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DUALITIES AND EQUIVALENCES INDUCED BY ADJOINT FUNCTORS

Ph.D. Thesis Summary

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Contents

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Introduction

A history of the study of equivalences and dualities induced by pairs of adjoint functors, as an important topic in Module Theory, has its starting point in the 1950s. Back then, Morita [38] and Azumaya [6] proved some important results which generalize some classical properties of modules over rings of matrices over fields, respectively the classical duality theorem for vector spaces. Their results characterized:

(1) an equivalence between two categories of right (or left) modules over two rings as being represented by the covariant Hom and tensor functors, induced by a balanced bimodule that is a progenerator on either side, and

(2) a duality between some subcategories of right and left modules over two rings as being represented by the contravariant Hom functors induced by a balanced bimodule that is an injective cogenerator on both sides.

The study of equivalences and dualities developed important concepts in Module Theory, such as *tilting module* (introduced by Brenner and Butler [17]), *star module* (introduced by Menini and Orsatti [36]), respectively *cotilting module* (introduced by Colby [20] and Happel [31]) and *costar module* (introduced by Colby and Fuller [22]). For complete surveys on the subjects, we refer to the books [23] and [51] and the papers [21], [25], [52], [53] and [54]. All these mentioned notions are used by many authors to generalize clasical results proved by Morita and Azumaya.

This kind of study is also useful for a more general setting, in order to apply these results to other kind of categories. For instance, Castaño-Iglesias, Gómez-Torrecillas and Wisbauer applied the study of adjoint pairs of functors between Grothendieck categories to special categories of graded modules or comodules [19]. Marcus and Modoi [35] also used other kind of equivalences in order to study categories of graded modules. Colpi [24], Gregorio [30] and Rump [47] constructed a general theory of tilting objects in various kind of categories. Recently, Bazzoni [7] considers some particular categories of fractions (which, in general, have no infinite direct sums) in order to describe the classes involved in a tilting theorem [7, Theorem 4.5], while Breaz [8], [10] studied functors and equivalences between similar categories of fractions in order to apply these results to the category of abelian groups and quasi-homomorphisms [1].

However, starting with a pair of adjoint functors between some abelian categories, in particular Grothendieck categories, we can construct other useful pairs of adjoint functors. It would be also nice to know if concepts developed for module categories work in this case. For example, Castaño-Iglesias, Gómez-Torrecillas and Wisbauer [19] have extended the study of equivalences induced by covariant Hom and tensor functors to Grothendieck categories and Castaño-Iglesias [18] proved that the notion of costar module can be also extended to Grothendieck categories.

Following this line, in this thesis we will extend notions and we will generalizes some results from module categories to general abelian categories, starting from a pair $\mathrm{F} : \mathcal{A} \rightleftarrows \mathcal{B} : \mathrm{G}$ of (adjoint) functors between abelian categories. More precisely, on one hand, if the considered functors are contravariant and right adjoint, we extend the study of dualities induced by contravariant Hom functors and, on the other hand, if the considered functors are covariant such that G is left adjoint for F, we extend the study of equivalences induced by covariant Hom and tensor functors.

The paper is divided in two chapters, each chapter containing more sections, as we will present in the following:

Chapter 1. Dualities Induced by Adjoint Functors. This chapter, dedicated to the study of dualities, consists in five sections, as follows:

1.1 Introduction, in which we present the framework and give examples of pairs of additive contravariant adjoint functors.

1.2 Preliminaries, which is dedicated to the presentation of the basic notions and basic results, used throughout this chapter. For example, basic properties related to the class $\mathrm{add}(X)$, for some object X, are proved and some characterizations of reflexive terms of short exact sequences are given. Most of these results can be found in [15], [16] and [41].

1.3 Costar Objects. Finitistic-1-F-cotilting, in which we first characterize the situation when F is U-w-π_f-exact through a duality between some full subcategories of $\mathcal{A}$ and $\mathcal{B}$. Secondly, we introduce a new version of the notion of *costar object*, similar to the one introduced by Colby and Fuller in module categories and we also prove a result that characterizes this notion. Finally, we present two other results, one of them inspired by the Wisbauer's paper [54] and the other result is a generalization of [12, Theorem 2.8] to abelian categories. Except for the last result, which is proved in [16], all the other results of this section are given and proved in [15].

1.4 Dominant Resolutions, in which we introduce the notions of *dominant resolutions* and we give a general theorem for abelian categories which exhibits some dualities induced by a pair of right adjoint contravariant functors. We will also use this result to generalize some known dualities obtained by Wakamatsu [50] and by Breaz [12]. All the same, it is introduced the notion of *finitistic-n-F-cotilting object* and it is characterized this notion in terms of a duality. All the results from this section are published in [41].

1.5 The U-coplex Category, in which is defined the notion of U-coplex, for a reflexive object U, and it is also defined the category of U-coplexes. Then, starting from the given pair of functors $\mathrm{F} : \mathcal{A} \rightleftarrows \mathcal{B} : \mathrm{G}$, we define a new pair of functors (F^U, G^U) and we will show that this new pair of functors induce a duality. This duality is a generalization of the duality given by Faticoni [27].

Chapter 2. Equivalences Induced by Adjoint Functors. This chapter, focused on the study of equivalences induced by a pair of additive covariant adjoint functors, is structured as follows:

2.1 Introduction, in which we present the framework and give some examples of pairs of additive and covariant functors, which are adjoint, between some categories.

2.2 Preliminaries, in which basic notions and results are presented in order to be used in this chapter. We refer here to [42].

2.3 Closure Properties with Respect to θ-Faithful Factors, in which we are interested about the closure properties of some full subcategories $\overline{\mathcal{A}}$ and $\overline{\mathcal{B}}$ such that the restrictions $\mathrm{F} : \overline{\mathcal{A}} \rightleftarrows \overline{\mathcal{B}} : \mathrm{G}$ are equivalences. The first important result is Proposition 2.3.3, where we characterize the situation when $\overline{\mathcal{B}}$ is closed with respect to faithful factors by a closure property of $\overline{\mathcal{A}}$ and by an exactness property of F. The main result of the first part of this section is Theorem 2.3.4, where we characterize the the situation when $\mathrm{F} : \overline{\mathcal{A}} \rightleftarrows \overline{\mathcal{B}} : \mathrm{G}$ is an equivalence with the class $\overline{\mathcal{B}}$ closed under θ-faithful factors. Then this result is applied for closure properties of some classes, constructed starting with the class $\mathrm{add}(V)$, where $V = \mathrm{F}(U)$, for some static object U. Next, we continue and developed this study setting a new condition for $\overline{\mathcal{A}}$.

We obtain new versions of the results presented above, the most important of them is Theorem 2.3.14, and then we will apply these new results to the particular class add(V). All the results presented here can be found in [42].

Finally, I would like to mention that, for categories theory we refer to the books [32], [37], [44], [45], for module theory we refer to the books [4], [46], [48]. We also refer, for theory of graded modules, to [34] and [40]. Another books which are useful are [11], [13].

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1. Dualities Induced by Adjoint Functors

1.1. **Introduction.** Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and let $\mathrm{F} : \mathcal{A} \rightleftarrows \mathcal{B} : \mathrm{G}$ be a pair of additive and contravariant functors which are adjoint on the right, i.e. there are natural isomorphisms

$$\eta_{X,Y} : \mathrm{Hom}_{\mathcal{A}}(X, \mathrm{G}(Y)) \to \mathrm{Hom}_{\mathcal{B}}(Y, \mathrm{F}(X)),$$

for all $X \in \mathcal{A}$ and for all $Y \in \mathcal{B}$. The natural transformations associated to the right adjunction $\eta_{X,Y}$ are defined as follows:

$$\delta : 1_{\mathcal{A}} \to \mathrm{GF},\ \ \delta_X = \eta^{-1}_{X,\mathrm{F}(X)}(1_{\mathrm{F}(X)}) \text{ and } \zeta : 1_{\mathcal{B}} \to \mathrm{FG},\ \ \zeta_Y = \eta^{-1}_{\mathrm{G}(Y),Y}(1_{\mathrm{G}(Y)}).$$

An object X is called δ-*faithful* (respectively, ζ-*faithful*) if δ_X (respectively, ζ_X) is a monomorphism and we will denote by Faith_δ (respectively, Faith_ζ) the class of all δ-faithful (respectively, ζ-faithful) objects. We mention that some authors use the term *torsionless* instead of *faithful*. An object X is called δ-*reflexive* (respectively, ζ-*reflexive*) if δ_X (respectively, ζ_X) is an isomorphism and we will denote by Refl_δ (respectively, Refl_ζ) the class of all δ-reflexive (respectively, ζ-reflexive) objects.

The typical example of such functors is the following:

Example 1.1.1. Let R and S be unital associative rings and let Q be a (S, R)-bimodule. If we denote by Mod-R (respectively, by S-Mod) the category of all right R- (respectively, left S-) modules, then the contravariant Hom functors

$$\Delta = \mathrm{Hom}_R(-, Q) : \text{Mod-}R \rightleftarrows S\text{-Mod} : \mathrm{Hom}_S(-, Q) = \Delta'$$

are right adjoint. Both natural transformations δ and ζ represent the evaluation maps

$$X \longrightarrow \mathrm{Hom}(\mathrm{Hom}(X, Q), Q)$$

defined by

$$x \mapsto (f \mapsto f(x)).$$

Moreover, if S is the endomorphism ring of Q, namely $S = \mathrm{End}_R(Q)$, then Q is δ-reflexive and S is ζ-reflexive under the pair (Δ, Δ'). □

Another important example was exhibited by Castaño-Iglesias in [18].

Example 1.1.2. Let G be a group. If $R = \bigoplus_{x\in G} R_x$ and $S = \bigoplus_{x\in G} {}_xS$ are two G-graded unital rings, we will denote by $\mathrm{Mod}_{\mathrm{gr}}$-$R$ (respectively, by S-$\mathrm{Mod}_{\mathrm{gr}}$) the category of all G-graded unital right R- (respectively, left S-) modules. If $M, N \in \mathrm{Mod}_{\mathrm{gr}}$-$R$ we consider the G-graded abelian group $\mathrm{HOM}_R(M,N)$ whose homogeneous component at x is the subgroup of $\mathrm{Hom}_R(M,N)$ consisting of all R-homomorphisms $f : M \to N$ such that $f(M_y) \subseteq N_{xy}$ for all $y \in G$. We note that $\mathrm{HOM}_R(M,M) = \mathrm{END}_R(M)$ has a canonical structure of G-graded unital ring. If $M, N \in S$-$\mathrm{Mod}_{\mathrm{gr}}$, we consider the G-graded abelian group $\mathrm{HOM}_S(M,N)$ whose homogeneous component at x is the subgroup of $\mathrm{Hom}_S(M,N)$ consisting of all S-homomorphisms $f : M \to N$ such that $f({}_yM) \subseteq {}_{yx}N$ for all $y \in G$. For more properties of the categories of G-graded modules we refer [40].

If $Q \in \mathrm{Mod}_{\mathrm{gr}}$-$R$ and $S = \mathrm{END}_R(Q)$, then

$$\mathrm{H}_R^{\mathrm{gr}} = \mathrm{HOM}_R(-, Q_R) : \mathrm{Mod}_{\mathrm{gr}}\text{-}R \rightleftarrows S\text{-}\mathrm{Mod}_{\mathrm{gr}} : \mathrm{HOM}_S(-, {}_SQ) = {}_S\mathrm{H}^{\mathrm{gr}}$$

is a pair of right adjoint contravariant functors. □

Năstăsescu and Torrecillas give another example (see [39]).

Example 1.1.3. Let C be a coalgebra over a field k. We denote by M^C (respectively, ${}^C\mathrm{M}$) the (Grothendieck) category of right (respectively, left) comodules over C. The dual space $C^* = \mathrm{Hom}_k(C,k)$ is endowed with a canonical algebra structure. We note that the category M^C (respectively, ${}^C\mathrm{M}$) is isomorphic to a closed subcategory of the category C^*-Mod (respectively, Mod-C^*) of all left (respectively, right) modules over C^*. More exactly, this full subcategory is the category of rational left (respectively, right) C^*-modules which is denoted by $\mathrm{Rat}(C^*\text{-Mod})$ (respectively, $\mathrm{Rat}(\text{Mod-}C^*)$). For this reason, we can identify these categories. If we denote by Rat the rational functors, i.e.

$$\mathrm{Rat} : C^*\text{-Mod} \to \mathrm{M}^C \text{ and } \mathrm{Rat} : \text{Mod-}C^* \to {}^C\mathrm{M},$$

then

$$\mathrm{Rat} \circ (-)^* : \mathrm{M}^C \rightleftarrows {}^C\mathrm{M} : \mathrm{Rat} \circ (-)^*$$

is a pair of right adjoint contravariant functors, where

$$(-)^* = \mathrm{Hom}_k(-,k) : \text{Mod-}C^* \rightleftarrows C^*\text{-Mod} : \mathrm{Hom}_k(-,k) = (-)^*.$$

□

Since our results works in general abelian categories (without infinite direct sums or products), let us recall here another example taken from [8] and [10].

Example 1.1.4. Let R be a ring and Σ be a multiplicatively closed set of non-zero integers. We consider the class $\mathcal{S}$ of all right R-modules B which are Σ-bounded as abelian groups (i.e. there is $n \in \Sigma$ such that $nB = 0$). This is a (complete) Serre class. Hence the quotient (abelian) category Mod-$R/\mathcal{S}$ exists and it is equivalent to the category $\mathbb{Z}[\Sigma^{-1}]$Mod-R which has as objects all the right R-modules and if $M, N \in$ Mod-R, then $\mathrm{Hom}_{\mathbb{Z}[\Sigma^{-1}]\text{Mod-}R}(M, N) = \mathbb{Z}[\Sigma^{-1}] \otimes_{\mathbb{Z}} \mathrm{Hom}_R(M, N)$. We refer to [14] for basic properties of this category. We will denote by $\mathbf{q} : \text{Mod-}R \to \mathbb{Z}[\Sigma^{-1}]\text{Mod-}R$ the canonical functor. Note that $\mathbf{q}(M) = M$ for any $M \in$ Mod-R and $\mathbf{q}(f) = 1 \otimes f$ for all R-homomorphisms f.

