BABEŞ - BOLYAI UNIVERSITY OF CLUJ - NAPOCA

Faculty of Mathematics and Computer Science

DIFFERENTIAL AND INTEGRAL OPERATORS ON SPACES OF FUNCTIONS OF ONE COMPLEX VARIABLE

Ph.D. Thesis Summary

by ELISABETA - ALINA TOTOI

Scientific Supervisor Professor Ph.D. GRIGORE ŞTEFAN SĂLĂGEAN

Cluj - Napoca

2011

Contents

Introduction **3**

1 Concepts and preliminary results. Applications **6**

1.1 The classes $H[a,n]$, A, S, Σ_u and Σ_0 . . . 6
1.2 Starlike and convex functions . . . 7
1.3 Starlike and convex functions of some order . . . 8
1.4 Close-to-convex functions . . . 8
1.5 Subordination . . . 8
1.6 Differential subordinations method. General form . . . 9
1.7 The class of admissible functions. Fundamental theorems . . . 9
1.8 Two subclasses of special functions . . . 10
1.9 "Open Door" theorem; the starlikeness order of the class $I_{\beta,\gamma}(S^*(\alpha))$ 10

2 Briot-Bouquet differential subordinations and superordinations **13**

2.1 Definitions and notations . . . 13
2.2 Dominants for the Briot-Bouquet differential subordinations . . . 13
2.3 Univalent solutions of the Briot-Bouquet differential equation . . . 14
2.4 The general theory of differential superordinations . . . 15
2.5 Briot-Bouquet differential superordinations . . . 16

3 Integral operators on special subclasses of the class Σ of meromorphic functions **18**

3.1 Starlikeness and convexity conditions for meromorphic functions . . . 18
3.2 Integral operators on spaces of meromorphic starlike functions . . . 19
3.3 The subclass of inverse-convex functions . . . 20
3.4 The subclass of close-to-inverse-convex functions . . . 23
3.5 Starlikeness properties for the integral operator $J_{\beta,\gamma}$ . . . 24

4 Integral operators on the class Σ_p of meromorphic multivalent functions **29**

4.1 The operator $J^{\Phi,\varphi}_{p,\alpha,\beta,\gamma,\delta}$ . . . 29
4.2 The operator $J_{p,\beta,\gamma}$ applied to the class $\Sigma^*_p(\alpha,\delta)$ . . . 31
4.3 The operator $J_{p,\gamma}$ applied to the classes $\Sigma K_p(\alpha,\delta)$ and $\Sigma\mathcal{C}_{p,0}(\alpha,\delta;\varphi)$. 34
4.4 Subclasses of the class Σ_p defined through a multiplier transformation 35

5 Applications of the Briot-Bouquet differential subordinations and superordinations **39**
5.1 The operator $J_{p,\beta,\gamma}$ and the class $\Sigma S_p(h_1, h_2)$ 39
5.2 The operator $J_{p,\gamma}$ and the class $\Sigma K_p(h_1, h_2)$ 45
5.3 The operator $J_{p,\gamma}$ applied to the classes $\Sigma\mathcal{C}_{p,0}(h_1, h_2; \varphi, h)$ and $\Sigma\mathcal{C}_{p,0}(h_2; h)$ 49

Bibliography **52**

Introduction

The complex analysis is one of the classical branches of mathematics and has its roots in the XVIII century. The two important directions of complex analysis are the theory of conformal representations and the geometric theory of analytic functions.

The geometric theory of one variable functions became a special branch of the complex analysis in the twenty century when the first important papers of this branch appeared, due to P. Koebe [34] (1907), T.H. Gronwall [27] (1914), [28] (1916), J.W. Alexander [4] (1915), L. Bieberbach [8], [9] (1916).

Important romanian mathematicians had a great role in the developing of this domain of mathematics. We remember here just two of them, from Cluj. One is G. Călugăreanu, who obtained important results in the field of univalent functions and in the field of meromorphic functions and the other is P.T. Mocanu, who introduced the concept of α-convexity, obtained univalence criteria for non-analytic functions, initiated and developed (in collaboration with S.S. Miller) the differential subordinations method and, more recently, the differential superordinations method.

In the first chapter we present the classes $H[a, n]$, A, S, Σ_u and Σ_0, some special subclasses of the class of univalent functions (starlike, convex, close-to-convex functions), the subordination, the method of differential subordinations, the "Open Door" ţheorem and the theorem relative to the order of starlikeness of the class $I_{\beta,\gamma}(S^*(\alpha))$. The penultimate section of this chapter contains two original results which are relative to a subclass of starlike functions and to a subclass of convex functions of order α. These results are published in [89].

In Chapter 2 we present some classical results and concepts regarding the Briot-Bouquet differential subordinations and superordinations.

In Chapter 3 we study the meromorphic functions with the unique simple pole $z = 0$.

In section 3.1 are presented the starlikeness and the convexity conditions for these functions and in section 3.2 are presented some starlikeness properties for a well-known integral operator.

The last three sections of this chapter contains only original results which are published in [90] and [92].

In section 3.3 are defined the class of inverse-starlike functions and the class of inverse-convex functions and are given analytical characterizations, duality theorem and distortion theorem.

In section 3.4 we define the class of close-to-convex functions and we state and prove two theorems which give a connection between this new class and the class of inverse-convex functions.

In section 3.5 is defined an integral operator, denoted by $J^{\Phi,\varphi}_{\alpha,\beta,\gamma,\delta}$ and some important properties regarding this operator are presented. There are also presented some starlikeness properties for the particular integral operator $J^{1,1}_{\beta,\beta,\gamma,\gamma}$, more simply denoted by $J_{\beta,\gamma}$.

In Chapter 4 we study the meromorphic multivalent functions and all the results presented in this chapter are original and are published in [91] and [93].

In section 4.1 is defined an integral operator, denoted by $J^{\Phi,\varphi}_{p,\alpha,\beta,\gamma,\delta}$, and an existence theorem, with respect to this operator, is given.

In the sections 4.2 and 4.3 are studied the preserving properties of some new subclasses, when we apply the particular integral operators $J_{p,\beta,\gamma}$ and $J_{p,\gamma}$.

In section 4.4 is considered a multiplier transformation, denoted by $J^{n}_{p,\lambda}$, and a new subclass of meromorphic multivalent functions is defined using this transformation and the condition from starlikeness. Then the preserving properties of this subclass are studied, when the integral operator $J_{p,\gamma}$ is applied.

Finally, in the last chapter we define new subclasses of meromorphic multivalent functions using the subordination and the superordination and we establish the conditions such that when we apply one of the integral operators $J_{p,\beta,\gamma}$ or $J_{p,\gamma}$ to a function which belongs to one of these subclasses, we get a function which belongs to a similar class.

Also, the full bibliography is included.

Keywords: starlikeness, convexity, close-to-convexity, subordination, superordination, integral operators, meromorphic functions, meromorphic multivalent functions

Chapter 1

Concepts and preliminary results. Applications

In this chapter we present the classes $H[a,n]$, A, S, Σ_u and Σ_0, some special subclasses of the class of univalent functions (starlike, convex, close-to-convex functions), the subordination, the method of differential subordination, the "Open Door" ţheorem and the theorem relative to the order of starlikeness of the class $I_{\beta,\gamma}(S^*(\alpha))$.

The penultimate section of this chapter contains two original results which are relative to a subclass of starlike functions and to a subclass of convex functions of order α. These results are published in [89].

We mention that the concepts and the results presented in this chapter may be also find in the first three chapters of the book "Differential Subordinations. Theory and Applications" written by S.S. Miller and P.T. Mocanu (see [52]).

1.1 The classes $H[a,n]$, A, S, Σ_u and Σ_0

First we present some of the notations which we use in this paper.
The disc of center a and radius r is denoted by $U(a,r)$, where $a \in \mathbb{C}$ and $r > 0$, so

$$U(a,r) = \{z \in \mathbb{C} : |z-a| < r\}.$$

We denote by U the unit disc, $U(0,1)$, and by U_R the disc $U(0,R)$.
Let $H(U)$ be the set of holomorphic functions in U, $H_u(U)$ the set of holomorphic and univalent functions in U and for $a \in \mathbb{C}$ and $n \in \mathbb{N}^*$ we denote

$$H[a,n] = \{f \in H(U) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots\}$$

and

$$A_n = \{f \in H(U) : f(z) = z + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \ldots\}.$$

We denote by A the set A_1.

Let S be the class

$$S = \{f \in A : f \,\text{is univalent in}\, U\}.$$

Let Σ_u be the class of meromorphic univalent functions defined on U^-, where $U^- = \{\zeta \in \mathbb{C}_\infty : |\zeta| > 1\}$, of the form

$$\varphi(\zeta) = \zeta + \alpha_0 + \frac{\alpha_1}{\zeta} + \cdots + \frac{\alpha_n}{\zeta^n} + \cdots, |\zeta| > 1$$

and $\Sigma_0 = \{\varphi \in \Sigma_u : \varphi(\zeta) \neq 0, \zeta \in U^-\}$.

1.2 Starlike and convex functions

Definition 1.2.1. *[64] Let $f \in H(U)$ be the function with $f(0) = 0$. We say that f is starlike in U with respect to zero (or more simply, starlike)if the function f is univalent in U and $f(U)$ is a starlike domain with respect to zero, this meaning that for each $z \in U$ the line segment joining 0 and $f(z)$ lies entirely in $f(U)$.*

Theorem 1.2.1. *[64]***(the theorem of analytical characterization of starlikeness)***Let $f \in H(U)$ be a function with $f(0) = 0$. Then f is starlike if and only if $f'(0) \neq 0$ and*

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > 0, \; z \in U.$$

Definition 1.2.2. *We denote by S^* the class of normalized starlike functions on the unit disc U, so*

$$S^* = \left\{ f \in A : \operatorname{Re} \frac{zf'(z)}{f(z)} > 0, \; z \in U \right\}.$$

Definition 1.2.3. *[64] Let $f : U \to \mathbb{C}$ be a holomorphic function. We say that f is convex on U (or more simply, convex) if f is univalent in U and $f(U)$ is a convex domain.*

Theorem 1.2.2. *[64]***(the theorem of analytical characterization of convexity)***Let $f \in H(U)$. Then f is convex if and only if $f'(0) \neq 0$ and*

$$\operatorname{Re} \frac{zf''(z)}{f'(z)} + 1 > 0, \, z \in U.$$

Definition 1.2.4. *[64] We denote by K the class of normalized convex functions on the unit disc U, so*

$$K = \left\{ f \in A : \operatorname{Re} \frac{zf''(z)}{f'(z)} + 1 > 0, \; z \in U \right\}.$$

1.3 Starlike and convex functions of some order

Definition 1.3.1. *[64] Let $\alpha < 1$. We define:*

- *The class of starlike functions of order α as*

$$S^*(\alpha) = \left\{ f \in A : \operatorname{Re} \frac{zf'(z)}{f(z)} > \alpha,\ z \in U \right\}.$$

- *The class of convex functions of order α as*

$$K(\alpha) = \left\{ f \in A : \operatorname{Re} \frac{zf''(z)}{f'(z)} + 1 > \alpha,\ z \in U \right\}.$$

It is obvious that for $\alpha = 0$ we have $S^*(0) = S^*$ and $K(0) = K$.

1.4 Close-to-convex functions

Definition 1.4.1. *[64] The function $f \in H(U)$ is said to be close-to-convex if there exists a convex function φ (defined on U), such that*

$$\operatorname{Re} \frac{f'(z)}{\varphi'(z)} > 0,\ z \in U.$$

Definition 1.4.2. *[64] We denoted by $\mathcal{C}$ the class of normalized close-to-convex functions, defined on U, so*

$$\mathcal{C} = \left\{ f \in A : (\exists)\varphi \in K,\ \operatorname{Re} \frac{f'(z)}{\varphi'(z)} > 0,\ z \in U \right\}.$$

1.5 Subordination

Definition 1.5.1. *[64] Let f and F be members of $H(U)$. The function f is said to be subordinate to F, written $f \prec F$ or $f(z) \prec F(z)$, if there exists a function w analytic in U, with $w(0) = 0$ and $|w(z)| < 1$, and such that $f(z) = F(w(z))$.*

Theorem 1.5.1. *[73], [64] Let $f, g \in H(U)$ and suppose that g is univalent in U. Then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subseteq g(U)$.*

1.6 Differential subordinations method. General form

In the papers [44] and [45], S.S. Miller and P.T. Mocanu introduced the theory of differential subordinations, which was developed in many other papers.

Definition 1.6.1. *[64] Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and let h be univalent in U. If p is analytic in U and satisfies the (second order) differential subordination*

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z), \tag{1.1}$$

then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, or more simply, a dominant, if $p \prec q$ for all p satisfying (1.1). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (1.1) is said to be the best dominant of (1.1). (Note that the best dominant is unique up to a rotation of U).
If we require the more restrictive condition $p \in H[a, n]$, then p will be called an (a, n)-solution, q an (a, n)-dominant, and $\tilde{q}$ the best (a, n)-dominant.

1.7 The class of admissible functions. Fundamental theorems

Definition 1.7.1. *[64] We denote by Q the set of functions f that are analytic and injective on $\overline{U} \setminus E(f)$, where*

$$E(f) = \left\{ \zeta \in \partial U : \lim_{z \to \zeta} f(z) = \infty \right\},$$

and they are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(f)$. The subclass of Q for which $f(0) = a$, is denoted by $Q(a)$.

Definition 1.7.2. *[45], [46], [64] Let $\Omega \subset \mathbb{C}$, $q \in Q$ and $n \in \mathbb{N}$, $n \geq 1$. We denote by $\Psi_n[\Omega, q]$ the class of the functions $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ which satisfies the condition*

$$(A) \quad \psi(r, s, t; z) \notin \Omega \quad \text{when}$$

$$r = q(\zeta),\ s = m\zeta q'(\zeta),\ \text{Re}\left[\frac{t}{s} + 1\right] \geq m\text{Re}\left[\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right],$$

$$\text{where} \quad z \in U,\ \zeta \in \partial U \setminus E(q),\ m \geq n.$$

The set $\Psi_n[\Omega, q]$ is said to be the class of admissible functions and the condition (A) is said to be the admissibility condition.

Theorem 1.7.1. *[47], [64] Let $\psi \in \Psi_n[\Omega, q]$ where $q(0) = a$. If the function $p \in H[a, n]$ satisfies the condition*

$$\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega,\ z \in U,$$

then $p(z) \prec q(z)$.

1.8 Two subclasses of special functions

The results presented in this section are original and are published in [89]. First we present a subclass of the class of starlike functions.

Definition 1.8.1. *[89] Let $\alpha \geq 0$ and $f \in A$ such that*

$$\frac{f(z)f'(z)}{z} \neq 0,\ \alpha + \frac{zf'(z)}{f(z)} \neq 0, z \in U.$$

We say that the function f is in the class N_α if the function $F : U \to \mathbb{C}$ given by

$$F(z) = zf'(z)\left(\alpha + \frac{zf'(z)}{f(z)}\right)$$

is starlike in U.

Theorem 1.8.1. *[89] For each real number $\alpha \geq 0$ we have*

$$N_\alpha \subset S^*.$$

We next present a subclass of the class of convex functions of order α.

Definition 1.8.2. *[89] Let $\alpha \in [0, 1)$ and $f \in A$ with*

$$\frac{f(z)f'(z)}{z} \neq 0, \quad 1 + \frac{zf''(z)}{f'(z)} \neq 0,\ z \in U.$$

We say that the function f is in the class $N(\alpha)$ if the function $F : U \to \mathbb{C}$ given by

$$F(z) = zf'(z)\left(1 + \frac{zf''(z)}{f'(z)}\right),$$

is starlike of order α.

Theorem 1.8.2. *[89] For $\alpha \in [0, 1)$ we have*

$$N(\alpha) \subset K(\alpha).$$

1.9 "Open Door" theorem; the starlikeness order of the class $I_{\beta,\gamma}(S^*(\alpha))$

First we define *the "Open Door" function.*

Definition 1.9.1. *[64] Let c be a complex number such that $\operatorname{Re} c > 0$, let n be a positive integer, and let*

$$C_n = C_n(c) = \frac{n}{\operatorname{Re} c}\left[|c|\sqrt{1+\frac{2\operatorname{Re} c}{n}} + \operatorname{Im} c\right]. \tag{1.2}$$

If $R(z)$ is the univalent function defined in U by $R(z) = \dfrac{2C_n z}{1-z^2}$, then the "Open Door" function is defined by

$$R_{c,n}(z) = R\left(\frac{z+b}{1+\bar{b}z}\right) = 2C_n \frac{(z+b)(1+\bar{b}z)}{(1+\bar{b}z)^2-(z+b)^2}, \tag{1.3}$$

where $b = R^{-1}(c)$.

Theorem 1.9.1. *[64] [64], [48], [49]* **("Open Door" Theorem or Integral Existence Theorem)** *Let $\Phi, \varphi \in H[1,n]$ with $\Phi(z)\cdot\varphi(z) \neq 0$ in U. Let α, β, γ and δ be complex numbers with $\beta \neq 0$, $\alpha+\delta = \beta+\gamma$, and $\operatorname{Re}(\alpha+\delta) > 0$. Let $f \in A_n$ and suppose that*

$$\alpha\frac{zf'(z)}{f(z)} + \frac{z\varphi'(z)}{\varphi(z)} + \delta \prec R_{\alpha+\delta,n}(z),$$

where $R_{c,n}$ is defined by (1.3). If $F = I^{\Phi,\varphi}_{\alpha,\beta,\gamma,\delta}(f)$ is defined by

$$F(z) = \left[\frac{\beta+\gamma}{z^{\gamma}\Phi(z)}\int_0^z f^{\alpha}(t)\varphi(t)t^{\delta-1}dt\right]^{\frac{1}{\beta}} = z + A_{n+1}z^{n+1} + \cdots, \tag{1.4}$$

then $F \in A_n$, $\dfrac{F(z)}{z} \neq 0$, $z \in U$, and

$$\operatorname{Re}\left[\beta\frac{zF'(z)}{F(z)} + \frac{z\Phi'(z)}{\Phi(z)} + \gamma\right] > 0,\ z \in U.$$

(All powers in (3.2) are principal ones.)

