max(X,Y)$) is well-posed if and only if*

$$\inf_{\sigma>0} \operatorname{diam}(L_{XY}(\sigma)) = 0 \quad \textit{and} \quad \inf_{\sigma>0} \operatorname{diam}(L_{YX}(\sigma)) = 0,$$

$$(\textit{resp.} \inf_{\sigma>0} \operatorname{diam}(M_{XY}(\sigma)) = 0 \quad \textit{and} \quad \inf_{\sigma>0} \operatorname{diam}(M_{YX}(\sigma)) = 0).$$

In order to state the results proved in [13], consider E a uniformly convex Banach space, $A \in \mathcal{P}_{b,cl}(E)$ and denote

$$\mathcal{P}^A_{b,cl,cv}(E) = \overline{\{X \in \mathcal{P}_{b,cl,cv}(E) : \lambda_{AX} > 0\}}.$$

Then, $(\mathcal{P}^A_{b,cl,cv}(E), H)$ is a complete metric space. The following minimization and maximization results are given in [13].

Theorem 5.3.1 (De Blasi, Myjak, Papini [13]). *Let E be a uniformly convex Banach space and $A \in \mathcal{P}_{b,cl}(E)$. Then,*

$$\{X \in \mathcal{P}^A_{b,cl,cv}(E) : \min(A, X) \text{ is well-posed}\}$$

is a dense G_δ-set in $\mathcal{P}^A_{b,cl,cv}(E)$.

Theorem 5.3.2 (De Blasi, Myjak, Papini [13]). *Let E be a uniformly convex Banach space and $A \in \mathcal{P}_{b,cl}(E)$. Then,*

$$\{X \in \mathcal{P}_{b,cl,cv}(E) : \max(A, X) \text{ is well-posed}\}$$

is a dense G_δ-set in $\mathcal{P}_{b,cl,cv}(E)$.

In the next two subsections we study minimization and maximization problems between sets in particular geodesic metric spaces.

5.3.1 Results in Busemann convex spaces with curvature bounded below globally

We begin this subsection by giving an estimation for $\operatorname{dist}(y, X)$, where $X \in \mathcal{P}_{b,cv}(E)$, $x' \in E$ such that $\operatorname{dist}(x', X) > 0$ and $y \in \overline{\operatorname{co}}\,(X \cup \{x'\})$. It is easy to see that in a Busemann convex geodesic metric space, $\operatorname{dist}(y, X) < \operatorname{dist}(x', X)$ for every $y \in \operatorname{co}(X \cup \{x'\})$ with $y \neq x'$. We sharpen this upper bound in the following way.

Lemma 5.3.1. *Let E be a Busemann convex metric space and $X \in \mathcal{P}_{b,cv}(E)$. Suppose $x' \in E$ such that $\operatorname{dist}(x', X) > 0$. Then, for every $y \in \overline{\operatorname{co}}\,(X \cup \{x'\})$,*

$$\operatorname{dist}(y, X) \leq \operatorname{dist}(x', X) - \frac{\operatorname{dist}(x', X)}{\operatorname{dist}(x', X) + \operatorname{diam}(X)} d(x', y). \tag{5.1}$$

We give next a property of Banach spaces which was used in [13] to prove minimization and maximization problems between two sets in Banach spaces.

Proposition 5.3.2 (De Blasi, Myjak, Papini [13]). *Let E be a Banach space, $X \in \mathcal{P}_{b,cl,cv}(E)$ and $\epsilon, r > 0$. Then there exists $0 < \tau_0 < r$ such that for every $u \in E$ with $\operatorname{dist}(u, X) \geq r$ and for every $0 < \tau \leq \tau_0$ we have*

$$\operatorname{diam}(C_{X,u}(\tau)) < \epsilon,$$

where

$$C_{X,u}(\tau) = [\overline{\operatorname{co}}(X \cup \{u\})] \setminus [X + (\operatorname{dist}(u, X) - \tau)B(0, 1)].$$

The following lemma is an analogue in the metric setting of the above proposition.

Lemma 5.3.2. *Let E be a Busemann convex metric space and $X \in \mathcal{P}_{b,cv}(E)$. For $r > 0$, $x' \in E$ with $\operatorname{dist}(x', X) \geq r$ and $n \in \mathbb{N}$ with $1/n < r$ define*

$$C_n = \overline{\operatorname{co}}\,(X \cup \{x'\}) \setminus \bigcup_{x \in X} B(x, \operatorname{dist}(x', X) - 1/n).$$

Then, the sequence $(\operatorname{diam}(C_n))$ converges to 0 uniformly with respect to $x' \in E$ such that $\operatorname{dist}(x', X) \geq r$.

In order to state our main results, we introduce the following notations. Let $A \in \mathcal{P}_{b,cl}(E)$ be fixed. Then, we denote $\lambda_X = \lambda_{XA}$ and $\mu_X = \mu_{XA}$ for $X \in \mathcal{P}_{b,cl}(E)$. Following [13], set

$$\mathcal{P}^A_{b,cl,cv}(E) = \overline{\{X \in \mathcal{P}_{b,cl,cv}(E) : \lambda_X > 0\}}.$$

Endowed with the Pompeiu-Hausdorff distance, $\mathcal{P}^A_{b,cl,cv}(E)$ is a complete metric space if E is Busemann convex.