Using [29, Corollaire 3.2], we observe that if $F : \text{Mod-}R \to \text{Mod-}S$ is an additive functor, then it induces a canonical functor $qF : \mathbb{Z}[\Sigma^{-1}]\text{Mod-}R \to \mathbb{Z}[\Sigma^{-1}]\text{Mod-}S$ such that $\mathbf{q}F = (qF)\mathbf{q}$ (here $\mathbf{q}$ denotes both canonical functors $\text{Mod-}R \to \mathbb{Z}[\Sigma^{-1}]\text{Mod-}R$ and $\text{Mod-}S \to \mathbb{Z}[\Sigma^{-1}]\text{Mod-}S$). In [10], the author proved a version of Morita's theorem for some equivalences between these categories.

Starting with the setting presented in Example 1.1.1 we have that

$$q\Delta : \mathbb{Z}[\Sigma^{-1}]\text{Mod-}R \rightleftarrows \mathbb{Z}[\Sigma^{-1}]S\text{-Mod} : q\Delta'$$

is a pair of right adjoint contravariant functors. □

We note that the contravariant functors F and G are left exact. Moreover, the natural transformations of right adjunctions, δ and ζ, associated to the considered pair satisfy the identities

$$\mathrm{F}(\delta_X) \circ \zeta_{\mathrm{F}(X)} = 1_{\mathrm{F}(X)} \text{ for all } X \in \mathcal{A}$$

and

$$\mathrm{G}(\zeta_Y) \circ \delta_{\mathrm{G}(Y)} = 1_{\mathrm{G}(Y)} \text{ for all } Y \in \mathcal{B}.$$

Furthermore, the restrictions of F and G to the classes of reflexive objects induce a duality $\mathrm{F} : \mathrm{Refl}_\delta \rightleftarrows \mathrm{Refl}_\zeta : \mathrm{G}$. Moreover, if $\mathrm{F} : \mathcal{A} \rightleftarrows \mathcal{B} : \mathrm{G}$ is a duality then $\mathcal{A} \subseteq \mathrm{Refl}_\delta$ and $\mathcal{B} \subseteq \mathrm{Refl}_\zeta$ ([51, Theorem 47.11]).

Recall that add(X) denotes the class of all direct summands of finite direct sums of copies of X. We denote by Proj$(\mathcal{A})$ the class of all projective objects in $\mathcal{A}$. Throughout this chapter we assume that all considered subcategories are isomorphically closed.

1.2. **Preliminaries.** Throughout this section, we consider a pair of additive and contravariant functors $\mathrm{F} : \mathcal{A} \rightleftarrows \mathcal{B} : \mathrm{G}$ which are adjoint on the right, between abelian categories. All the same, throughout this section, let U be a δ-reflexive object with $\mathrm{F}(U) = V$.

Lemma 1.2.1. *The following assertion hold:*

(a) *V is ζ-reflexive;*
(b) $\mathrm{add}(U) \subseteq \mathrm{Refl}_\delta$ *and* $\mathrm{add}(V) \subseteq \mathrm{Refl}_\zeta$*;*
(c) $\mathrm{F}(\mathrm{add}(U)) = \mathrm{add}(V)$ *and* $\mathrm{G}(\mathrm{add}(V)) = \mathrm{add}(U)$*;*
(d) *If V is a projective object in $\mathcal{B}$ then* $\mathrm{add}(V) \subseteq \mathrm{Proj}(\mathcal{B})$ *(here is not necessarily for U to be δ-reflexive).*

Remark 1.2.2. By Lemma 1.2.1, we have $\mathrm{F}(\mathrm{add}(U)) = \mathrm{add}(V)$, $\mathrm{add}(U) \subseteq \mathrm{Refl}_\delta$ and $\mathrm{G}(\mathrm{add}(V)) = \mathrm{add}(U)$, $\mathrm{add}(V) \subseteq \mathrm{Refl}_\zeta$. It follows that

$$\mathrm{F} : \mathrm{add}(U) \rightleftarrows \mathrm{add}(V) : \mathrm{G}$$

is a duality. The natural isomorphisms corresponding to this duality are:

- $\delta : 1_{\mathrm{add}(U)} \to \mathrm{GF}$, i.e. the restriction of $\delta : 1_\mathcal{A} \to \mathrm{GF}$ to the class $\mathrm{add}(U)$;
- $\zeta : 1_{\mathrm{add}(V)} \to \mathrm{FG}$, i.e. the restriction of $\zeta : 1_\mathcal{B} \to \mathrm{FG}$ to the class $\mathrm{add}(V)$.

Lemma 1.2.3. *The following statements hold:*

(a) $\mathrm{F}(\mathcal{A}) \subseteq \mathrm{Faith}_\zeta$ *and* $\mathrm{G}(\mathcal{B}) \subseteq \mathrm{Faith}_\delta$*;*
(b) *The classes* Faith_δ *and* Faith_ζ *are closed with respect to subobjects.*

Lemma 1.2.4. *If* $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ *is an exact sequence in $\mathcal{A}$ then the unique morphism α, for which the following diagram with exact rows*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\delta_Y} & & \downarrow{\scriptstyle\delta_Z} & & \\
0 & \longrightarrow & \mathrm{G}(\mathrm{Im}\mathrm{F}(f)) & \xrightarrow{\mathrm{G}(\pi)} & \mathrm{GF}(Y) & \xrightarrow{\mathrm{GF}(g)} & \mathrm{GF}(Z) & &
\end{array}$$

is commutative, is given by the formula $\alpha = \mathrm{G}(\sigma) \circ \delta_X$*, where* $\mathrm{F}(f) = \sigma \circ \pi$ *is the canonical decomposition.*

Now we will display some characterizations of reflexive terms of short exact sequences. In the case of category of modules, these lemmas could be found in [28].

Lemma 1.2.5. *Let* $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ *be an exact sequence with* $Y \in \mathrm{Refl}_\delta$ *and* $\mathrm{F}(f)$ *an epimorphism. Then* $X \in \mathrm{Refl}_\delta$ *if and only if* $Z \in \mathrm{Faith}_\delta$*.*

Lemma 1.2.6. *Let* $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ *be an exact sequence with* $Y \in \mathrm{Refl}_\delta$ *and* $Z \in \mathrm{Faith}_\delta$*. Then* $\mathrm{F}(f)$ *is an epimorphism if and only if* $\mathrm{Im}\mathrm{F}(f) \in \mathrm{Refl}_\zeta$*.*

In other words, F *is exact with respect to the considered sequence if and only if* $\mathrm{Im}\mathrm{F}(f)$ *is a* ζ*-reflexive object.*

Lemma 1.2.7. *Let* $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ *be an exact sequence with* $Y \in \mathrm{Refl}_\delta$ *and* $Z \in \mathrm{Faith}_\delta$*. Then* $Z \in \mathrm{Refl}_\delta$ *if and only if* $\mathrm{GF}(g)$ *is an epimorphism.*

Lemma 1.2.8. *Let* $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ *be an exact sequence with* $Y \in \mathrm{Refl}_\delta$ *and* $Z \in \mathrm{Faith}_\delta$*. Then* $X \in \mathrm{Refl}_\delta$ *if and only if* $\mathrm{GF}(f)$ *is a monomorphism.*

Let A be an object in $\mathcal{A}$. We say that Y is *finitely-A-generated* if there is an epimorphism $A^n \to Y \to 0$, for some positive integer n. We denote by $\mathrm{gen}(A)$ the class of all finitely-A-generated objects. We say that Y is *finitely-A-presented* if there is an exact sequence $A^m \to A^n \to Y \to 0$, for some positive integers m and n. We denote by $\mathrm{pres}(A)$ the class of all finitely-A-presented objects. We say that X is *finitely-A-cogenerated* if there is a monomorphism $0 \to X \to A^n$, for some positive integer n. We denote by $\mathrm{cog}(A)$ the class of all finitely-A-cogenerated objects. We say that X is *finitely-A-copresented* if there is an exact sequence $0 \to X \to A^m \to A^n$, for some positive integers m and n. We denote by $\mathrm{cop}(A)$ the class of all finitely-A-copresented objects.

Lemma 1.2.9. *An object* $X \in \mathcal{A}$ *is* δ*-faithful with* $\mathrm{F}(X) \in \mathrm{gen}(V)$ *if and only if there exists a monomorphism* $f : X \to U^n$ *such that* $\mathrm{F}(f)$ *is an epimorphism.*

We will denote by $\mathrm{cop}_\delta(U)$ the class of all objects $X \in \mathcal{A}$ such that there exists an exact sequence $0 \to X \to U^n \to Z \to 0$ with $Z \in \mathrm{Faith}_\delta$. We will say that F is

U-w-π_f-exact if it is exact with respect to the short exact sequences $0 \to X \to U^n \to Z \to 0$ with $Z \in \mathrm{Faith}_\delta$.

Lemma 1.2.10. *If* F *is U-w-π_f-exact then the following assertions hold:*

(a) ζ_Y *is an epimorphism, for all* $Y \in \mathrm{gen}(V)$*;*

(b) $\mathrm{gen}(V) \cap \mathrm{Faith}_\zeta \subseteq \mathrm{Refl}_\zeta$.

1.3. **Costar Objects. Finitistic-1-F-cotilting.**

Lemma 1.3.1. *If* F *is* U-w-π_f-exact *then the following assertions hold:*

(a) $\mathrm{cop}_\delta(U) \subseteq \mathrm{Refl}_\delta$*;*

(b) $\mathrm{F}(\mathrm{cop}_\delta(U)) \subseteq \mathrm{gen}(V)$.

Now, we will characterize the situation when F is U-w-π_f-exact, through a duality induced by the considered pair $\mathrm{F} : \mathcal{A} \rightleftarrows \mathcal{B} : \mathrm{G}$.

Theorem 1.3.2. *The following statements are equivalent:*

(a) F *is U-w-π_f-exact;*

(b) $\mathrm{F} : \mathrm{cop}_\delta(U) \rightleftarrows \mathrm{gen}(V) \cap \mathrm{Faith}_\zeta : \mathrm{G}$ *is a duality.*

In 2001, Colby and Fuller introduced the notion of *costar module*, which is the dual notion of *star module*, and characterized this kind of notion. Inspired by their work, we define the notion of *costar object*, which extend to abelian categories the notion of costar module.

We say that the triple $\mathfrak{D} = (U, \mathrm{F}, \mathrm{G})$ is *costar* (or, *U is a costar object with respect to* F *and* G) if

$$\mathrm{F} : \mathrm{F}^{-1}(\mathrm{gen}(V)) \cap \mathrm{Faith}_\delta \rightleftarrows \mathrm{gen}(V) \cap \mathrm{Faith}_\zeta : \mathrm{G}$$

is a duality.

Now, we will give equivalent conditions for the triple $\mathfrak{D}$ to be costar.

Theorem 1.3.3. *The following statements are equivalent:*

(a) $\mathfrak{D}$ *is costar;*

(b) (1) $\mathrm{F} : \mathrm{cop}_\delta(U) \rightleftarrows \mathrm{gen}(V) \cap \mathrm{Faith}_\zeta : \mathrm{G}$ *is a duality;*

(2) $\mathrm{cop}_\delta(U) = \mathrm{F}^{-1}(\mathrm{gen}(V)) \cap \mathrm{Faith}_\delta$*;*

(c) (1) δ_X *is an epimorphism for all* $X \in \mathrm{F}^{-1}(\mathrm{gen}(V))$*;*

(2) ζ_Y *is an epimorphism for all* $Y \in \mathrm{gen}(V)$*;*

(d) F *preserves the exactness of an exact sequence of the form*

$$0 \to X \to U^n \to Z \to 0$$

if and only if $Z \in \mathrm{Faith}_\delta$.

The next result describes another kind of dualities induced by a pair of right adjoint contravariant functors. If it is happens in the classical context of contravariant functors induced by a module Q, Wisbauer called this module *f-cotilting* (see [54]).

Theorem 1.3.4. *The following statements are equivalent:*

(a) $\mathrm{F} : \mathrm{cog}(U) \rightleftarrows \mathrm{gen}(V) \cap \mathrm{Faith}_\zeta : \mathrm{G}$ *is a duality;*

(b) (1) $\mathrm{cog}(U) = \mathrm{cop}_\delta(U)$*;*

(2) F *is* U*-*w*-*π_f*-exact.*

The next result is a extension to abelian categories of [12, Corrolary 2.8].

Theorem 1.3.5. *The following statements are equivalent:*

(a) $\mathrm{F} : \mathrm{cog}(U) \rightleftarrows \mathrm{pres}(V) \cap \mathrm{Faith}_\zeta : \mathrm{G}$ *is a duality;*

(b) (1) $\mathrm{cog}(U) = \mathrm{cop}(U)$*;*

(2) F *is exact with respect to the short exact sequences*

$$0 \to X \to U^n \to Z \to 0$$

with $Z \in \mathrm{cog}(U)$.

We say that the object U is *finitistic-1-*F*-cotilting* if it satisfies the both conditions of (b) from the above Theorem. Hence, Theorem 1.3.5 characterizes finitistic-1-F-cotilting objects in terms of a duality.

1.4. **Dominant Resolutions.** Throughout this section we fix a positive integer n.

Now, we will define the notions of (finitely) dominant resolutions, using the Wakamatsu terminology (see [50]). Let $\mathcal{C}$ be a class in $\mathcal{A}$.