Let $I_{\beta,\gamma}$ be the operator defined by the identity

$$I_{\beta,\gamma}(f)(z) = \left[\frac{\beta+\gamma}{z^{\gamma}}\int_0^z f^{\beta}(t)t^{\gamma-1}dt\right]^{\frac{1}{\beta}}. \tag{1.5}$$

Definition 1.9.2. *[64] Let $\beta > 0$ and $\gamma \in \mathbb{R}$ with $\beta+\gamma > 0$. For a given number $\alpha \in \left[-\dfrac{\gamma}{\beta}, 1\right)$ we define the order of starlikeness of the class $I_{\beta,\gamma}(S^*(\alpha))$ as the biggest number $\delta = \delta(\alpha;\beta,\gamma)$ such that $I_{\beta,\gamma}(S^*(\alpha)) \subset S^*(\delta)$.*

The order of starlikeness of the class $I_{\beta,\gamma}(S^*(\alpha))$ was determined în 1981 by P.T. Mocanu, D. Ripeanu and I. Şerb în [65] and this result is next presented.

Theorem 1.9.2. *[65], [64]* **(the theorem regarding the order of starlikeness of the class** $I_{\beta,\gamma}(S^*(\alpha))$**)** *Let* $\beta > 0$, $\beta + \gamma > 0$ *and let* $I_{\beta,\gamma}$ *be the integral operator defined by (1.5).*

If $\alpha \in \left[-\frac{\gamma}{\beta}, 1\right)$, *then the order of starlikeness of the class* $I_{\beta,\gamma}(S^*(\alpha))$ *is*

$$\delta(\alpha; \beta, \gamma) = \inf\{\operatorname{Re} q(z) : z \in U\},$$

where

$$q(z) = \frac{1}{\beta Q(z)} - \frac{\gamma}{\beta} \text{ și } Q(z) = \int_0^1 \left(\frac{1-z}{1-tz}\right)^{2\beta(1-\alpha)} t^{\beta+\gamma-1} dt. \tag{1.6}$$

Moreover, if $\alpha \in [\alpha_0, 1)$, *where*

$$\alpha_0 = \max\left\{\frac{\beta - \gamma - 1}{2\beta}, -\frac{\gamma}{\beta}\right\}$$

and $g = I_{\beta,\gamma}(f)$ *with* $f \in S^*(\alpha)$, *then*

$$\operatorname{Re} \frac{zg'(z)}{g(z)} \geq q(-r) = \frac{1}{\beta}\left[\frac{\gamma+\beta}{{}_2F_1(1, 2\beta(1-\alpha), \gamma+1+\beta; \frac{r}{r+1})} - \gamma\right],$$

for $|z| \leq r < 1$ *and*

$$\delta(\alpha; \beta, \gamma) = q(-1) = \frac{1}{\beta}\left[\frac{\gamma+\beta}{{}_2F_1(1, 2\beta(1-\alpha), \gamma+1+\beta; \frac{1}{2})} - \gamma\right],$$

where the function q *is given by (1.6) and* ${}_2F_1(a, b, c; z)$ *is the second order hypergeometric function.*

The extremal function is $g = I_{\beta,\gamma}(k)$, *where* $k(z) = z(1-z)^{2(\alpha-1)}$.

Chapter 2

Briot-Bouquet differential subordinations and superordinations

In this chapter we present some classical results and concepts regarding the Briot-Bouquet differential subordinations and superordinations.

2.1 Definitions and notations

Definition 2.1.1. *[64]*

1. *A differential Briot-Bouquet operator is an operator which has the form*

$$\Phi(p(z), zp'(z)), \quad \textit{where} \quad \Phi(r,s) = r + \frac{s}{\beta r + \gamma}.$$

2. *Let be $h \in H_u(U)$ and $p \in H(U)$ with $p(0) = h(0)$. A Briot-Bouquet differential subordination is a differential subordination of the form*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z).$$

2.2 Dominants for the Briot-Bouquet differential subordinations

Theorem 2.2.1. *[51], [64] Let $\beta, \gamma \in \mathbb{C}, \beta \neq 0$ and let h be convex in U, with $h(0) = a$. Let n be a positive integer. Suppose that the differential equation*

$$q(z) + \frac{nzq'(z)}{\beta q(z) + \gamma} = h(z), \quad [q(0) = h(0) = a] \tag{2.1}$$

has a univalent solution q that satisfies $q(z) \prec h(z)$. If $p \in H[a,n]$ satisfies

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z), \tag{2.2}$$

then $p(z) \prec q(z)$, and q is the best (a,n)-dominant of (2.2).

Theorem 2.2.2. *[51], [64] Let β, $\gamma \in \mathbb{C}$, $\beta \neq 0$ and let h be a convex function in U such that*

$$\operatorname{Re}[\beta h(z) + \gamma] > 0,\ z \in U.$$

If the Briot-Bouquet differential equation

$$q(z) + \frac{nzq'(z)}{\beta q(z) + \gamma} = h(z), \quad [q(0) = h(0) = a]$$

has a univalent solution $q \in H_u(U)$, then

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z) \Rightarrow p(z) \prec q(z)$$

and the function q is the best (a,n) dominant.

2.3 Univalent solutions of the Briot-Bouquet differential equation

Theorem 2.3.1. *[51], [64] Let $\beta, \gamma \in \mathbb{C}$ and let h be a convex function in U, with*

$$\operatorname{Re}[\beta h(z) + \gamma] > 0,\ z \in U.$$

Let q_m and q_k be the univalent solutions of the Briot-Bouquet differential equation

$$q(z) + \frac{nzq'(z)}{\beta q(z) + \gamma} = h(z),\ z \in U, \quad q(0) = h(0),$$

for $n = m$ and $n = k$ respectively. If m/k, then $q_k(z) \prec q_m(z) \prec h(z)$. So, $q_k(z) \prec q_1(z) \prec h(z)$.

Theorem 2.3.2. *[51], [64]Let $\beta, \gamma \in \mathbb{C}$ with $\beta \neq 0$, and let n be a positive integer. Let $R_{\beta a+\gamma,n}$ be as given in (1.3), let h be analytic in U with $h(0) = a$, and let $\operatorname{Re}[\beta a + \gamma] > 0$. If*

$$\beta h(z) + \gamma \prec R_{\beta a+\gamma,n}(z),$$

then the solution q of

$$q(z) + \frac{nzq'(z)}{\beta q(z) + \gamma} = h(z), \tag{2.3}$$

with $q(0) = a$, is analytic in U and satisfies $\operatorname{Re}[\beta q(z) + \gamma] > 0$.

If $a \neq 0$, then the solution for (2.3) is given by

$$q(z) = z^{\frac{\gamma}{n}} H^{\frac{\beta a}{n}}(z) \left[\frac{\beta}{n} \int_0^z H^{\frac{\beta a}{n}}(t) t^{\frac{\gamma}{n}-1} dt \right]^{-1} - \frac{\gamma}{\beta} =$$

$$= \left[\frac{\beta}{n} \int_0^1 \left[\frac{H(tz)}{H(z)} \right]^{\frac{\beta a}{n}} t^{\frac{\gamma}{n}-1} dt \right]^{-1} - \frac{\gamma}{\beta}, \tag{2.4}$$

where

$$H(z) = z \exp \int_0^z \frac{h(t) - a}{at} dt.$$

If $a = 0$, then the solution is given by

$$q(z) = H^{\frac{\gamma}{n}}(z) \left[\frac{\beta}{n} \int_0^z H^{\frac{\gamma}{n}}(t) t^{-1} dt \right]^{-1} - \frac{\gamma}{\beta} =$$

$$= \left[\frac{\beta}{n} \int_0^1 \left[\frac{H(tz)}{H(z)} \right]^{\frac{\gamma}{n}} t^{-1} dt \right]^{-1} - \frac{\gamma}{\beta},$$

where

$$H(z) = z \exp \frac{\beta}{\gamma} \int_0^z \frac{h(t)}{t} dt.$$

2.4 The general theory of differential superordinations

We mention that the results presented in this section were first published by S.S. Miller and P.T. Mocanu in 2003 in [53].

Definition 2.4.1. *[16], [53] Let $\varphi : \mathbb{C}^3 \times U \to \mathbb{C}$ and let h be analytic in U. If p and $\varphi(p(z), zp'(z), z^2p''(z); z)$ are univalent in U and satisfy the second order differential superordination*

$$h(z) \prec \varphi(p(z), zp'(z), z^2p''(z); z), \tag{2.5}$$

then p is called a solution of the differential superordination . An analytic function q is called a subordinant of the solutions of the differential superordination, or more simply, a subordinant, if $q \prec p$ for all p satisfying (2.5). An univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants q of (2.5) is said to be the best subordinant. Note that the best subordinant is unique up to a rotation of U.

Definition 2.4.2. *[16], [53] Let Ω be a set in $\mathbb{C}$ and $q \in H[a, n]$. The class of admissible functions $\Phi_n[\Omega, q]$, consists of those functions $\varphi : \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$ that satisfy the admissibility condition*

$$\varphi(r, s, t; \zeta) \in \Omega \quad \textit{whenever} \tag{2.6}$$

$$r = q(z),\ s = \frac{zq'(z)}{m},\ \mathrm{Re}\, \frac{t}{s} + 1 \leq \frac{1}{m} \mathrm{Re} \left[\frac{zq''(z)}{q'(z)} + 1 \right],$$

where $\zeta \in \partial U$, $z \in U$ and $m \geq n \geq 1$. When $n = 1$ we write $\Phi_1[\Omega, q]$ as $\Phi[\Omega, q]$.

If $\varphi : \mathbb{C}^2 \times \overline{U} \to \mathbb{C}$, then the admissibility condition (2.6) reduces to

$$\varphi\left(q(z), \frac{zq'(z)}{m}; \zeta\right) \in \Omega,$$

where $z \in U, \zeta \in \partial U$ and $m \geq n \geq 1$.

The next theorem is a key result in the theory of first and second order differential superordinations.

Theorem 2.4.1. *[16], [53] Let $\Omega \subset \mathbb{C}$, let $q \in H[a,n]$ and let $\varphi \in \Phi_n[\Omega, q]$. If $p \in Q(a)$ and $\varphi(p(z), zp'(z), z^2p''(z); z)$ is univalent in U, then*

$$\Omega \subset \{\varphi(p(z), zp'(z), z^2p''(z); z) : z \in U\}$$

implies $q(z) \prec p(z)$.

2.5 Briot-Bouquet differential superordinations

Let $\beta, \gamma \in \mathbb{C}$, Ω_1, $\Delta_1 \subset \mathbb{C}$, and $p \in H(U)$. In this section we study the implication:

$$\Omega_1 \subset \left\{p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} : z \in U\right\} \Rightarrow \Delta_1 \subset p(U), \tag{2.7}$$

and we are interested in determining the largest set $\Delta_1 \subset \mathbb{C}$ for which (2.7) holds.

If the sets Ω_1, $\Delta_1 \subset \mathbb{C}$ are simply connected domains not equal to $\mathbb{C}$, then it is possible to rephrase the above expression in terms of superordination in the form:

$$h_1(z) \prec p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \Rightarrow q_1(z) \prec p(z). \tag{2.8}$$

The left side of (2.8) is called a Briot-Bouquet differential superordination.

Corollary 2.5.1. *[16], [54] Let $\beta, \gamma \in \mathbb{C}$ and let h be convex in U with $h(0) = a$. Suppose that the differential equation*

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = h(z), \quad z \in U,$$

has the univalent solution q with $q(0) = a$, and $q(z) \prec h(z)$. If $p \in H[a,1] \cap Q$ and $p(z) + \dfrac{zp'(z)}{\beta p(z) + \gamma}$ is univalent in U, then

$$h(z) \prec p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \Rightarrow q(z) \prec p(z).$$

The function q is the best subordinant.

Corollary 2.5.2. *[16], [54] Let* $\beta, \gamma \in \mathbb{C}$ *and let the function* $h \in H(U)$ *with* $h(0) = a$ *and* $\operatorname{Re} c > 0$, *where* $c = \beta a + \gamma$ *and suppose that*

$$(i) \quad \beta h(z) + \gamma \prec R_{c,1}(z).$$

Let q *be the analytic solution of the Briot-Bouquet differential equation*

$$h(z) = q(z) + \frac{zq'(z)}{\beta q(z) + \gamma}$$

and suppose that

$$(ii) \quad \frac{zq'(z)}{\beta q(z) + \gamma} \quad \text{is starlike in } U.$$

If $p \in H[a,1] \cap Q$ *and* $p(z) + \dfrac{zp'(z)}{\beta p(z) + \gamma}$ *is univalent in* U, *then*

$$h(z) \prec p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \Rightarrow q(z) \prec p(z)$$

and the function q *is the best subordinant.*

Chapter 3

Integral operators on special subclasses of the class Σ of meromorphic functions

In this chapter we study the meromorphic functions with the unique simple pole $z = 0$.

In section 3.1 are presented the starlikeness and convexity conditions for these functions and in section 3.2 are presented some starlikeness properties of a well-known integral operator.

The last three sections of this chapter contains only original results which are published in [90] and [92].

In section 3.3 are defined the class of inverse-starlike functions and the class of inverse-convex functions and are given analytical characterizations, duality theorem and distortion theorem.

In section 3.4 we define the class of close-to-convex functions and we state and prove two theorems which give a connection between this new class and the class of inverse-convex functions.

In section 3.5 is defined an integral operator, denoted by $J^{\Phi,\varphi}_{\alpha,\beta,\gamma,\delta}$ and some important properties regarding this operator are presented. There are also presented some starlikeness properties for the particular integral operator $J^{1,1}_{\beta,\beta,\gamma,\gamma}$, more simply denoted by $J_{\beta,\gamma}$.

3.1 Starlikeness and convexity conditions for meromorphic functions

Let

$$\varphi(\zeta) = \zeta + \alpha_0 + \frac{\alpha_1}{\zeta} + \cdots + \frac{\alpha_n}{\zeta^n} + \cdots,\ \zeta \in U^-, \tag{3.1}$$

be a meromorphic function in U^-, with the unique simple pole $\zeta = \infty$. We denote, as usual, the set $E(\varphi) = \mathbb{C} \setminus \varphi(U^-)$, where $U^- = \{\zeta \in \mathbb{C}_\infty : |\zeta| > 1\}$.

Definition 3.1.1. *[64] We say that the function φ of the form (3.1) is starlike in U^- if φ is univalent in U^- and the set $E(\varphi)$ is starlike with respect to 0.*

Definition 3.1.2. *[64] We denote by Σ^* the class*

$$\Sigma^* = \{\varphi \in \Sigma_0 : \varphi \text{ is starlike in } U^-\}.$$

Definition 3.1.3. *[64] Let $g(z) = \frac{1}{z} + \alpha_0 + \alpha_1 z + \cdots + \alpha_n z^n + \cdots$, $0 < |z| < 1$, be a meromorphic function in U. We say that g is starlike in $\dot{U}$ if the function $\varphi(\zeta) = g\left(\frac{1}{\zeta}\right)$, $\zeta \in U^-$, is starlike in U^-.*

Theorem 3.1.1. *[64]* **(the theorem of analytical characterization of the starlikeness for meromorphic functions)** *Let $g : \dot{U} \to \mathbb{C}$ be a meromorphic function in U of the form*

$$g(z) = \frac{1}{z} + \alpha_0 + \alpha_1 z + \cdots,$$

such that $g(z) \neq 0$, $z \in \dot{U}$. Then the function g is starlike in $\dot{U}$ if and only if g is univalent in $\dot{U}$ and

$$\operatorname{Re}\left[-\frac{zg'(z)}{g(z)}\right] > 0,\ z \in \dot{U}.$$

Definition 3.1.4. *[64] We say that the function φ of the form (3.1) is convex in U^- if φ is univalent in U^- and the set $E(\varphi)$ is convex.*

Definition 3.1.5. *Let $g(z) = \frac{1}{z} + \alpha_0 + \alpha_1 z + \cdots + \alpha_n z^n + \cdots$, $0 < |z| < 1$, be a meromorphic function in U. We say that g is convex in $\dot{U}$ if the function $\varphi(\zeta) = g\left(\frac{1}{\zeta}\right)$, $\zeta \in U^-$, is convex in U^-.*

Theorem 3.1.2. *[64]* **(the theorem of analytical characterization of the convexity for meromorphic functions)** *Let $g : \dot{U} \to \mathbb{C}$ be a meromorphic function in U of the form*

$$g(z) = \frac{1}{z} + \alpha_0 + \alpha_1 z + \cdots,$$

such that $g(z) \neq 0$, $z \in \dot{U}$. Then the function g is convex in $\dot{U}$ if and only if g is univalent in $\dot{U}$ and

$$\operatorname{Re}\left[-\left(\frac{zg''(z)}{g'(z)} + 1\right)\right] > 0,\ z \in \dot{U}.$$

3.2 Integral operators on spaces of meromorphic starlike functions

We denote by Σ the class of meromorphic functions in the unit disc of the form

$$f(z) = \frac{1}{z} + a_0 + a_1 z + \cdots + a_n z^n + \cdots,\ z \in \dot{U},$$

and with Σ_0 we denote the class of functions $f \in \Sigma$ which are univalent in $\dot{U}$ and with $f(z) \neq 0$, $z \in \dot{U}$.

