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LINEAR OPERATORS AND WAVELETS ANALYSIS WITH APPLICATIONS

Ph.D. THESIS SUMMARY

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Contents

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Positive linear operators, A-statistical convergence, Korovkin type approximation theorem, wavelets analysis, signal, white noise, nonparametric regression, nonlinear wavelets estimators, thresholding procedure, subdivision schemes, , subdivision operator.

Introduction

The development of wavelets is fairly recent in applied mathematics, but wavelets have already had a remarkable impact, being used in a lot of situations. A wavelet is a wave function carefully constructed so that it has certain mathematical properties. An entire set of wavelets is constructed from a mother wavelet function and this set provides building block functions that can be used to describe anything in a large class of functions.

Wavelets analysis is a refinement of Fourier analysis. The Fourier transform analyzes a signal in terms of frequency components. If the analyzed signal reveals aspects like trends, breakdown points, discontinuities in higher derivatives, the Fourier analysis is ineffective for capturing the details. In this case we will analyze the signal with a flexible time-frequency window that is automatically adapted, in the sense that narrow time-window is needed to examine high-frequency content and wide time-window is allowed when investigating low-frequency components. This good time-frequency localization is the most advantage that wavelets have over other methods, especially in statistics. Standard methods in statistical function estimation (kernel smoothers or orthogonal series methods) require certain assumptions about the smoothness of the function being estimated. With wavelets, which have a spatial adaptivity property, these requirements are considerably reduced. Wavelets are connected to the notion of multiresolution analysis, in that, signals, functions, data, can be examined using a wide variety of resolution levels. Wavelets have an interdisciplinary flavor. Many of the founders of wavelets analysis concept: Yves Meyer, Jean Morlet and Alex Grossman were a mathematician, a geophysicist and physicist respectively.

This thesis presents the advantages of wavelets analysis in the context of statistics in mathematics, approach being sustained by the following remarkable wavelets properties: good time frequency localization, fast algorithms in that a big set of data could be represented through a small amount of wavelets coefficients and their simplicity of form. Also the thesis contains the development of some classes of linear and positive operators and studies their statistical approximation properties.

The thesis is structured in 3 chapters: *Statistical Concepts and Wavelets Analysis*, *Nonparametric Regression* and *Wavelets Estimators* being close to many research domains such as: numerical analysis, statistics, functional analysis, mathematical modeling and linear algebra. The correlation between these domains is realized through wavelets functions as an instrument of nonparametric regression.

The first chapter presents several general aspects of the mathematical regression model, the concept of statistical convergence and wavelets analysis and is composed of five paragraphs.

The first paragraph of this chapter contains most of the notions and notations that will be used throughout the entire work and the most important results that assumed to be known.

The second paragraph presents the mathematical regression model in the context of parametric regression. It also describes the linear model of the regression from the statistical point of view as well the performance criteria of estimators for a linear model. The estimators are evaluated from a statistical point of view by studying both their quality and inferences that can be realized on them, assuming that errors are normal, zero in average, independent and identically distributed. The estimators are statistically by studying...

The next two paragraphs introduce the concept of linear and positive operator in the context of functions approximation. They also include the summability matrix concept, the statistical convergence concept and approximation theorems of type Bohman-Korovkin in $C[a, b]$. space. The purpose of these concepts (topics?) is to build different classes of linear and positive operators of integral or discrete type and to study their statistical approximation properties.

The last paragraph of this chapter is entirely dedicated to the wavelets analysis concept. Here, essential elements related to wavelets are described: ortonormated wavelets basis, multiple resolution analysis, wavelet decomposition and reconstruction, direct and inverse wavelets transformation.

The second chapter studies the regression model using techniques of the nonparametric regression based on wavelets functions. This chapter has five paragraphs, the last four concentrating on its applicability. Nonparametric regression based on wavelets functions is a significant area of the modern statistics being studied in well known monographies: Härdle (1992), Green and Silverman (1993), Wand and Jones (1994), Donoho and Johnstone (1994), Fan (1996), Bowman and Azzalini (1997), Eubank (1999), Wasserman (2005), Antoniadis (2007).

The first paragraph describes the mathematical model of a noisy signal and introduces several methods to eliminate the noise through nonlinear wavelets estimators. These methods are based on the *thresholding* technique and the *Penalized least-squares wavelets* method.

The paragraph *A Comparative Study of Two Noise Removing Methods* presents the comparison of the following two wavelets noise removing methods: *Minimax* and *VisuShrink*. These methods are applied to a signal represented by the relative humidity. The risk is evaluated for both considered cases.

In the paragraph *Analysis of Reconstruction Methods with Wavelets Technique* it is estimated a discontinuous function affected by a white noise at levels $SNR = 4$ and $SNR = 10$. The reconstruction was made using the *penalized least-squares* method, the *cross validation* method and the *SureShrink* method. A comparative analysis of the committed errors is also done for each of the reconstruction case.

In the *Application* paragraph three signals represented by respiratory rate, heart rate and blood antioxidant enzymes level are processed using two types of *Fast Haar Wavelets Transforms*. The analyzed signals have been recorded for a group of eight cows exposed to solar radiation. The algorithms for wavelets transforms were implemented as VBA Macros in Microsoft Excel.

In the last paragraph of this chapter one can see the implementation of the *Fast Daubechies Wavelets Transform*. VBA Macros have been used for the implementation of this algorithm also. This algorithm is then applied for the processing of a real data of atmospheric temperature.

The third and last chapter presents in a more detail the wavelets estimators. In order to underline the way in which wavelets estimators are used in the nonparametric regression the following functions spaces will be used: $L^2(\mathbb{R})$, *Hölder* C^δ, $0 < \delta \leq 1$, *Sobolev* space, as well as the *Besov* and *Triebel* spaces. These last two spaces model the concept of "different smooth levels degree" in different locations more efficiently than classes of smooth functions, having a high statistical importance. Nonlinear wavelets estimators study the nonparametric regression from the *minimax* point of view having an optimal asymptotic character while the classical linear wavelets estimators are suboptimal in the case of estimation from the particular *Besov* and *Triebel* spaces. This chapter has five paragraphs. The first three paragraphs contain a more general approach of the *denoising* term taking into account the following aspects: smoothness and adaptability of the estimator reconstructed with *soft thresholding* technique.

The fourth paragraph describes a certain type of wavelets transforms that characterize the smoothness between different spaces of functions.

The last paragraph of this chapter presents a nonlinear wavelets transforms type based on the subdivision scheme. These transforms are tightly related to the construction of wavelets functions through multiresolution analysis. In this paragraph it is built also a new subdivision scheme that generates the set of quadratic polynomials.

Chapter 1

Statistical Concepts and Wavelets Analysis

1.1 Preliminaries

The purpose of this section is to collect information about the functions spaces that will be used throughout the thesis.

1.1.1 Basic Concepts and Functions Spaces

In this subsection we establish most of the notions and notations that will be used throughout the work and list, without proof, the most important results that are assumed to be known.

1.2 Mathematical Regression Model

Regression analysis has its origins in various practical tests that emerge when we look to understand cause and effect in the study of phenomena. Suppose that each element of a statistical population possesses a numerical feature X and another Y. To determine how the values of X affects variable Y accomplishments, it is necessary to study the possible existing correlations between the two variables.

Consider, as it happens in practice, many exogenous variables or several predictors $X_1, X_2, \ldots X_p$, for the variable effect Y. Mathematical regression model will be written as

$$Y = f\left(X_1, X_2, \ldots X_p\right) + \varepsilon,$$

where ε is a random variable that satisfies the following properties relative to the mean and variance: $E(\varepsilon) = 0$ and $Var(\varepsilon)$ small.

1.2.1 Linear Model. Performance Criteria

Definition 1.2.1 *The linear regressional model between the variables Y and $X_1, X_2, \dots X_p$, is defined below*

$$Y = \sum_{k=1}^{p} \alpha_k X_k + \varepsilon. \tag{1.2.1}$$

Let's consider the standard observational regression model, $y_i = f(x_i) + \varepsilon_i,\ i = \overline{1,n}$, where, $(x_i, y_i),\ i = \overline{1,n}$, are sampling data, the sample size n is a power of two $n = 2^J$, for some positive integer J, $x_i = \dfrac{i}{n}$ are equidistant, $f(x_i)$ are the values of the unknown function f, and $\varepsilon = (\varepsilon_1, \varepsilon_2, \ \dots, \ \varepsilon_n)^T$ is the white noise. Suppose that the errors are $N(0,1)$ distributed. In the context of signal theory $f(x_i)$ will be the original considered signal and y_i will be the noisy signal. We estimate the function f using regression techniques. The estimator of function f will be denoted by $\hat{f}$.

There are several performance criteria that optimized provide good estimators. These include: the loss, the risk and the risk prediction.

Definition 1.2.2 *The loss of an estimate $\hat{f}$ from f is defined as follows*

$$L(\hat{f}, f)_{l^2(\mathbb{Z})} = n^{-1} \left\| \hat{f} - f \right\|^2_{l^2(\mathbb{Z})} = n^{-1} \sum_{i=1}^{n} \left(\hat{f}(x_i) - f(x_i) \right)^2. \tag{1.2.2}$$

$$L(\hat{f}, f)_{L^2(\mathbb{R})} = \left\| \hat{f} - f \right\|^2_{L^2(\mathbb{R})} = \int_{-\infty}^{+\infty} \left(\hat{f}(x) - f(x) \right)^2 dt. \tag{1.2.3}$$

Definition 1.2.3 *The risk is the mean value of the loss*

$$R\left(\hat{f}, f\right) = E\left(L(\hat{f}, f)\right). \tag{1.2.4}$$

Definition 1.2.4 *The risk prediction is defined as folows*

$$P\left(\hat{f}, f\right) = n^{-1} \sum_{i=1}^{n} E\left(y_i^* - f(x_i))\right)^2, \tag{1.2.5}$$

where $y^* = (y_1^*, \ \dots \ y_n^*)$ are n new observations which we intend to do, $y^* = f + \varepsilon^*$, and ε^* are independently distributed as $N(0, \sigma^2)$, uncorrelated with the errors of the ε.