An exact sequence

$$0 \to X \to A_0 \to A_1 \to \dots$$

in $\mathcal{A}$ is called *dominant-left-$\mathcal{C}$-resolution of X* if $A_i \in \mathcal{C}$ for all $i \geq 0$ and the induced sequence

$$\cdots \to \mathrm{F}(A_1) \to \mathrm{F}(A_0) \to \mathrm{F}(X) \to 0$$

is also exact. We denote by $\mathrm{cog}^{\star}(\mathcal{C})$ the class of all objects $X \in \mathcal{A}$ such that there is a dominant-left-$\mathcal{C}$-resolution of X.

An exact sequence

$$0 \to X \to A_0 \to A_1 \to \cdots \to A_{n-1} \to A_n$$

in $\mathcal{A}$ is called *n-dominant-left-$\mathcal{C}$-resolution of X* if $A_i \in \mathcal{C}$ for all $i = \overline{0,n}$ and the induced sequence

$$\mathrm{F}(A_n) \to \mathrm{F}(A_{n-1}) \to \cdots \to \mathrm{F}(A_1) \to \mathrm{F}(A_0) \to \mathrm{F}(X) \to 0$$

is also exact. We denote by n-$\mathrm{cog}^{\star}(\mathcal{C})$ the class of all objects $X \in \mathcal{A}$ such that there is a n-dominant-left-$\mathcal{C}$-resolution of X.

An exact sequence

$$\cdots \to B_1 \to B_0 \to Y \to 0$$

in $\mathcal{A}$ is called *dominant-right-$\mathcal{C}$-resolution of Y* if $B_i \in \mathcal{C}$ for all $i \geq 0$ and the induced sequence

$$0 \to \mathrm{F}(Y) \to \mathrm{F}(B_0) \to \mathrm{F}(B_1) \to \dots$$

is also exact. We denote by $\mathrm{gen}^{\star}(\mathcal{C})$ the class of all objects $Y \in \mathcal{A}$ such that there is a dominant-right-$\mathcal{C}$-resolution of Y.

An exact sequence

$$B_n \to B_{n-1} \to \cdots \to B_1 \to B_0 \to Y \to 0$$

in $\mathcal{A}$ is called *n-dominant-right-$\mathcal{C}$-resolution of Y* if $B_i \in \mathcal{C}$ for all $i = \overline{0,n}$ and the induced sequence

$$0 \to \mathrm{F}(Y) \to \mathrm{F}(B_0) \to \mathrm{F}(B_1) \to \cdots \to \mathrm{F}(B_{n-1}) \to \mathrm{F}(B_n)$$

is also exact. We denote by n-$\mathrm{gen}^{\star}(\mathcal{C})$ the class of all objects $Y \in \mathcal{A}$ such that there is a n-dominant-right-$\mathcal{C}$-resolution of Y.

The main result of this section is the following theorem:

Theorem 1.4.1. *If* $\mathcal{C} \subseteq \mathrm{Refl}_\delta$ *then*

$$\mathrm{F} : n\text{-cog}^\star(\mathcal{C}) \rightleftarrows n\text{-gen}^\star(\mathrm{F}(\mathcal{C})) \cap \mathrm{Refl}_\zeta : \mathrm{G}$$

is a duality.

Example 1.4.2. Let Q be a right R-module with $S = \mathrm{End}(Q_R)$. As we saw in Example 1.1.1, $\Delta : \text{Mod-}R \rightleftarrows S\text{-Mod} : \Delta'$ is a pair of right adjoint contravariant functors. Both of natural transformations δ and ζ represent the evaluation maps.

(i) Since Q_R is δ-reflexive we can consider $\mathcal{C} = \{Q_R\}$. Then $\Delta(\mathcal{C}) = \{_SS\}$. With these settings, the corresponding duality of Theorem 1.4.1 is

$$\Delta : n\text{-cog}^\star(\{Q_R\}) \rightleftarrows n\text{-gen}^\star(\{_SS\}) \cap \mathrm{Refl}_\zeta : \Delta'$$

The class $n\text{-cog}^\star(\{Q_R\})$ consists in all right R-modules M_R of which there exist an exact sequence

$$0 \to M \xrightarrow{f_0} Q \xrightarrow{f_1} Q \xrightarrow{f_2} \dots \xrightarrow{f_{n-1}} Q \xrightarrow{f_n} Q$$

in Mod-R which stays exact under Δ. If we denote $\Delta(f_i) = \mathrm{Hom}_R(f_i, Q)$ by f_i^*, the induced sequence is

$$S \xrightarrow{f_n^*} S \xrightarrow{f_{n-1}^*} \dots \xrightarrow{f_2^*} S \xrightarrow{f_1^*} S \xrightarrow{f_0^*} \Delta(M) \to 0.$$

On the other hand, the class $n\text{-gen}^\star(\{_SS\})$ consists of all left S-modules $_SN$ for which there exists an exact sequence

$$S \xrightarrow{g_n} S \xrightarrow{g_{n-1}} \dots \xrightarrow{g_2} S \xrightarrow{g_1} S \xrightarrow{g_0} N \to 0$$

in S-Mod with the induced sequence

$$0 \to \Delta'(N) \xrightarrow{g_0^*} Q \xrightarrow{g_1^*} Q \xrightarrow{g_2^*} \dots \xrightarrow{g_{n-1}^*} Q \xrightarrow{g_n^*} Q$$

exact in Mod-R.

(ii) We can view S as an (S, R)-bimodule. We set $\mathcal{C} = \{_SS\}$. Then $\mathcal{C} \subseteq \mathrm{Refl}_\zeta$ and $\Delta'(\mathcal{C}) = \{Q_R\}$. In this case, the corresponding duality of Theorem 1.4.1 is

$$\Delta' : n\text{-cog}^\star(\{_SS\}) \rightleftarrows n\text{-gen}^\star(\{Q_R\}) \cap \mathrm{Refl}_\delta : \Delta$$

The class n-$\mathrm{cog}^\star(\{{}_SS\})$ consists of all left S-modules ${}_SN$ for which there exists an exact sequence

$$0 \to N \xrightarrow{f_0} S \xrightarrow{f_1} S \xrightarrow{f_2} \dots \xrightarrow{f_{n-1}} S \xrightarrow{f_n} S$$

in S-Mod with the induced sequence

$$Q \xrightarrow{f_n^*} Q \xrightarrow{f_{n-1}^*} \dots \xrightarrow{f_2^*} Q \xrightarrow{f_1^*} Q \xrightarrow{f_0^*} \Delta'(N) \to 0$$

exact in Mod-R. The class n-$\mathrm{gen}^\star(\{Q_R\})$ consists of all right R-modules M_R for which exists an exact sequence

$$Q \xrightarrow{g_n} Q \xrightarrow{g_{n-1}} \dots \xrightarrow{g_2} Q \xrightarrow{g_1} Q \xrightarrow{g_0} M \to 0$$

such that the induced sequence

$$0 \to \Delta(M) \xrightarrow{g_0^*} S \xrightarrow{g_1^*} S \xrightarrow{g_2^*} \dots \xrightarrow{g_{n-1}^*} S \xrightarrow{g_n^*} S$$

is also exact. □

Example 1.4.3. Let G be a group and let R be a G-graded unital ring. We consider $Q \in \mathrm{Mod}_{\mathrm{gr}}$-$R$ with $S = \mathrm{END}_R(Q)$. As we saw in Example 1.1.2, $\mathrm{H}_R^{\mathrm{gr}} : \mathrm{Mod}_{\mathrm{gr}}\text{-}R \rightleftarrows S\text{-}\mathrm{Mod}_{\mathrm{gr}} : {}_S\mathrm{H}^{\mathrm{gr}}$ is a pair of right adjoint contravariant functors. The evaluation maps are $\delta^{Q_R} : 1_{\mathrm{Mod}_{\mathrm{gr}}\text{-}R} \longrightarrow \mathrm{HOM}_S(\mathrm{HOM}_R(-, Q), Q)$ and $\zeta^{{}_SQ} : 1_{S\text{-}\mathrm{Mod}_{\mathrm{gr}}} \longrightarrow \mathrm{HOM}_R(\mathrm{HOM}_S(-, Q), Q)$. A graded right R-module M is called Q_R-gr-*reflexive* (respectively, Q_R-gr-*tosionless*) if $\delta_M^{Q_R}$ is an isomorphism (respectively, a monomorphism). A graded left S-module N is called ${}_SQ$-gr-*reflexive* (respectively, ${}_SQ$-gr-*tosionless*) if $\zeta_N^{{}_SQ}$ is an isomorphism (respectively, a monomorphism). We denote by $\mathrm{Ref}^{\mathrm{gr}}(Q_R)$ (respectively, by $\mathrm{Ref}^{\mathrm{gr}}({}_SQ)$) the class of all Q_R-gr-reflexive right R-modules (respectively, ${}_SQ$-gr-reflexive left S-modules).

Let M_R be a graded right R-module which is Q_R-gr-reflexive. If we set the class $\mathcal{C}$ to be $\mathrm{add}(M_R)$, then $\mathcal{C} \subseteq \mathrm{Ref}^{\mathrm{gr}}(Q_R)$. The corresponding duality of Theorem 1.4.1 is

$$\mathrm{H}_R^{\mathrm{gr}} : n\text{-}\mathrm{cog}^\star(\mathrm{add}(M_R)) \rightleftarrows n\text{-}\mathrm{gen}^\star(\mathrm{add}(\mathrm{HOM}_R(M, Q))) \cap \mathrm{Ref}^{\mathrm{gr}}({}_SQ) : {}_S\mathrm{H}^{\mathrm{gr}}.$$

□

Next, we will apply the Theorem 1.4.1, for some particular class $\mathcal{C}$, in order to obtain generalizations of some known dualities.

The Case $\mathcal{C} = \mathrm{add}(U)$. Setting $\mathcal{C} = \mathrm{add}(U)$ we have, by Lemma 1.2.1, that $\mathcal{C} \subseteq \mathrm{Refl}_\delta$ and $\mathrm{F}(\mathcal{C}) = \mathrm{add}(V)$. Now Theorem 1.4.1 becomes:

Corollary 1.4.4. *The functors* F *and* G *induce the following duality:*

$$\mathrm{F} : n\text{-cog}^\star(\mathrm{add}(U)) \rightleftarrows n\text{-gen}^\star(\mathrm{add}(V)) \cap \mathrm{Refl}_\zeta : \mathrm{G}.$$

According to Lemma 1.2.10, if F is U-w-π_f-exact, we have the following equality

$$n\text{-gen}^\star(\mathrm{add}(V)) \cap \mathrm{Refl}_\zeta = n\text{-gen}^\star(\mathrm{add}(V)) \cap \mathrm{Faith}_\zeta.$$

Corollary 1.4.5. *If* F *is U-w-π_f-exact, then*

$$\mathrm{F} : n\text{-cog}^\star(\mathrm{add}(U)) \rightleftarrows n\text{-gen}^\star(\mathrm{add}(V)) \cap \mathrm{Faith}_\zeta : \mathrm{G}$$

is a duality.

Theorem 1.4.6. *The following statements are equivalent:*

(a) $\mathrm{F} : \mathrm{F}^{-1}(n\text{-gen}^\star(\mathrm{add}(V))) \cap \mathrm{Faith}_\delta \rightleftarrows n\text{-gen}^\star(\mathrm{add}(V)) \cap \mathrm{Faith}_\zeta : \mathrm{G}$ *is a duality;*

(b) (1) $\mathrm{F} : n\text{-cog}^\star(\mathrm{add}(U)) \rightleftarrows n\text{-gen}^\star(\mathrm{add}(V)) \cap \mathrm{Faith}_\zeta : \mathrm{G}$ *is a duality;*

(2) $n\text{-cog}^\star(\mathrm{add}(U)) = \mathrm{F}^{-1}(n\text{-gen}^\star(\mathrm{add}(V))) \cap \mathrm{Faith}_\delta$*;*

(c) (1) δ_X *is an epimorphism, for all* $X \in \mathrm{F}^{-1}(n\text{-gen}^\star(\mathrm{add}(V)))$*;*

(2) ζ_Y *is an epimorphism, for all* $Y \in n\text{-gen}^\star(\mathrm{add}(V)) \cap \mathrm{Faith}_\zeta$*;*

Moreover, when the above statements hold then

(d) F *is exact with respect to the short exact sequences*

$$0 \to X \to Y \to Z \to 0$$

with $Y \in \mathrm{add}(U)$ *and* $Z \in \mathrm{F}^{-1}(n\text{-gen}^\star(\mathrm{add}(V))) \cap \mathrm{Faith}_\delta$*.*

We also have the following theorem which characterizes the duality from the Corollary 1.4.5 in the case $n = 0$. We remember that $\mathrm{cop}_\delta(U)$ denotes the class of all objects $X \in \mathcal{A}$ such that there exists an exact sequence $0 \to X \to U^m \to Z \to 0$ with $Z \in \mathrm{Faith}_\delta$.

Theorem 1.4.7. *The following statements are equivalent:*

(a) (1) $\mathrm{F} : 0\text{-cog}^\star(\mathrm{add}(U)) \rightleftarrows 0\text{-gen}^\star(\mathrm{add}(V)) \cap \mathrm{Faith}_\zeta : \mathrm{G}$ *is a duality;*

(2) F *is U-w-π_f-exact.*

(b) (1) $\mathrm{F} : \mathrm{cop}_\delta(U) \rightleftarrows 0\text{-gen}^\star(\mathrm{add}(V)) \cap \mathrm{Faith}_\zeta : \mathrm{G}$ *is a duality;*

(2) δ_X *is an epimorphism, for all* $X \in 0\text{-cog}^\star(\mathrm{add}(U))$.