Definition 3.2.1. *[64] For $\alpha < 1$ let*

$$\Sigma^*(\alpha) = \left\{ f \in \Sigma : \operatorname{Re}\left[-\frac{zf'(z)}{f(z)} \right] > \alpha,\ z \in \dot{U} \right\}$$

be the class of meromorphic starlike functions of order α.

For $\gamma \in \mathbb{C}$ with $\operatorname{Re}\gamma > 0$ we consider the integral operator

$$I_\gamma : \Sigma \to \Sigma$$

defined by

$$I_\gamma(f)(z) = \frac{\gamma}{z^{\gamma+1}} \int_0^z t^\gamma f(t)dt = \gamma \int_0^1 u^\gamma f(uz)du. \tag{3.2}$$

Properties of this integral operator were studied in many papers such as: [6], [25], [66], [77], [83], [84].

Theorem 3.2.1. *[64], [66] Let $0 \leq \alpha < 1$ and $0 < \gamma \leq 1$. Then $I_\gamma[\Sigma^*(\alpha)] \subset \Sigma^*(\beta)$, where*

$$\beta = \beta(\alpha, \gamma) = \frac{1}{4}\left[2\alpha + 2\gamma + 3 - \sqrt{[2(\gamma - \alpha) + 1]^2 + 8\gamma} \right] \tag{3.3}$$

and the operator I_γ is defined by (3.2).

If we consider the condition $F(z) = I_\gamma(f)(z) \neq 0$, $z \in \dot{U}$, we have the next result.

Theorem 3.2.2. *[64], [66] Let $\alpha < 1$ and $\gamma > 0$. Let $f \in \Sigma^*(\alpha)$ and $F = I_\gamma(f)$, where the operator I_γ is defined by (3.2) and we suppose that $F(z) \neq 0$, $z \in \dot{U}$. Then $F \in \Sigma^*(\beta)$, where $\beta = \beta(\alpha, \gamma)$ is given by (3.3).*

3.3 The subclass of inverse-convex functions

We next present a special class of meromorphic functions, the class of inverse-convex functions which was defined in [92] and we will study some properties relative to this class.

The results presented in this section are original and are published in [92].

Definition 3.3.1. *[92] Let $g : \dot{U} \to \mathbb{C}$ be a meromorphic function in U of the form*

$$g(z) = \frac{1}{z} + \alpha_0 + \alpha_1 z + \cdots,\ z \in \dot{U}.$$

We say that the function g is inverse-starlike in $\dot{U}$ if there exists a starlike function $f \in S^*$ such that $f(z)g(z) = 1$ for each $z \in \dot{U}$. We denote by S_i^* the class of these functions.

Remark 3.3.1. It is obvious that for $g \in S_i^*$ we have $g(z) \neq 0$, $z \in \dot{U}$ and g univalent in $\dot{U}$.

Theorem 3.3.1. (the theorem of analytical characterization of the inverse-starlikeness for meromorphic functions) *Let $g : \dot{U} \to \mathbb{C}$ be a meromorphic function in U of the form*

$$g(z) = \frac{1}{z} + \alpha_0 + \alpha_1 z + \cdots,$$

such that $g(z) \neq 0$, $z \in \dot{U}$. Then the function g is inverse-starlike in $\dot{U}$ if and only if g is univalent on $\dot{U}$ and

$$\operatorname{Re}\left[-\frac{zg'(z)}{g(z)}\right] > 0,\ z \in \dot{U}.$$

Remark 3.3.2.

1. From the above theorem and Theorem 3.1.1 we have that $g \in S_i^*$ if and only if g is starlike in $\dot{U}$.
2. If we consider the Koebe function, $K_\tau(z) = \dfrac{z}{(1+e^{i\tau}z)^2}$, $z \in U$, we have that the function $g_\tau(z) = \dfrac{1}{K_\tau(z)} = \dfrac{1}{z} + 2e^{i\tau} + e^{2i\tau}z \in S_i^*, \tau \in \mathbb{R}$, (since $K_\tau \in S^*$), so g_τ is starlike in $\dot{U}$.

Definition 3.3.2. *[92] Let $g : \dot{U} \to \mathbb{C}$ be a meromorphic function in U of the form*

$$g(z) = \frac{\alpha_{-1}}{z} + \alpha_0 + \alpha_1 z + \cdots,\ z \in \dot{U}.$$

We say that the function g is inverse-convex in $\dot{U}$ if there exists a convex function f defined on U with $f(0) = 0$ such that $f(z)g(z) = 1$ for each $z \in \dot{U}$.

Remark 3.3.3. *[92]*

1. From the above definition we notice that if g is inverse-convex, then $g(z) \neq 0$, $z \in \dot{U}$ and g is univalent in $\dot{U}$.
2. If $\alpha_{-1} = 1$, we can easily see that the function f from the above definition is also normalized, hence a function $g(z) = \dfrac{1}{z} + \alpha_0 + \alpha_1 z + \cdots$, $0 < |z| < 1$, is inverse-convex in $\dot{U}$ if there exists a function $f \in K$ such that $f(z)g(z) = 1$ for each $z \in \dot{U}$. We will denote the class of these functions by K_i (the class of normalized inverse-convex functions in $\dot{U}$).
3. If g is inverse-convex in $\dot{U}$ and $\lambda \in \mathbb{C}^*$, then the meromorphic function λg is also inverse-convex in $\dot{U}$.

Theorem 3.3.2. *[92]***(the theorem of analytical characterization of the inverse-convexity for meromorphic functions)** *Let $g : \dot{U} \to \mathbb{C}$ be a meromorphic function in U of the form*

$$g(z) = \frac{1}{z} + \alpha_0 + \alpha_1 z + \cdots,$$

such that $g(z) \neq 0$, $z \in \dot{U}$. Then the function g is inverse-convex on $\dot{U}$ if and only if g is univalent on $\dot{U}$ and

$$\mathrm{Re}\left[\frac{zg''(z)}{g'(z)} - 2\frac{zg'(z)}{g(z)} + 1\right] > 0,\ z \in \dot{U}.$$

We denote by Σ^* the class of meromorhic starlike normalized functions in $\dot{U}$, so

$$\Sigma^* = \left\{g \in \Sigma : \mathrm{Re}\left[-\frac{zg'(z)}{g(z)}\right] > 0,\ z \in \dot{U}\right\},$$

and with Σ^c we denote the class of meromorhic convex normalized functions in $\dot{U}$, so

$$\Sigma^c = \left\{g \in \Sigma_0 : \mathrm{Re}\left[-\left(\frac{zg''(z)}{g'(z)} + 1\right)\right] > 0,\ z \in \dot{U}\right\}.$$

Remark 3.3.4. *[92] 1. An easy computation shows that the function*

$$f(z) = \log(1+z), z \in U \left(\text{with} \quad \log(1+z)\Big|_{z=0} = 0\right)$$

is convex on U and normalized, so the function $g(z) = \dfrac{1}{f(z)}$, $z \in \dot{U}$ belongs to the class K_i.
On the other hand we have

$$\frac{zg''(z)}{g'(z)} + 1 = \frac{\log(1+z) + 2z}{(1+z)\log(1+z)},$$

and it is easy to see that the inequality

$$\mathrm{Re}\left[-\left(\frac{zg''(z)}{g'(z)} + 1\right)\right] > 0$$

does not hold for each $z \in \dot{U}$ $\left(\text{for exemple we can take } z = \dfrac{1}{2}\right)$, so $g \notin \Sigma^c$. In other words, $K_i \neq \Sigma^c$.

2. We know that the function $f(z) = \dfrac{z}{1 + e^{i\tau}z} \in K$, so

$$g(z) = \frac{1}{f(z)} = \frac{1}{z} + e^{i\tau} \in K_i.$$

But on the other hand, it is easy to show that $g \in \Sigma^c$, hence $K_i \cap \Sigma^c \neq \emptyset$.

3. If $g \in K_i$, then $f = \dfrac{1}{g} \in K \subset S^*(1/2)$, so $g \in \Sigma^*(1/2)$. Therefore, we have $K_i \subset \Sigma^*(1/2)$.

Theorem 3.3.3. *[92]***(the duality theorem between the classes** Σ^* **and** K_i**)** *Let* $g : \dot{U} \to \mathbb{C}$ *be a function in* Σ*. Then* $g \in K_i$ *if and only if the function*

$$G(z) = -\frac{g^2(z)}{zg'(z)} \in \Sigma^*.$$

Theorem 3.3.4. *[92]***(the distortion theorem for the class** K_i**)** *If the function* g *belongs to the class* K_i*, then we have:*

$$\frac{1}{r}-1 \leq |g(z)| \leq \frac{1}{r}+1,\ |z| = r \in (0,1) \quad \left(\text{equivalent with } \left||g(z)| - \frac{1}{|z|}\right| \leq 1,\ z \in \dot{U}\right),$$

$$\left(\frac{1-r}{r+r^2}\right)^2 \leq |g'(z)| \leq \left(\frac{1+r}{r-r^2}\right)^2,\ |z| = r \in (0,1).$$

For $|g(z)|$ *these estimates are sharp and we have equality for* $g(z) = \frac{1}{z} + e^{i\tau}$, $\tau \in \mathbb{R}$.

3.4 The subclass of close-to-inverse-convex functions

In this section we present a new class of meromorphic functions, named the class of close-to-inverse-convex functions, which was defined in [92] and we will state and prove two theorems regarding this class. The first theorem gives a connection between the two concepts, inverse-convexity and close-to-inverse-convexity.

The results presented in this section are original and are published in [92].

Definition 3.4.1. *[92] Let* $g : \dot{U} \to \mathbb{C}$ *be a meromorphic function in* U *of the form*

$$g(z) = \frac{\alpha_{-1}}{z} + \alpha_0 + \alpha_1 z + \cdots.$$

We say that the function g *is close-to-inverse-convex in* $\dot{U}$ *if there exists an inverse-convex function* ψ *on* $\dot{U}$ *such that*

$$\operatorname{Re} \frac{g'(z)}{\psi'(z)} > 0,\ z \in \dot{U}.$$

We denote by C_i the class of normalized close-to-inverse-convex functions on $\dot{U}$, so

$$C_i = \left\{ g \in \Sigma : (\exists)\psi \in K_i \text{ such that } \operatorname{Re} \frac{g'(z)}{\psi'(z)} > 0,\ z \in U \right\}.$$

Let $\alpha < 1 < \beta$. We consider the next classes of meromorphic functions:

$$\Sigma^*(\alpha, \beta) = \left\{ g \in \Sigma : \alpha < \operatorname{Re}\left[-\frac{zg'(z)}{g(z)}\right] < \beta,\ z \in U \right\},$$

$$C_{i;\beta} = \left\{ g \in \Sigma : (\exists)\psi \in K_i \cap \Sigma^*(0,\beta) \text{ astfel încât } \operatorname{Re} \frac{g'(z)}{\psi'(z)} > 0,\ z \in U \right\}.$$

Theorem 3.4.1. *[92] Let* $\lambda \in \mathbb{C} \setminus \{0\}$ *with* $\operatorname{Re}\lambda > 2|\lambda|^2$, $\beta = \dfrac{\operatorname{Re}\lambda}{2|\lambda|^2}$ *and* $g \in K_i$ *with* $\operatorname{Re}\left[-\dfrac{zg'(z)}{g(z)}\right] < \beta$, $z \in U$ *(i.e.* $g \in K_i \cap \Sigma^*(0,\beta)$*), then the function*

$$h_\lambda(z) = g(z) + \lambda z g'(z),\ z \in \dot{U},$$

is close-to-inverse-convex.

We note that in the hypothesis of the above theorem we need $\operatorname{Re}\lambda > 2|\lambda|^2$ because we must have $\beta > 1$.

It is easy to see that $\operatorname{Re}\lambda > 2|\lambda|^2$ implies $|\lambda| < 1/2$, so the above theorem can not be used for the complex numbers λ with $|\lambda| \geq 1/2$.

We mention that a similar result, but which regards the analytic functions, was given by B.N. Rahmanov in [74].

For $\gamma \in \mathbb{C}$ with $\operatorname{Re}\gamma > 0$ we consider the integral operator $I_\gamma : \Sigma \to \Sigma$ given by

$$I_\gamma(g)(z) = \frac{\gamma}{z^{\gamma+1}} \int_0^z t^\gamma g(t)dt. \tag{3.4}$$

We have the next theorem with respect to the classes K_i, C_i and to the integral operator I_γ.

Theorem 3.4.2. *[92] Let be* $\gamma \in \mathbb{C}$ *with* $\operatorname{Re}\gamma > 1$ *and* $\beta = \dfrac{\operatorname{Re}\gamma + 1}{2}$.
If $I_\gamma[K_i] \subset K_i$*, then* $I_\gamma[C_{i;\beta}] \subset C_i$.

3.5 Starlikeness properties for the integral operator $J_{\beta,\gamma}$

The results presented in this section are original and are published in [90].

For $\Phi, \varphi \in H[1,1]$ with $\Phi(z)\varphi(z) \neq 0$, $z \in U$, and $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ with $\beta \neq 0$, we consider the integral operator $J^{\Phi,\varphi}_{\alpha,\beta,\gamma,\delta} : H \subset \Sigma \to \Sigma$, defined by:

$$J^{\Phi,\varphi}_{\alpha,\beta,\gamma,\delta}(g)(z) = \left[\frac{\gamma-\beta}{z^\gamma \Phi(z)} \int_0^z g^\alpha(t)\varphi(t)t^{\delta-1}dt\right]^{\frac{1}{\beta}}.$$

The first result of this section present some important properties of the integral operator $J^{\Phi,\varphi}_{\alpha,\beta,\gamma,\delta}$.

Theorem 3.5.1. *[90] Let* $\Phi, \varphi \in H[1,1]$ *with* $\Phi(z)\varphi(z) \neq 0$, $z \in U$*. Let* $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ *with* $\beta \neq 0$, $\alpha + \gamma = \beta + \delta$ *and* $\operatorname{Re}(\gamma - \beta) > 0$*. If* $g \in \Sigma$ *and*

$$\alpha\frac{zg'(z)}{g(z)} + \frac{z\varphi'(z)}{\varphi(z)} + \delta \prec R_{\delta-\alpha,1}(z),$$

then

$$G(z) = J^{\Phi,\varphi}_{\alpha,\beta,\gamma,\delta}(g)(z) = \left[\frac{\gamma-\beta}{z^{\gamma}\Phi(z)}\int_0^z g^{\alpha}(t)\varphi(t)t^{\delta-1}dt\right]^{\frac{1}{\beta}} \in \Sigma, \tag{3.5}$$

with $zG(z) \neq 0$, $z \in U$, *and*

$$\mathrm{Re}\left[\beta\frac{zG'(z)}{G(z)} + \frac{z\Phi'(z)}{\Phi(z)} + \gamma\right] > 0,\ z \in U.$$

All powers in (3.5) are chosen as principal ones.

We next consider a particular case of Theorem 3.5.1. Considering $\Phi = \varphi \equiv 1$, $\alpha = \beta$, $\gamma = \delta$ and using the notation $J_{\beta,\gamma}$ instead of $J^{1,1}_{\beta,\beta,\gamma,\gamma}$, we have:

Corollary 3.5.1. *[90] Let* $\beta, \gamma \in \mathbb{C}$ *with* $\beta \neq 0$ *and* $\mathrm{Re}\,(\gamma-\beta) > 0$. *If* $g \in \Sigma$ *and*

$$\beta\frac{zg'(z)}{g(z)} + \gamma \prec R_{\gamma-\beta,1}(z),$$

then

$$G(z) = J_{\beta,\gamma}(g)(z) = \left[\frac{\gamma-\beta}{z^{\gamma}}\int_0^z g^{\beta}(t)t^{\gamma-1}dt\right]^{\frac{1}{\beta}} \in \Sigma, \tag{3.6}$$

with $zG(z) \neq 0$, $z \in U$, *and*

$$\mathrm{Re}\left[\beta\frac{zG'(z)}{G(z)} + \gamma\right] > 0,\ z \in U.$$

(All powers in (3.6) are chosen as principal ones).

Remark 3.5.1.

1. If we define the classes $\mathcal{K}_{\beta,\gamma}$ as

$$\mathcal{K}_{\beta,\gamma} = \left\{g \in \Sigma : \gamma + \beta\frac{zg'(z)}{g(z)} \prec R_{\gamma-\beta,1}(z)\right\},$$

   from Corollary 3.5.1, we have that $J_{\beta,\gamma} : \mathcal{K}_{\beta,\gamma} \to \Sigma$ with $zJ_{\beta,\gamma}(g)(z) \neq 0$, $z \in U$ and

$$\mathrm{Re}\left[\gamma + \beta\frac{zJ_{\beta,\gamma}(g)'(z)}{J_{\beta,\gamma}(g)(z)}\right] > 0,\ z \in U.$$

2. Let be

$$\tilde{\mathcal{K}}_{\beta,\gamma} = \left\{g \in \Sigma : \mathrm{Re}\left[\gamma + \beta\frac{zg'(z)}{g(z)}\right] > 0,\ z \in U\right\}.$$

   Using the above corollary we have $J_{\beta,\gamma}(\mathcal{K}_{\beta,\gamma}) \subset \tilde{\mathcal{K}}_{\beta,\gamma}$, so $J_{\beta,\gamma}(\tilde{\mathcal{K}}_{\beta,\gamma}) \subset \tilde{\mathcal{K}}_{\beta,\gamma}$, where β, $\gamma \in \mathbb{C}$ with $\beta \neq 0$ and $\mathrm{Re}\,(\gamma-\beta) > 0$.