1.2.2 Performance Criteria Estimators

Definition 1.2.5 *An estimator $\hat{f}$ of the function f is said to be unbiased if we have*

$$E\left(\hat{f}\right) = f.$$

Definition 1.2.6 *The cross validation function of the estimator $\hat{f}$ is defined below*

$$CV(f) = n^{-1}\sum_{i=1}^{n} E\left(y_i - \hat{f}_i(x_i)\right)^2, \tag{1.2.6}$$

where $\hat{f}_i$ is the sample estimator of order i, $i = \overline{1,n}$, obtained by eliminating point (x_i, y_i) from the sample.

Definition 1.2.7 *An estimator $\hat{f}$ of the function f is said to be unbiased if we have*

$$\lim_{n\to\infty} P\left(\left|\hat{f}_n - f\right| < \varepsilon\right) = 1,\ \forall\varepsilon > 0, \tag{1.2.7}$$

where the notation $\hat{f}_n$ shows that the estimator depends on the sample size n.

Definition 1.2.8 *An estimator $\hat{f}'$ is called minimax if its maximal risk is minimal among all estimators, meaning it satisfies*

$$\sup_{f\in\mathcal{F}} R(\hat{f}', f) = \inf_{\hat{f}} \sup_{f\in\mathcal{F}} R\left(\hat{f}, f\right),\ \forall f \in \mathcal{F}, \tag{1.2.8}$$

where $\mathcal{F}$ represents a certain class of functions. This type of risk is denoted by $R(n, \mathcal{F})$.

1.3 Statistical Convergence

Statistical convergence was introduced in connection with problems of series summation. The main idea of statistical convergence of a sequence $(x_n)_{n\in\mathbb{N}}$ is that the majority, in a certain sense, of its elements converge and we do not care what happens with other elements. At the same time, it is known that sequences that come from real life sources are not convergent in the strictly mathematical sense. This way, the advantage of replacing the uniform convergence by statistical convergence consists in the fact that the second convergence models and improves the technique of signal approximation in different functions spaces.

The idea of statistical convergence was introduced independently by Steinhaus [111], Fast [49] and Schöenberg [98]. Over the years, statistical convergence has been discussed in the theory of Fourier analysis [117], ergodic theory and number theory [25]. Later on, it was further investigated from the sequence of spaces point of view and linked with the summability theory [55]. Also, it has been studied in connection with trigonometric series [117], measure theory [82, 83] and Banach space theory [24].

The study of the statistical convergence for sequences of linear positive operators was attempted in the year 2002 by A.D. Gadjiev and C. Orhan [57]. The research field was proved to be extremely fertile, many researchers approaching this subject. Motivated by this research direction, our interest is to construct different classes of linear positive operators of discrete or integral type and to study their statistical approximation properties. We know that any convergent sequence is statistically convergent but the reverse is not true. The aim is to construct such sequences of operators that approximate the functions in the statistical sense, but not in the classical sense.

1.3.1 Linear Positive Operators

Given a non-empty set X, we denote by $B(X)$ the space of all real-valued bounded functions defined on X, endowed with the norm of the uniform convergence (or the sup-norm) defined by

$$\|f\| := \sup_{x \in X} |f(x)|, \quad f \in B(X).$$

The set $B(X)$ is a linear subspace of $\mathbb{R}^X$. If X is a topological space, C(X) denotes the space of all real-valued continuous functions on X. Furthermore, we set

$$C_B(X) := C(X) \cap B(X).$$

If X is a topological space, then $B(X)$ and $C_B(X)$ endowed with the sup-norm, are Banach spaces.

If X is a topological compact space, then $C(X) = C_B(X)$.

Definition 1.3.1 *Let X, Y be two linear spaces of real functions. The mapping $L : X \to Y$ is called linear operator if and only if*

$$L(\alpha f + \beta g) = \alpha L(f) + \beta L(g), \quad \forall f, g \in X, \quad \text{and } \alpha, \beta \in \mathbb{R}.$$

The operator L is called positive if and only if $\forall f \in X,\ f \geq 0$ implies $Lf \geq 0$.

Remark 1.3.2 *The set* $\mathcal{L} := \{L : X \to Y | \ L$ *is a linear operator*$\}$ *is a real vector space.*

Proposition 1.3.3 *Let* $L : X \to Y$ *be a linear positive operator.*

(*i*) *If* $f, g \in X$ *with* $f \leq g$, *then* $Lf \leq Lg$;

(*ii*) $\forall f \in X$ *we have* $|Lf| \leq L|f|$.

The next result provides a necessary and sufficient condition for the convergence of a sequence of linear positive operators towards the identity operator. It was independently discovered and proved by three mathematicians in three consecutive years: T. Popoviciu [94] in 1951, H. Bohman [18] in 1952 and P.P. Korovkin [71] în 1953. This classical result of approximation theory is mostly known under the name of Bohman- Korovkin Theorem, because T. Popoviciu's contribution in [94] remained unknown for a long time.

Theorem 1.3.4 *Let* $L_n : C([a,b]) \to C([a,b])$ *be a sequence of linear positive operators,* $n \in \mathbb{N}$. *Suppose that* $(L_n e_j)_{n\geq 1}$ *converges uniformly to* e_j *for* $j \in \{0, 1, 2\}$, *where* $e_0 = 1$, $e_1(x) = x$, $e_2(x) = x^2$, $x \in [a,b]$. *Then the sequence* $(L_n f)_{n\geq 1}$ *converges uniformly to* f *on* $[a,b]$, *for all functions* $f \in C([a,b])$.

1.3.2 Matrix Summability

Definition 1.3.5 *Let* $A = (a_{j,n})_{j,n\in\mathbb{N}}$ *be an infinite real matrix. A sequence* $(x_n)_{n\in\mathbb{N}}$ *is said to be A-summable to a real number* $s \in \mathbb{R}$ *if*

1^0 $\forall j \in \mathbb{N}$, *the series* $\sum\limits_{n=1}^{\infty} a_{j,n} x_n$ *converges; let* s_j *be the limit;*

2^0 $\lim\limits_{n\to\infty} s_j = s$.

Definition 1.3.6 *A summability matrix A is said to be regular if any convergent sequence is A-summable to its limit.*

1.3.3 Convergence in Statistical Sense

Definition 1.3.7 *A sequence of real numbers* $x := (x_n)_{n\in\mathbb{N}}$ *is said to be statistically convergent to a real number L if for every* $\varepsilon > 0$,

$$\delta\left(\{n \in \mathbb{N} : \ |x_n - L| \geq \varepsilon\}\right) = 0,$$

where

$$\delta(S) := \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \chi_S(k),$$

is the density of $S \subseteq \mathbb{N}$. Here χ_S represents the characteristic function of S.

We denote this limit by $st - \lim_n x_n = L$ ([57]).

Remark 1.3.8 *Any convergent sequence is statistically convergent. The converse is not true. This statement can be easily illustrated by the following example:*

Example 1.3.9 *Let us consider the sequence $(x_n)_{n\in\mathbb{N}}$,*

$$x_n = \begin{cases} i, & \text{for} \quad i = n^3, \; n = 1, 2, 3 \ \dots \\ \dfrac{1}{i^2+1}, & \text{otherwise.} \end{cases}$$

The limit $\lim_n x_n$ does not exist, but $st - \lim_n x_n = 0$ because $\delta(S) = 0$, where $S = \{n^3, \; n = 1, 2, 3, \ \dots\}$.

Definition 1.3.10 *Let $A = (a_{j,n})_{j,n\in\mathbb{N}}$ be a non-negative regular summability matrix. A real sequence $(x_n)_{n\in\mathbb{N}}$ is said to be A-statistically convergent to the real number L if, $\forall\varepsilon > 0$,*

$$\lim_{j\to\infty} \sum_{n:|x_n - L|\geq\varepsilon} a_{j,n} = 0.$$

We denote this limit by $st_A - \lim_n x_n = L$ ([45]).

1.3.4 Bohman-Korovkin Type Theorems

In this section we enunciate two Bohman-Korovkin type statistical approximation theorems. This theorems was proved by A.D. Gadjiev and C. Orhan [57].

1.4 Statistical Convergence by Positive Linear Operators

We consider the following sequence of positive linear operators defined in [93]

$$(T_n f)(x) = \frac{u_n}{F_n(x,t)} \sum_{v=0}^{\infty} f\left(\frac{v}{a_n(v)}\right) C_v^{(n)}(t) x^v, \quad f \in C[0,b], \tag{1.4.1}$$

where $u_n \geq 0$ for any $n \in \mathbb{N}$ and

$$st_A - \lim_n u_n = 1, \tag{1.4.2}$$

$x \in [0, b]$, $t \in (-\infty, 0]$ and $\{F_n(x,t)\}$ is the set of generating functions for the sequence of functions $\{C_v^{(n)}(t)\}_{v \in \mathbb{N}_0}$, in the form ,

$$F_n(x,t) = \sum_{v=0}^{\infty} C_v^{(n)}(t) x^v, \tag{1.4.3}$$

and $C_v^{(n)}(t) \geq 0$ for $t \in (-\infty, 0]$.

Assume that the following conditions hold

$$(i) F_{n+1}(x,t) = p(x) F_n(x,t),\ p(x) < M < \infty,\ x \in (0,1), \tag{1.4.4}$$

$(ii) BtC_{v-1}^{(n+1)}(t) = a_n(v) C_{v-1}^{(n)}(t) - vC_v^{(n)}(t),\ B \in [0, a],\ C_v^{(n)}(t) = 0$ for $v \in \mathbb{Z}^- := \{\dots, -3, -2, -1\}$,

$(iii) \max\{v, n\} \leq a_n(v) \leq a_n(v+1)$.