For the rest of the section, we assume that $\mathcal{B}$ *has enough projectives*. Let $B \in \mathcal{B}$. A *projective resolution*

$$\cdots \to P_1 \to P_0 \to Y \to 0$$

of Y is called *finitely*-add(B)-*generated* if $P_i \in \mathrm{add}(B)$ for all $i \geq 0$. We will denote by $\mathrm{gen}^\bullet(\mathrm{add}(B))$ the class of all objects $Y \in \mathcal{B}$ such that there exists a finitely-add(B)-generated projective resolution of Y.

A *projective resolution*

$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to Y \to 0$$

of Y is called n-*finitely*-add(B)-*generated* if $P_i \in \mathrm{add}(B)$ for all $i = \overline{0,n}$. We will denote by $n\text{-gen}^\bullet(\mathrm{add}(B))$ the class of all objects $Y \in \mathcal{B}$ such that there exists a n-finitely-add(B)-generated projective resolution of Y.

We also denote by $\mathrm{R}^j\mathrm{G}$ the j-th right derived functor of G. We consider the following orthogonal classes:

$$^{\perp_{<n}}\mathcal{B} = \{Y \in \mathcal{B} \mid \mathrm{R}^j\mathrm{G}(Y) = 0, \text{ for all } 0 < j < n\}$$

and

$$^{\perp}\mathcal{B} = \{Y \in \mathcal{B} \mid \mathrm{R}^j\mathrm{G}(Y) = 0, \text{ for all } j \geq 1\}.$$

If V is projective in $\mathcal{B}$, it is easy to show that the equality

$$n\text{-gen}^\star(\mathrm{add}(V)) = {}^{\perp_{<n}}\mathcal{B} \cap n\text{-gen}^\bullet(\mathrm{add}(V))$$

holds. By Corollary 1.4.4, we obtain the following result:

Corollary 1.4.8. *If V is a projective object in $\mathcal{B}$, then*

$$\mathrm{F} : n\text{-cog}^\star(\mathrm{add}(U)) \rightleftarrows {}^{\perp_{<n}}\mathcal{B} \cap n\text{-gen}^\bullet(\mathrm{add}(V)) \cap \mathrm{Refl}_\zeta : \mathrm{G}$$

is a duality.

Using the above corollary, we obtain the following dualities, which are generalizations of [50, Proposition 4.1 and Theorem 4.2] to abelian categories. Now we suppose that both abelian categories $\mathcal{A}$ and $\mathcal{B}$ have *enough projectives*. We also consider the perpendicular class ${}^{\perp}\mathcal{A} = \{X \in \mathcal{A} \mid \mathrm{R}^j\mathrm{F}(X) = 0, \text{ for all } j \geq 1\}$, where $\mathrm{R}^j\mathrm{F}$ is the j-th right derived functor of F.

Corollary 1.4.9. *Suppose that V is a projective object in $\mathcal{B}$. Let A be a δ-reflexive and projective object in $\mathcal{A}$. Then:*

(a) $\mathrm{F} : \mathrm{cog}^{\star}(\mathrm{add}(U)) \rightleftarrows {}^{\perp}\mathcal{B} \cap \mathrm{gen}^{\bullet}(\mathrm{add}(V)) \cap \mathrm{Refl}_{\zeta} : \mathrm{G}$ *is a duality;*
(b) $\mathrm{G} : \mathrm{cog}^{\star}(\mathrm{add}(\mathrm{F}(A))) \rightleftarrows {}^{\perp}\mathcal{A} \cap \mathrm{gen}^{\bullet}(\mathrm{add}(A)) \cap \mathrm{Refl}_{\delta} : \mathrm{F}$ *is a duality.*

Corollary 1.4.10. *Suppose that V is a projective object in $\mathcal{B}$. Let A be a δ-reflexive and projective object in $\mathcal{A}$. Then*

$$\mathrm{F} : {}^{\perp}\mathcal{A} \cap \mathrm{gen}^{\bullet}(\mathrm{add}(A)) \cap \mathrm{cog}^{\star}(\mathrm{add}(U)) \rightleftarrows$$

$$\rightleftarrows {}^{\perp}\mathcal{B} \cap \mathrm{gen}^{\bullet}(\mathrm{add}(V)) \cap \mathrm{cog}^{\star}(\mathrm{add}(\mathrm{F}(A))) : \mathrm{G}$$

is a duality.

Finitistic-n-F-cotilting. Let A be an object in $\mathcal{A}$.
We say that X is *n-finitely-A-copresented* if there is an exact sequence

$$0 \to X \to A^{m_0} \to A^{m_1} \to \cdots \to A^{m_{n-2}} \to A^{m_{n-1}},$$

where all m_k are positive integers. We denote by n-cop(A) the class of all n-finitely-A-copresented objects. In particular, 1-cop$(A) = \mathrm{cog}(A)$ and 2-cop$(A) = \mathrm{cop}(A)$.
We say that Y is *n-finitely-A-presented* if there is an exact sequence

$$A^{m_{n-1}} \to A^{m_{n-2}} \to \cdots \to A^{m_1} \to A^{m_0} \to Y \to 0,$$

where all m_k are positive integers. We denote by $\mathrm{FP}_n(A)$ the class of all n-finitely-A-presented objects. In particular, $\mathrm{FP}_1(A) = \mathrm{gen}(A)$ and $\mathrm{FP}_2(A) = \mathrm{pres}(A)$.

An object A is called *n-w_f-F-exact* if every short exact sequence in $\mathcal{A}$ of the form $0 \to X \to A^m \to Z \to 0$, with $Z \in n$-cop(A), stays exact under F. An object A is called *finitistic-n-F-cotilting* if A is n-w_f-F-exact and n-cop$(A) = (n+1)$-cop(A).

We remind that is assumed that $\mathcal{B}$ has enough projectives.

Lemma 1.4.11. *Assume that U is finitistic-n-F-cotilting. If $X \in n\text{-cop}(U)$ then there is a long exact sequence*

$$0 \to X \longrightarrow U^{m_0} \longrightarrow U^{m_1} \longrightarrow U^{m_2} \longrightarrow \dots$$

with the induced sequence

$$\dots \longrightarrow \mathrm{F}(U^{m_2}) \longrightarrow \mathrm{F}(U^{m_1}) \longrightarrow \mathrm{F}(U^{m_0}) \longrightarrow \mathrm{F}(X) \to 0$$

being also exact.

Assume that $V = \mathrm{F}(U)$ is a projective object in $\mathcal{B}$. We set $\mathcal{C} = \{U^k \mid k \in \mathbb{N}^*\}$. It follows that $\mathcal{C} \subseteq \mathrm{Refl}_\delta$ and $\mathrm{F}(\mathcal{C}) = \{V^k \mid k \in \mathbb{N}^*\}$.

Lemma 1.4.12. *If U is finitistic-n-F-cotilting, then we have:*

(a) $n\text{-cog}^\star(\mathcal{C}) = n\text{-cop}(U)$;
(b) $n\text{-gen}^\star(\mathrm{F}(\mathcal{C})) \cap \mathrm{Refl}_\zeta = {}^{\perp_{<n}}\mathcal{B} \cap \mathrm{FP}_{(n+1)}(V) \cap \mathrm{Faith}_\zeta$;
(c) $n\text{-gen}^\star(\mathrm{F}(\mathcal{C})) \cap \mathrm{Refl}_\zeta = {}^{\perp}\mathcal{B} \cap \mathrm{FP}_{(n+1)}(V) \cap \mathrm{Faith}_\zeta$.

For the next result, it is not necessary for the abelian category $\mathcal{B}$ to have enough projectives.

Proposition 1.4.13. *If $\mathcal{C} \subseteq \mathrm{Refl}_\delta$, then the following statements hold:*

(a) *If $X \in \mathrm{Refl}_\delta$ with $\mathrm{F}(X) \in n\text{-gen}^\star(\mathrm{F}(\mathcal{C}))$ then $X \in n\text{-cog}^\star(\mathcal{C})$;*
(b) *If $n\text{-gen}^\star(\mathrm{F}(\mathcal{C})) \cap \mathrm{Faith}_\zeta \subseteq \mathrm{Refl}_\zeta$ then F is exact with respect to the short exact sequences*

$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

with $Y \in \mathcal{C}$ and $Z \in \mathrm{Refl}_\delta \cap \mathrm{F}^{-1}(n\text{-gen}^\star(\mathrm{F}(\mathcal{C})))$.

According to Proposition 1.4.13 and since $n\text{-gen}^\star(\mathrm{F}(\mathcal{C})) = {}^{\perp_{<n}}\mathcal{B} \cap \mathrm{FP}_{(n+1)}(V)$, where $\mathcal{C} = \{U^k \mid k \in \mathbb{N}^*\}$, we have the following corollary:

Corollary 1.4.14. *Let $X \in \mathrm{Refl}_\delta$ such that $\mathrm{F}(X) \in {}^{\perp_{<n}}\mathcal{B} \cap \mathrm{FP}_{(n+1)}(V)$. Then $X \in (n+1)\text{-cop}(U)$.*

The next theorem is a generalization of [12, Theorem 2.7].

Theorem 1.4.15. *The following statements are equivalent for an object $U \in \mathrm{Refl}_\delta$ with $\mathrm{F}(U) = V$ projective object in $\mathcal{B}$ and a positive integer n:*

(a) *U is finitistic-n-F-cotilting;*

(b) $\mathrm{F} : n\text{-cop}(U) \rightleftarrows {}^{\perp_{<n}}\mathcal{B} \cap \mathrm{FP}_{(n+1)}(V) \cap \mathrm{Faith}_\zeta : \mathrm{G}$ *is a duality;*

(c) $\mathrm{F} : n\text{-cop}(U) \rightleftarrows {}^{\perp}\mathcal{B} \cap \mathrm{FP}_{(n+1)}(V) \cap \mathrm{Faith}_\zeta : \mathrm{G}$ *is a duality.*

1.5. **The** add(U)-coplex **Category.** By $\mathrm{Comp}_\mathcal{A}$ will be denoted the category of all complexes in $\mathcal{A}$. We also denote by $H_n(\mathcal{C})$ the *n-th homology of $\mathcal{C}$*, for some complex $\mathcal{C} \in \mathrm{Comp}_\mathcal{A}$ and for some integer n. For basic properties of the category $\mathrm{Comp}_\mathcal{A}$ we refer to [46, Chapter 10]. Throughout this section we suppose that the abelian category $\mathcal{B}$ has enough projectives. We also assume that $V = \mathrm{F}(U)$ is a projective object in $\mathcal{B}$.

Consider an object $A \in \mathcal{A}$.

Definition 1.5.1. A complex

$$\mathcal{C} : C_0 \xrightarrow{\sigma_1} C_1 \xrightarrow{\sigma_2} C_2 \xrightarrow{\sigma_3} C_3 \xrightarrow{\sigma_4} \dots$$

in $\mathcal{A}$ is called add(A)-*coplex* (or, *semi-dominant-right*-add(A)-*resolution*) if the following condition are satisfied:

(1) $C_k \in \mathrm{add}(A)$, for all $k \geq 0$;

(2) The induced complex

$$\mathrm{F}(\mathcal{C}) : \dots \xrightarrow{\mathrm{F}(\sigma_4)} \mathrm{F}(C_3) \xrightarrow{\mathrm{F}(\sigma_3)} \mathrm{F}(C_2) \xrightarrow{\mathrm{F}(\sigma_2)} \mathrm{F}(C_1) \xrightarrow{\mathrm{F}(\sigma_1)} \mathrm{F}(C_0)$$

is an exact sequence in $\mathcal{B}$.

Definition 1.5.2. Let $\mathcal{C}$ and $\mathcal{C}'$ be two add(A)-coplexes in $\mathcal{A}$. A sequence of morphisms $f = (f_0, f_1, f_2, f_3, \dots)$, where $f_k \in \mathrm{Hom}_\mathcal{A}(C_k, C'_k)$, is called *chain map between* add(A)-*coplexes $\mathcal{C}$ and $\mathcal{C}'$* if the following diagram is commutative

$$\begin{array}{ccccccccc}
C_0 & \xrightarrow{\sigma_1} & C_1 & \xrightarrow{\sigma_2} & C_2 & \xrightarrow{\sigma_3} & C_3 & \xrightarrow{\sigma_4} & \cdots \\
\downarrow f_0 & & \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & \\
C'_0 & \xrightarrow{\sigma'_1} & C'_1 & \xrightarrow{\sigma'_2} & C'_2 & \xrightarrow{\sigma'_3} & C'_3 & \xrightarrow{\sigma'_4} & \cdots
\end{array}$$

i.e. $f_k \circ \sigma_k = \sigma'_k \circ f_{k-1}$, for all integers $k \geq 1$.

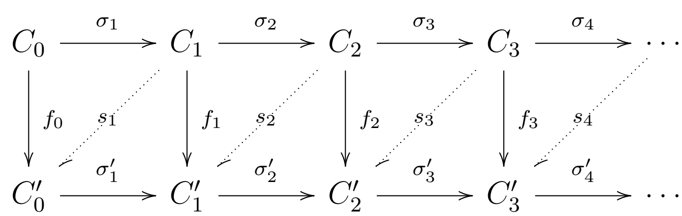
Definition 1.5.3. Let $\mathcal{C}$ and $\mathcal{C}'$ be two add(A)-coplexes.

(a) Let $f = (f_0, f_1, f_2, f_3, \dots) : \mathcal{C} \to \mathcal{C}'$ be a chain map between add(A)-coplexes $\mathcal{C}$ and $\mathcal{C}'$. We say that *f is null homotopic* (or, *f is homotopic to zero*) if there are, for all $k \geq 1$, morphisms $s_k : C_k \to C'_{k-1}$ in $\mathcal{A}$ such that:

(1) $f_k = s_{k+1} \circ \sigma_{k+1} + \sigma'_k \circ s_k$, for all integers $k \geq 1$;

(2) $f_0 = s_1 \circ \sigma_1$.