3. If $\beta < 0$, $\gamma \in \mathbb{C}$ with $\beta < \mathrm{Re}\,\gamma$ and if we consider $\dfrac{\mathrm{Re}\,\gamma}{\beta} \leq \alpha < 1$, then from $J_{\beta,\gamma}(\tilde{\mathcal{K}}_{\beta,\gamma}) \subset \tilde{\mathcal{K}}_{\beta,\gamma}$ we deduce $J_{\beta,\gamma}(\Sigma^*(\alpha)) \subset \Sigma^*\left(\dfrac{\mathrm{Re}\,\gamma}{\beta}\right)$.

If

$$G(z) = \left[\frac{\gamma - \beta}{z^\gamma}\int_0^z t^{\gamma-1} g^\beta(t)dt\right]^{\frac{1}{\beta}}, \; z \in \dot{U},$$

and if we suppose that $p(z) = -\dfrac{zG'(z)}{G(z)}$ is analytic in U, we have

$$p(z) + \frac{zp'(z)}{\gamma - \beta p(z)} = -\frac{zg'(z)}{g(z)}, \; z \in U. \tag{3.7}$$

Next we find the conditions over the complex numbers $\alpha, \beta, \gamma, \delta$ such that

$$J_{\beta,\gamma}(g) \in \Sigma^*(\alpha, \delta)$$

when $g \in \Sigma^*(\alpha, \delta)$.

We mention that the integral operator $J_{\beta,\gamma}$ is defined by (3.6) and the class $\Sigma^*(\alpha, \delta)$ is defined as:

$$\Sigma^*(\alpha, \delta) = \left\{g \in \Sigma : \alpha < \operatorname{Re}\left[-\frac{zg'(z)}{g(z)}\right] < \delta, \; z \in U\right\}.$$

Similar results to those that follows, with respect to the operator $J_{1,\gamma}$, may also be found in [1].

Theorem 3.5.2. *[90] Let $\beta > 0$, $\gamma \in \mathbb{C}$ and $0 \le \alpha < 1 < \delta \le \dfrac{\operatorname{Re}\gamma}{\beta}$.
If $g \in \Sigma^*(\alpha, \delta)$, then $G = J_{\beta,\gamma}(g) \in \Sigma^*(\alpha, \delta)$.*

Taking $\beta = 1$ in the above theorem we obtain:

Corollary 3.5.2. *[90] Let $\gamma \in \mathbb{C}$ and $0 \le \alpha < 1 < \delta \le \operatorname{Re}\gamma$. If $g \in \Sigma^*(\alpha, \delta)$ and*

$$G(z) = J_{1,\gamma}(g)(z) = \frac{\gamma - 1}{z^\gamma}\int_0^z t^{\gamma-1} g(t)dt,$$

then $G \in \Sigma^*(\alpha, \delta)$.

Theorem 3.5.3. *[90] Let $\beta < 0$, $\gamma \in \mathbb{C}$ and $\dfrac{\operatorname{Re}\gamma}{\beta} \le \alpha < 1 < \delta$.
If $g \in \Sigma^*(\alpha, \delta)$, then $G = J_{\beta,\gamma}(g) \in \Sigma^*(\alpha, \delta)$.*

Remark 3.5.2. *If we consider $\delta \to \infty$ in the above theorem we obtain that for $\beta < 0$, $\gamma \in \mathbb{C}$ with $\operatorname{Re}\gamma > \beta$ and $\dfrac{\operatorname{Re}\gamma}{\beta} \le \alpha < 1$,*

$$g \in \Sigma^*(\alpha) \Rightarrow G = J_{\beta,\gamma}(g) \in \Sigma^*(\alpha).$$

Definition 3.5.1. *Let $\beta < 0$ and $\gamma \in \mathbb{C}$ with $\operatorname{Re}\gamma > \beta$. For a given number $\alpha \in \left[\dfrac{\operatorname{Re}\gamma}{\beta}, 1\right)$, we will define the order of starlikeness of the class $J_{\beta,\gamma}(\Sigma^*(\alpha))$ as the biggest number $\mu = \mu(\alpha; \beta, \gamma)$ such that $J_{\beta,\gamma}(\Sigma^*(\alpha)) \subset \Sigma^*(\mu)$.*

Theorem 3.5.4. *[90]* **(the order of starlikeness of the class** $J_{\beta,\gamma}(\Sigma^*(\alpha))$**)**

Let $\beta < 0$, $\gamma - \beta > 0$ *and let* $J_{\beta,\gamma}$ *be given by (3.6). If* $\alpha \in [\alpha_0, 1)$, *where* $\alpha_0 = \max\left\{\dfrac{\beta+\gamma+1}{2\beta}, \dfrac{\gamma}{\beta}\right\}$, *then the order of starlikeness of the class* $J_{\beta,\gamma}(\Sigma^*(\alpha))$ *is given by*

$$\mu(\alpha;\beta,\gamma) = -\frac{1}{\beta}\left[\frac{\gamma-\beta}{{}_2F_1(1, 2\beta(\alpha-1), \gamma+1-\beta; \frac{1}{2})} - \gamma\right],$$

where ${}_2F_1$ *represents the hypergeometric function.*

Further we will find some conditions for α, β, γ and $\delta = \delta(\alpha,\beta,\gamma)$ such that

$$J_{\beta,\gamma}[\Sigma^*(\alpha) \cap \mathcal{K}_{\beta,\gamma}] \subset \Sigma^*(\delta).$$

Theorem 3.5.5. *[90] Let* $0 \le \alpha < 1$, $0 < \beta < \gamma$. *Let us denote*

$$\begin{aligned}
\beta_1(\alpha,\gamma) &= \frac{2\sqrt{2\gamma(\alpha-1)^2+\alpha}-\alpha-1}{2(\alpha-1)^2}\\
\delta_1(\alpha,\beta,\gamma) &= \frac{2\alpha\beta+2\gamma+1-\sqrt{(1+2\alpha\beta-2\gamma)^2+8(\gamma-\beta)}}{4\beta}\\
\delta_2(\alpha,\beta,\gamma) &= \frac{2\alpha\beta+2\beta+1-\sqrt{(1+2\alpha\beta-2\beta)^2+8(\beta-\gamma)}}{4\beta}
\end{aligned}$$

If $\gamma > \dfrac{1}{8}$ *and* $\beta < \beta_1(\alpha,\gamma)$, *then* $J_{\beta,\gamma}[\Sigma^*(\alpha) \cap \mathcal{K}_{\beta,\gamma}] \subset \Sigma^*(\delta_1(\alpha,\beta,\gamma))$.

If $\gamma \le \dfrac{1}{8}$ *or* $\begin{cases} \gamma > \dfrac{1}{8} \\ \beta \ge \beta_1(\alpha,\gamma) \end{cases}$, *then* $J_{\beta,\gamma}[\Sigma^*(\alpha) \cap \mathcal{K}_{\beta,\gamma}] \subset \Sigma^*(\delta(\alpha,\beta,\gamma))$, *where*

$$\delta(\alpha,\beta,\gamma) = \min\{\delta_1(\alpha,\beta,\gamma), \delta_2(\alpha,\beta,\gamma).$$

The operator $J_{\beta,\gamma}$ *is defined by (3.6).*

We see that if we consider, in the above theorem, the condition $zJ_{\alpha,\beta}(g)(z) \neq 0$, $z \in U$, we have:

Theorem 3.5.6. *[90] Let* $0 \le \alpha < 1$, $0 < \beta < \gamma$, $g \in \Sigma^*(\alpha)$ *and* $G(z) = J_{\alpha,\beta}(g)(z)$. *We suppose that* $zG(z) \neq 0$, $z \in U$, *and we consider:*

$$\begin{aligned}
\beta_1(\alpha,\gamma) &= \frac{2\sqrt{2\gamma(\alpha-1)^2+\alpha}-\alpha-1}{2(\alpha-1)^2},\\
\delta_1(\alpha,\beta,\gamma) &= \frac{2\alpha\beta+2\gamma+1-\sqrt{(1+2\alpha\beta-2\gamma)^2+8(\gamma-\beta)}}{4\beta},\\
\delta_2(\alpha,\beta,\gamma) &= \frac{2\alpha\beta+2\beta+1-\sqrt{(1+2\alpha\beta-2\beta)^2+8(\beta-\gamma)}}{4\beta}.
\end{aligned}$$

If $\gamma > \dfrac{1}{8}$ *and* $\beta < \beta_1(\alpha,\gamma)$, *then* $G \in \Sigma^*(\delta_1(\alpha,\beta,\gamma))$.

If $\gamma \le \dfrac{1}{8}$ *or* $\begin{cases} \gamma > \dfrac{1}{8} \\ \beta \ge \beta_1(\alpha,\gamma) \end{cases}$, *then* $G \in \Sigma^*(\delta(\alpha,\beta,\gamma))$, *where*

$$\delta(\alpha,\beta,\gamma) = \min\{\delta_1(\alpha,\beta,\gamma), \delta_2(\alpha,\beta,\gamma)\}.$$

Results which are similar to these two last results may also be found in [66] and [79].

Chapter 4

Integral operators on the class Σ_p of meromorphic multivalent functions

In this chapter we study the meromorphic multivalent functions and all the results presented in this chapter are original and are published in [91] and [93].

In section 4.1 is defined an integral operator, denoted by $J^{\Phi,\varphi}_{p,\alpha,\beta,\gamma,\delta}$, and an existence theorem, with respect to this operator, is given.

In the sections 4.2 and 4.3 are studied the preserving properties of some new subclasses, when we apply the particular integral operators $J_{p,\beta,\gamma}$ and $J_{p,\gamma}$.

In section 4.4 is considered a multiplier transformation, denoted by $J^n_{p,\lambda}$, and a new subclass of meromorphic multivalent functions is defined using this transformation and the condition from starlikeness. Then the preserving properties of this subclass are studied, when the integral operator $J_{p,\gamma}$ is applied.

4.1 The operator $J^{\Phi,\varphi}_{p,\alpha,\beta,\gamma,\delta}$

The results from this section are original and are published in [91].
For $p \in \mathbb{N}^*$ we denote by Σ_p the class of meromorphic functions in U of the form

$$g(z) = \frac{a_{-p}}{z^p} + a_0 + a_1 z + \cdots + a_n z^n + \cdots,\ z \in \dot{U},\ a_{-p} \neq 0.$$

Let $p \in \mathbb{N}^*$, $\Phi, \varphi \in H[1,p]$ with $\Phi(z)\varphi(z) \neq 0$, $z \in U$, $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ with $\beta \neq 0$ and $g \in \Sigma_p$. We consider the integral operator

$$J^{\Phi,\varphi}_{p,\alpha,\beta,\gamma,\delta}(g)(z) = \left[\frac{\gamma - p\beta}{z^\gamma \Phi(z)} \int_0^z g^\alpha(t)\varphi(t)t^{\delta-1}dt\right]^{\frac{1}{\beta}},$$

where all powers are chosen as principal ones.

The first result of this section presents some important properties for the operator considered above.

Theorem 4.1.1. *[91] Let $p \in \mathbb{N}^*$, $\Phi, \varphi \in H[1,p]$ with $\Phi(z)\varphi(z) \neq 0$, $z \in U$. Let $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ with $\beta \neq 0$, $\delta + p\beta = \gamma + p\alpha$ and $\mathrm{Re}\,(\gamma - p\beta) > 0$. Let $g \in \Sigma_p$ and suppose that*

$$\alpha \frac{zg'(z)}{g(z)} + \frac{z\varphi'(z)}{\varphi(z)} + \delta \prec R_{\delta - p\alpha, p}(z).$$

If $G = J^{\Phi,\varphi}_{p,\alpha,\beta,\gamma,\delta}(g)$ is defined by

$$G(z) = J^{\Phi,\varphi}_{p,\alpha,\beta,\gamma,\delta}(g)(z) = \left[\frac{\gamma - p\beta}{z^{\gamma}\Phi(z)} \int_0^z g^{\alpha}(t)\varphi(t)t^{\delta-1}dt\right]^{\frac{1}{\beta}}, \; z \in \dot{U}, \tag{4.1}$$

then $G \in \Sigma_p$ with $z^p G(z) \neq 0$, $z \in U$, and

$$\mathrm{Re}\left[\beta \frac{zG'(z)}{G(z)} + \frac{z\Phi'(z)}{\Phi(z)} + \gamma\right] > 0, \; z \in U.$$

All powers in (4.1) are principal ones.

Taking $\alpha = \beta$ and $\gamma = \delta$ in the above theorem and using the notation $J^{\Phi,\varphi}_{p,\beta,\gamma}$ instead of $J^{\Phi,\varphi}_{p,\beta,\beta,\gamma,\gamma}$, we obtain the next corollary:

Corollary 4.1.1. *[91] Let $p \in \mathbb{N}^*$, $\Phi, \varphi \in H[1,p]$ with $\Phi(z)\varphi(z) \neq 0$, $z \in U$. Let $\beta, \gamma \in \mathbb{C}$ with $\beta \neq 0$ and $\mathrm{Re}\,(\gamma - p\beta) > 0$. If $g \in \Sigma_p$ and*

$$\beta \frac{zg'(z)}{g(z)} + \frac{z\varphi'(z)}{\varphi(z)} + \gamma \prec R_{\gamma - p\beta, p}(z),$$

then

$$G(z) = J^{\Phi,\varphi}_{p,\beta,\gamma}(g)(z) = \left[\frac{\gamma - p\beta}{z^{\gamma}\Phi(z)} \int_0^z g^{\beta}(t)\varphi(t)t^{\gamma-1}dt\right]^{\frac{1}{\beta}} \in \Sigma_p,$$

with $z^p G(z) \neq 0$, $z \in U$, and

$$\mathrm{Re}\left[\beta \frac{zG'(z)}{G(z)} + \frac{z\Phi'(z)}{\Phi(z)} + \gamma\right] > 0, \; z \in U.$$

Considering $\Phi = \varphi \equiv 1$ in Corollary 4.1.1, and using the notation $J_{p,\beta,\gamma}$ instead of $J^{1,1}_{p,\beta,\beta,\gamma,\gamma}$, we obtain:

Corollary 4.1.2. *[91] Let $p \in \mathbb{N}^*$, $\beta, \gamma \in \mathbb{C}$ with $\beta \neq 0$ and $\mathrm{Re}\,(\gamma - p\beta) > 0$. If $g \in \Sigma_p$ and*

$$\beta \frac{zg'(z)}{g(z)} + \gamma \prec R_{\gamma - p\beta, p}(z),$$

then

$$G(z) = J_{p,\beta,\gamma}(g)(z) = \left[\frac{\gamma - p\beta}{z^{\gamma}} \int_0^z g^{\beta}(t)t^{\gamma-1}dt\right]^{\frac{1}{\beta}} \in \Sigma_p,$$

with $z^p G(z) \neq 0$, $z \in U$, and

$$\mathrm{Re}\left[\beta \frac{zG'(z)}{G(z)} + \gamma\right] > 0, \; z \in U.$$

Let $p \in \mathbb{N}^*$, $\beta, \gamma \in \mathbb{C}$ with $\beta \neq 0$, $g \in \Sigma_p$, $G = J_{p,\beta,\gamma}(g)$ and let us denote $P(z) = -\dfrac{zG'(z)}{G(z)}$, $z \in U$. If we suppose that $P \in H(U)$, we obtain from

$$G(z) = \left[\frac{\gamma - p\beta}{z^\gamma} \int_0^z t^{\gamma-1} g^\beta(t) dt\right]^{\frac{1}{\beta}}, \; z \in \dot{U},$$

that

$$P(z) + \frac{zP'(z)}{\gamma - \beta P(z)} = -\frac{zg'(z)}{g(z)}, \; z \in U. \tag{4.2}$$

Considering $\beta = 1$ in the above corollary and using the notation $J_{p,\gamma}$ instead of $J_{p,1,\gamma}$, we get:

Corollary 4.1.3. *Let $p \in \mathbb{N}^*$ and $\gamma \in \mathbb{C}$ with $\operatorname{Re}\gamma > p$. Let $g \in \Sigma_p$ satisfying the condition:*

$$\frac{zg'(z)}{g(z)} + \gamma \prec R_{\gamma-p,p}(z).$$

Then

$$G(z) = J_{p,\gamma}(g)(z) = \frac{\gamma - p}{z^\gamma} \int_0^z g(t) t^{\gamma-1} dt \in \Sigma_p,$$

with $z^p G(z) \neq 0$, $z \in U$, and $\operatorname{Re}\left[\gamma + \dfrac{zG'(z)}{G(z)}\right] > 0$, $z \in U$.