By using (1.4.3) we observe that

$$(T_n e_0)(x) = u_n \text{ cu } e_0(y) = 1. \tag{1.4.5}$$

We will study the A-statistical convergence of the sequence of positive linear operators defined by (1.4.1). The obtained results have been published in the paper R. Sobolu [100].

Theorem 1.4.1 (R. Sobolu, [100]) *Let $A = (a_{j,n})_{j,n \in \mathbb{N}}$ be a non-negative regular summability matrix. Then we have*

$$st_A - \lim_{n \to \infty} \|T_n e_1 - e_1\|_{C[0,b]} = 0, \tag{1.4.6}$$

where the operator T_n is defined by (1.4.1).

Theorem 1.4.2 (R. Sobolu, [100]) *Let $A = (a_{j,n})_{j,n \in \mathbb{N}}$ be a non-negative regular summability matrix. Then we have*

$$st_A - \lim_{n \to \infty} \|T_n e_2 - e_2\|_{C[0,b]} = 0, \tag{1.4.7}$$

where the operator T_n is defined by (1.4.1).

Using Theorem 1.4.1 and Theorem 1.4.2 one obtains the following A-statistical approximation theorem for the sequence $(T_n)_{n\in\mathbb{N}}$ given by (1.4.1).

Theorem 1.4.3 (R. Sobolu, [100]) *Let $A = (a_{j,n})_{j,n\in\mathbb{N}}$ be a non-negative regular summability matrix. Then, for all $f \in C[0,b]$ we have*

$$st_A - \lim_n \|T_n f - f\|_{C[0,b]} = 0. \tag{1.4.8}$$

Next we will construct an integral type generalization of the positive linear operators defined by (1.4.1) and present an A- statistical approximation result for these operators. These results can be found in the paper R. Sobolu [101].

We introduce the sequence of operators, $(T_n^*)_{n\in\mathbb{N}}$ as follows

$$(T_n^* f)(x) = \frac{u_n}{F_n(x,t)} \sum_{v=0}^{\infty} C_v^{(n)}(t) x^v \int_v^{v+c_{n,v}} f\left(\frac{\xi}{a_n(v)}\right) d\xi, \quad n \in \mathbb{N}, \tag{1.4.9}$$

where f is an integrable function on the interval $(0,1)$ and $(c_{n,v})_{n,v\in\mathbb{N}}$ is a sequence such that

$$0 < c_{n,v} \le 1 \tag{1.4.10}$$

for every $n, v \in \mathbb{N}$.

For any $n \in \mathbb{N}$ we have $u_n \ge 0$ and

$$st_A - \lim_n u_n = 1. \tag{1.4.11}$$

$\{F_n(x,t)\}$ is the set of generating functions for the sequence of functions $\{C_v^{(n)}(t)\}_{v\in\mathbb{N}_0}$, in the form

$$F_n(x,t) = \sum_{v=0}^{\infty} C_v^{(n)}(t) x^v, \tag{1.4.12}$$

$C_v^{(n)}(t) \ge 0$ for $t \in (-\infty, 0]$.

Assume that the next conditions hold

$$(i) F_{n+1}(x,t) = p(x) F_n(x,t),\ p(x) < M < \infty,\ x \in (0,1), \tag{1.4.13}$$

$(ii) BtC_{v-1}^{(n+1)}(t) = a_n(v) C_{v-1}^{(n)}(t) - vC_v^{(n)}(t),\ B \in [0,a], C_v^{(n)}(t) = 0$ for $v \in \mathbb{Z}^- := \{\ldots, -3, -2, -1\}$,

$(iii) \max\{v, n\} \le a_n(v) \le a_n(v+1)$.

Following, for use in our main result, we prove inequalities for the sequence of operators $(T_n^*)_{n\in\mathbb{N}}$ given by (1.4.9)

Theorem 1.4.4 (R. Sobolu, [101]) *Let $(T_n^*)_{n\in\mathbb{N}}$ be the positive linear operator given by (1.4.9). Then, for each $x \in [0,b]$, $t \in (-\infty, 0]$ and $n \in \mathbb{N}$ we have*

$$\|T_n^* e_1 - e_1\|_{C[0,b]} \leq \frac{u_n}{2n} + abM|t|\frac{u_n}{n} + b|u_n - 1|,$$

where M is given as in (1.4.13).

Theorem 1.4.5 (R. Sobolu, [101]) *For each $x \in [0,b]$, $t \in (-\infty, 0]$ and $n \in \mathbb{N}$ we have*

$$\|T_n^* e_2 - e_2\|_{C[0,b]} \leq \frac{u_n}{3n^2} + abM|t|\frac{u_n}{n^2} + \frac{u_n}{n} b(abM|t| + aM|t| + 2) + b^2|u_n - 1|,$$

where the sequence $(T_n^*)_{n\in\mathbb{N}}$ and M are defined as in Theorem 1.4.4.

Theorem 1.4.6 (R. Sobolu, [101]) *Let $A = (a_{j,n})_{j,n\in\mathbb{N}}$ be a non-negative regular summability matrix. Then we have*

$$st_A - \lim_{n\to\infty} \|T_n^* e_1 - e_1\|_{C[0,b]} = 0,$$

where the operator T_n^* is defined by (1.4.9).

Theorem 1.4.7 (R. Sobolu, [101]) *Let $A = (a_{j,n})_{j,n\in\mathbb{N}}$ be a non-negative regular summability matrix. Then we have*

$$st_A - \lim_{n} \|T_n^* e_2 - e_2\|_{C[0,b]} = 0,$$

where T_n^* is defined by (1.4.9).

Now we provide a Korovkin type approximation theorem for the operators T_n^* via A-statistical convergence.

Theorem 1.4.8 (R. Sobolu, [101]) *Let $A = (a_{j,n})_{j,n\in\mathbb{N}}$ be a non-negative regular summability matrix. Then, for all $f \in C[0,b]$, we have*

$$st_A - \lim_{n} \|T_n^* f - f\|_{C[0,b]} = 0.$$

1.5 Wavelets Analysis Concept

Wavelets analysis breaks down a signal (i.e. a sequence of numerical measurements) in a small wave components type called wavelets. A wavelet is a waveform of effectively limited duration that has an average of zero. Wavelets analysis is a refinement of Fourier analysis. Wavelets analysis represents a windowing technique with variable-sized regions. It allows the use of long time intervals where we want more precise low-frequency information, and shorter regions where we want high-frequency information.

In the mathematical sense, wavelets concept describes a category of orthonormal basis of space $L^2(\mathbb{R})$, with remarkable approximation properties. The Fourier orthonormal basis are sine waves .The purpose of wavelets analysis is to build orthonormal bases thats consist of wavelets.

1.5.1 Haar System

This subsection presents the simplest example of orthonormal wavelet function, the *Haar* function.

1.5.2 Multiresolution Analysis

The concept of multiresolution analysis is related to the study of signals f of different levels of resolution, each of them being a finer resolution of f. In the presentation of this section we used monographs [6, pages 65-76] and [29, I. Daubechies, pages 129-156].

Wavelets analysis is based on the decomposition of a piecewise constant approximation function into a coarser approximation and a detail function. The approximation can be written as a sum of the next coarser approximation, say f_{j-1} and a detail function, say g_{j-1}, this meaning $f_j = f_{j-1} + g_{j-1}$. Each detail function can be written as a linear combination of the corresponding mother wavelet functions, $\psi_{j,k},\ \psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k), \quad x \in \mathbb{R}$, where $j \in \mathbb{Z}$, is the dilation index and $k \in \mathbb{Z}$, is the translation index. As the index j runs from small to large, the corresponding approximations run from coarse to fine. For each resolution $j \in \mathbb{Z}$, we have a space basis functions $(\psi_{j,k})_{k\in\mathbb{Z}}$. Consequently, we work with several spaces at different resolutions, this meaning *multiresolution*.

For each $j \in \mathbb{Z}$, we define a function space V_j,

$$V_j = \left\{ f \in L^2(\mathbb{R}) :\ f \text{ is piecewise constant on } [k2^{-j}, (k+1)2^{-j}],\ j \in \mathbb{Z} \right\}.$$

This ladder of subspaces enjoys the following properties

(P_1) $\ldots \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \ldots$;

(P_2) $\bigcap_{j\in\mathbb{Z}} V_j = 0$, $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$;

(P_3) $f \in V_j$ if and only if $f(2\,\cdot) \in V_{j+1}$, where $j \in \mathbb{Z}$;

(P_4) $f \in V_0$ implies $f(\cdot - k) \in V_0$, for all $k \in \mathbb{Z}$;

(P_5) There exists a function $\varphi \in V_0$ such that the set $\varphi_{0,k} = \{\varphi(\cdot - k) : \; k \in \mathbb{Z}\}$ forms an orthonormal basis for V_0.

If the central space V_0 is generated by a single function $\varphi \in V_0$, $V_0 = \overline{sp\,\{\varphi_{0,k} : \; k \in \mathbb{Z}\}}$, then each subspace V_j is generated by the same function φ, $V_j = \overline{sp\,\{\varphi_{j,k} : \; k \in \mathbb{Z}\}}$, $j \in \mathbb{Z}$. Here

$$\varphi_{j,k}(x) = 2^{j/2}\varphi(2^j x - k), \; x \in \mathbb{R}, \; (j,k) \in \mathbb{Z} \times \mathbb{Z}. \tag{1.5.1}$$

The function $\varphi \in V_0$, which verifies (P_5) is called *scaling function* or *father wavelet*.