The sequence $s = (s_1, s_2, s_3, \dots)$ is called *a homotopy of f* (or, *a homotopy between f and* 0). The morphisms are illustrated in the following diagram:



The condition for s to be a homotopy of f says that each vertical map is the sum of the sides of the parallelogram containing it.

(b) Let $f = (f_0, f_1, f_2, f_3, \dots) : \mathcal{C} \to \mathcal{C}'$ and $g = (g_0, g_1, g_2, g_3, \dots) : \mathcal{C} \to \mathcal{C}'$ be two chain maps. We say that *f and g are homotopic* (or, *f is homotopic to g*), written $f \simeq g$, if

$$f - g = (f_0 - g_0, f_1 - g_1, f_2 - g_2, f_3 - g_3, \dots) : \mathcal{C} \to \mathcal{C}'$$

is a null homotopic chain map. A homotopy between $f - g$ and 0 is also called *a homotopy between f and g*. The homotopic relation " $\simeq$ " is a additive equivalence relation on the set of chain maps $f : \mathcal{C} \to \mathcal{C}'$. We denote by $[f]$ the homotopy (equivalence) class of f.

(c) We say that *$\mathcal{C}$ and $\mathcal{C}'$ have the same homotopy type* if there exists two chain maps $f = (f_0, f_1, f_2, f_3, \dots) : \mathcal{C} \to \mathcal{C}'$ and $g = (g_0, g_1, g_2, g_3, \dots) : \mathcal{C}' \to \mathcal{C}$ such that

$$[g \circ f] = [(g_0 \circ f_0, g_1 \circ f_1, g_2 \circ f_2, g_3 \circ f_3, \dots)] \simeq [(1_{C_0}, 1_{C_1}, 1_{C_2}, 1_{C_3}, \dots)] = [1_{\mathcal{C}}]$$

and

$$[f \circ g] = [(f_0 \circ g_0, f_1 \circ g_1, f_2 \circ g_2, f_3 \circ g_3, \dots)] \simeq [(1_{C'_0}, 1_{C'_1}, 1_{C'_2}, 1_{C'_3}, \dots)] = [1_{\mathcal{C}'}].$$

Now we define *the category of* add(A)*-coplexes*, denoted by add(A)-coplex, as follows:

- The Objects consisting in the class of all add(A)-coplexes $\mathcal{C}$;
- The Morphisms, $[f] : \mathcal{C} \to \mathcal{C}'$, consisting in the set of all homotopy classes of chain maps $f : \mathcal{C} \to \mathcal{C}'$. More exactly,

$$\mathrm{Hom}_{\mathrm{add}(A)\text{-coplex}}(\mathcal{C}, \mathcal{C}') = \{[f] \mid f : \mathcal{C} \to \mathcal{C}' \text{ is a chain map}\}$$

Lemma 1.5.4. *Let* $\mathcal{C} : C_0 \xrightarrow{\sigma_1} C_1 \xrightarrow{\sigma_2} C_2 \xrightarrow{\sigma_3} C_3 \xrightarrow{\sigma_4} \dots$ *be a complex in* $\mathcal{A}$ *with* $C_k \in \mathrm{add}(U)$*, for all* $k \geq 0$*. Then the following statements hold:*

(a) $\mathcal{C}$ *is an* add(U)*-coplex if and only if* $\mathrm{F}(\mathcal{C})$ *is a finitely-*add(V)*-generated projective resolution of* $H_0(\mathrm{F}(\mathcal{C}))$*;*

(b) *If* $f, g : \mathcal{C} \to \mathcal{C}'$ *are homotopic chain maps between complexes* $\mathcal{C}$ *and* $\mathcal{C}'$*, then* $H_0(\mathrm{F}(f)) = H_0(\mathrm{F}(g))$*.*

Definition 1.5.5. We define the functor $\mathrm{F}^U : \mathrm{add}(U)\text{-coplex} \to \mathrm{gen}^\bullet(\mathrm{add}(V))$ as follows:

- On Objects: $\mathrm{F}^U(\mathcal{C}) = H_0(\mathrm{F}(\mathcal{C}))$, for each $\mathcal{C} \in \mathrm{add}(U)$-coplex;
- On Morphisms: $\mathrm{F}^U([f]) = H_0(\mathrm{F}(f))$, for each $[f] \in \mathrm{add}(U)$-coplex.

Theorem 1.5.6. *The functor* F^U *is a well defined contravariant functor.*

Definition 1.5.7. We define the functor $\mathrm{G}^U : \mathrm{gen}^\bullet(\mathrm{add}(V)) \to \mathrm{add}(U)\text{-coplex}$ as follows:

- On objects. Let $Y \in \mathrm{gen}^\bullet(\mathrm{add}(V))$. Then Y has a finitely-add(V)-generated projective resolution

$$\mathcal{P}(Y) = \dots \xrightarrow{\partial_3} P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} Y \to 0.$$

Applying the functor G to the deleted projective resolution $\mathcal{P}(Y)$, we have the following complex in $\mathcal{A}$

$$\mathrm{G}(\mathcal{P}(Y)) = \mathrm{G}(P_0) \xrightarrow{\mathrm{G}(\partial_1)} \mathrm{G}(P_1) \xrightarrow{\mathrm{G}(\partial_2)} \mathrm{G}(P_2) \xrightarrow{\mathrm{G}(\partial_3)} \dots$$

Since $\mathcal{P}(Y)$ is finitely-add(V)-generated, we have $P_k \in \mathrm{add}(V)$, for all $k \geq 0$, and, since $\zeta : 1_{\mathrm{add}(V)} \to \mathrm{FG}$ is a natural isomorphism, the following diagram is commutative with the vertical maps being isomorphisms

$$\begin{array}{ccccccc}
\mathcal{P}(Y) = \dots & \xrightarrow{\partial_3} & P_2 & \xrightarrow{\partial_2} & P_1 & \xrightarrow{\partial_1} & P_0 \\
 & & \downarrow \zeta_{P_2} & & \downarrow \zeta_{P_1} & & \downarrow \zeta_{P_0} \\
\mathrm{FG}(\mathcal{P}(Y)) = \dots & \xrightarrow{\mathrm{FG}(\partial_3)} & \mathrm{FG}(P_2) & \xrightarrow{\mathrm{FG}(\partial_2)} & \mathrm{FG}(P_1) & \xrightarrow{\mathrm{FG}(\partial_1)} & \mathrm{FG}(P_0)
\end{array}$$

Since the top row is an exact sequence, it follows that the bottom row is an exact sequence. By Lemma 1.2.1, $\mathrm{G}(P_k) \in \mathrm{add}(U)$, for all $k \geq 0$. Thus $\mathrm{G}(\mathcal{P}(Y))$ is a complex in $\mathcal{A}$ with all terms $\mathrm{G}(P_k) \in \mathrm{add}(U)$ and the induced sequence $\mathrm{F}(\mathrm{G}(\mathcal{P}(Y)))$ being an exact sequence. Therefore $\mathrm{G}(\mathcal{P}(Y))$ is a $\mathrm{add}(U)$-coplex. We set

$$\mathrm{G}^U(Y) := \mathrm{G}(\mathcal{P}(Y)) \in \mathrm{add}(U)\text{-coplex}.$$

- On morphisms. Let $\phi \in \mathrm{Hom}_{\mathrm{gen}^\bullet(\mathrm{add}(V))}(Y, Y')$. Since $Y \in \mathrm{gen}^\bullet(\mathrm{add}(V))$, Y has a finitely-add(V)-generated projective resolution

$$\mathcal{P}(Y) = \dots \xrightarrow{\partial_4} P_3 \xrightarrow{\partial_3} P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} Y \to 0.$$

Since $Y' \in \mathrm{gen}^\bullet(\mathrm{add}(V))$, Y' has a finitely-add(V)-generated projective resolution

$$\mathcal{P}(Y') = \dots \xrightarrow{\partial'_4} P'_3 \xrightarrow{\partial'_3} P'_2 \xrightarrow{\partial'_2} P'_1 \xrightarrow{\partial'_1} P'_0 \xrightarrow{\partial'_0} Y' \to 0.$$

Since $\phi \in \mathrm{Hom}_{\mathrm{gen}^\bullet(\mathrm{add}(V))}(Y, Y')$, we have $\phi \in \mathrm{Hom}_{\mathcal{B}}(Y, Y')$, hence ϕ lifts to a chain map

$$f = (\dots, f_3, f_2, f_1, f_0) : \mathcal{P}(Y) \to \mathcal{P}(Y'),$$

as in the following diagram

$$\begin{array}{ccccccccccccc}
\cdots & \xrightarrow{\partial_4} & P_3 & \xrightarrow{\partial_3} & P_2 & \xrightarrow{\partial_2} & P_1 & \xrightarrow{\partial_1} & P_0 & \xrightarrow{\partial_0} & Y & \longrightarrow & 0 \\
 & & \downarrow f_3 & & \downarrow f_2 & & \downarrow f_1 & & \downarrow f_0 & & \downarrow \phi & & \\
\cdots & \xrightarrow{\partial'_4} & P'_3 & \xrightarrow{\partial'_3} & P'_2 & \xrightarrow{\partial'_2} & P'_1 & \xrightarrow{\partial'_1} & P'_0 & \xrightarrow{\partial'_0} & Y' & \longrightarrow & 0
\end{array}$$

Applying the functor G to the chain map f, we get a chain map in $\mathcal{A}$

$$\mathrm{G}(f) = (\mathrm{G}(f_0), \mathrm{G}(f_1), \mathrm{G}(f_2), \mathrm{G}(f_3), \dots) : \mathrm{G}(\mathcal{P}(Y')) \to \mathrm{G}(\mathcal{P}(Y))$$

illustrated in the diagram below

$$\begin{array}{ccccccccc}
G(P'_0) & \xrightarrow{G(\partial'_1)} & G(P'_1) & \xrightarrow{G(\partial'_2)} & G(P'_2) & \xrightarrow{G(\partial'_3)} & G(P'_3) & \xrightarrow{G(\partial'_4)} & \cdots \\
\downarrow{\scriptstyle G(f_0)} & & \downarrow{\scriptstyle G(f_1)} & & \downarrow{\scriptstyle G(f_2)} & & \downarrow{\scriptstyle G(f_3)} & & \\
G(P_0) & \xrightarrow{G(\partial_1)} & G(P_1) & \xrightarrow{G(\partial_2)} & G(P_2) & \xrightarrow{G(\partial_3)} & G(P_3) & \xrightarrow{G(\partial_4)} & \cdots
\end{array}$$

Since $G(\mathcal{P}(Y))$ and $G(\mathcal{P}(Y'))$ are add(U)-coplexes, it follows that $[G(f)]$ is a morphism in add(U)-coplex, i.e. $[G(f)] \in \mathrm{Hom}_{\mathrm{add}(U)\text{-coplex}}(G(\mathcal{P}(Y')), G(\mathcal{P}(Y)))$. We set

$$G^U(\phi) = [G(f)] \in \mathrm{Hom}_{\mathrm{add}(U)\text{-coplex}}(G^U(Y'), G^U(Y)).$$

Theorem 1.5.8. *The functor* $G^U : \mathrm{gen}^\bullet(\mathrm{add}(V)) \to \mathrm{add}(U)\text{-coplex}$ *is a well-defined and contravariant functor.*

Let $H : \mathcal{A} \longrightarrow \mathcal{B}$ be an additive and contravariant functor. Then, for every pair of objects $X, Y \in \mathcal{A}$, H induces a map $H_{X,Y} : \mathrm{Hom}_{\mathcal{A}}(X, Y) \longrightarrow \mathrm{Hom}_{\mathcal{B}}(H(Y), H(X))$, defined by $H_{X,Y}(\alpha) := H(\alpha)$.

- We say that H is *faithful*, if the map $H_{X,Y}$ is injective, for every pair of objects $X, Y \in \mathcal{A}$.
- We say that H is *full*, if the map $H_{X,Y}$ is surjective, for every pair of objects $X, Y \in \mathcal{A}$.
- We say that H is *dense*, if it satisfies the following condition, denoted by (#):

For any object $Y \in \mathcal{B}$, there is an object $X \in \mathcal{A}$

and an isomorphism $H(X) \cong Y$.

Theorem 1.5.9. *The functor* $F^U : \mathrm{add}(U)\text{-coplex} \to \mathrm{gen}^\bullet(\mathrm{add}(V))$ *is full, faithful and satisfies condition (#).*

Theorem 1.5.10. *The functor* $G^U : \mathrm{gen}^\bullet(\mathrm{add}(V)) \to \mathrm{add}(U)\text{-coplex}$ *is full, faithful and satisfies condition (#).*

Theorem 1.5.11. *The functors* F^U *and* G^U *induce the following duality*

$$F^U : \mathrm{add}(U)\text{-coplex} \rightleftarrows \mathrm{gen}^\bullet(\mathrm{add}(V)) : G^U$$

2. Equivalences Induced by Adjoint Functors

2.1. **Introduction.** Throughout this chapter, we consider a pair of additive and covariant functors $\mathrm{F} : \mathcal{A} \rightleftarrows \mathcal{B} : \mathrm{G}$ between abelian categories such that G is a left adjoint to F, i.e. there are natural isomorphisms

$$\varphi_{X,M} : \mathrm{Hom}_{\mathcal{A}}(\mathrm{G}(X), M) \to \mathrm{Hom}_{\mathcal{B}}(X, \mathrm{F}(M)),$$

for all $M \in \mathcal{A}$ and for all $X \in \mathcal{B}$. Then, they induce two natural transformations

$$\phi : \mathrm{GF} \to 1_{\mathcal{A}}, \text{ defined by } \phi_M = \varphi^{-1}_{\mathrm{F}(M),M}(1_{\mathrm{F}(M)})$$

and

$$\theta : 1_{\mathcal{B}} \to \mathrm{FG}, \text{ defined by } \theta_X = \varphi_{X,\mathrm{G}(X)}(1_{\mathrm{G}(X)}).$$

We note that F is left exact and G is right exact. Moreover, they satisfy the identities

$$\mathrm{F}(\phi_M) \circ \theta_{\mathrm{F}(M)} = 1_{\mathrm{F}(M)}$$

and

$$\phi_{\mathrm{G}(X)} \circ \mathrm{G}(\theta_X) = 1_{\mathrm{G}(X)},$$

for all $M \in \mathcal{A}$ and for all $X \in \mathcal{B}$. We also assume that all considered subcategories are isomorphically closed.