Theorem 4.1.2. *[91] Let $p \in \mathbb{N}^*$, $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > p$. If $g \in \Sigma_p$, then $J_{p,\lambda}(g) \in \Sigma_p$, where $J_{p,\lambda}(g)(z) = \dfrac{\lambda - p}{z^\lambda} \displaystyle\int_0^z g(t) t^{\lambda-1} dt$.*

Remark 4.1.1. *[91] Let $p \in \mathbb{N}^*$, $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > p$ and $\Sigma_{p,0} = \{g \in \Sigma_p : a_{-p} = 1\}$. From the above theorem, it is easy to see that we have $J_{p,\lambda}(g) \in \Sigma_{p,0}$, when $g \in \Sigma_{p,0}$.*

4.2 The operator $J_{p,\beta,\gamma}$ applied to the class $\Sigma_p^*(\alpha, \delta)$

The results presented in this section are original and are published in [91].
Let $p \in \mathbb{N}^*$ and $\alpha, \delta \in \mathbb{R}$ with $\alpha < p < \delta$. We consider the classes:

$$\Sigma_p^*(\alpha) = \left\{g \in \Sigma_p : \operatorname{Re}\left[-\frac{zg'(z)}{g(z)}\right] > \alpha, \; z \in U\right\},$$

$$\Sigma_p^*(\alpha, \delta) = \left\{g \in \Sigma_p : \alpha < \operatorname{Re}\left[-\frac{zg'(z)}{g(z)}\right] < \delta, \; z \in U\right\}.$$

We remark that $\Sigma_1^*(\alpha)$ is the class of meromorphic starlike functions of order α.
For the next results we need the following lemmas:

Lemma 4.2.1. *[91] Let $n \in \mathbb{N}^*$, $\alpha, \beta \in \mathbb{R}$ and $\gamma \in \mathbb{C}$ with $\operatorname{Re}\gamma > \alpha\beta$. If $P \in H[P(0), n]$ with $P(0) \in \mathbb{R}$ and $P(0) > \alpha$, then we have*

$$\operatorname{Re}\left[P(z) + \frac{zP'(z)}{\gamma - \beta P(z)}\right] > \alpha \Rightarrow \operatorname{Re} P(z) > \alpha, \; z \in U.$$

Lemma 4.2.2. *[91] Let $n \in \mathbb{N}^*$, $\delta, \beta \in \mathbb{R}$ and $\gamma \in \mathbb{C}$ with $\operatorname{Re}\gamma > \delta\beta$. If $P \in H[P(0), n]$ with $P(0) \in \mathbb{R}$ and $P(0) < \delta$, then we have*

$$\operatorname{Re}\left[P(z) + \frac{zP'(z)}{\gamma - \beta P(z)}\right] < \delta \Rightarrow \operatorname{Re} P(z) < \delta,\ z \in U.$$

Next we will find conditions over the numbers $\alpha, \beta, \gamma, \delta$ such that $J_{p,\beta,\gamma}(g) \in \Sigma_p^*(\alpha, \delta)$ when $g \in \Sigma_p^*(\alpha, \delta)$, where $J_{p,\beta,\gamma}$ is the integral operator defined by

$$J_{p,\beta,\gamma}(g)(z) = \left[\frac{\gamma - p\beta}{z^\gamma} \int_0^z g^\beta(t) t^{\gamma-1} dt\right]^{\frac{1}{\beta}}. \tag{4.3}$$

Theorem 4.2.1. *[91] Let $p \in \mathbb{N}^*$, $\beta > 0, \gamma \in \mathbb{C}$ and $\alpha < p < \delta < \dfrac{\operatorname{Re}\gamma}{\beta}$.
If $g \in \Sigma_p^*(\alpha, \delta)$, then $G = J_{p,\beta,\gamma}(g) \in \Sigma_p^*(\alpha, \delta)$.*

Taking $\beta = 1$ in the above theorem and using the notation $J_{p,\gamma}$ instead of $J_{p,1,\gamma}$, we obtain:

Corollary 4.2.1. *[91] Let $p \in \mathbb{N}^*$, $\gamma \in \mathbb{C}$ and $\alpha < p < \delta < \operatorname{Re}\gamma$. If $g \in \Sigma_p^*(\alpha, \delta)$, then*

$$G = J_{p,\gamma}(g) \in \Sigma_p^*(\alpha, \delta),$$

where $J_{p,\gamma}(g)(z) = \dfrac{\gamma - p}{z^\gamma} \displaystyle\int_0^z t^{\gamma-1} g(t) dt$, $z \in \dot{U}$.

Theorem 4.2.2. *[91] Let $p \in \mathbb{N}^*$, $\beta > 0, \gamma \in \mathbb{C}$ and $\alpha < p < \dfrac{\operatorname{Re}\gamma}{\beta} \leq \delta$.
If $g \in \Sigma_p^*(\alpha, \delta)$, with*

$$\beta \frac{zg'(z)}{g(z)} + \gamma \prec R_{\gamma - p\beta, p}(z),$$

then $G = J_{p,\beta,\gamma}(g) \in \Sigma_p^*(\alpha, \delta)$.

If we consider $\delta \to \infty$, in the above theorem, we obtain the next corollary:

Corollary 4.2.2. *[91] Let $p \in \mathbb{N}^*$, $\beta > 0, \gamma \in \mathbb{C}$ and $\alpha < p < \dfrac{\operatorname{Re}\gamma}{\beta}$.
If $g \in \Sigma_p^*(\alpha)$, with*

$$\beta \frac{zg'(z)}{g(z)} + \gamma \prec R_{\gamma - p\beta, p}(z),$$

then $G = J_{p,\beta,\gamma}(g) \in \Sigma_p^*(\alpha)$.

We make the remark that we can obtain a similar result, without the condition $\beta \dfrac{zg'(z)}{g(z)} + \gamma \prec R_{\gamma - p\beta, p}(z)$, as it follows:

Theorem 4.2.3. *[91] Let $p \in \mathbb{N}^*$, $\beta > 0, \gamma \in \mathbb{C}$, $\alpha < p < \dfrac{\operatorname{Re}\gamma}{\beta}$ and $g \in \Sigma_p^*(\alpha)$. Let $G = J_{p,\beta,\gamma}(g)$. If $G \in \Sigma_p$ and $z^p G(z) \neq 0$, $z \in U$, then $G \in \Sigma_p^*(\alpha)$.*

Since we know from Theorem 4.1.2 that for $p \in \mathbb{N}^*$, $\gamma \in \mathbb{C}$ with $\operatorname{Re}\gamma > p$, we have $J_{p,\gamma}(g) \in \Sigma_p$ when $g \in \Sigma_p$, we obtain for the above theorem, taking $\beta = 1$, the next corollary:

Corollary 4.2.3. *[91] Let $p \in \mathbb{N}^*$, $\gamma \in \mathbb{C}$ and $\alpha < p < \operatorname{Re}\gamma$.*
If $g \in \Sigma_p^*(\alpha)$ with $z^p J_{p,\gamma}(g)(z) \neq 0$, $z \in U$, then $G = J_{p,\gamma}(g) \in \Sigma_p^*(\alpha)$.

Taking $\beta = 1$ in Theorem 4.2.2, we get:

Corollary 4.2.4. *[91] Let $p \in \mathbb{N}^*$, $\gamma \in \mathbb{C}$ and $\alpha < p < \operatorname{Re}\gamma \leq \delta$.*
If $g \in \Sigma_p^*(\alpha, \delta)$, with

$$\frac{zg'(z)}{g(z)} + \gamma \prec R_{\gamma-p,p}(z),$$

then $G = J_{p,\gamma}(g) \in \Sigma_p^*(\alpha, \delta)$.

Theorem 4.2.4. *[91] Let $p \in \mathbb{N}^*$, $\beta < 0$, $\gamma \in \mathbb{C}$ and $\dfrac{\operatorname{Re}\gamma}{\beta} < \alpha < p < \delta$.*
If $g \in \Sigma_p^*(\alpha, \delta)$, then $G = J_{p,\beta,\gamma}(g) \in \Sigma_p^*(\alpha, \delta)$.

If we consider $\delta \to \infty$, in the above theorem, we obtain the next corollary:

Corollary 4.2.5. *[91] Let $p \in \mathbb{N}^*$, $\beta < 0$, $\gamma \in \mathbb{C}$ and $\dfrac{\operatorname{Re}\gamma}{\beta} < \alpha < p$. Then we have*

$$g \in \Sigma_p^*(\alpha) \Rightarrow G = J_{p,\beta,\gamma}(g) \in \Sigma_p^*(\alpha).$$

Theorem 4.2.5. *[91] Let $p \in \mathbb{N}^*$, $\beta < 0, \gamma \in \mathbb{C}$ and $\alpha \leq \dfrac{\operatorname{Re}\gamma}{\beta} < p < \delta$.*
If $g \in \Sigma_p^*(\alpha, \delta)$, with

$$\beta\frac{zg'(z)}{g(z)} + \gamma \prec R_{\gamma-p\beta,p}(z),$$

then $G = J_{p,\beta,\gamma}(g) \in \Sigma_p^*(\alpha, \delta)$.

If we consider $\delta \to \infty$, in the above theorem, we obtain the next corollary:

Corollary 4.2.6. *[91] Let $p \in \mathbb{N}^*$, $\beta < 0, \gamma \in \mathbb{C}$ and $\alpha \leq \dfrac{\operatorname{Re}\gamma}{\beta} < p$.*
If $g \in \Sigma_p^*(\alpha)$, with

$$\beta\frac{zg'(z)}{g(z)} + \gamma \prec R_{\gamma-p\beta,p}(z),$$

then $G = J_{p,\beta,\gamma}(g) \in \Sigma_p^*(\alpha)$.

We remark that we can obtain a similar result, without the condition

$$\beta\frac{zg'(z)}{g(z)} + \gamma \prec R_{\gamma-p\beta,p}(z),$$

as it follows:

Theorem 4.2.6. *[91] Let $p \in \mathbb{N}^*$, $\beta < 0, \gamma \in \mathbb{C}$, $\alpha \leq \dfrac{\operatorname{Re}\gamma}{\beta} < p$ and $g \in \Sigma_p^*(\alpha)$. Let $G = J_{p,\beta,\gamma}(g)$. If $G \in \Sigma_p$ and $z^p G(z) \neq 0$, $z \in U$, then $G \in \Sigma_p^*(\alpha)$.*

4.3 The operator $J_{p,\gamma}$ applied to the classes $\Sigma K_p(\alpha,\delta)$ and $\Sigma\mathcal{C}_{p,0}(\alpha,\delta;\varphi)$

The results presented in this section are original and are published in [91].
Let $p \in \mathbb{N}^*$ and $\alpha, \delta \in \mathbb{R}$. We consider the next subclasses of meromorphic functions:
$\Sigma K_p(\alpha) = \left\{ g \in \Sigma_p : \operatorname{Re}\left[1 + \frac{zg''(z)}{g'(z)}\right] < -\alpha,\ z \in U \right\}$, $\alpha < p$,
$\Sigma K_{p,0}(\alpha) = \Sigma K_p(\alpha) \cap \Sigma_{p,0}$,
$\Sigma K_p(\alpha,\delta) = \left\{ g \in \Sigma_p : \alpha < \operatorname{Re}\left[-1 - \frac{zg''(z)}{g'(z)}\right] < \delta,\ z \in U \right\}$, $\alpha < p < \delta$,
$\Sigma K_{p,0}(\alpha,\delta) = \Sigma K_p(\alpha,\delta) \cap \Sigma_{p,0}$,
$\Sigma\mathcal{C}_{p,0}(\alpha,\delta;\varphi) = \left\{ g \in \Sigma_{p,0} : \alpha < \operatorname{Re}\left[\frac{g'(z)}{\varphi'(z)}\right] < \delta,\ z \in U \right\}$, where $\alpha < 1 \le p < \delta$ and $\varphi \in \Sigma K_{p,0}(\alpha,\delta)$.

We remark that $\Sigma K_1(\alpha) \cap \Sigma_0$ is the class of normalized meromorphic convex functions of order α.

Let $\gamma \in \mathbb{C}$ with $\operatorname{Re}\gamma > p$. We consider the integral operator $J_{p,\gamma}$ defined by

$$J_{p,\gamma}(g)(z) = \frac{\gamma - p}{z^\gamma}\int_0^z t^{\gamma-1} g(t)dt,\ g \in \Sigma_p.$$

It is easy to see that if $g \in \Sigma_p$ has the form

$$g(z) = \frac{a_{-p}}{z^p} + \sum_{k=0}^{\infty} a_k z^k,\ z \in \dot{U},$$

then

$$J_{p,\gamma}(g)(z) = \frac{a_{-p}}{z^p} + \sum_{k=0}^{\infty} \frac{\gamma - p}{\gamma + k} a_k z^k,\ z \in \dot{U}.$$

Moreover, $J_{p,\gamma}(zg'(z)) = z\left[J_{p,\gamma}(g)(z)\right]'$, $z \in \dot{U}$.

Theorem 4.3.1. *[91] Let $p \in \mathbb{N}^*$, $\gamma \in \mathbb{C}$ with $\operatorname{Re}\gamma > p$ and let $\alpha < p < \delta < \operatorname{Re}\gamma$. If $g \in \Sigma K_p(\alpha,\delta)$ and $z^{p+1}J'_{p,\gamma}(g)(z) \neq 0$, $z \in U$, then*

$$J_{p,\gamma}(g) \in \Sigma K_p(\alpha,\delta).$$

Theorem 4.3.2. *[91] Let $p \in \mathbb{N}^*$, $\alpha \in \mathbb{R}$ and $\gamma \in \mathbb{C}$ with $\alpha < p < \operatorname{Re}\gamma$. If $g \in \Sigma K_p(\alpha)$ and $z^{p+1}J'_{p,\gamma}(g)(z) \neq 0$, $z \in U$, then*

$$J_{p,\gamma}(g) \in \Sigma K_p(\alpha).$$

Theorem 4.3.3. *[91] Let $p \in \mathbb{N}^*$, $\gamma \in \mathbb{C}$ with $\operatorname{Re}\gamma > p$ and $\alpha < 1 \le p < \delta < \operatorname{Re}\gamma$. Let φ be a function in $\Sigma K_{p,0}(\alpha,\delta)$ and $g \in \Sigma\mathcal{C}_{p,0}(\alpha,\delta;\varphi)$ such that $z^{p+1}J'_{p,\gamma}(\varphi) \neq 0$, $z \in U$, then*

$$J_{p,\gamma}(g) \in \Sigma\mathcal{C}_{p,0}(\alpha,\delta;\Phi),$$

where $\Phi = J_{p,\gamma}(\varphi)$.

We remark that Theorem 4.3.1 and Theorem 4.3.3 can be improved as it follows:

Theorem 4.3.4. *Let $p \in \mathbb{N}^*$, $\gamma \in \mathbb{C}$ with $\operatorname{Re}\gamma > p$ and $\alpha < p < \delta < \operatorname{Re}\gamma$. If $g \in \Sigma K_p(\alpha, \delta)$, then*

$$J_{p,\gamma}(g) \in \Sigma K_p(\alpha, \delta).$$

Theorem 4.3.5. *Let $p \in \mathbb{N}^*$, $\gamma \in \mathbb{C}$ with $\operatorname{Re}\gamma > p$ and $\alpha < 1 \leq p < \delta < \operatorname{Re}\gamma$. Let $\varphi \in \Sigma K_{p,0}(\alpha, \delta)$ and $g \in \Sigma \mathcal{C}_{p,0}(\alpha, \delta; \varphi)$, then*

$$J_{p,\gamma}(g) \in \Sigma \mathcal{C}_{p,0}(\alpha, \delta; \Phi),$$

where $\Phi = J_{p,\gamma}(\varphi)$.

Theorem 4.3.4 and Theorem 4.3.5 are not published yet.

4.4 Subclasses of the class Σ_p defined through a multiplier transformation

The results presented in this section are original and are published in [93].
Let $n \in \mathbb{Z}$, $p \in \mathbb{N}^*$ and $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > p$. We consider the operator $J^n_{p,\lambda}$ on Σ_p defined by:

$$J^n_{p,\lambda} g(z) = \frac{a_{-p}}{z^p} + \sum_{k=0}^{\infty} \left(\frac{\lambda - p}{k + \lambda} \right)^n a_k z^k, \text{ where } g(z) = \frac{a_{-p}}{z^p} + \sum_{k=0}^{\infty} a_k z^k.$$

We mention that this operator may also be found in [5]. We have the next properties for the above operator:

1. $J^{-1}_{p,\lambda} g(z) = \dfrac{1}{\lambda - p} z g'(z) + \dfrac{\lambda}{\lambda - p} g(z), \ g \in \Sigma_p,$
2. $J^0_{p,\lambda} g(z) = g(z), \ g \in \Sigma_p,$
3. $J^1_{p,\lambda} g(z) = \dfrac{\lambda - p}{z^\lambda} \displaystyle\int_0^z t^{\lambda - 1} g(t) dt = J_{p,\lambda}(g)(z), \ g \in \Sigma_p,$
4. If $g \in \Sigma_p$ with $J^n_{p,\lambda} g \in \Sigma_p$, then $J^m_{p,\lambda}(J^n_{p,\lambda} g) = J^{n+m}_{p,\lambda} g$, for $m, n \in \mathbb{Z}$.

Remark 4.4.1. *[93]* Let $p \in \mathbb{N}^*$ and $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > p$. We know that if $g \in \Sigma_p$, then $J_{p,\lambda}(g) \in \Sigma_p$, hence, using item 4 and the induction, we obtain

$$J^n_{p,\lambda} g \in \Sigma_p \ \text{ for all } n \in \mathbb{N}^*.$$

We notice from item 1 that for $g \in \Sigma_p$ we have $J^{-1}_{p,\lambda} g \in \Sigma_p$, so

$$J^{-n}_{p,\lambda} g \in \Sigma_p \ \text{ for all } n \in \mathbb{N}^*.$$

Therefore, for $n \in \mathbb{Z}$, $p \in \mathbb{N}^*$, $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > p$, we have $J^n_{p,\lambda} : \Sigma_p \to \Sigma_p$.

Now it is easy to see that we have the next properties for $J^n_{p,\lambda}$, when $\operatorname{Re}\lambda > p$:

1. $J^n_{p,\lambda}(J^m_{p,\lambda}g(z)) = J^{n+m}_{p,\lambda}g(z)$, $n, m \in \mathbb{Z}$, $g \in \Sigma_p$,
2. $J^n_{p,\gamma}(J^m_{p,\lambda}g(z)) = J^m_{p,\lambda}(J^n_{p,\gamma}g(z))$, $n, m \in \mathbb{Z}$, $g \in \Sigma_p$, $\operatorname{Re}\gamma > p$,
3. $J^n_{p,\lambda}(g_1 + g_2)(z) = J^n_{p,\lambda}g_1(z) + J^n_{p,\lambda}g_2(z)$, for $g_1, g_2 \in \Sigma_p$, $n \in \mathbb{Z}$,
4. $J^n_{p,\lambda}(cg)(z) = cJ^n_{p,\lambda}g(z)$, $c \in \mathbb{C}^*$, $n \in \mathbb{Z}$,
5. $J^n_{p,\lambda}(zg'(z)) = z(J^n_{p,\lambda}g(z))' = (\lambda - p)J^{n-1}_{p,\lambda}g(z) - \lambda J^n_{p,\lambda}g(z)$, $n \in \mathbb{Z}$, $g \in \Sigma_p$.