We prove next the two main results of this subsection, which are counterparts in the geodesic case of Theorems 5.3.1 and 5.3.2 respectively.

Theorem 5.3.3. *Let E be a complete Busemann convex metric space with curvature bounded below globally by $\kappa < 0$. Suppose $A \in \mathcal{P}_{b,cl}(E)$. Then,*

$$\mathcal{W}_{min} = \left\{X \in \mathcal{P}^A_{b,cl,cv}(E) : \min(A, X) \text{ is well-posed}\right\}$$

is a dense G_δ-set in $\mathcal{P}^A_{b,cl,cv}(E)$.

Theorem 5.3.4. *Let E be a complete Busemann convex metric space with the geodesic extension property and curvature bounded below globally by $\kappa < 0$. Suppose $A \in \mathcal{P}_{b,cl}(E)$. Then,*

$$\mathcal{W}_{max} = \{X \in \mathcal{P}_{b,cl,cv}(E) : \max(A, X) \text{ is well-posed}\}$$

is a dense G_δ-set in $\mathcal{P}_{b,cl,cv}(E)$.

We conclude this subsection by giving a characterization of the well-posedness of the minimization problem $\min(X, Y)$ in complete CAT(0) spaces. We prove that in the following particular context, the conditions in Proposition 5.3.1 can be relaxed.

Proposition 5.3.3. *Let E be a complete CAT(0) space, $X \in \mathcal{P}_{b,cl,cv}(E)$ and $Y \in \mathcal{P}_{b,cl}(E)$. The problem $\min(X, Y)$ is well-posed if and only if*

$$\inf_{\sigma > 0} \operatorname{diam}\left(L_{YX}(\sigma)\right) = 0.$$

5.3.2 Results involving compactness

In this subsection we study the same problems but we modify conditions we imposed in our results. More particularly, we focus on the situation in which the set A is compact. We show that under this stronger assumption on the set we can weaken the condition on the geodesic space from being of curvature bounded below globally to not having bifurcating geodesics. However, in the first theorem we need to add the reflexivity condition on the space. Before stating the theorem we give the following property of reflexive Busemann convex geodesic spaces.

Lemma 5.3.3. *Let (E, d) be a reflexive Busemann convex metric space. Then E is complete.*

Theorem 5.3.5. *Let E be a reflexive Busemann convex metric space with no bifurcating geodesics. Suppose $A \in \mathcal{P}_{cp}(E)$. Then,*

$$\mathcal{W}_{min} = \left\{X \in \mathcal{P}^A_{b,cl,cv}(E) : \min(A, X) \text{ is well-posed}\right\}$$

is a dense G_δ-set in $\mathcal{P}^A_{b,cl,cv}(E)$.

Corollary 5.3.1. *Let E be a complete $CAT(0)$ space with no bifurcating geodesics. Suppose $A \in \mathcal{P}_{cp}(E)$. Then,*

$$\mathcal{W}_{min} = \left\{X \in \mathcal{P}^A_{b,cl,cv}(E) : \min(A, X) \text{ is well-posed}\right\}$$

is a dense G_δ-set in $\mathcal{P}^A_{b,cl,cv}(E)$.

Remark 5.3.1. The proof of Theorem 5.3.5 relies on the fact that $\min(A, X)$ always has a solution. In fact, the reflexivity of the space is mainly used to ensure this condition. Therefore, it is natural to ask whether it is possible to drop the condition that the problem has a solution.

Next we focus on the maximization problem for A compact. In order to follow the same line of argument as in the previous result we need the fact that the problem $\max(A, X)$ has a solution. However, in [70], it is proved that in a reflexive Banach space, the remotal distance from a point to a bounded, closed and convex set is guaranteed to be reached if and only if the space is finite dimensional. This is why it is natural to impose the compactness condition on the set X in our next result.

Theorem 5.3.6. *Let E be a complete geodesic space with no bifurcating geodesics and the geodesic extension property. Suppose $A \in \mathcal{P}_{cp}(E)$. Then,*

$$\mathcal{W}_{max} = \{X \in \mathcal{P}_{cp}(E) : \max(A, X) \text{ is well-posed}\}$$

is a dense G_δ-set in $\mathcal{P}_{cp}(E)$.

Remark 5.3.2. Regarding the problem $\max(A, X)$, where the fixed set A is compact, we raise the following question: is

$$\mathcal{W}_{max} = \{X \in \mathcal{P}_{cp,cv}(E) : \max(A, X) \text{ is well-posed}\}$$

a dense G_δ-set in $\mathcal{P}_{cp,cv}(E)$? The Hopf-Rinow Theorem (see [5, Chapter I.3, Proposition 3.7]) states that if E is complete and locally compact, then it is proper. Hence, if the space is additionally locally compact and Busemann convex then we can answer the question in the positive.