Definition 1.5.1 *A multiresolution analysis generated by the scaling function* φ *consists of a sequence* V_j, $j \in \mathbb{Z}$, *of closed subspaces of* $L^2(\mathbb{R})$ *that satisfy the properties* (P_1), (P_2), (P_3), (P_4), (P_5).

1.5.3 On the Mother Wavelet

Let $(V_j)_{j\in\mathbb{Z}}$ be a multiresolution analysis of $L^2(\mathbb{R})$. Since $V_j \subset V_{j+1}$, we define W_j the orthogonal complement of V_j in V_{j+1}, for every integer j.

Hence $V_{j+1} = V_j \oplus W_j$, $j \in \mathbb{Z}$. We define the *mother wavelet* function as follows

$$\psi_{j,k} = 2^{j/2}\psi(2^j x - k), \; x \in \mathbb{R}, \; (j,k) \in \mathbb{Z} \times \mathbb{Z}. \tag{1.5.2}$$

As the *father wavelet* generates orthonormal bases in $V_j, j \in \mathbb{Z}$ the *mother wavelet* generates orthonormal bases in $W_j, j \in \mathbb{Z}$.

1.5.4 Wavelet Decomposition and Reconstruction

Every signal $f \in L^2(\mathbb{R})$ can be uniquely decomposed as

$$f(x) = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} (f, \psi_{j,k})\psi_{j,k}(x), \; x \in \mathbb{R}, \tag{1.5.3}$$

where $(f, \psi_{j,k}) = \int_{\mathbb{R}} f(x)\psi_{j,k}(x)dx$.

1.5.5 The Discrete Wavelet Transform

Let $(V_j)_{j\in\mathbb{Z}}$ be a multiresolution analysis of $L^2(\mathbb{R})$. Since $(\varphi_{J,k})_{k\in\mathbb{Z}}$ forms an orthonormal basis for V_J, we have $f_J(x) = \sum_{k\in\mathbb{Z}} \alpha_{J,k}\varphi_{J,k}(x), \quad x \in \mathbb{R}$, where $\alpha_{J,k} = (f, \varphi_{J,k})$. Starting at the finest scale J and repeating the decomposition until a certain level j' we can write

$$f_J(x) = \sum_{k\in\mathbb{Z}} \alpha_{j\prime,k}\varphi_{j\prime,k}(x) + \sum_{j=j\prime}^{J-1}\sum_{k\in\mathbb{Z}} w_{j,k}\psi_{j,k}(x), \quad x \in \mathbb{R}. \tag{1.5.4}$$

The coefficients $(\alpha_{j\prime,k})_{k\in\mathbb{Z}}$ are called *approximations coefficients* and the coefficients $(w_{j,k})_{(j,k)\in\mathbb{Z}\times\mathbb{Z}}$ are called *wavelets coefficients*.

Chapter 2

Nonparametric Regression

The nonparametric regression will approximate a regression function without forcing a particular analytical form over it. We will suppose that function f belongs to a certain class of functions and that it has certain properties such as smoothness.

2.1 Denoising by Wavelet Thresholding

The noisy signal model can be written so that *noisy signal* = *original signal* + *noise*. We consider the regression model

$$Y = f(X) + \varepsilon. \tag{2.1.1}$$

Suppose we have noisy sampled data

$$y_i = f(x_i) + \sigma\varepsilon_i,\ i = \overline{1, n}, \tag{2.1.2}$$

where $\varepsilon = (\varepsilon_1, \varepsilon_2,\ \dots\ \varepsilon_n)^T$, ε_i are independent random variables, distributed as $N(0, 1)$ and σ is the noise level that can be known or unknown. We suppose, without losing generality, that x_i are equidistant within the unit interval $[0, 1]$, of type $\frac{i}{n}$, and n, the sample size, is a power of two, $n = 2^J$, for some positive integer J. The goal is to recover the underlying function f from the noisy data, $y = (y_1, y_2,\ \cdots,\ y_n)^T$, without assuming any particular parametric structure for f.

The nonparametric regression algorithm based on wavelets transforms consists of the following steps:

1. Calculate the wavelet transform of the noisy signal.
2. Modify the noisy wavelet coefficients according to some rule.

3. Compute the inverse transform using the modified coefficients.

For the second step of the above approach there are two kinds of denoising methods: linear and nonlinear techniques. A wavelet based linear approach, simply extends the spline smoothing estimation methods. This method is appropriate for estimating relatively regular functions, but is not designed to handle spatially inhomogeneous functions with low regularity. For such functions one usually relies upon nonlinear thresholding or shrinkage methods. The first mathematical treatment of wavelet shrinkage and wavelet thresholding was done by Donoho [35], [36], [37]. He analyzed wavelet thresholding and shrinkage methods in the context of *minimax* estimation and showed that wavelet shrinkage generates asymptotically optimal estimates for noisy data that outperform any linear estimator.

Mathematically, wavelet coefficients are estimated using *thresholding* methods. These methods modify wavelets coefficients values w at the value $\hat{w}$, by removing the coefficients with small absolute value considered to represent the noise, respectively by keeping the coefficients with high absolute value that are used for reconstruction. The choice of threshold value is a fundamental issue. Donoho and Johnstone [36], [37], [38], Nason and Silverman [85], [86] established a variety of threshold methods. These are split in two main categories: global *threshold* methods and *threshold* methods that depend on resolution level. The first category applies to all empirical wavelets coefficients.

2.1.1 Thresholding Procedure

Let w be the wavelets coefficients corresponding to threshold value λ. The function *hard thresholding* is defined as follows

$$\eta_{hard}(w;\lambda) = wI(|w| > \lambda), \tag{2.1.3}$$

and the function *soft thresholding* can be expressed as

$$\eta_{soft}(w;\lambda) = sign(w)(|w| - \lambda)I(|w| > \lambda) \tag{2.1.4}$$

$$= \begin{cases} w+\lambda \quad , & w < -\lambda, \\ 0 \quad , & |w| \leq \lambda, \\ w-\lambda, & w > \lambda \, . \end{cases}$$

The technique *firm thresholding* is based on a continuous function of the form

$$\eta_F(w;\lambda_1,\lambda_2) = \begin{cases} 0 & , \quad |w| < \lambda_1, \\ sign(w)\dfrac{\lambda_2(|w|-\lambda_1)}{\lambda_2-\lambda_1} & , \quad \lambda_1 < w \le \lambda_2, \\ w, & \quad |w| > \lambda_2. \end{cases}$$

The technique *SCAD thresholding* is based on piecewise linear function.

$$\eta_{SCAD}(w;\lambda) = \begin{cases} sign(w)\; max(0,|w|-\lambda) & , \quad |w| < 2\lambda, \\ \dfrac{(a-1)w - a\lambda sign(w)}{a-2} & , \quad 2\lambda < w \le a\lambda, \\ w, & \quad |w| > a\lambda. \end{cases}$$

2.1.2 Thresholding Selecting Rules

Donoho and Johnstone established in [36] the *universal* threshold. Its value is given by $\lambda = \hat{\sigma}\sqrt{2\log n}$, where n is selection size and $\hat{\sigma}$ is an estimate of the noise level σ. The particular case $\lambda = \sqrt{2\log n}$ corresponds to the *VisuShrink* procedure.

The *minimax* threshold is another global method developed by Donoho and Johnstone in [36]. Suppose that we are given observations

$$w_i = \theta_i + \varepsilon z_i, \;\; i = 1, \; \ldots \; n, \tag{2.1.5}$$

where $z_i \sim N(0,1),\; \varepsilon > 0$, and $w = (w_i),\; i = 1, \; \ldots \; n$.

We wish to estimate the risk

$$R(\hat{\theta},\theta) = E\left\|\hat{\theta}-\theta\right\|^2_{l^2(\mathbb{Z})}. \tag{2.1.6}$$

Let $\eta_S(w,\lambda)$ be the soft thresholding function defined in (2.1.4). Suppose we have a single obeservation $y \sim N(0,1),\; \varepsilon > 0$. Define the function

$$\rho_S(\lambda,\mu) = E\left\{\eta_S(y,\mu)-\mu\right\}^2 \tag{2.1.7}$$

and the *minimax* quantities

$$\Lambda_n^* \equiv \inf_\lambda \sup_\mu \frac{\rho_S(\lambda,\mu)}{n^{-1}+min(\mu^2,1)}, \tag{2.1.8}$$

where λ_n^* the largest λ attaining Λ_n^* above.

Theorem 2.1.1 *Assume the models (2.1.5) and (2.1.6). The minimax threshold* λ_n^*

defined by (2.1.8) yields an estimator

$$\hat{\theta}^* = \eta_S(w_i, \lambda_n^* \varepsilon),\ i = 1,\ \ldots\ldots,\ n,$$

which satisfies

$$E \left\| \hat{\theta}^* - \theta \right\|_{l^2}^2 \leq \Lambda_n^* \left\{ \varepsilon^2 + \sum_{i-1}^{n} min \left(\theta_i^2, \varepsilon^2 \right) \right\},$$

for all $\theta \in \mathbb{R}^n$, *where*

$$\Lambda_n^* \leq 2 \log n + 1 \ \ \text{și} \ \ \lim_{n \to \infty} \Lambda_n^* = 2 \log n,$$

$$\lambda_n^* \leq \sqrt{2 \log n} \ \ \text{și} \ \ \lim_{n \to \infty} \lambda_n^* = \sqrt{2 \log n}.$$

Compared with universal threshold, the minimax thresholding is more conservative and is more proper when small details of function f lie in the noise range.

The *SureShrink* method chooses a threshold λ_j by minimizing the Stein Unbiased Risk Estimate [36], [38], for each wavelet level j.