Remark 4.4.2. *[93]*

1. When $\lambda = 2$ and $p = 1$, we have

$$J^n_{1,2}g(z) = \frac{a_{-1}}{z} + \sum_{k=0}^{\infty}(k+2)^{-n}a_k z^k,$$

and this operator was studied by Cho and Kim [17] for $n \in \mathbb{Z}$ and by Uralegaddi and Somanatha [97] for $n < 0$.

2. We also have the relation

$$z^2 J^n_{1,2}g(z) = D^n(z^2 g(z)),\ g \in \Sigma_{1,0}\,,$$

where D^n is the well-known Sălăgean differential operator of order n [82], defined by $D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k$, $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$.

3. $J^n_{p,\lambda}$ is an extension to the meromorphic functions of the operator K^n_p, defined on $A(p) = \left\{ f \in H(U) : f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \right\}$, introduced in [87]. Also, for $n \geq 0$ we find that K^n_p is the Komatu linear operator, defined in [35].

4. It is easy to see that for $n > 0$, $J^n_{p,\lambda}$ is an integral operator while $J^{-n}_{p,\lambda}$ is a differential operator with the property $J^{-n}_{p,\lambda}(J^n_{p,\lambda}g) = g$, $g \in \Sigma_p$.

Definition 4.4.1. *[93] For $p \in \mathbb{N}^*$, $n \in \mathbb{Z}$, $\lambda \in \mathbb{C}$, $\operatorname{Re}\lambda > p$ and $\alpha < p < \delta$ we define*

$$\Sigma S^n_{p,\lambda}(\alpha) = \left\{ g \in \Sigma_p : \operatorname{Re}\left[-\frac{z\left(J^n_{p,\lambda}g(z)\right)'}{J^n_{p,\lambda}g(z)} \right] > \alpha,\ z \in U \right\},$$

$$\Sigma S^n_{p,\lambda}(\alpha, \delta) = \left\{ g \in \Sigma_p : \alpha < \operatorname{Re}\left[-\frac{z\left(J^n_{p,\lambda}g(z)\right)'}{J^n_{p,\lambda}g(z)} \right] < \delta,\ z \in U \right\}.$$

Remark 4.4.3. *[93]*

1. We have $g \in \Sigma S^n_{p,\lambda}(\alpha)$ if and only if $J^n_{p,\lambda} g \in \Sigma^*_p(\alpha)$, respectively $g \in \Sigma S^n_{p,\lambda}(\alpha,\delta)$ if and only if $J^n_{p,\lambda} g \in \Sigma^*_p(\alpha,\delta)$.

2. Using the equality $z(J^n_{p,\lambda} g(z))' = (\lambda - p) J^{n-1}_{p,\lambda} g(z) - \lambda J^n_{p,\lambda} g(z)$, we can easy see that for $\operatorname{Re}\lambda > p$ the condition

$$\alpha < \operatorname{Re}\left[-\frac{z\left(J^n_{p,\lambda} g(z)\right)'}{J^n_{p,\lambda} g(z)}\right] < \delta,\ z \in U,$$

is equivalent to

$$\operatorname{Re}\lambda - \delta < \operatorname{Re}\left[(\lambda - p)\frac{J^{n-1}_{p,\lambda} g(z)}{J^n_{p,\lambda} g(z)}\right] < \operatorname{Re}\lambda - \alpha,\ z \in U. \tag{4.4}$$

3. We have

$$\Sigma S^0_{p,\lambda}(\alpha,\delta) = \Sigma^*_p(\alpha,\delta),$$

$$\Sigma S^1_{p,\lambda}(\alpha,\delta) = \left\{ g \in \Sigma_p : G(z) = \frac{\lambda - p}{z^\lambda}\int_0^z t^{\lambda-1} g(t)dt \in \Sigma^*_p(\alpha,\delta)\right\}.$$

The following theorem gives us a connection between the sets $\Sigma S^n_{p,\lambda}(\alpha)$ and $\Sigma S^{n-1}_{p,\lambda}(\alpha)$, respectively between $\Sigma S^n_{p,\lambda}(\alpha,\delta)$ and $\Sigma S^{n-1}_{p,\lambda}(\alpha,\delta)$.

Theorem 4.4.1. *[93] Let $p \in \mathbb{N}^*$, $n \in \mathbb{Z}$, $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > p$, $\alpha < p < \delta$ and $g \in \Sigma_p$. Then*

$$g \in \Sigma S^n_{p,\lambda}(\alpha) \Leftrightarrow J_{p,\lambda}(g) \in \Sigma S^{n-1}_{p,\lambda}(\alpha),$$

$$g \in \Sigma S^n_{p,\lambda}(\alpha,\delta) \Leftrightarrow J_{p,\lambda}(g) \in \Sigma S^{n-1}_{p,\lambda}(\alpha,\delta),$$

where $J_{p,\lambda}(g)(z) = \dfrac{\lambda - p}{z^\lambda}\displaystyle\int_0^z t^{\lambda-1} g(t)dt.$

Theorem 4.4.2. *[93] Let $p \in \mathbb{N}^*$, $n \in \mathbb{Z}$, $\lambda, \gamma \in \mathbb{C}$ with $\operatorname{Re}\lambda > p$ and $\alpha < p < \delta < \operatorname{Re}\gamma$. Then*

$$g \in \Sigma S^n_{p,\lambda}(\alpha,\delta) \Rightarrow J_{p,\gamma}(g) \in \Sigma S^n_{p,\lambda}(\alpha,\delta).$$

Corollary 4.4.1. *[93] Let $n \in \mathbb{Z}$, $p \in \mathbb{N}^*, \lambda \in \mathbb{C}$ and $\alpha < p < \delta < \operatorname{Re}\lambda$. Then we have*

$$\Sigma S^n_{p,\lambda}(\alpha,\delta) \subset \Sigma S^{n+1}_{p,\lambda}(\alpha,\delta).$$

Theorem 4.4.3. *[93] Let $n \in \mathbb{Z}$, $p \in \mathbb{N}^*, \lambda, \gamma \in \mathbb{C}$ with $\operatorname{Re}\lambda > p$ and $\alpha < p < \operatorname{Re}\gamma \le \delta$. If $g \in \Sigma S^n_{p,\lambda}(\alpha,\delta)$ and satisfies the condition*

$$\frac{z\left[J^n_{p,\lambda}(g)(z)\right]'}{J^n_{p,\lambda}(g)(z)} + \gamma \prec R_{\gamma - p, p}(z),$$

then $J_{p,\gamma}(g) \in \Sigma S^n_{p,\lambda}(\alpha,\delta)$.

If we consider in Theorem 4.4.3 that $\delta \to \infty$ we get:

Theorem 4.4.4. *[93] Let $n \in \mathbb{Z}$, $p \in \mathbb{N}^*$, $\lambda, \gamma \in \mathbb{C}$ with $\operatorname{Re}\lambda > p$ and $\alpha < p < \operatorname{Re}\gamma$. If $g \in \Sigma S^n_{p,\lambda}(\alpha)$ and satisfies the condition*

$$\frac{z\left[J^n_{p,\lambda}(g)(z)\right]'}{J^n_{p,\lambda}(g)(z)} + \gamma \prec R_{\gamma-p,p}(z),$$

then $J_{p,\gamma}(g) \in \Sigma S^n_{p,\lambda}(\alpha)$.

Theorem 4.4.5. *[93] Let $n \in \mathbb{N}$, $p \in \mathbb{N}^*$, $\lambda, \gamma \in \mathbb{C}$ and $\alpha < p < \operatorname{Re}\gamma \leq \operatorname{Re}\lambda \leq \delta$. If $h \in \Sigma S^n_{p,\lambda}(\alpha,\delta)$ and satisfies the condition*

$$\frac{zh'(z)}{h(z)} + \gamma \prec R_{\gamma-p,p}(z),$$

then $J_{p,\gamma}(h) \in \Sigma S^n_{p,\lambda}(\alpha,\delta)$.

If we consider $\gamma = \lambda$, in the above theorem, we obtain:

Corollary 4.4.2. *[93] Let $n \in \mathbb{N}$, $p \in \mathbb{N}^*$, $\lambda \in \mathbb{C}$ and $h \in \Sigma S^n_{p,\lambda}(\alpha,\delta)$ with $\alpha < p < \operatorname{Re}\lambda \leq \delta$. If*

$$\frac{zh'(z)}{h(z)} + \lambda \prec R_{\lambda-p,p}(z).$$

Then $J_{p,\lambda}(h) \in \Sigma S^n_{p,\lambda}(\alpha,\delta)$.

Taking $n = 0$ in Corollary 4.4.2, we get:

Corollary 4.4.3. *[93]Let $p \in \mathbb{N}^*$, $\lambda \in \mathbb{C}$ and $\alpha < p < \operatorname{Re}\lambda \leq \delta$. If $h \in \Sigma^*_p(\alpha,\delta)$ with*

$$\frac{zh'(z)}{h(z)} + \lambda \prec R_{\lambda-p,p}(z),$$

then $J_{p,\lambda}(h) \in \Sigma^*_p(\alpha,\delta)$.

If in Corollary 4.4.3 we consider $\delta \mapsto \infty$ we have the next result:

Corollary 4.4.4. *[93] Let $p \in \mathbb{N}^*$, $\lambda \in \mathbb{C}$ and $\alpha < p < \operatorname{Re}\lambda$. If $h \in \Sigma^*_p(\alpha)$ with*

$$\frac{zh'(z)}{h(z)} + \lambda \prec R_{\lambda-p,p}(z),$$

then $J_{p,\lambda}(h) \in \Sigma^*_p(\alpha)$.

We remark that Corollary 4.4.3 and Corollary 4.4.4 were also obtained in section 2 of this chapter.

From the proof of Theorem 4.4.5 we remark that we also have:

Theorem 4.4.6. *[93] Let $n \in \mathbb{Z}$, $p \in \mathbb{N}^*$, $\lambda, \gamma \in \mathbb{C}$ with $\operatorname{Re}\lambda > p$ and $\alpha < p < \operatorname{Re}\gamma$. If $g \in \Sigma S^n_{p,\lambda}(\alpha)$ with $J_{p,\gamma}(J^n_{p,\lambda}(g)(z)) \neq 0$, $z \in \dot{U}$, then*

$$J_{p,\gamma}(g) \in \Sigma S^n_{p,\lambda}(\alpha).$$

Taking $n = 0$ in the above theorem we get the next corollary, which was also obtained in section 2 of this chapter.

Corollary 4.4.5. *[93] Let $p \in \mathbb{N}^*$, $\gamma \in \mathbb{C}$ and $\alpha < p < \operatorname{Re}\gamma$. If $g \in \Sigma^*_p(\alpha)$ with $z^p J_{p,\gamma}(g)(z) \neq 0$, $z \in U$, then $J_{p,\gamma}(g) \in \Sigma^*_p(\alpha)$.*

Chapter 5

Applications of the Briot-Bouquet differential subordinations and superordinations

In this chapter we define new subclasses of meromorphic multivalent functions using the subordination and the superordination and we establish the conditions such that when we apply one of the integral operators $J_{p,\beta,\gamma}$ or $J_{p,\gamma}$ to a function which belongs to one of these subclasses, we get a function which belongs to a similar class.

The results presented in this chapter are original and were sent for publishing.

5.1 The operator $J_{p,\beta,\gamma}$ and the class $\Sigma S_p(h_1, h_2)$

The results presented in this section will be published in [94]. The first result is a simple lemma and we will use it latter to present some examples for the results included in this paper.

Lemma 5.1.1. *[94] Let $\alpha, \beta, \gamma \in \mathbb{C}$ with $\gamma \neq 0$, $\alpha + \gamma \neq 0$ and $|\beta| < |\gamma|$. Let h be the function*

$$h(z) = z + \frac{\alpha z}{\beta z + \gamma},\ z \in U.$$

If we have

$$4|\alpha\beta\gamma^2| \leq (|\gamma| - |\beta|)^3|\alpha + \gamma|, \tag{5.1}$$

then h is convex in U.

Remark 5.1.1. *[94] 1. It is obvious that if h is a convex function in U (with $h'(0) \neq 0$), then $\delta_1 + \delta_2 h(rz)$ is also a convex function, when $r \in (0, 1]$, $\delta_1, \delta_2 \in \mathbb{C}$, $\delta_2 \neq 0$.*
2. If we consider $\alpha = |\beta| = 1$ in the above lemma, then the condition (5.1) becomes

$$4|\gamma|^2 \leq |\gamma + 1|(|\gamma| - 1)^3. \tag{5.2}$$

It is not difficult to verify that the condition (5.2) holds for each real number $\gamma \geq 3,2$. *In other words, the functions*

$$z + \frac{z}{\gamma + z},\ z + \frac{z}{\gamma - z},\ z \in U,$$

are convex functions when $\gamma \geq 3,2$.
We mention here that in [70] the authors proved that the function

$$h(z) = 1 + z + \frac{z}{z+2},\ z \in U,$$

is convex in U*, so the function* $z + \dfrac{z}{2+z}$ *is also a convex function.*

Next, we define some new subclasses of the class Σ_p, associated with superordination and subordination, such that, in some particular cases, these new subclasses are the well-known classes of meromorphic starlike functions.

Definition 5.1.1. *[94] Let* $p \in \mathbb{N}^*$ *and* $h_1, h_2, h \in H(U)$ *with* $h_1(0) = h_2(0) = h(0) = p$ *and* $h_1(z) \prec h_2(z)$. *We define:*

$$\Sigma S_p(h_1, h_2) = \left\{ g \in \Sigma_p : h_1(z) \prec -\frac{zg'(z)}{g(z)} \prec h_2(z) \right\},$$

$$\Sigma S_p(h) = \left\{ g \in \Sigma_p : -\frac{zg'(z)}{g(z)} \prec h(z) \right\}.$$

We remark that if we consider $h(z) = h_{p,\alpha}(z) = \dfrac{p + (p - 2\alpha)z}{1 - z}$, $z \in U, 0 \leq \alpha < p$, since $h_{p,\alpha}(U) = \{z \in \mathbb{C} : \operatorname{Re} z > \alpha\}$, we have $\Sigma S_p(h_{p,\alpha}) = \Sigma_p^*(\alpha)$.

We have the integral operator $J_{p,\beta,\gamma}$ defined by

$$J_{p,\beta,\gamma}(g)(z) = \left[\frac{\gamma - p\beta}{z^\gamma} \int_0^z g^\beta(t) t^{\gamma - 1} dt \right]^{\frac{1}{\beta}},\ g \in \Sigma_p,$$

and it was introduced at section 4.1.

Theorem 5.1.1. *[94] Let* $p \in \mathbb{N}^*$ *and* $\beta, \gamma \in \mathbb{C}$ *with* $\beta \neq 0$ *and* $\operatorname{Re}(\gamma - p\beta) > 0$. *Let* h_1 *and* h_2 *be convex functions in* U *with* $h_1(0) = h_2(0) = p$ *and let* $g \in \Sigma S_p(h_1, h_2)$ *such that*

$$\beta \frac{zg'(z)}{g(z)} + \gamma \prec R_{\gamma - \beta p, p}(z).$$

Suppose that the Briot-Bouquet differential equations

$$q(z) + \frac{zq'(z)}{\gamma - \beta q(z)} = h_1(z) \quad \text{and} \quad q(z) + \frac{pzq'(z)}{\gamma - \beta q(z)} = h_2(z),\ z \in U, \tag{5.3}$$

have the univalent solutions q_1^1 *and* q_2^p *, respectively, with* $q_1^1(0) = q_2^p(0) = p$ *and* $q_1^1 \prec h_1$, $q_2^p \prec h_2$.
Let $G = J_{p,\beta,\gamma}(g)$. *If* $\dfrac{zg'(z)}{g(z)}$ *is univalent in* U *and* $\dfrac{zG'(z)}{G(z)} \in Q$, *then*

$$G \in \Sigma S_p(q_1^1, q_2^p).$$

The functions q_1^1 and q_2^p are the best subordinant and the best (p,p)-dominant, respectively.

If we consider in the hypothesis of Theorem 5.1.1 the condition

$$\mathrm{Re}\,[\gamma - \beta h_2(z)] > 0,\ z \in U,$$

instead of

$$\beta \frac{zg'(z)}{g(z)} + \gamma \prec R_{\gamma-\beta p,p}(z)$$

we get the next result.

Theorem 5.1.2. *[94] Let $p \in \mathbb{N}^*$ and $\beta, \gamma \in \mathbb{C}$ with $\beta \neq 0$ and $\mathrm{Re}\,(\gamma - p\beta) > 0$. Let h_1 and h_2 be convex functions in U with $h_1(0) = h_2(0) = p$, $h_1 \prec h_2$ and*

$$\mathrm{Re}\,[\gamma - \beta h_2(z)] > 0,\ z \in U.$$

Let $g \in \Sigma S_p(h_1, h_2)$ and $G = J_{p,\beta,\gamma}(g)$. If $\dfrac{zg'(z)}{g(z)}$ is univalent in U and $\dfrac{zG'(z)}{G(z)} \in Q$, then

$$G \in \Sigma S_p(q_1^1, q_2^p),$$

where q_1^1 and q_2^p are the univalent solutions of the Briot-Bouquet differential equations

$$q(z) + \frac{zq'(z)}{\gamma - \beta q(z)} = h_1(z),\ z \in U, \tag{5.4}$$

and, respectively,

$$q(z) + \frac{pzq'(z)}{\gamma - \beta q(z)} = h_2(z),\ z \in U, \tag{5.5}$$

with $q_1^1(0) = q_2^p(0) = p$.