The classical example of such a pair of functors is the following:

Example 2.1.1. Let R and S be two unital associative rings and let ${}_SQ_R$ be a (S,R)-bimodule. Then

$$\mathrm{F}(-) = \mathrm{Hom}_R(Q, -) : \text{Mod-}R \rightleftarrows \text{Mod-}S : - \otimes_S Q = \mathrm{G}(-)$$

and

$$\mathrm{F}(-) = \mathrm{Hom}_S(Q, -) : S\text{-Mod} \rightleftarrows R\text{-Mod} : Q \otimes_R - = \mathrm{G}(-)$$

are pairs of additive and covariant functors. Moreover, the tensor functor $- \otimes_S Q$ (respectively, $Q \otimes_R -$) is an adjoint on the left of the functor $\mathrm{Hom}_R(Q, -)$ (respectively, $\mathrm{Hom}_S(Q, -)$). □

The next two examples were presented by Castaño-Iglesias, Gómez-Torrecillas and Wisbauer in [19]:

Example 2.1.2. Let G be a group. If $R = \bigoplus_{x\in G} R_x$ is a G-graded ring, we will denote by R-$\mathrm{Mod}_{\mathrm{gr}}$ the category of all G-graded unital left R-modules. If $M, N \in R$-$\mathrm{Mod}_{\mathrm{gr}}$, we consider the G-graded abelian group $\mathrm{HOM}_R(M,N)$ whose homogeneous component at x is the subgroup of $\mathrm{Hom}_R(M,N)$ consisting of all R-homomorphisms $f : M \to N$ such that $f(M_y) \subseteq N_{yx}$, for all $y \in G$. We note that $S = \mathrm{HOM}_R(M,M) = \mathrm{END}_R(M)$ is a G-graded ring and M has a G-graded (R,S)-bimodule structure, in sense that $R_x \cdot M_y \cdot S_z \subseteq M_{xyz}$, for every $x, y, z \in G$. If T is a G-graded unital left S-module, then the unital left R-module $M \otimes_S T$ has a G-graded left R-module structure, where the homogeneous component at x is $(M \otimes_S T)_x = \{\sum_{yz=x} m_y \otimes t_z \mid m_y \in M_y, t_z \in T_z\}$. For $x \in G$, we denote by M^x the left R-module M endowed with a new grading given by $(M^x)_y = M_{yx}$, for all $y \in G$.

If $Q \in R$-$\mathrm{Mod}_{\mathrm{gr}}$ with $\mathrm{END}_R(Q) = S$, then

$$\mathrm{F}(-) = \mathrm{HOM}_R(Q,-) : R\text{-}\mathrm{Mod}_{\mathrm{gr}} \rightleftarrows S\text{-}\mathrm{Mod}_{\mathrm{gr}} : Q \otimes_S - = \mathrm{G}(-)$$

is a pair of additive and covariant functors. Moreover, the functor $Q \otimes_S -$ is a left adjoint to the functor $\mathrm{HOM}_R(Q,-)$. □

Example 2.1.3. Let C be a coalgebra over a commutative ring R with identity. We denote by M^C the category of all right C-comodules. This category is a Grothendieck category if and only if C is flat as R-module. A right C-comodule M is called *quasi-finite* if the functor $- \otimes_R M : \text{Mod-}R \to M^C$ has a left adjoint.

If Q is a quasi-finite right C-comodule and $D = h(Q,Q)$ is the coendomorphism coalgebra, then

$$\mathrm{F}(-) = - \boxdot_D Q : M^D \rightleftarrows M^C : H_C(Q,-) = \mathrm{G}(-)$$

is a pair of additive and covariant functors, where $- \boxdot_D Q$ is the cotensor functor and $H_C(Q,-)$ is the cohom functor induced by Q. Moreover, $H_C(Q,-)$ is left adjoint to $- \boxdot_D Q$. □

The following example is used by Breaz in [10].

Example 2.1.4. Let R be a ring and Σ be a multiplicatively closed set of non-zero integers. We consider the category of fractions $\mathbb{Z}[\Sigma^{-1}]\text{Mod-}R$ which has as objects all the right R-modules and if $M, N \in$ Mod-R, then $\mathrm{Hom}_{\mathbb{Z}[\Sigma^{-1}]\text{Mod-}R}(M,N) = \mathbb{Z}[\Sigma^{-1}] \otimes_{\mathbb{Z}} \mathrm{Hom}_R(M,N)$. There is a canonical functor $\mathbf{q}$: Mod-$R \to \mathbb{Z}[\Sigma^{-1}]$Mod-$R$. By [29], every pair of adjoint functors F : Mod-$R \rightleftarrows$ Mod-S : G induces a canonical pair of adjoint functors $qF : \mathbb{Z}[\Sigma^{-1}]\text{Mod-}R \rightleftarrows \mathbb{Z}[\Sigma^{-1}]\text{Mod-}S : qG$ such that $\mathbf{q}F = (qF)\mathbf{q}$ and $\mathbf{q}G = (qG)\mathbf{q}$ (here $\mathbf{q}$ denotes both the canonical functors Mod-$R \to \mathbb{Z}[\Sigma^{-1}]$Mod-$R$ and Mod-$S \to \mathbb{Z}[\Sigma^{-1}]$Mod-$S$).

Starting with the setting presented in Example 2.1.1, we have that

$$\mathrm{F}(-) = q\mathrm{Hom}_R(Q,-) : \mathbb{Z}[\Sigma^{-1}]\text{Mod-}R \rightleftarrows \mathbb{Z}[\Sigma^{-1}]\text{Mod-}S : q(- \otimes_S Q) = \mathrm{G}(-)$$

is a pair of adjoint covariant functors. □

An object $M \in \mathcal{A}$ (respectively, $X \in \mathcal{B}$) is called *ϕ-faithful* (respectively, *θ-faithful*) if ϕ_M (respectively, θ_X) is a monomorphism. We denote by Faith_ϕ (respectively, Faith_θ) the class of all ϕ-faithful (respectively, θ-faithful) objects. An object $M \in \mathcal{A}$ (respectively, $X \in \mathcal{B}$) is called *ϕ-generated* (respectively, *θ-generated*) if ϕ_M (respectively, θ_X) is an epimorphism. We denote by Gen_ϕ (respectively, Gen_θ) the class of all ϕ-generated (respectively, θ-generated) objects. An object $M \in \mathcal{A}$ (respectively, $X \in \mathcal{B}$) is called F-*static* (respectively, F-*adstatic*) if ϕ_M (respectively, θ_X) is an isomorphism. We denote by Stat_F (respectively, by $\mathrm{Adstat}_\mathrm{F}$) the class of all F-static (respectively, F-adstatic) objects.

2.2. **Preliminaries.** In the following lemma, we prove some closure properties of the classes defined in Section 2.1. This result is quite often used throughout this chapter.

Lemma 2.2.1. *The following assertions hold:*

(a) $\mathrm{F}(\mathcal{A}) \subseteq \mathrm{Faith}_\theta$ *and* $\mathrm{G}(\mathcal{B}) \subseteq \mathrm{Gen}_\phi$*;*
(b) $\mathrm{F}(\mathrm{Stat}_\mathrm{F}) = \mathrm{Adstat}_\mathrm{F}$ *and* $\mathrm{G}(\mathrm{Adstat}_\mathrm{F}) = \mathrm{Stat}_\mathrm{F}$*;*
(c) *The class* Gen_ϕ *is closed with respect to factors;*
(d) *The class* Faith_θ *is closed with respect to subobjects;*
(e) Stat_F *and* $\mathrm{Adstat}_\mathrm{F}$ *are closed with respect to finite direct sums and direct summands.*

Moreover, if U is an F*-static object with* $\mathrm{F}(U) = V$ *then:*

(f) $\mathrm{add}(U) \subseteq \mathrm{Stat}_\mathrm{F}$ *and* $\mathrm{add}(V) \subseteq \mathrm{Adstat}_\mathrm{F}$*;*

(g) $\mathrm{F}(\mathrm{add}(U)) = \mathrm{add}(V)$ *and* $\mathrm{G}(\mathrm{add}(V)) = \mathrm{add}(U)$.

Corollary 2.2.2. *If* $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ *is an exact sequence in* $\mathcal{B}$*, then* $\mathrm{ImG}(f) = \mathrm{KerG}(g) \in \mathrm{Gen}_\phi$.

Lemma 2.2.3. *If* $0 \to K \xrightarrow{f} M \xrightarrow{g} N \to 0$ *is an exact sequence in* $\mathcal{A}$*, then the unique morphism* β*, for which the following diagram with exact rows*

$$\begin{array}{ccccccccc} & & \mathrm{GF}(K) & \xrightarrow{\mathrm{GF}(f)} & \mathrm{GF}(M) & \xrightarrow{\mathrm{G}(\pi)} & \mathrm{G}(\mathrm{ImF}(g)) & \longrightarrow & 0 \\ & & {\scriptstyle \phi_K}\downarrow & & \downarrow{\scriptstyle \phi_M} & & \downarrow{\scriptstyle \beta} & & \\ 0 & \longrightarrow & K & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0 \end{array}$$

is commutative, is given by the formula $\beta = \phi_N \circ \mathrm{G}(\sigma)$*, where* π *and* σ *comes from the canonical decomposition of* $\mathrm{F}(g)$.

Lemma 2.2.4. *If* $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ *is an exact sequence in* $\mathcal{B}$*, then the unique morphism* α*, for which the following diagram with exact rows*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & 0 \\ & & {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \theta_Y} & & \downarrow{\scriptstyle \theta_Z} & & \\ 0 & \longrightarrow & \mathrm{F}(\mathrm{ImG}(f)) & \xrightarrow{\mathrm{F}(\sigma)} & \mathrm{FG}(Y) & \xrightarrow{\mathrm{FG}(g)} & \mathrm{FG}(Z) & & \end{array}$$

is commutative, is given by the formula $\alpha = \mathrm{F}(\pi) \circ \theta_X$*, where* π *and* σ *comes from the canonical decomposition of* $\mathrm{G}(f)$.

Next, we list some lemmas which characterizes F-static (respectively, F-adstatic) terms of short exact sequences.

Lemma 2.2.5. *Let* $0 \to K \xrightarrow{f} M \xrightarrow{g} N \to 0$ *be an exact sequence in* $\mathcal{A}$*, with* $M \in \mathrm{Stat}_\mathrm{F}$ *and* $\mathrm{F}(g)$ *an epimorphism. Then* $K \in \mathrm{Gen}_\phi$ *if and only if* $N \in \mathrm{Stat}_\mathrm{F}$.

Lemma 2.2.6. *Let* $0 \to K \xrightarrow{f} M \xrightarrow{g} N \to 0$ *be an exact sequence in* $\mathcal{A}$*, with* $M \in \mathrm{Stat}_\mathrm{F}$ *and* $K \in \mathrm{Gen}_\phi$*. Then* $\mathrm{F}(g)$ *is an epimorphism if and only if* $\mathrm{ImF}(g) \in \mathrm{Adstat}_\mathrm{F}$.

Lemma 2.2.7. *Let* $0 \to K \xrightarrow{f} M \xrightarrow{g} N \to 0$ *be an exact sequence in* $\mathcal{A}$*, with* $M \in \mathrm{Stat}_\mathrm{F}$ *and* $K \in \mathrm{Gen}_\phi$*. Then* $K \in \mathrm{Stat}_\mathrm{F}$ *if and only if* $\mathrm{GF}(f)$ *is a monomorphism.*

Lemma 2.2.8. *Let* $0 \to K \xrightarrow{f} M \xrightarrow{g} N \to 0$ *be an exact sequence in* $\mathcal{A}$*, with* $M \in \mathrm{Stat}_\mathrm{F}$ *and* $K \in \mathrm{Gen}_\phi$*. Then* $N \in \mathrm{Stat}_\mathrm{F}$ *if and only if* $\mathrm{GF}(g)$ *is an epimorphism.*

Lemma 2.2.9. *Let* $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ *be an exact sequence in* $\mathcal{B}$ *such that* $Y \in \mathrm{Adstat}_{\mathrm{F}}$ *and* $\mathrm{G}(f)$ *is a monomorphism. Then* $Z \in \mathrm{Faith}_{\theta}$ *if and only if* $X \in \mathrm{Adstat}_{\mathrm{F}}$*.*

Lemma 2.2.10. *Let* $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ *be an exact sequence in* $\mathcal{B}$ *such that* $Y \in \mathrm{Adstat}_{\mathrm{F}}$ *and* $Z \in \mathrm{Faith}_{\theta}$*. Then* $\mathrm{G}(f)$ *is a monomorphism if and only if* $\mathrm{ImG}(f) = \mathrm{KerG}(g) \in \mathrm{Stat}_{\mathrm{F}}$*.*

Lemma 2.2.11. *Let* $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ *be an exact sequence in* $\mathcal{B}$*, with* $Y \in \mathrm{Adstat}_{\mathrm{F}}$ *and* $Z \in \mathrm{Faith}_{\theta}$*. Then* $Z \in \mathrm{Adstat}_{\mathrm{F}}$ *if and only if* $\mathrm{FG}(g)$ *is an epimorphism.*

Lemma 2.2.12. *Let* $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ *be an exact sequence in* $\mathcal{B}$*, with* $Y \in \mathrm{Adstat}_{\mathrm{F}}$ *and* $Z \in \mathrm{Faith}_{\theta}$*. Then* $X \in \mathrm{Adstat}_{\mathrm{F}}$ *if and only if* $\mathrm{FG}(f)$ *is a monomorphism.*

If M is an object in $\mathcal{A}$, then the object $\mathrm{Im}(\phi_M)$ is called F-*socle of* M and it is denoted by $\mathrm{S}_{\mathrm{F}}(M)$. If X is an object in $\mathcal{B}$, then the object $\mathrm{Ker}(\theta_X)$ is called F-*radical of* X and it is denoted by $\mathrm{R}_{\mathrm{F}}(X)$.