2.1.3 Penalized Least-Squares Wavelet Estimators

When estimate a signal that is corrupted by additive noise by wavelets based methods, the traditional smoothing problem can be formulated in the wavelet domain by finding the minimum in θ of the penalized least-squares functional $l(\theta)$ defined by

$$l(\theta) = \|Wy - \theta\|_n^2 + 2\lambda \sum_{i > i_0} p(|\theta_i|), \tag{2.1.9}$$

where θ is the vector of the wavelet coefficients of the unknown regression function f and p is a given penalty function. The value i_0 is a given integer corresponding to penalizing wavelet coefficients above certain resolution level j_0.

The performance of the resulting wavelet estimator depends on the penalty and the smoothing parameter λ.

2.2 A Comparative Study of Two Noise Removing Methods

In this section we will present wavelet thresholding estimators in nonparametric regression for denoising data modeled as observations of a signal contaminated with added Gaussian noise. A case study, referring to the daily averages of RH recorded in Cluj-Napoca (Romania) area is presented in order to compare performance of two nonparametric regression techniques: the *Minimax thresholding rule* and *VisuShrink thresholding rule* in the context of the mean-squared error. RH is an array of size 256. The data processing were carried out by using the *Wavelab* software library of *Matlab* routines for wavelets analysis. The 1-d signal, *humidity*, is stored as a single column of ASCII text, in the directory */Wavelab850/Datasets*. This study was published in the paper Sobolu R. [108].

The denoising algorithm based on nonparametric regression in the case of *Minimax thresholding rule* respectively *VisuShrink thresholding rule* consists of three steps:

1. Process first the data by using the *Most Nearly Symmetric Daubechies* wavelet, $N = 8$.

2. Apply a soft thresholding nonlinearity with threshold set to λ_n^* respectively $\lambda^V = \sqrt{2 \log n}$.

3. Apply an inverse wavelet transform to the wavelets coefficients obtained in the second step and reconstruct the original signal.

In order to compare the efficiency of applied techniques we calculated in both cases the risk looking that it is the minimal risk possible.

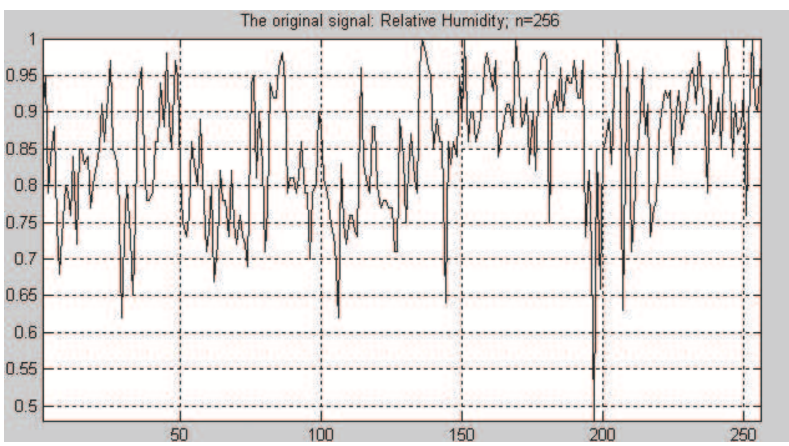


Figure 2.1: The original signal

The threshold	The risk
λ_n^*	0.0119
$\lambda^V = \sqrt{2 \log n}$	0.0040

The threshold λ^V provides a better visual quality of reconstruction than the procedure based on the *Minimax* procedure (see Figure 2.3 and Figure 2.4).

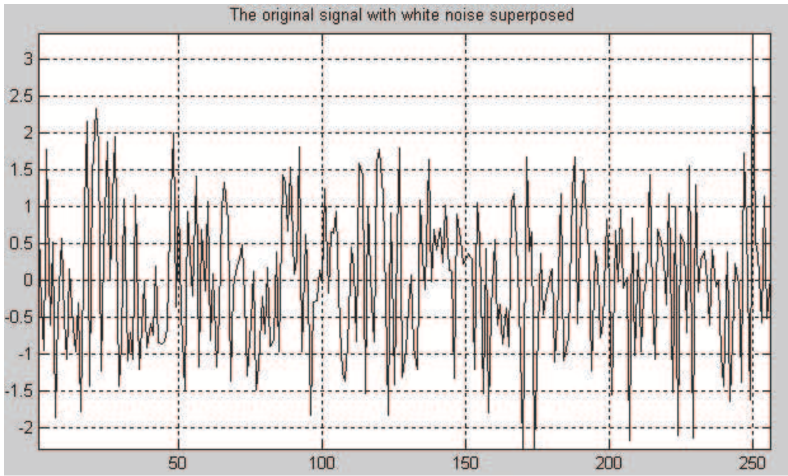


Figure 2.2: The noisy signal

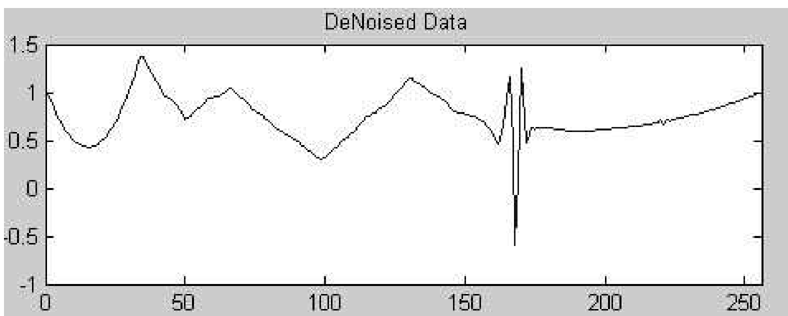


Figure 2.3: The *Minimax thresholding* case

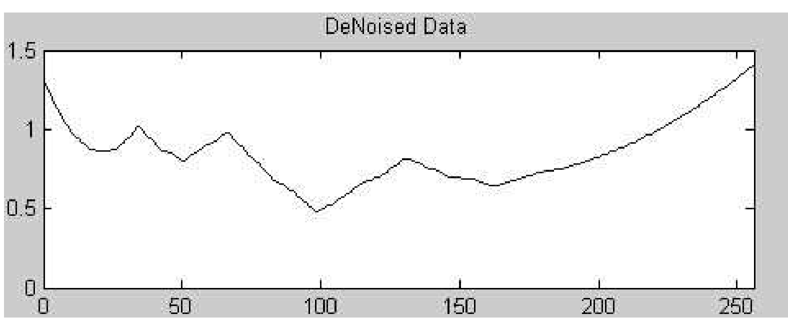


Figure 2.4: The *VisuShrink thresholding* case

2.3 Analysis of Reconstruction Methods with Wavelets Techniques

This paragraph reveals the performance of some reconstruction methods: the *penalized least-squares* method, the *cross-validation (hard and soft)* method and *SureShrink* method, by using a simulated data sets composed by the test function

$$f : [0,1] \to \mathbb{R}, \quad f(x) = x + exp(-39(x-0.5)^2) - I(x \geq 0.5).$$

This function presents a discontinuous jump at $x = 0.5$. For the signal simulation we have used two noise levels corresponding to signal-to-noise ratios $SNR = 4$ and $SNR = 10$. For each simulation we have used an equidistant design of size 512 within the interval [0, 1] and a Gaussian noise was added to obtain the observed data. The noisy sampling data is

$$y_i = f(t_i) + \varepsilon_i, \quad i = 1, 2, 3, \ldots, \ 512,$$

where $t_i = \dfrac{i}{512}$, and ε_i are independent random errors, distributed as $N(0,1)$.

The processings were made by means of some functions implemented in *Matlab*.

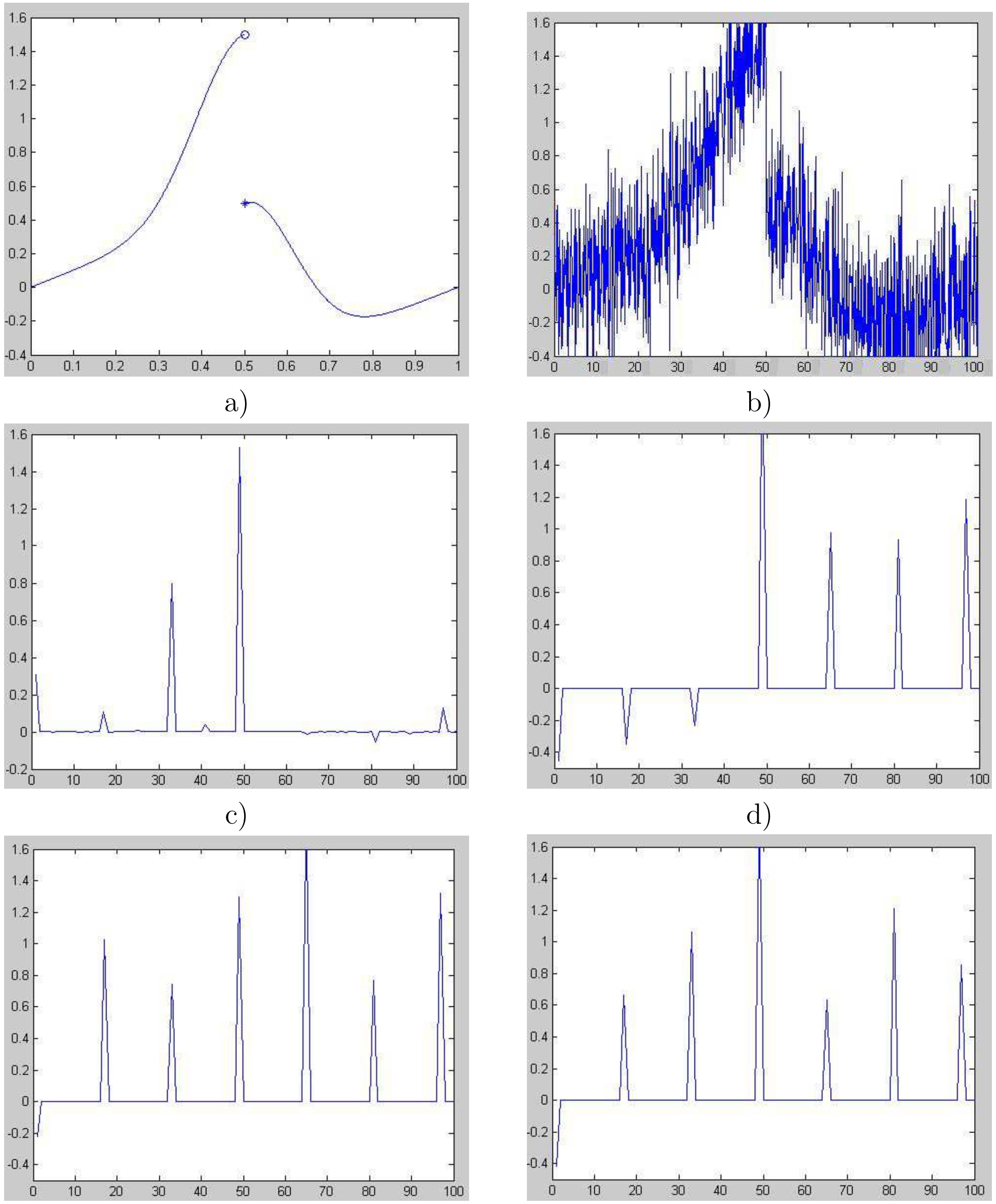
The Figure 2.5 shows the applied techniques performance. The following table displays the mean square error evaluated in each considered case and for every SNR setting.