The functions q_1^1 and q_2^p are the best subordinant and the best (p,p)-dominant, respectively.

Remark 5.1.2. *[94]* Let the conditions from the hypothesis of Theorem 5.1.2 be fulfilled. If we consider, in addition, that q_1^p and q_2^1 are the univalent solutions of the Briot-Bouquet differential equations

$$q(z) + \frac{pzq'(z)}{\gamma - \beta q(z)} = h_1(z),\ z \in U,$$

and, respectively,

$$q(z) + \frac{zq'(z)}{\gamma - \beta q(z)} = h_2(z),\ z \in U,$$

with $q_1^p(0) = q_2^1(0) = p$, we have from the above theorem and Theorem 2.3.1, that

$$q_1^p(z) \prec q_1^1(z) \prec -\frac{zG'(z)}{G(z)} \prec q_2^p(z) \prec q_2^1(z).$$

Hence $G \in \Sigma S_p(q_1^1, q_2^p)$ is the best choice.

If we consider for Theorem 5.1.1 only the subordination, we obtain the next result.

Theorem 5.1.3. *Let $p \in \mathbb{N}^*$ and $\beta, \gamma \in \mathbb{C}$ with $\beta \neq 0$ and $\operatorname{Re}(\gamma - p\beta) > 0$. Let h be a convex function in U with $h(0) = p$ and $g \in \Sigma S_p(h)$ such that*

$$\beta \frac{zg'(z)}{g(z)} + \gamma \prec R_{\gamma-\beta p,p}(z).$$

Suppose that the Briot-Bouquet differential equation

$$q(z) + \frac{pzq'(z)}{\gamma - \beta q(z)} = h(z),\ z \in U,$$

has the univalent solution q with $q(0) = p$ and $q \prec h$. Then

$$G = J_{p,\beta,\gamma}(g) \in \Sigma S_p(q).$$

The function q is the best (p,p)-dominant.

Theorem 5.1.4. *[94] Let $p \in \mathbb{N}^*$ and $\beta, \gamma \in \mathbb{C}$ with $\beta \neq 0$ and $\operatorname{Re}[\gamma - p\beta] > 0$. Let h_1 and h_2 be analytic functions in U with $h_1(0) = h_2(0) = p$, $h_1 \prec h_2$ and*

$$(i) \quad \gamma - \beta h_2(z) \prec R_{\gamma-p\beta,1}(z).$$

If q_1 and q_2 are the analytic solutions of the Briot-Bouquet differential equations

$$q(z) + \frac{zq'(z)}{\gamma - \beta q(z)} = h_1(z),\ z \in U, \tag{5.6}$$

and, respectively,

$$q(z) + \frac{pzq'(z)}{\gamma - \beta q(z)} = h_2(z),\ z \in U, \tag{5.7}$$

with $q_1(0) = q_2(0) = p$ and if

$$(ii) \quad \frac{zq_1'(z)}{\gamma - \beta q_1(z)} \quad \text{is starlike in } U,$$

$$(iii) \quad h_2 \quad \text{is convex or} \quad \frac{zq_2'(z)}{\gamma - \beta q_2(z)} \quad \text{is starlike},$$

then q_1 and q_2 are univalent in U.

Moreover, if $g \in \Sigma S_p(h_1, h_2)$ such that $\frac{zg'(z)}{g(z)}$ is univalent in U and $\frac{zG'(z)}{G(z)} \in Q$, where $G = J_{p,\beta,\gamma}(g)$, then

$$G \in \Sigma S_p(q_1, q_2).$$

The functions q_1 and q_2 are the best subordinant and the best (p,p)-dominant, respectively.

From Theorem 2.3.2, since $p \neq 0$, we have that the solutions q_1 and q_2 (from the above theorem) are given by:

$$q_1(z) = z^{\gamma} H_1^{-p\beta}(z)\left[-\beta \int_0^z H_1^{-p\beta}(t)t^{\gamma-1}dt\right]^{-1} + \frac{\gamma}{\beta} = \left[-\beta \int_0^1 \left[\frac{H_1(tz)}{H_1(z)}\right]^{-p\beta} t^{\gamma-1}dt\right]^{-1} + \frac{\gamma}{\beta}, \tag{5.8}$$

$$q_2(z) = z^{\frac{\gamma}{p}} H_2^{-\beta}(z)\left[\frac{-\beta}{p} \int_0^z H_2^{-\beta}(t)t^{\frac{\gamma}{p}-1}dt\right]^{-1} + \frac{\gamma}{\beta} = \left[\frac{-\beta}{p} \int_0^1 \left[\frac{H_2(tz)}{H_2(z)}\right]^{-\beta} t^{\frac{\gamma}{p}-1}dt\right]^{-1} + \frac{\gamma}{\beta}, \tag{5.9}$$

where

$$H_k(z) = z \exp \int_0^z \frac{h_k(t)-p}{pt}dt,\ k = 1,2.$$

If we consider only the subordination for Theorem 5.1.4 we obtain the next result.

Theorem 5.1.5. *[94] Let $p \in \mathbb{N}^*$ and $\beta, \gamma \in \mathbb{C}$ with $\beta \neq 0$ and $\mathrm{Re}\,(\gamma - p\beta) > 0$. Also let $h \in H(U)$ with $h(0) = p$ such that*

$$(i) \quad \gamma - \beta h(z) \prec R_{\gamma-\beta p,p}(z).$$

If q is the analytic solution of the Briot-Bouquet differential equation

$$q(z) + \frac{pzq'(z)}{\gamma - \beta q(z)} = h(z),\ z \in U,$$

with $q(0) = p$, given by (5.9) and if

$$(ii) \quad h \quad \text{is convex or } \frac{zq'(z)}{\gamma - \beta q(z)} \quad \text{is starlike},$$

then q is univalent in U.
Moreover, if $g \in \Sigma S_p(h)$ and $G = J_{p,\beta,\gamma}(g)$, then $G \in \Sigma S_p(q)$.
The function q is the best (p,p)-dominant.

If we consider, in the above theorem, that the function h is convex we obtain the corollary:

Corollary 5.1.1. *[94] Let $p \in \mathbb{N}^*$ and $\beta, \gamma \in \mathbb{C}$ with $\beta \neq 0$ and $\mathrm{Re}\,(\gamma - p\beta) > 0$. Also let $g \in \Sigma S_p(h)$ with h convex in U, $h(0) = p$. If the function h satisfies the condition*

$$\gamma - \beta h(z) \prec R_{\gamma-\beta p,p}(z),$$

then

$$G = J_{p,\beta,\gamma}(g) \in \Sigma S_p(q),$$

where q is the univalent solution of the Briot-Bouquet differential equation

$$q(z) + \frac{pzq'(z)}{\gamma - \beta q(z)} = h(z)\,,\ z \in U,$$

with $q(0) = p$.
The function q is the best (p,p)-dominant.

Next, we present an application for the above corollary, when $\beta = 1, \gamma \in \mathbb{R}$, for a particular function h. We will use the notation $J_{p,\gamma}$ instead of $J_{p,1,\gamma}$.

Corollary 5.1.2. *[94] Let $p \in \mathbb{N}^*$ and $\gamma \geq p+3$ such that $4p(\gamma-p)^2 \leq \gamma(\gamma-p-1)^3$. If $g \in \Sigma S_p(h)$ with $h(z) = p + z + \dfrac{pz}{\gamma - p - z}$, then*

$$G = J_{p,\gamma}(g) \in \Sigma S_p(p+z),$$

which is equivalent to $\left|\dfrac{zG'(z)}{G(z)} + p\right| < 1$, $z \in U$. Therefore,

$$p - 1 < \mathrm{Re}\left[-\frac{zG'(z)}{G(z)}\right] < p + 1,\ z \in U,$$

this meaning that $G \in \Sigma_p^*(p-1, p+1)$.

If we consider for Corollary 5.1.1 the condition $\mathrm{Re}\,[\gamma - \beta h(z)] > 0$, $z \in U$, instead of $\gamma - \beta h(z) \prec R_{\gamma-\beta p,p}(z)$, we get:

Corollary 5.1.3. *[94] Let $p \in \mathbb{N}^*$ and $\beta, \gamma \in \mathbb{C}$ with $\beta \neq 0$ and $\mathrm{Re}\,(\gamma - p\beta) > 0$. Also let $g \in \Sigma S_p(h)$ with h convex in U and $h(0) = p$. If*

$$\mathrm{Re}\,[\gamma - \beta h(z)] > 0,\ z \in U,$$

then

$$G = J_{p,\beta,\gamma}(g) \in \Sigma S_p(q),$$

where q is the univalent solution of the Briot-Bouquet differential equation

$$q(z) + \frac{pzq'(z)}{\gamma - \beta q(z)} = h(z)\,,\ z \in U,\ q(0) = p.$$

The function q is the best (p,p)-dominant.

Since for Corollary 5.1.3 we have $q \prec h$ (see Theorem 2.3.1), we get the next corollary:

Corollary 5.1.4. *[94] Let $p \in \mathbb{N}^*$ and $\beta, \gamma \in \mathbb{C}$ with $\beta \neq 0$ and $\mathrm{Re}\,(\gamma - p\beta) > 0$. Also let $g \in \Sigma S_p(h)$ with h convex in U and $h(0) = p$. If*

$$\mathrm{Re}\,[\gamma - \beta h(z)] > 0,\ z \in U,$$

then

$$G = J_{p,\beta,\gamma}(g) \in \Sigma S_p(h).$$

Furthermore, using Corollary 5.1.4 for a particular function h, we present a result which was also obtained in section 4.2.

We consider $h(z) = h_{p,\alpha}(z) = \dfrac{p + (p - 2\alpha)z}{1 - z}$, $z \in U$, where $p \in \mathbb{N}^*$ and $0 \leq \alpha < p$. It is not difficult to see that $h_{p,\alpha}(U) = \{z \in \mathbb{C} / \mathrm{Re}\, z > \alpha\}$ and $h_{p,\alpha}(0) = p$.

Hence

$$g \in \Sigma S_p(h_{p,\alpha}) \Leftrightarrow g \in \Sigma_p^*(\alpha).$$

We get now the next result:

Corollary 5.1.5. *[94] Let* $p \in \mathbb{N}^*$, $\beta < 0$, $\gamma \in \mathbb{C}$ *and* $\frac{\operatorname{Re}\gamma}{\beta} \le \alpha < p$. *Then we have*

$$g \in \Sigma_p^*(\alpha) \Rightarrow G = J_{p,\beta,\gamma}(g) \in \Sigma_p^*(\alpha).$$

5.2 The operator $J_{p,\gamma}$ and the class $\Sigma K_p(h_1, h_2)$

The results presented in this section are original and will be published in [95].

We introduce some new subclasses of the class of meromorphic multivalent functions, which are defined by subordination and superordination, such that, in some particular cases, these new subclasses are the well-known classes of meromorphic convex functions.

Definition 5.2.1. *[95] Let* $p \in \mathbb{N}^*$ *and* $h_1, h_2, h \in H(U)$, *with* $h_1(0) = h_2(0) = h(0) = p$ *and* $h_1(z) \prec h_2(z)$. *We define:*

$$\Sigma K_p(h_1, h_2) = \left\{ g \in \Sigma_p : h_1(z) \prec -\left[1 + \frac{zg''(z)}{g'(z)}\right] \prec h_2(z) \right\},$$

$$\Sigma K_p(h) = \left\{ g \in \Sigma_p : -\left[1 + \frac{zg''(z)}{g'(z)}\right] \prec h(z) \right\},$$

$$\Sigma K_{p,0}(h_1, h_2) = \Sigma K_p(h_1, h_2) \cap \Sigma_{p,0}, \Sigma K_{p,0}(h) = \Sigma K_p(h) \cap \Sigma_{p,0}.$$

We remark that if we consider $h(z) = h_{p,\alpha}(z) = \frac{p + (p - 2\alpha)z}{1 - z}$, $z \in U, 0 \le \alpha < p$, since $h(U) = \{z \in \mathbb{C} : \operatorname{Re} z > \alpha\}$, we have $\Sigma K_p(h_{p,\alpha}) = \Sigma K_p(\alpha)$.

Theorem 5.2.1. *[95] Let* $p \in \mathbb{N}^*$ *and* $\gamma \in \mathbb{C}$ *with* $\operatorname{Re}\gamma > p$. *Let* h_1, h_2 *be convex functions in* U *with* $h_1(0) = h_2(0) = p$ *and* $g \in \Sigma K_p(h_1, h_2)$ *with* $z^{p+1} J'_{p,\gamma}(g)(z) \neq 0$, $z \in U$. *Suppose that the Briot-Bouquet differential equations*

$$q(z) + \frac{zq'(z)}{\gamma - q(z)} = h_1(z),\ z \in U,$$

and

$$q(z) + \frac{(p+1)zq'(z)}{\gamma - q(z)} = h_2(z),\ z \in U,$$

have the univalent solutions q_1^1 *and* q_2^{p+1}, *respectively, with* $q_1^1(0) = q_2^{p+1}(0) = p$ *and* $q_1^1 \prec h_1$, $q_2^{p+1} \prec h_2$.

Let $G = J_{p,\gamma}(g)$. *If* $\frac{zg''(z)}{g'(z)}$ *is univalent in* U *and* $\frac{zG''(z)}{G'(z)} \in Q$, *then* $G \in \Sigma K_p(q_1^1, q_2^{p+1})$.

The functions q_1^1 *and* q_2^{p+1} *are the best subordinant and the best (p,p+1)-dominant, respectively.*

If we consider in the hypothesis of Theorem 5.2.1 the condition

$$\operatorname{Re}[\gamma - h_2(z)] > 0,\ z \in U,$$

instead of $z^{p+1} J'_{p,\gamma}(g)(z) \neq 0$, $z \in U$, we get the next result.

Theorem 5.2.2. *[95] Let $p \in \mathbb{N}^*$ and $\gamma \in \mathbb{C}$ with $\operatorname{Re}\gamma > p$. Let h_1 and h_2 be convex functions in U with $h_1(0) = h_2(0) = p$ and $g \in \Sigma K_p(h_1, h_2)$. Suppose that*

$$\operatorname{Re}[\gamma - h_2(z)] > 0,\ z \in U.$$

Let $G = J_{p,\gamma}(g)$. If $\dfrac{zg''(z)}{g'(z)}$ is univalent in U and $\dfrac{zG''(z)}{G'(z)} \in Q$, then

$$G \in \Sigma K_p(q_1^1, q_2^{p+1}),$$

where q_1^1 and q_2^{p+1} are the univalent solutions of the Briot-Bouquet differential equations

$$q(z) + \frac{zq'(z)}{\gamma - q(z)} = h_1(z) \text{ and } \quad q(z) + \frac{(p+1)zq'(z)}{\gamma - q(z)} = h_2(z), \text{ respectively,}$$

with $q_1^1(0) = q_2^{p+1}(0) = p$.
The functions q_1^1 and q_2^{p+1} are the best subordinant and the best (p,p+1)-dominant, respectively.

Remark 5.2.1. *[95]* Let the conditions from the hypothesis of Theorem 5.2.2 be fulfilled. If we consider, in addition, that q_1^{p+1} and q_2^1 are the univalent solutions of the Briot-Bouquet differential equations

$$q(z) + \frac{(p+1)zq'(z)}{\gamma - q(z)} = h_1(z) \text{ and } \quad q(z) + \frac{zq'(z)}{\gamma - q(z)} = h_2(z), \text{ respectively,}$$

with $q_1^{p+1}(0) = q_2^1(0) = p$, we have from the above theorem and Theorem 2.3.1, that

$$q_1^{p+1}(z) \prec q_1^1(z) \prec -1 - \frac{zG''(z)}{G'(z)} \prec q_2^{p+1}(z) \prec q_2^1(z).$$

Hence $G \in \Sigma K_p(q_1^1, q_2^{p+1})$ is the best choice.

If we consider for Theorem 5.2.1 only the subordination, we obtain the next result.

Theorem 5.2.3. *[95] Let $p \in \mathbb{N}^*$ and $\gamma \in \mathbb{C}$ with $\operatorname{Re}\gamma > p$. Let h be a convex function in U with $h(0) = p$ and $g \in \Sigma K_p(h)$ with $z^{p+1}J'_{p,\gamma}(g)(z) \neq 0$, $z \in U$. Suppose that the Briot-Bouquet differential equation*

$$q(z) + \frac{(p+1)zq'(z)}{\gamma - q(z)} = h(z),\ z \in U,$$

has the univalent solution q with $q(0) = p$ and $q \prec h$.
Then

$$G = J_{p,\gamma}(g) \in \Sigma K_p(q).$$

The function q is the best (p,p+1)-dominant.

Theorem 5.2.4. *[95] Let $p \in \mathbb{N}^*$ and $\gamma \in \mathbb{C}$ with $\operatorname{Re}\gamma > p$. Let $h_1, h_2 \in H(U)$ with $h_1(0) = h_2(0) = p$, $h_1 \prec h_2$ and*

$$(i) \quad \gamma - h_2(z) \prec R_{\gamma-p,1}(z),\ z \in U.$$

If q_1 and q_2 are the analytic solutions of the Briot-Bouquet differential equations

$$q(z) + \frac{zq'(z)}{\gamma - q(z)} = h_1(z),\ q(z) + \frac{(p+1)zq'(z)}{\gamma - q(z)} = h_2(z),\ z \in U, \tag{5.10}$$

respectively, and if

$$(ii) \quad \frac{zq_1'(z)}{\gamma - q_1(z)} \quad \text{is starlike in } U,$$

$$(iii) \quad h_2 \quad \text{is convex or} \quad \frac{zq_2'(z)}{\gamma - q_2(z)} \quad \text{is starlike},$$

then q_1 and q_2 are univalent in U.