From the identities $\mathrm{F}(\phi_M) \circ \theta_{\mathrm{F}(M)} = 1_{\mathrm{F}(M)}$ and $\phi_{\mathrm{G}(X)} \circ \mathrm{G}(\theta_X) = 1_{\mathrm{G}(X)}$, we have that $\theta_{\mathrm{F}(M)}$, $\mathrm{G}(\theta_X)$ are monomorphisms and $\mathrm{F}(\phi_M)$, $\phi_{\mathrm{G}(X)}$ are epimorphisms.

Lemma 2.2.13. *Let* $M \in \mathcal{A}$ *and* $X \in \mathcal{B}$*. The following assertions hold:*

(a) *If* $i : \mathrm{S}_{\mathrm{F}}(M) \to M$ *is the canonical inclusion, then* $\mathrm{F}(i) : \mathrm{F}(\mathrm{S}_{\mathrm{F}}(M)) \to \mathrm{F}(M)$ *is an isomorphism;*

(b) *If* $q : X \to X/\mathrm{R}_{\mathrm{F}}(X)$ *is the canonical epimorphism, then* $\mathrm{G}(q) : \mathrm{G}(X) \to \mathrm{G}(X/\mathrm{R}_{\mathrm{F}}(X))$ *is an isomorphism.*

Lemma 2.2.14. *Let* $M \in \mathcal{A}$ *and* $Y \in \mathcal{B}$*. The following assertions hold:*

(a) *If* K *is a subobject of* M*, with* $K \in \mathrm{Gen}_{\phi}$*, then* K *is a subobject of* $\mathrm{S}_{\mathrm{F}}(M)$*;*

(b) *If* X *is a subobject of* Y*, with* $Y/X \in \mathrm{Faith}_{\theta}$*, then* $\mathrm{R}_{\mathrm{F}}(Y)$ *is a subobject of* X*;*

(c) *If* $f : X \to Y$ *is a monomorphism, i.e.* X *is a subobject of* Y*, such that* $\mathrm{G}(f) = 0$*, then* X *is a subobject of* $\mathrm{R}_{\mathrm{F}}(Y)$*.*

Lemma 2.2.15. *If* $M \in \mathcal{A}$ *and* $X \in \mathcal{B}$*, then:*

(a) (1) $\mathrm{S}_{\mathrm{F}}(M) \in \mathrm{Gen}_{\phi}$*;*

(2) $\mathrm{S_F}(M) \in \mathrm{Faith}_\phi$ *if and only if* $M \in \mathrm{Faith}_\phi$;

(b) (1) $X/\mathrm{R_F}(X) \in \mathrm{Faith}_\theta$;

(2) $X/\mathrm{R_F}(X) \in \mathrm{Gen}_\theta$ *if and only if* $X \in \mathrm{Gen}_\theta$.

Remark 2.2.16. If $M \in \mathcal{A}$ and $X \in \mathcal{B}$ then:

(i) $\mathrm{S_F}(M)$ is the biggest ϕ-generated subobject of M;

(ii) $\mathrm{R_F}(X)$ is the smallest subobject of X such that $X/\mathrm{R_F}(X) \in \mathrm{Faith}_\theta$.

Lemma 2.2.17. *Let* $M \in \mathcal{A}$ *and* $X \in \mathcal{B}$. *Then:*

(a) *If* $M \in \mathrm{Gen}_\phi$, *then* $M/\mathrm{S_F}(M) \in \mathrm{Gen}_\phi$;

(b) *If* $X \in \mathrm{Faith}_\theta$, *then* $\mathrm{R_F}(X) \in \mathrm{Faith}_\theta$.

2.3. **Closure Properties with Respect to θ-Faithful Factors.**

Proposition 2.3.1. *Let* Y *be an* F-*adstatic object. The following statements are equivalent:*

(a) *If* Z *is a* θ-*faithful factor of* Y, *then* $Z \in \mathrm{Adstat_F}$;

(b) *If* $0 \to K \xrightarrow{f} \mathrm{G}(Y) \xrightarrow{g} N \to 0$ *is an exact sequence in* $\mathcal{A}$ *with* $K \in \mathrm{Gen}_\phi$, *then* $\mathrm{F}(g)$ *is an epimorphism.*

Corollary 2.3.2. *Let* Y *be an* F-*adstatic object which satisfies the equivalent conditions from the previous result. If* $K \in \mathrm{Gen}_\phi$ *is a subobject of* $\mathrm{G}(Y)$ *then* $\mathrm{G}(Y)/K$ *is* F-*static.*

Proposition 2.3.3. *Let* $\mathrm{F} : \overline{\mathcal{A}} \rightleftarrows \overline{\mathcal{B}} : \mathrm{G}$ *be an equivalence between the full additive subcategories* $\overline{\mathcal{A}}$ *and* $\overline{\mathcal{B}}$ *of* $\mathcal{A}$ *and* $\mathcal{B}$, *respectively. The following statements are equivalent:*

(a) $\overline{\mathcal{B}}$ *is closed under* θ-*faithful factors;*

(b) (1) $\overline{\mathcal{A}}$ *is closed with respect to factors modulo* ϕ-*generated subobjects;*

(2) F *is exact with respect to the short exact sequences* $0 \to K \xrightarrow{f} M \xrightarrow{g} N \to 0$ *with* $M \in \overline{\mathcal{A}}$ *and* $K \in \mathrm{Gen}_\phi$.

Theorem 2.3.4. *Let* $\mathcal{B}_0$ *be a full additive subcategory of* $\mathcal{B}$ *consisting in* F-*adstatic objects and let* $\mathcal{A}_0 = \mathrm{G}(\mathcal{B}_0)$. *Let* $\overline{\mathcal{B}}$ *be the class of all* θ-*faithful factors of objects in*

$\mathcal{B}_0$ *and let* $\overline{\mathcal{A}} = \{M/K \mid M \in \mathcal{A}_0,\ K \in \mathrm{Gen}_\phi\}$. *Then the following statements are equivalent:*

(a) $\mathrm{F} : \overline{\mathcal{A}} \rightleftarrows \overline{\mathcal{B}} : \mathrm{G}$ *is an equivalence and* $\overline{\mathcal{B}}$ *is closed under* θ*-faithful factors;*

(b) $\overline{\mathcal{B}} \subseteq \mathrm{Adstat}_{\mathrm{F}}$*;*

(c) *If* $0 \to K \xrightarrow{f} M \xrightarrow{g} N \to 0$ *is an exact sequence in* $\mathcal{A}$*, with* $M \in \mathcal{A}_0$ *and* $K \in \mathrm{Gen}_\phi$*, then* $\mathrm{F}(g)$ *is an epimorphism.*

Example 2.3.5. Let U be an F-static object with $\mathrm{F}(U) = V$. Since V is F-adstatic, hence V^k is F-adstatic, for all positive integers k, we could consider $\mathcal{B}_0 = \{V^k \mid k \in \mathbb{N}^*\}$. Then $\mathcal{A}_0 = \{U^k \mid k \in \mathbb{N}^*\}$. We observe that $\overline{\mathcal{B}} = \mathrm{gen}(V) \cap \mathrm{Faith}_\theta$ and the class $\overline{\mathcal{A}}$ consists in all objects $N \in \mathcal{A}$ such that $N = U^n/K$ with $K \in \mathrm{Gen}_\phi$ and $n \in \mathbb{N}^*$.

Example 2.3.6. Let R be an unital associative ring and let Q be a right R-module. If $S = \mathrm{End}_R(Q)$ is the endomorphism ring of Q, then Q has a structure of (S, R)-bimodule and, as we seen in Example 2.1.1, we have the pair $\mathrm{F}(-) = \mathrm{Hom}_R(Q, -) : \text{Mod-}R \rightleftarrows \text{Mod-}S : - \otimes_S Q = \mathrm{G}(-)$ of additive and covariant functors. Moreover, the right R-module Q_R is $\mathrm{Hom}_R(Q, -)$-static and the right S-module S_S is $\mathrm{Hom}_R(Q, -)$-adstatic.

(i) If we set $\mathcal{B}_0 = \{S^k \mid k \in \mathbb{N}^*\}$ then we have $\mathcal{A}_0 = \{Q^k \mid k \in \mathbb{N}^*\}$, $\overline{\mathcal{B}} = \mathrm{gen}(S) \cap \mathrm{Faith}_\theta$ and $\overline{\mathcal{A}}$ consists in the class of all right R-modules N such that $N = Q^n/K$, for some $K \in \mathrm{Gen}_\phi$ and $n \in \mathbb{N}^*$;

(ii) If we set $\mathcal{B}_0 = \{S\}$ then we have $\mathcal{A}_0 = \{Q\}$, $\overline{\mathcal{B}} = \{Z \in \mathcal{B} \mid Z = S/X\} \cap \mathrm{Faith}_\theta$ and $\overline{\mathcal{A}}$ consists in the class of all right R-modules N such that $N = Q/K$, for some $K \in \mathrm{Gen}_\phi$.

Example 2.3.7. Let G be a group and let $R = \bigoplus_{x \in G} R_x$ be a G-graded ring. Let $Q \in R\text{-Mod}_{\mathrm{gr}}$ with $S = \mathrm{END}_R(Q)$. Then S is a G-graded ring and Q is a G-graded (R, S)-bimodule. As we saw in Example 2.1.2, we have the pair $\mathrm{F}(-) = \mathrm{HOM}_R(Q, -) : R\text{-Mod}_{\mathrm{gr}} \rightleftarrows S\text{-Mod}_{\mathrm{gr}} : Q \otimes_S - = \mathrm{G}(-)$ of additive and covariant functors. If Q is *gr-self-small*, i.e. $\mathrm{HOM}_R(Q, -)$ preserves coproducts of $\bigoplus_{x \in G} Q^x$, then $\bigoplus_{x \in G} Q^x$ is F-static. Moreover, $\mathrm{HOM}_R(Q, \bigoplus_{x \in G} Q^x) = \bigoplus_{x \in G} S^x$. Denoting $\bigoplus_{x \in G} Q^x$ by U and $\bigoplus_{x \in G} S^x$ by V, we have:

(i) If we set $\mathcal{B}_0 = \{V^k \mid k \in \mathbb{N}^*\}$ then we have $\mathcal{A}_0 = \{U^k \mid k \in \mathbb{N}^*\}$, $\overline{\mathcal{B}} = \mathrm{gen}(V) \cap \mathrm{Faith}_\theta$ and $\overline{\mathcal{A}}$ consists in the class of all G-graded unital left R-modules N such that $N = U^n/K$, for some $K \in \mathrm{Gen}_\phi$ and $n \in \mathbb{N}^*$;

(ii) If we set $\mathcal{B}_0 = \{V\}$ then we have $\mathcal{A}_0 = \{U\}$, $\overline{\mathcal{B}} = \{Z \in \mathcal{B} \mid Z = V/X\} \cap \mathrm{Faith}_\theta$ and $\overline{\mathcal{A}}$ consists in the class of all G-graded unital left R-modules N such that $N = U/K$, for some $K \in \mathrm{Gen}_\phi$.

Application. The case add(U).

Let $U \in \mathrm{Stat}_\mathrm{F}$ with $\mathrm{F}(U) = V$. If we set $\mathcal{B}_0 = \mathrm{add}(V)$ we have, by Lemma 2.2.1, that $\mathcal{B}_0 \subseteq \mathrm{Adstat}_\mathrm{F}$ and $\mathcal{A}_0 = \mathrm{add}(U)$. It is easy to show that $\overline{\mathcal{B}} = \mathrm{gen}(V) \cap \mathrm{Faith}_\theta$. Moreover, in this setting, we can see that $\overline{\mathcal{A}} = \{M/K \mid M \in \mathrm{add}(U),\ K \in \mathrm{Gen}_\phi\}$.

Example 2.3.8. (1) With the settings presented in Example 2.3.6, we could consider U to be the right R-module Q. Then V is the right S-module S. It follows that $\mathcal{B}_0 = \mathrm{add}(S)$, $\mathcal{A}_0 = \mathrm{add}(Q)$, $\overline{\mathcal{B}} = \mathrm{gen}(S) \cap \mathrm{Faith}_\theta$ and $\overline{\mathcal{A}} = \{M/K \mid M \in \mathrm{add}(Q),\ K \in \mathrm{Gen}_\phi\}$;

(2) Using the settings from Example 2.3.7, we have $\mathcal{B}_0 = \mathrm{add}(V)$, $\mathcal{A}_0 = \mathrm{add}(U)$, $\overline{\mathcal{B}} = \mathrm{gen}(V) \cap \mathrm{Faith}_\theta$ and $\overline{\mathcal{A}} = \{M/K \mid M \in \mathrm{add}(U),\ K \in \mathrm{Gen}_\phi\}$.