The method	$SNR = 4$	$SNR = 10$
The *penalized least-squares* method	0.9512	0.9049
The *cross-validation* method (hard thresholding)	0.9200	0.8900
The *cross-validation* method (soft thresholding)	0.9306	0.8955
The *SureShrink* method (soft thresholding)	0.9562	0.8855

2.4 Application

This paragraph includes practical results published in the papers Sobolu R. [104], Sobolu R. [105], Sobolu R. [110].

The aim of this study is to determine the changes of the respiratory rate, the heart rates and the antioxidant enzymes level in the blood for a group of cows exposed to solar radiation compared to a group of the same number of cows kept in stable during the hot summer days. The determinations were carried out during the period May - October 2006.



a) b) c) d) e) f)

Figure 2.5: *Thresholding* methods applied to the signal $f(x) = x + exp(-39(x - 0.5)^2) - I(x \geq 0.5)$ disturbed by white Gaussian noise
a) The original function
b) The function contaminated by white Gaussian noise , $SNR = 4$
c) Reconstruction with the *penalized least-squares* method
d) Reconstruction with the *cross-validation* method (hard thresholding)
e) Reconstruction with the *cross-validation* method (soft thresholding)
f) Reconstruction with the *SureShrink* method.

The recorded individual data (the signals) were processed using two types of wavelets transforms: The Ordered Fast Haar Wavelet Transform - OFHWT and The In Place Fast Haar Wavelet Transform - PFHWT [89]. These transforms allow us to calculate the wavelets coefficients in order to assess the obtained results. The transforms algorithms have been implemented in VBA in Microsoft Excel.

The signal representing the respiratory rate measurements of the cows exposed to solar radiation respectively of the cows kept in stable over the period May - October (May, July, August, October) was analyzed by using *The In-Place Fast Haar Wavelet Transform*. The signal contains $32 = 2^5$ observed data. The sequence s(5-0) includes the initial data and the sequence s(5-5) contains the result, i.e. the wavelets coefficients (see Figure 2.6) .

The Figure 2.6 a) displays the wavelets coefficients in the case of exposure to solar radiation. The first coefficient, 49.776 represents the average respiratory rate for the whole four-month period. The second coefficient, -5.969 means that the respiratory rate changed by $(-5.969) * (-2) = 11.938 \approx 12$, an increase of 12 resp/min from May to October, so there were significant changes of the respiratory rate from May to October. The next two coefficients -11.021 and 10.416 represent similar changes of the respiratory rate over the first half and the second half of the period. The coefficient -11.021 corresponds to a change of $(-11.021) \cdot (-2) = 22.0142 \approx 22$, an increase of 22 resp/min from May to July. The coefficient 10.416 corresponds to a change of $10.416 \cdot (-2) = -20.832 \approx 21$, a decrease of 21 resp/min from August to October.

Similarly, we analyzed the wavelets coefficients obtained by applying *The In-Place Fast Haar Wavelet Transform* to the signal represented by the respiratory rate in a group of dairy cows maintained in the stable. The processing reveals not significant changes in this case (see Figure 2.6 b)).

Using similar methods we performed comparative studies of the signals represented by the heart rates and by the antioxidant enzymes level in blood (the SOD, the catalase activity and the peroxidase activity) for both cases: exposure to solar radiation and maintenance in stable. There were also recorded the main meteorological indexes in the days when the specified signals was performed: temperature, humidity, solar radiation intensity and so on. When the determinations were made the following maximum THI values were calculated: 64 in May, 77 in July, 87 in August and 60 in October. In the stable the maximum THI was varying between 60 and 70.

The studies, previously made by us, show an increase of the main physiological indexes (such as respiratory rate, heart rate, internal and cutaneous temperature, a)

n	5

s(5-0)	s(5-1)	s(5-2)	s(5-3)	s(5-4)	s(5-5)
31.660	30.995	32.078	32.786	43.807	49.776
30.330	33.160	33.495	54.828	55.745	-5.969
29.660	34.660	53.328	66.161	-11.021	-11.021
36.660	32.330	56.328	45.329	10.416	10.416
35.660	54.160	66.413	-0.709	-0.709	-0.709
33.660	52.495	65.910	-1.500	-1.500	-1.500
33.330	59.660	45.245	0.251	0.251	0.251
31.330	52.995	45.413	-0.084	-0.084	-0.084
58.660	66.330	-1.083	-1.083	-1.083	-1.083
49.660	66.495	1.165	1.165	1.165	1.165
46.660	66.660	0.833	0.833	0.833	0.833
58.330	65.160	3.333	3.333	3.333	3.333
63.660	43.995	-0.083	-0.083	-0.083	-0.083
55.660	46.495	0.750	0.750	0.750	0.750
50.660	46.330	-1.250	-1.250	-1.250	-1.250
55.330	44.495	0.918	0.918	0.918	0.918
72.000	0.665	0.665	0.665	0.665	0.665
60.660	-3.500	-3.500	-3.500	-3.500	-3.500
64.660	1.000	1.000	1.000	1.000	1.000
68.330	1.000	1.000	1.000	1.000	1.000
70.660	4.500	4.500	4.500	4.500	4.500
62.660	-5.835	-5.835	-5.835	-5.835	-5.835
60.660	4.000	4.000	4.000	4.000	4.000
69.660	-2.335	-2.335	-2.335	-2.335	-2.335
45.330	5.670	5.670	5.670	5.670	5.670
42.660	-1.835	-1.835	-1.835	-1.835	-1.835
44.330	4.000	4.000	4.000	4.000	4.000
48.660	-4.500	-4.500	-4.500	-4.500	-4.500
50.330	1.335	1.335	1.335	1.335	1.335
42.330	-2.165	-2.165	-2.165	-2.165	-2.165
40.660	4.000	4.000	4.000	4.000	4.000
48.330	-3.835	-3.835	-3.835	-3.835	-3.835

b)

n	5

s(5-0)	s(5-1)	s(5-2)	s(5-3)	s(5-4)	s(5-5)
28.660	27.495	26.995	26.495	29.641	30.662
26.330	26.495	25.995	32.788	31.683	-1.021
27.660	25.995	32.495	34.536	-3.146	-3.146
25.330	25.995	33.080	28.829	2.854	2.854
26.330	33.495	36.078	0.500	0.500	0.500
25.660	31.495	32.995	-0.293	-0.293	-0.293
26.330	30.665	29.498	1.541	1.541	1.541
25.660	35.495	28.160	0.669	0.669	0.669
36.330	38.495	0.500	0.500	0.500	0.500
30.660	33.660	0.000	0.000	0.000	0.000
32.660	32.495	1.000	1.000	1.000	1.000
30.330	33.495	-2.415	-2.415	-2.415	-2.415
31.330	29.830	2.418	2.418	2.418	2.418
30.000	29.165	-0.500	-0.500	-0.500	-0.500
31.660	28.660	0.333	0.333	0.333	0.333
39.330	27.660	0.500	0.500	0.500	0.500
42.330	1.165	1.165	1.165	1.165	1.165
34.660	1.165	1.165	1.165	1.165	1.165
35.660	0.335	0.335	0.335	0.335	0.335
31.660	0.335	0.335	0.335	0.335	0.335
33.660	2.835	2.835	2.835	2.835	2.835
31.330	1.165	1.165	1.165	1.165	1.165
35.330	0.665	0.665	0.665	0.665	0.665
31.660	-3.835	-3.835	-3.835	-3.835	-3.835
30.330	3.835	3.835	3.835	3.835	3.835
29.330	2.000	2.000	2.000	2.000	2.000
30.000	1.165	1.165	1.165	1.165	1.165
28.330	1.835	1.835	1.835	1.835	1.835
28.660	0.500	0.500	0.500	0.500	0.500
28.660	0.835	0.835	0.835	0.835	0.835
26.660	0.000	0.000	0.000	0.000	0.000
28.660	-1.000	-1.000	-1.000	-1.000	-1.000

Figure 2.6:

a) The wavelets coefficients obtained by using *The In-Place Fast Haar Wavelet Transform* to process respiratory rate (in the group exposed to solar radiation)
b) The wavelets coefficients obtained by using *The In-Place Fast Haar Wavelet Transform* to process respiratory rate (in the group maintained in the stable)

variation of the blood indexes) that determined us to study the reaction of the cows' organism at the cell level, regarding the thermal stress. The increase of the antioxidant enzymes level proved that, during the hot summer day, when the values of the THI index are higher then 72, which is the limit value for the thermal comfort the cows were submitted to the heat stress. We can also certainly say that between the increasing of the blood level of the antioxidants enzymes and the THI values exists a direct correlated relation that means that when the values of THI exceed 72, the production of the antioxidant enzymes at the blood level starts to increase. These changes reflect both the decrease in milk production as well as its qualitative change (protein and lactose decrease).