5.4 The Drop Theorem in Busemann convex spaces

In [10], Daneš proved the following geometric result known as the Drop Theorem.

Theorem 5.4.1 (Drop Theorem). *Let $(E, \|\cdot\|)$ be a Banach space and $A \in \mathcal{P}_{cl}(E)$ be such that* $\inf\{\|x\| : x \in A\} > 1$. *Then there exists $a \in A$ such that*

$$\operatorname{co}(B(0, 1) \cup \{a\}) \cap A = \{a\}.$$

The name of this theorem has its origin in the fact that the set $\operatorname{co}(B(0,1) \cup \{a\})$ was called a *drop*. Equivalences of this result or of its generalized versions with other fundamental theorems in nonlinear analysis and various areas of their applications are discussed, for instance, in [25, 65].

In this section we prove a variant of the Drop Theorem in the setting of Busemann convex metric spaces.

Theorem 5.4.2. *Let (E, d) be a complete Busemann convex metric space and let $A \in \mathcal{P}_{cl}(E)$ and $B \in \mathcal{P}_{b,cl,cv}(E)$ be such that $\lambda_{AB} > 0$. Suppose $\epsilon > 0$. Then there exists $a \in A$ such that*

(i) $\operatorname{dist}(a, B) < \lambda_{AB} + \epsilon$;

(ii) $\overline{\operatorname{co}}\,(B \cup \{a\}) \cap A = \{a\}$;

(iii) $x_n \to a$ *for every sequence* (x_n) *in* $\overline{\operatorname{co}}\,(B \cup \{a\})$ *with* $\operatorname{dist}(x_n, A) \to 0$.

As an application of this version of the Drop Theorem we obtain an analogue of an optimization result proved by Georgiev [25, Theorem 4.2] in the context of Banach spaces. In order to state this result we need to briefly introduce some notions which can also be found in [25].

Let (E, d) be a complete metric space, $f : E \to \mathbb{R}$ a lower semi-continuous function which is bounded below, and $A \in \mathcal{P}_{b,cl}(E)$. The *minimization problem* denoted by $\min(A, f)$ consists in finding $x_0 \in A$ (the solution of the problem) such that $f(x_0) = \inf\{f(x) : x \in A\}$.
For $\sigma > 0$, let

$$L_{A,f}(\sigma) = \left\{ x \in E : f(x) \leq \inf_{y \in A} f(y) + \sigma \text{ and } \operatorname{dist}(x, A) \leq \sigma \right\}.$$

The problem $\min(A, f)$ is *well-posed in the sense of Levitin-Polyak* (see [55, 68]) if

$$\inf_{\sigma > 0} \operatorname{diam}\left(L_{A,f}(\sigma)\right) = 0.$$

This is equivalent to requesting that it has a unique solution $x_0 \in A$ and every sequence (x_n) in E converges to x_0 provided $f(x_n) \to f(x_0)$ and $\operatorname{dist}(x_n, A) \to 0$.

The following lemma is the counterpart of [25, Lemma 4.1] for geodesic metric spaces.

Lemma 5.4.1. *Let E be a geodesic space, $X \in \mathcal{P}_b(E)$ and $f : E \to \mathbb{R}$ continuous and convex. For $c \in \mathbb{R}$, let $A = \{x \in E : f(x) \leq c\}$. Suppose there exists $z \in E$ such that $f(z) < c$. Then for every $\epsilon > 0$ there exists $\delta > 0$ such that* $\operatorname{dist}(x, A) < \epsilon$ *for each $x \in X$ with $f(x) < c + \delta$.*

We prove next the optimization result.

Theorem 5.4.3. *Let E be a complete Busemann convex metric space and let $f : E \to \mathbb{R}$ be continuous, convex, bounded below on bounded sets and satisfying one of the following conditions:*

(i) $\inf_{x \in E} f(x) = -\infty$;

(ii) *there exists $z_0 \in E$ such that $f(z_0) = \inf_{x \in E} f(x)$ and every sequence (x_n) in E converges to z_0 if $f(x_n) \to f(z_0)$.*

Then,

$$\mathcal{W}_{min} = \{X \in \mathcal{P}_{b,cl,cv}(E) : \min(X, f) \text{ is well-posed in the sense of Levitin-Polyak}\}$$

is a dense G_δ-set in $\mathcal{P}_{b,cl,cv}(E)$.

Theorem 5.4.3 is not only interesting by itself, but it is also important because several best approximation results follow as simple consequences thereof. We finish our exposition by deriving such a consequence which is, in fact, an extension of a result proved in [11].

Corollary 5.4.1. *Let E be a complete Busemann convex metric space and suppose $y \in E$. Then,*

$$\mathcal{W}_{min} = \{X \in \mathcal{P}_{b,cl,cv}(E) : \min(y, X) \text{ is well-posed}\}$$

is a dense G_δ-set in $\mathcal{P}_{b,cl,cv}(E)$.

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