Corollary 2.3.9. *Let* $U \in \mathrm{Stat}_\mathrm{F}$ *with* $\mathrm{F}(U) = V$. *Let* $\mathcal{B}^\star = \mathrm{gen}(V) \cap \mathrm{Faith}_\theta$ *and let* $\mathcal{A}^\star = \{M/K \mid M \in \mathrm{add}(U),\ K \in \mathrm{Gen}_\phi\}$. *Then the following statements are equivalent:*

(a) $\mathrm{F} : \mathcal{A}^\star \rightleftarrows \mathcal{B}^\star : \mathrm{G}$ *is an equivalence and* $\mathcal{B}^\star$ *is closed under* θ*-faithful factors;*

(b) $\mathcal{B}^\star \subseteq \mathrm{Adstat}_\mathrm{F}$*;*

(c) *If* $0 \to K \xrightarrow{f} M \xrightarrow{g} N \to 0$ *is an exact sequence in* $\mathcal{A}$ *with* $M \in \mathrm{add}(U)$ *and* $K \in \mathrm{Gen}_\phi$*, then* $\mathrm{F}(g)$ *is an epimorphism;*

(d) *If* $0 \to K \xrightarrow{f} U^n \xrightarrow{g} N \to 0$ *is an exact sequence in* $\mathcal{A}$ *with* $K \in \mathrm{Gen}_\phi$*, then* $\mathrm{F}(g)$ *is an epimorphism.*

Corollary 2.3.10. *Let* $\overline{\mathcal{A}}$ *and* $\overline{\mathcal{B}}$ *be full additive subcategories of* $\mathcal{A}$ *and* $\mathcal{B}$*, respectively. Let* $U \in \overline{\mathcal{A}}$ *with* $\mathrm{F}(U) = V$. *Assume that* $V^k \in \overline{\mathcal{B}}$*, for all positive integers* k. *Let* $\mathcal{B}^\star = \mathrm{gen}(V) \cap \mathrm{Faith}_\theta$ *and let* $\mathcal{A}^\star = \{M/K \mid M \in \mathrm{add}(U),\ K \in \mathrm{Gen}_\phi\}$. *If* $\mathrm{F} : \overline{\mathcal{A}} \rightleftarrows \overline{\mathcal{B}} : \mathrm{G}$ *is an equivalence with* $\overline{\mathcal{B}}$ *closed under* θ*-faithful factors, then* $\mathrm{F} : \mathcal{A}^\star \rightleftarrows \mathcal{B}^\star : \mathrm{G}$ *is an equivalence with* $\mathcal{B}^\star$ *closed under* θ*-faithful factors.*

For the next results, we assume that the right derived functors of F does exist. For example, we could consider that the category $\mathcal{A}$ has enough injectives or is a Grothendieck category. We denote by $\mathrm{R}^j\mathrm{F}$ the j-th right derived functor of F. If n is a positive integer, we also consider the perpendicular class ${}^{\perp_{=n}}\mathcal{A}$ of all objects $M \in \mathcal{A}$ for which $\mathrm{R}^n\mathrm{F}(M) = 0$.

Corollary 2.3.11. *Let $\overline{\mathcal{A}}$ and $\overline{\mathcal{B}}$ be full additive subcategories of $\mathcal{A}$ and $\mathcal{B}$, respectively. Let $U \in \mathcal{A}$. Assume that $\mathrm{R}^1\mathrm{F}(U) = 0$ and $U^k \in \overline{\mathcal{A}}$, for all positive integers k. If $\mathrm{F} : \overline{\mathcal{A}} \rightleftarrows \overline{\mathcal{B}} : \mathrm{G}$ is an equivalence with the class $\overline{\mathcal{B}}$ closed under θ-faithful factors, then $\mathrm{cog}(U) \cap \mathrm{Gen}_\phi \subseteq {}^{\perp_{=1}}\mathcal{A}$.*

Proposition 2.3.12. *Let $U \in \mathrm{Stat}_\mathrm{F}$ with $\mathrm{F}(U) = V$. Assume that $\mathrm{R}^1\mathrm{F}(U) = 0$. Let $\mathcal{B}^\star = \mathrm{gen}(V) \cap \mathrm{Faith}_\theta$ and let $\mathcal{A}^\star = \{M/K \mid M \in \mathrm{add}(U),\ K \in \mathrm{Gen}_\phi\}$. Then the following statements hold:*

(a) *$\mathcal{B}^\star \subseteq \mathrm{Adstat}_\mathrm{F}$ if and only if $\mathrm{cog}(U) \cap \mathrm{Gen}_\phi \subseteq {}^{\perp_{=1}}\mathcal{A}$;*

(b) *$\mathrm{F} : \mathcal{A}^\star \rightleftarrows \mathcal{B}^\star : \mathrm{G}$ is an equivalence and $\mathcal{B}^\star$ is closed under θ-faithful factors if and only if $\mathrm{cog}(U) \cap \mathrm{Gen}_\phi \subseteq {}^{\perp_{=1}}\mathcal{A}$.*

Proposition 2.3.13. *Let $\mathrm{F} : \overline{\mathcal{A}} \rightleftarrows \overline{\mathcal{B}} : \mathrm{G}$ be an equivalence between the full additive subcategories $\overline{\mathcal{A}}$ and $\overline{\mathcal{B}}$ of $\mathcal{A}$ and $\mathcal{B}$, respectively. The following statements are equivalent:*

(a) (1) *$\overline{\mathcal{B}}$ is closed under θ-faithful factors;*

(2) *$\overline{\mathcal{A}} = \mathrm{Gen}_\phi \cap \mathrm{F}^{-1}(\overline{\mathcal{B}})$;*

(b) (1) *$\overline{\mathcal{A}}$ is closed with respect to ϕ-faithful factors modulo ϕ-generated subobjects;*

(2) *If $0 \to K \xrightarrow{f} M \xrightarrow{g} N \to 0$ is an exact sequence in $\mathcal{A}$, with $M \in \overline{\mathcal{A}}$, then $\mathrm{F}(g)$ is an epimorphism if and only if $K \in \mathrm{Gen}_\phi$.*

Theorem 2.3.14. *Let $\mathcal{B}_0$ be a full additive subcategory of $\mathcal{B}$, consisting of F-adstatic objects and let $\mathcal{A}_0 = \mathrm{G}(\mathcal{B}_0)$. Assume that $\mathcal{B}_0$ is closed under finite direct sums. Let $\overline{\mathcal{B}}$ be the class of all θ-faithful factors of objects in $\mathrm{gen}(\mathcal{B}_0)$ and let $\overline{\mathcal{A}} = \mathrm{Gen}_\phi \cap \mathrm{F}^{-1}(\overline{\mathcal{B}})$. Let $\overline{\overline{\mathcal{A}}} = \{M/K \mid M \in \mathcal{A}_0, K \in \mathrm{Gen}_\phi\}$. The following statements are equivalent:*

(a) *$\mathrm{F} : \overline{\mathcal{A}} \rightleftarrows \overline{\mathcal{B}} : \mathrm{G}$ is an equivalence and $\overline{\mathcal{B}}$ is closed under θ-faithful factors;*

(b) (1) $\mathrm{F} : \overline{\overline{\mathcal{A}}} \rightleftarrows \overline{\mathcal{B}} : \mathrm{G}$ *is an equivalence and* $\overline{\mathcal{B}}$ *is closed under* θ*-faithful factors;*

(2) $\overline{\mathcal{A}} = \overline{\overline{\mathcal{A}}}$*;*

(c) (1) ϕ_M *is a monomorphism, for all* $M \in \mathcal{A}$ *with* $\mathrm{F}(M) \in \overline{\mathcal{B}}$*;*

(2) θ_X *is an epimorphism, for all* $X \in \mathcal{B}$ *with* $X \in \mathrm{gen}(\mathcal{B}_0)$*;*

(d) (1) $\overline{\mathcal{A}} \subseteq \overline{\overline{\mathcal{A}}}$*;*

(2) *If* $0 \to K \xrightarrow{f} M \xrightarrow{g} N \to 0$ *is an exact sequence in* $\mathcal{A}$*, with* $M \in \mathcal{A}_0$ *and* $K \in \mathrm{Gen}_\phi$*, then* $\mathrm{F}(g)$ *is an epimorphism;*

(e) (1) $\overline{\mathcal{A}} \subseteq \overline{\overline{\mathcal{A}}}$*;*

(2) *If* $0 \to K \xrightarrow{f} M \xrightarrow{g} N \to 0$ *is an exact sequence in* $\mathcal{A}$*, with* $M \in \mathcal{A}_0$*, then* $\mathrm{F}(g)$ *is an epimorphism if and only if* $K \in \mathrm{Gen}_\phi$*.*

Applications Let $U \in \mathrm{Stat}_{\mathrm{F}}$ with $\mathrm{F}(U) = V$. Since $\mathrm{add}(V) \subseteq \mathrm{Adstat}_{\mathrm{F}}$ and $\mathrm{add}(V)$ is closed under finite direct sums, we could consider $\mathcal{B}_0 = \mathrm{add}(V)$. Then $\mathcal{A}_0 = \mathrm{add}(U)$. One can show that $\overline{\mathcal{B}} = \mathrm{gen}(V) \cap \mathrm{Faith}_\theta$. Moreover, we have that $\overline{\mathcal{A}} = \mathrm{Gen}_\phi \cap \mathrm{F}^{-1}(\overline{\mathcal{B}})$ and $\overline{\overline{\mathcal{A}}} = \{M/K \mid M \in \mathrm{add}(U), K \in \mathrm{Gen}_\phi\}$. Now, Theorem 2.3.14 becomes:

Theorem 2.3.15. *Let* $U \in \mathrm{Stat}_{\mathrm{F}}$ *with* $\mathrm{F}(U) = V$*. Let* $\mathcal{B}^\star = \mathrm{gen}(V) \cap \mathrm{Faith}_\theta$ *and let* $\mathcal{A}^\star = \{M/K \mid M \in \mathrm{add}(U), K \in \mathrm{Gen}_\phi\}$*. Let* $\overline{\mathcal{A}} = \mathrm{Gen}_\phi \cap \mathrm{F}^{-1}(\mathcal{B}^\star)$*. Then the following statements are equivalent:*

(a) $\mathrm{F} : \overline{\mathcal{A}} \rightleftarrows \mathcal{B}^\star : \mathrm{G}$ *is an equivalence and* $\mathcal{B}^\star$ *is closed under* θ*-faithful factors;*

(b) (1) $\mathrm{F} : \mathcal{A}^\star \rightleftarrows \mathcal{B}^\star : \mathrm{G}$ *is an equivalence and* $\mathcal{B}^\star$ *is closed under* θ*-faithful factors;*

(2) $\overline{\mathcal{A}} = \mathcal{A}^\star$*;*

(c) (1) ϕ_M *is a monomorphism, for all* $M \in \mathcal{A}$ *with* $\mathrm{F}(M) \in \mathrm{gen}(V)$*;*

(2) θ_X *is an epimorphism, for all* $X \in \mathcal{B}$ *with* $X \in \mathrm{gen}(V)$*;*

(d) (1) $\overline{\mathcal{A}} \subseteq \mathcal{A}^\star$*;*

(2) *If* $0 \to K \xrightarrow{f} U^n \xrightarrow{g} N \to 0$ *is an exact sequence in* $\mathcal{A}$ *with* $K \in \mathrm{Gen}_\phi$*, then* $\mathrm{F}(g)$ *is an epimorphism;*

(e) (1) $\overline{\mathcal{A}} \subseteq \mathcal{A}^\star$*;*

(2) *If* $0 \to K \xrightarrow{f} U^n \xrightarrow{g} N \to 0$ *is an exact sequence in* $\mathcal{A}$ *then* $K \in \mathrm{Gen}_\phi$ *if and only if* $\mathrm{F}(g)$ *is an epimorphism.*

Corollary 2.3.16. *Let $\overline{\mathcal{A}}$ and $\overline{\mathcal{B}}$ be full additive subcategories of $\mathcal{A}$ and $\mathcal{B}$, respectively. Let $U \in \mathcal{A}$ with $\mathrm{F}(U) = V$. Assume that $U^k \in \overline{\mathcal{A}}$, for all positive integers k and assume that $\mathrm{R}^1\mathrm{F}(U) = 0$. If $\mathrm{F} : \overline{\mathcal{A}} \rightleftarrows \overline{\mathcal{B}} : \mathrm{G}$ is an equivalence such that $\overline{\mathcal{A}} = \mathrm{Gen}_\phi \cap \mathrm{F}^{-1}(\overline{\mathcal{B}})$ and $\overline{\mathcal{B}}$ is closed under θ-faithful factors, then $\mathrm{cog}(U) \cap \mathrm{Gen}_\phi = \mathrm{cog}(U) \cap {}^{\perp_{=1}}\mathcal{A}$.*

We denote by $\mathrm{L}_j\mathrm{G}$ the j-th left derived functor of G. We also consider the perpendicular class ${}^{=n\perp}\mathcal{B} = \{X \in \mathcal{B} \mid \mathrm{L}_n\mathrm{G}(X) = 0\}$, where n is an integer.

Corollary 2.3.17. *Let $\overline{\mathcal{A}}$ and $\overline{\mathcal{B}}$ be full additive subcategories of $\mathcal{A}$ and $\mathcal{B}$, respectively. Let $U \in \mathcal{A}$ with $\mathrm{F}(U) = V$. Assume that $\mathrm{L}_1\mathrm{G}(V) = 0$ and $V^k \in \overline{\mathcal{B}}$, for all positive integers k. Suppose that $\mathrm{F} : \overline{\mathcal{A}} \rightleftarrows \overline{\mathcal{B}} : \mathrm{G}$ is an equivalence with $\overline{\mathcal{A}} = \mathrm{Gen}_\phi \cap \mathrm{F}^{-1}(\overline{\mathcal{B}})$ and $\overline{\mathcal{B}}$ is closed under θ-faithful factors. Let $X \in \mathrm{pres}(V)$. Then $X \in \mathrm{Faith}_\theta$ if and only if $X \in {}^{=1\perp}\mathcal{B}$.*

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