The study's conclusions establish practical measures to be taken to prevent harmful effects of solar radiation in order to ensure the comfort of dairy cows .

2.5 The Fast Daubechies Wavelet Transform Algorithm Implementation

This section presents the practical significance of Daubechies wavelets coefficients with real data. The algorithm of Daubechies Wavelet Transform has been implemented in VBA in Microsoft Excel. The results for this application are included in the paper Sobolu R. [106].

We analyzed a signal represented by the semiweekly measurements of temperature, in Celsius degrees, for February 2008 and March 2008 at a fixed location in Cluj-Napoca. The sequence s contains the initial data and the sequences $a - Step0$ respectively $c - Step0$, $c - Step1$, $c - Step2$, $c - Step3$, $c - Step4$ display the result (the Daubechies wavelets coefficients), see Figure 2.7. Further on it was explained the practical significance of wavelets coefficients considering the Fast Daubechies Wavelet Transform applied on real data.

The coefficient $a - Step0 = a_0^{(n-5)} = 7.993^0C$ represents the average temperature for the whole two months period. The coefficient $c - Step0 = c_0^{(n-5)} = 1.105$ means that the temperature changed by $1.105 \cdot 2 = 2.210^0C$, an increase of $\approx 2^0C$ from March to February. The coefficient $c - Step1 = c_0^{(n-4)} = 1.214$ corresponds to a change of $1.214 \cdot 2 = 2.428^0C$, an increase of 2.428^0C from the first two weeks to the last two weeks in February. The coefficient $c_1^{(n-4)} = -1.045$ corresponds to a change of $(-1.045) \cdot 2 = -2.09^0C$, a decrease of 2.0903^oC from the first two weeks to the last two weeks in March.

Each of the next four coefficients $0.538, -0.057, -3.546, -2.874$ represent the temperature change over two weeks. The coefficient -0.057 means that the temperature decreased by $(-0.057) \cdot 2 = -0.114^0 C$ from the third week to the fourth week of February.

The next eight coefficients $-1.201, -1.598, -1.455, -2.400, -2.043, 0.751, 1.359, 3.205$ represent a temperature change over one week. The coefficient 0.751 means that the temperature changed by $0.751 \cdot 2 = 1.502^0 C$ during the second week of March.

n	4																
h	0.683	1.183	0.317	-0.183													
s	7.730	8.770	6.800	3.850	-0.330	3.750	8.770	10.170	11.200	7.750	8.360	12.670	10.000	7.000	10.270	11.130	...
a	9.491	7.880	5.380	-1.823	1.913	8.258	9.793	12.463	7.527	6.782	13.647	11.098	5.803	9.955	11.130	11.445	...
a-Step4	8.922	0.306	5.949	11.288	7.730	11.234	8.587	11.848	7.533	12.276	7.626	11.474	6.176	1.130	8.041	7.770	
a-Step3	3.138	8.906	9.562	10.011	9.993	10.267	3.341	8.728									
a-Step2	6.939	9.831	9.216	5.986													
a-Step1	9.098	6.888															
a-Step0	7.993																
c-Step4	1.688	-1.892	0.053	-0.736	2.519	-2.975	0.566	1.935	-1.513	-0.222	1.386	1.731	-3.848	0.202	0.056	1.051	
c-Step3	-1.201	-1.598	-1.455	-2.400	-2.043	0.751	1.359	3.205									
c-Step2	0.538	-0.057	-3.546	-2.874													
c-Step1	1.214	-1.045															
c-Step0	1.105																

Figure 2.7: The wavelets coefficients of Daubechies Wavelet Transform

Chapter 3

Wavelets Estimators

3.1 Preliminary Results

Nonlinear wavelet estimators study nonparametric regression from *minimax* point of view by using classes of functions not found in the case of linear estimators such as *Hölder* or *Sobolev* spaces. These estimators focus on the inhomogeneous functions spaces or functions of bounded variation. These classes of functions can be summarized in *Besov* or *Triebel* spaces. Meyer [80] develops the idea of multiresolution analysis and its use in the study of function spaces and integral operators. The research articles of I. Daubechies [30], Mallat [78] and the monograph of Frazier, Jawerth and Weiss [50], [51] provide a connection between the orthonormal wavelets bases and the *minimax* estimation in *Besov* spaces.

3.2 An abstract Denoising Model

Denoising concept intends to optimize the mean-squared error

$$n^{-1}E\left\|\hat{f}-f\right\|_{l^2}^2 = n^{-1}\sum_{i=0}^{n-1} E\left(\hat{f}\left(\frac{i}{n}\right)-f\left(\frac{i}{n}\right)\right)^2 \tag{3.2.1}$$

by fulfilling the condition that with high probability $\hat{f}$ is at least as smooth as f. This demands a tradeoff between bias and variance which keeps the two terms at about the same order of magnitude. The estimators which are optimal from a mean-square-error point of view exhibit considerable, undesirable, noise-induced structures-"ripples", "blips", and oscillations. Reconstruction methods are designed to avoid spurious oscillations demanding that the reconstruction do not oscillate essentially more than the true underlying function.

Donoho and Johnstone [36] proposed a kind of *thresholding* procedure for recovering a function f from noisy data taking into account the following aspects:

1.[[*Smooth*] With high probability, the estmator $\hat{f}_n^*$ is at least as smooth as f, with smoothness measured by any of a wide range of smoothness measures.

2.[*Adapt*] The estimator $\hat{f}_n^*$ achieves almost the minimax mean-square error over every one of a wide range of smoothness classes, including many classes where traditional linear estimators do not achieve the minimax rate.

The statistical theory focuses on the following abstract denoising model

$$y_I = \theta_I + \varepsilon \cdot z_I, \quad I \in \mathcal{I}_n, \tag{3.2.2}$$

where $z_i \sim N(0,1)$ is a Gaussian white noise and ε is the noise level. $\mathcal{I}_n$ is an index set, $|\mathcal{I}_n| = n$.

3.2.1 *Thresholding* Method and Optimal Recovery

We consider an abstract model, in which noise is deterministic

$$y_I = \theta_I + \delta \cdot u_I, \quad I \in \mathcal{I}, \text{ and } \mathcal{I} \text{ is an index set.} \tag{3.2.3}$$

In this case $\delta > 0$ is a known noise level and (u_I) is a noisy term that satisfy $|u_I| \le 1, \ \forall I \in \mathcal{I}$. We suppose that the noise is minimal. We will evaluate the performance

$$E_\delta(\hat{\theta}, \theta) = \sup_{|u_I| \le 1} \left\| \hat{\theta}(y) - \theta \right\|_{l^2}^2 . \tag{3.2.4}$$

We want that the error in the formula (3.2.4) to be as small as possible and, at the same time, we aim to ensure the uniform shrinkage condition

$$\left| \hat{\theta}_I \right| \le |\theta_I| \, , \ I \in \mathcal{I}. \tag{3.2.5}$$

Consider a reconstruction formula based on the *soft thresholding* nonlinearity

$$\eta_\lambda(y) = sgn(y)(|y| - t)_+. \tag{3.2.6}$$

Setting the threshold level $\lambda = \delta$ we define the estimator

$$\hat{\theta}^{(\delta)} = \eta_\lambda(y_I), \ I \in \mathcal{I}. \tag{3.2.7}$$

Theorem 3.2.1 ([35, *Theorem 3.1*]) *The soft thresholding estimator satisfies the uniform-shrinkage condition (3.2.5).*

3.3 *Thresholding* Method and Statistical Estimation

Let (z_I) be white noise, i.i.d., coresponding to abstract model (3.2.2). Then

$$\pi_n \equiv P\left\{\|(z_I)\|_{l^\infty} \le \sqrt{2\log n}\right\} \to 1,\ n \to \infty. \tag{3.3.1}$$

The above relation, motivates us to act as if (3.2.2) were an instance of the deterministic model (3.2.3), with noise level $\delta_n = \sqrt{2\log n}\cdot\varepsilon$.

3.3.1 Near Optimal Mean Squared Error

The following lower bound says that statistical estimation at noise level ε is at least as hard as optimal recovery at that same noise level δ_n.

Theorem 3.3.1 ([35, *Theorem 4.2*]) *Let Θ be solid and orthosymmetric. Then, the estimator $\hat{\theta}^{(n)}$ is nearly minimax and satisfies*

$$M_n\left(\hat{\theta}^{(n)}, \theta\right) \le n(2\log(n)+1)(\varepsilon^2 + 2.22M_n^*(\Theta)), \quad \theta \in \Theta, \tag{3.3.2}$$

i.e. $\hat{\theta}^{(n)}$ is uniformly within the same factor $4.44\log(n)$ of minimax for every solid orthosymmetric set.

3.3.2 Near Minimal Shrinkage

Let Y be a random variable with normal distribution $N(\mu, 1)$. Consider the class U_α of all monotone odd nonlinearities $u(y)$ which satisfy the probabilistic shrinkage property with probability at least $1-\alpha$..

$$P\left\{|u(Y)| \le |\mu| \ge 1-\alpha\right\}, \quad \forall \mu \in \mathbb{R}.$$

The *soft thresholding* $\eta_{\lambda(\alpha)}$ is a member of this class with the threshold $\lambda(\alpha) = \Phi^{-1}\left(1-\dfrac{\alpha}{2}\right)$, where $\Phi(y)$ is the standard normal distribution. The resulting vector estimate $\hat{\theta} = (u(y_I))_I,\ u \in U_\alpha$, satisfies

$$P\left\{\left|\hat{\theta}_I\right| \leq |\theta_I|\,,\ \ \forall I \in \mathcal{I}_n\right\} \geq 1-(1-\alpha)^n.$$

3.4 Interpolating Wavelet Transforms

In this section we will describe a kind of wavelets transforms that characterize smooth spaces and for which the coefficients are obtained by sampling rather than integration. Then, we use these transforms in order to reinterpret the empirical wavelet transform, i.e. we will apply pyramid filters to samples of a function.

The interpolating wavelets transforms represent the objects in terms of dilations and translations of a basic function, but for which the coefficients are obtained from linear combinations of samples rather than from integrals. The interpolating transform is optimal from the point of view of computing individual coeffcients: computational cost and coeffcient decay.

3.4.1 Empirical Wavelets Transforms

The empirical wavelet coefficients, which derive only from finite filtering calculations, are actually the first n theoretical coefficients for a nicely behaved transform of continuous functions. This interpretation shows us that empirical wavelet coefficients of a smooth function automatically obey the same type of decay estimates as theoretical orthogonal wavelet coefficients. It also shows that *shrinking* empirical wavelet coefficients towards zero (linear or nonlinear procedure) always acts as a smoothing operator in any of a wide range of smoothness measures. It also shows that sampling followed by appropriate interpolation of the sampled values is a smoothing operator in any of a wide range of smoothness classes. Finally it shows that the theoretical wavelet coefficients are close to the empirical wavelet coefficients in an exact sense. These facts are significant for the study of certain nonlinear methods for smoothing and denoising noisy, sampled data.

3.5 Subdivision Schemes

In a broader sense, the subdivision is a method of processing data available at a coarse scale by recursive generation of these data, more smooth, at a finer resolution. This method is useful in generating curves and surfaces, but is also closely linked with the construction of wavelets functions via multiresolution analysis.

3.5.1 Sampling, Interpolation and Smoothing

Suppose that we take samples $(2^{-j_1} f(k/2^{j_1}))_{k\in\mathbb{Z}}$. Using just those samples, obtain the interpolating wavelet coefficients of f at all levels up to and including $j_1 - 1$.

If the interpolating wavelet was a fundamental spline, this is a spline interpolation. If the interpolating wavelet was a Deslauriers-Dubuc fundamental function, this is a Deslauriers-Dubuc interpolation.

3.5.2 Nonlinear Wavelets Transforms based on Subdivision Schemes

The uniform subdivision schemes are defined as operators considered on a multivariate integer grid. Under certain conditions, a uniform subdivision scheme defines a refinable function, i.e. a function that can be expressed as a finite sum of dilates and translates of it. The integer translates of this function span over a space of functions that can be calculated through iterative application of the subdivision scheme. A subdivision scheme consists in applying repeatedly a refinement operator S to a given set of control points, $P^0 = \{P_0, P_1, P_3 \dots\}$. The control points determine the shape of the limit curve. Typically, each point of the curve is computed by taking a weighted sum of a number of control points. The weight of each point varies according to a subdivision rule. The set of weights is called the mask of the subdivision scheme. The control points at the kth level are generated by a subdivision rule,

$$P_i^k = (SP^{k-1})_i = (S^k P^0)_i = \sum_{j\in\mathbb{Z}} a_{i-2j}^{(k)} P_j^{k-1},\ i \in \mathbb{Z},\ k = 1, 2, 3, \dots$$

The set of coefficients, $a^{(k)} = \{a_i^{(k)} |\ i \in \mathbb{Z}\}$ is called the mask of the subdivision scheme at level k. The mask is always assumed to be of finite support so that the set $\{i \in \mathbb{Z} :\ a_i^{(k)} \neq 0\}$, called the support of the subdivision scheme, is finite for every $k = 1, 2, \dots$. The control points at different refinement levels converge to a curve called the limit curve. If the refinement rule is the same for all levels (thus independent of k), such a scheme is called stationary. A subdivision scheme is named local if the computation of a new control point at level $k+1$ only involves old control points of level k which lie in the topological neighbourhood of the new point.

3.5.3 A New Scheme in Study

We will construct a new non-uniform subdivision scheme combining the ternary 3-point interpolating scheme in the case $b = \frac{2}{9}$ [63] with Chaikin's scheme [68], [69]. This new subdivision scheme is presented in the paper Sobolu R. [109].

Our purpose is to define the operator $S : l(\mathbb{Z}) \to l(\mathbb{Z})$ that generates all quadratic polynomials, $\pi_2(\mathbb{R})$.

According to the results established by Levin in [74], it is sufficient to show that for some $Q : \pi_2(\mathbb{R}) \to l(\mathbb{Z})$, we have $SQ = Q\sigma$ and then to determine the corresponding S

We define $Q : \pi_2(\mathbb{R}) \to l(\mathbb{Z}),\ \forall\ f \in \pi_2(\mathbb{R}),\ \forall\ i \in \mathbb{Z}$, such that,

$$Qf(i) = \begin{cases} f(i),\ i \leq 0, \\ f\left(i - \frac{1}{2}\right) - \frac{1}{8} f''\left(i - \frac{1}{2}\right),\ i > 0, \end{cases} \tag{3.5.1}$$

and then we solve the equation $SQ = Q\sigma$.

For a given $P \in l(\mathbb{Z})$ let $SP(i)$ be defined by the Chaikin's scheme

$$(SP)_{2i} = \frac{P_i + 3P_{i-1}}{4} \quad \text{and} \quad (SP)_{2i+1} = \frac{3P_i + P_{i-1}}{4},\ \text{for } i = 2, 3, 4, \ldots, \tag{3.5.2}$$

and by the ternary 3-point interpolating scheme

$$(Sp)_j^{i+1} = \sum_k a_{3k-j} p_k^i,\ \text{for } i = 0, -2, -3, -4, \ldots, \tag{3.5.3}$$

where $a = (a_j)$ is the mask scheme and $P^i = (p_j^i)$ is the set of control points after i^{th} subdivision step.

We can see that $SQf(i) = Q\sigma f(i),\ \ \forall f \in \pi_2(\mathbb{R}),\ \ i \in \mathbb{Z} \setminus \{-1,\ 1\}$. We need to define $SP(-1)$, $SP(1)$, for arbitrary P such that S satisfies $SQ = Q\sigma$ for $f \in \pi_2(\mathbb{R})$. Therefore, we will look for S satisfying

$$\begin{aligned} SP(-1) &= a_0 P(-2) + a_1 P(-1) + a_2 P(0), \\ SP(1) &= b_0 P(-1) + b_1 P(0) + b_2 P(1). \end{aligned} \tag{3.5.4}$$

The parameters a_0, a_1, a_2 are calculated from condition $SQf(-1) = Q\sigma f(-1)$, $\forall\ f \in \pi_2(\mathbb{R})$. We take for f the monomials up to degree 2, $f(x) = x^k,\ k \in \{0, 1, 2\}$.

Then we obtain

$$
\begin{aligned}
SQ1 = Q1 \quad &\Longrightarrow \quad a_0 + a_1 + a_2 = 1,\\
SQx = \frac{1}{2}Qx \quad &\Longrightarrow \quad -2a_0 - a_1 = \frac{1}{2},\\
SQx^2 = \frac{1}{4}Qx^2 \quad &\Longrightarrow \quad 4a_0 + a_1 = \frac{1}{4}.
\end{aligned}
$$

Similarly, from condition $SQf(1) = Q\sigma f(1)$ for all $f \in \pi_2(\mathbb{R})$, we get

$$
\begin{aligned}
SQ1 = Q1 \quad &\Longrightarrow \quad b_0 + b_1 + b_2 = 1,\\
SQx = \frac{1}{2}Qx \quad &\Longrightarrow \quad -b_0 + \frac{1}{2}b_2 = \frac{1}{4},\\
SQx^2 = \frac{1}{4}Qx^2 \quad &\Longrightarrow \quad b_0 + b_2 = 0.
\end{aligned}
$$

The above systems have a unique solution given by,

$$
a = \left(\frac{3}{8}, -\frac{5}{4}, \frac{15}{8}\right) \quad \text{and} \quad b = \left(-\frac{1}{6}, 1, \frac{1}{6}\right).
$$

The rules (3.5.1) and (3.5.4) generate the new scheme$S$.

Subdivision operator Q defined above can be extended to an operator from $C(\mathbb{R})$ to $l(\mathbb{Z}^s)$ which is bounded and local.

We propose the following extension,

$Q : C(\mathbb{R}) \to l(\mathbb{Z}),\ \forall\, f \in C(\mathbb{R}),\ \forall\, i \in \mathbb{Z}$,

$$
Qf(i) = \begin{cases} f(i),\ \ i \leq 0,\\ f(i) - \dfrac{f(i+1) - f(i)}{2},\ \ i > 0. \end{cases} \tag{3.5.5}
$$

Numerical examples

We have tested this scheme for the functions $f_1, f_2 : \mathbb{R} \to \mathbb{R},\ f_1(x) = \cos x,\ f_2(x) = \frac{x}{x^2+1}$, by measuring the error $\|\sigma^{-n} S^{\infty} Q \sigma^n f(x) - f(x)\|_{\infty}$.

The following maximal errors were obtained:

n	maximal error for f_1	maximal error for f_2
0	0.47	0.05
1	0.38	$0.19996 \cdot 10^{-7}$
2	0.09	$0.799679 \cdot 10^{-8}$

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