Moreover, if $g \in \Sigma K_p(h_1, h_2)$ such that $\frac{zg''(z)}{g'(z)}$ is univalent in U and $\frac{zG''(z)}{G'(z)} \in Q$, where $G = J_{p,\gamma}(g)$, then

$$G \in \Sigma K_p(q_1, q_2).$$

The functions q_1 and q_2 are the best subordinant and the best (p,p+1)-dominant, respectively.

If we consider only the subordination we have the next result.

Theorem 5.2.5. *[95] Let $p \in \mathbb{N}^*, \gamma \in \mathbb{C}$ with $\operatorname{Re}\gamma > p$, $h \in H(U)$ with $h(0) = p$ and*

$$(i) \quad \gamma - h(z) \prec R_{\gamma-p,p}(z).$$

If q is the analytic solution of the Briot-Bouquet differential equation

$$q(z) + \frac{(p+1)zq'(z)}{\gamma - q(z)} = h(z),\ z \in U,$$

with $q(0) = p$ and if

$$(ii) \quad h \quad \text{is convex or} \quad \frac{zq'(z)}{\gamma - q(z)} \quad \text{is starlike},$$

then q is univalent in U.

Moreover, if $g \in \Sigma K_p(h)$ and $G = J_{p,\gamma}(g)$, then $G \in \Sigma K_p(q)$.
The function q is the best (p,p+1)-dominant.

If we consider, in the above theorem, that the function h is convex we obtain the corollary:

Corollary 5.2.1. *[95] Let $p \in \mathbb{N}^*, \gamma \in \mathbb{C}$ with $\operatorname{Re}\gamma > p$, and $g \in \Sigma K_p(h)$ with h convex in U. If*

$$\gamma - h(z) \prec R_{\gamma-p,p}(z),$$

then

$$G = J_{p,\gamma}(g) \in \Sigma K_p(q),$$

where q is the univalent solution of the Briot-Bouquet differential equation

$$q(z) + \frac{(p+1)zq'(z)}{\gamma - q(z)} = h(z)\,,\ z \in U,$$

with $q(0) = p$.
The function q is the best (p,p+1)-dominant.

We next present an application for the above corollary, when h is a particular function.

Corollary 5.2.2. *[95] Let $p \in \mathbb{N}^*$, $\gamma \geq p+4$ such that $4(p+1)(\gamma-p)^2 \leq (\gamma+1)(\gamma-p-1)^3$.*
If $g \in \Sigma K_p(h)$, with $h(z) = p + z + \dfrac{(p+1)z}{\gamma - p - z}$, then

$$G = J_{p,\gamma}(g) \in \Sigma K_p(p+z).$$

If we consider for Corollary 5.2.1 the condition $\operatorname{Re}[\gamma - h(z)] > 0$, $z \in U$, instead of $\gamma - h(z) \prec R_{\gamma-p,p}(z)$ we get:

Corollary 5.2.3. *Let $p \in \mathbb{N}^*, \gamma \in \mathbb{C}$ with $\operatorname{Re}\gamma > p$, and $g \in \Sigma K_p(h)$ with h convex in U. If*

$$\operatorname{Re}[\gamma - h(z)] > 0,\ z \in U,$$

then

$$G = J_{p,\gamma}(g) \in \Sigma K_p(q),$$

where q is the univalent solution of the Briot-Bouquet differential equation

$$q(z) + \frac{(p+1)zq'(z)}{\gamma - q(z)} = h(z)\,,\ z \in U,$$

with $q(0) = p$.
The function q is the best (p,p+1)-dominant.

Since for Corollary 5.2.3 we have $q \prec h$ (see Theorem 2.3.1), we get the next corollary:

Corollary 5.2.4. *[95] Let $p \in \mathbb{N}^*, \gamma \in \mathbb{C}$ with $\operatorname{Re}\gamma > p$, $g \in \Sigma K_p(h)$ with h convex in U. If*

$$\operatorname{Re}[\gamma - h(z)] > 0,\ z \in U,$$

then

$$G = J_{p,\gamma}(g) \in \Sigma K_p(h).$$

Furthermore, we present two simple examples which use the above corollary.

Exemple 5.2.1. *Let us consider* $h(z) = (\operatorname{Re}\gamma - p)z + p$, $z \in U$, *where* $p \in \mathbb{N}^*, \gamma \in \mathbb{C}$ *with* $\operatorname{Re}\gamma > p$ *and let* $g \in \Sigma K_p(h)$.

We have h *convex in* U *and*

$$\operatorname{Re}\left[\gamma - h(z)\right] > 0,\ z \in U,$$

so, we get from the above corollary that

$$G = J_{p,\gamma}(g) \in \Sigma K_p(h),$$

which is equivalent to

$$-1 - \frac{zG''(z)}{G'(z)} \prec (\operatorname{Re}\gamma - p)z + p,\ z \in U,$$

hence $G \in \Sigma K_p(2p - \operatorname{Re}\gamma, \operatorname{Re}\gamma)$.

The next example is a result which was also obtained in section 4.4.

Exemple 5.2.2. *Let* $p \in \mathbb{N}^*$, $\gamma \in \mathbb{C}$ *with* $\operatorname{Re}\gamma > p$ *and let* $\alpha < p < \delta \leq \operatorname{Re}\gamma$. *If* $g \in \Sigma K_p(\alpha, \delta)$ *, then*

$$J_{p,\gamma}(g) \in \Sigma K_p(\alpha, \delta).$$

5.3 The operator $J_{p,\gamma}$ applied to the classes $\Sigma\mathcal{C}_{p,0}(h_1, h_2; \varphi, h)$ and $\Sigma\mathcal{C}_{p,0}(h_2; h)$

The results presented in this section are original and have been sent for publishing. First we define some subclasses of $\Sigma_{p,0}$ associated with superordination and subordination, using the close-to-convexity condition.

Definition 5.3.1. *[96] Let* $p \in \mathbb{N}^*$, $h_1, h_2, h \in H(U)$ *with* $h_1(0) = h_2(0) = 1$, $h(0) = p$, $h_1 \prec h_2$ *and* $\varphi \in \Sigma K_{p,0}(h)$. *We define:*

$$\Sigma\mathcal{C}_{p,0}(h_1, h_2; \varphi, h) = \left\{ g \in \Sigma_{p,0} : h_1(z) \prec \frac{g'(z)}{\varphi'(z)} \prec h_2(z) \right\},$$

$$\Sigma\mathcal{C}_{p,0}(h_2; \varphi, h) = \left\{ g \in \Sigma_{p,0} : \frac{g'(z)}{\varphi'(z)} \prec h_2(z) \right\}.$$

Definition 5.3.2. *[96] Let* $p \in \mathbb{N}^*$ *and* $h_2, h \in H(U)$ *with* $h_2(0) = 1$, $h(0) = p$. *We define:*

$$\Sigma\mathcal{C}_{p,0}(h_2; h) = \left\{ g \in \Sigma_{p,0} : (\exists)\varphi \in \Sigma K_{p,0}(h)\ s.t.\ \frac{g'(z)}{\varphi'(z)} \prec h_2(z) \right\},$$

$$\Sigma\mathcal{C}_{p,0}(h) = \left\{ g \in \Sigma_{p,0} : (\exists)\varphi \in \Sigma K_{p,0}(h)\ s.t.\ \frac{g'(z)}{\varphi'(z)} \prec \frac{1}{p}h(z) \right\}.$$

Remark 5.3.1. *[96]*

1. *If $H \in H(U)$, $H(0) = p$ and $h \prec H$, then $\Sigma C_{p,0}(h_2; h) \subset \Sigma C_{p,0}(h_2; H)$.*
2. *If $H_2 \in H(U)$, $H_2(0) = 1$ and $h_2 \prec H_2$, then $\Sigma C_{p,0}(h_2; h) \subset \Sigma C_{p,0}(H_2; h)$.*
3. *If $h_1, h_2, h, H \in H(U)$ with $h_1(0) = h_2(0) = 1$, $h(0) = H(0) = p$, $h_1 \prec h_2$ and $\varphi \in \Sigma K_{p,0}(h) \cap \Sigma K_{p,0}(H)$, then*

$$\Sigma\mathcal{C}_{p,0}(h_1, h_2; \varphi, h) = \Sigma\mathcal{C}_{p,0}(h_1, h_2; \varphi, H),$$

$$\Sigma\mathcal{C}_{p,0}(h_2; \varphi, h) = \Sigma\mathcal{C}_{p,0}(h_2; \varphi, H).$$

Next we present some particular cases for the classes defined above.

If $p = 1$ and $h_2(z) = h(z) = \dfrac{1+z}{1-z}$, $z \in U$, then a function φ is in the class $\Sigma K_{1,0}(h)$ if and only if

$$\mathrm{Re}\left[-1 - \frac{z\varphi''(z)}{\varphi'(z)}\right] > 0,\ z \in U,$$

so, the class of meromorpic close-to-convex functions is included into the class $\Sigma\mathcal{C}_{1,0}\left(\dfrac{1+z}{1-z}\right)$.

Let $\alpha < 1 \leq p < \delta$. We consider $h_2 = h_{1,\alpha,\delta}$ and $h = h_{p,\alpha,\delta}$, where $h_{p,\alpha,\delta} : U \to \mathbb{C}$ is the convex function with $h_{p,\alpha,\delta}(U) = \{z \in \mathbb{C} : \alpha < \mathrm{Re}\, z < \delta\}$ and $h_{p,\alpha,\delta}(0) = p$.

We know that this function, $h_{p,\alpha,\delta}$, exists and it is obtained by composing different well-known elementary functions. It is not difficult to see that

$$\Sigma K_{p,0}(h_{p,\alpha,\delta}) = \Sigma K_{p,0}(\alpha, \delta), \tag{5.11}$$

$$\Sigma\mathcal{C}_{p,0}(h_{1,\alpha,\delta}; \varphi, h_{p,\alpha,\delta}) = \Sigma\mathcal{C}_{p,0}(\alpha, \delta; \varphi),\ \text{where}\, \varphi \in \Sigma K_{p,0}(\alpha, \delta). \tag{5.12}$$

We will also use the more simply notation

$$\Sigma\mathcal{C}_{p,0}(h_{1,\alpha,\delta}; h_{p,\alpha,\delta}) = \Sigma\mathcal{C}_{p,0}(\alpha, \delta). \tag{5.13}$$

Theorem 5.3.1. *[96] Let $p \in \mathbb{N}^*$ and $\gamma \in \mathbb{C}$ with $\mathrm{Re}\,\gamma > p$. Let h_2 and h be convex functions in U with $h_2(0) = 1$, $h(0) = p$ and let $g \in \Sigma\mathcal{C}_{p,0}(h_2; h)$. If we have $\mathrm{Re}\,[\gamma - h(z)] > 0$, $z \in U$, then*

$$G = J_{p,\gamma}(g) \in \Sigma\mathcal{C}_{p,0}(h_2; q),$$

where q is the univalent solution of the Briot-Bouquet differential equation

$$q(z) + \frac{(p+1)zq'(z)}{\gamma - q(z)} = h(z)\,,\ z \in U,$$

with $q(0) = p$.
The function q is the best (p,p+1)-dominant.

Theorem 5.3.2. *[96] Let $p \in \mathbb{N}^*$, $\gamma \in \mathbb{C}$ with $\operatorname{Re}\gamma > p$ and h_2, h be convex functions in U with $h_2(0) = 1$, $h(0) = p$ and $\operatorname{Re}[\gamma - h(z)] > 0$, $z \in U$. If $\varphi \in \Sigma K_{p,0}(h)$ and $g \in \Sigma\mathcal{C}_{p,0}(h_2; \varphi, h)$, then*

$$G = J_{p,\gamma}(g) \in \Sigma\mathcal{C}_{p,0}(h_2; J_{p,\gamma}(\varphi), q),$$

where q is the univalent solution of the Briot-Bouquet differential equation

$$q(z) + \frac{(p+1)zq'(z)}{\gamma - q(z)} = h(z)\,,\ z \in U,$$

with $q(0) = p$.
The function q is the best (p,p+1)-dominant.

If we consider that the conditions from the hypothesis of Theorem 5.3.1 and Theorem 5.3.2 respectively, are met, since we know from Theorem 2.3.1 that $q \prec h$, we obtain the next corollaries:

Corollary 5.3.1. *[96] Let $p \in \mathbb{N}^*$ and $\gamma \in \mathbb{C}$ with $\operatorname{Re}\gamma > p$. Let h_2, h be convex functions in U with $h_2(0) = 1$, $h(0) = p$ and let $g \in \Sigma\mathcal{C}_{p,0}(h_2; h)$. If $\operatorname{Re} h(z) < \operatorname{Re}\gamma$, $z \in U$, then*

$$G = J_{p,\gamma}(g) \in \Sigma\mathcal{C}_{p,0}(h_2; h).$$

Corollary 5.3.2. *[96] Let $p \in \mathbb{N}^*$ and $\gamma \in \mathbb{C}$ with $\operatorname{Re}\gamma > p$. Let h_2, h be convex functions in U with $h_2(0) = 1$, $h(0) = p$ and $\operatorname{Re} h(z) < \operatorname{Re}\gamma$, $z \in U$. If $\varphi \in \Sigma K_{p,0}(h)$ and $g \in \Sigma\mathcal{C}_{p,0}(h_2; \varphi, h)$, then*

$$G = J_{p,\gamma}(g) \in \Sigma\mathcal{C}_{p,0}(h_2; J_{p,\gamma}(\varphi), h).$$

Next we present two results which concern the particular classes $\Sigma\mathcal{C}_{p,0}(\alpha, \delta)$ and $\Sigma\mathcal{C}_{p,0}(\alpha, \delta; \varphi)$.

Theorem 5.3.3. *[96] Let $p \in \mathbb{N}^*$, $\alpha, \delta \in \mathbb{R}$ and $\gamma \in \mathbb{C}$ with $\alpha < 1 \le p < \delta \le \operatorname{Re}\gamma$. If $g \in \Sigma\mathcal{C}_{p,0}(\alpha, \delta)$, then*

$$J_{p,\gamma}(g) \in \Sigma\mathcal{C}_{p,0}(\alpha, \delta).$$

Theorem 5.3.4. *[96] Let $p \in \mathbb{N}^*$, $\alpha, \delta \in \mathbb{R}$ and $\gamma \in \mathbb{C}$ with $\alpha < 1 \le p < \delta \le \operatorname{Re}\gamma$. If $\varphi \in \Sigma K_{p,0}(\alpha, \delta)$ and $g \in \Sigma\mathcal{C}_{p,0}(\alpha, \delta; \varphi)$, then*

$$J_{p,\gamma}(g) \in \Sigma\mathcal{C}_{p,0}(\alpha, \delta; \Phi),$$

where $\Phi = J_{p,\gamma}(\varphi)$.

We mention that Theorem 5.3.4 was also met in section 4.3.

Lemma 5.3.1. *[96] Let $r > 0$ and let $\lambda : \overline{U} \to \mathbb{C}$ be an analytic function in U such that $\sup_{z \in \overline{U}} |\lambda(z)| = M < \infty$. If $p \in H[1,1] \cap Q$ and $p(z) + \lambda(z)zp'(z)$ is univalent in U, then*

$$U(1, r) \subset \{p(z) + \lambda(z)zp'(z) : z \in U\} \Rightarrow U\left(1, \frac{r}{1+M}\right) \subset p(U).$$

Theorem 5.3.5. *[96] Let* $m, r > 0$, $p \in \mathbb{N}^*$ *and* $\gamma \in \mathbb{C}$ *with* $\operatorname{Re}\gamma > p$. *Let* h_2 *and* h *be convex functions in* U *such that* $h_2(0) = 1$, $h(0) = p$ *and* $\operatorname{Re}[\gamma - h(z)] > m$, $z \in U$.

Let $\varphi \in \Sigma K_{p,0}(h)$ *and* $g \in \Sigma\mathcal{C}_{p,0}(h_1, h_2; \varphi, h)$, *where* $h_1(z) = rz + 1$, $z \in U$. *Suppose that* $\dfrac{g'}{\varphi'}$ *is univalent in* U *and* $\dfrac{J'_{p,\gamma}(g)}{J'_{p,\gamma}(\varphi)} \in Q$. *Then*

$$G = J_{p,\gamma}(g) \in \Sigma\mathcal{C}_{p,0}(q_1, h_2; \Phi, q),$$

where

$$\Phi = J_{p,\gamma}(\varphi)$$
$$q_1(z) = \frac{rm}{m+1}z + 1,\ z \in U,$$

and q *is the univalent solution of the Briot-Bouquet differential equation*

$$q(z) + \frac{(p+1)zq'(z)}{\gamma - q(z)} = h(z)\,,\ z \in U,$$

with $q(0) = p$.
The function q *is the best (p,p+1)-dominant.*

Bibliography

[1] M. Acu, *Some subclasses of meromorphic functions*, Filomat 21:2(2007),1-9.

[2] L.A. Aksentiev, *Sufficient conditions for univalence of regular functions*, Izv. Vyss. Ucebn. Zaved. Mat., 3(4)(1958), 3-7 (limba rusă)

[3] H.S. Al-Amiri, P.T. Mocanu, *Some simple criteria of starlikeness and convexity for meromorphic functions*, Mathematica (Cluj), 37(60), (1995), 11-21.

[4] J.W. Alexander, *Functions which map the interior of the unit circle upon simple regions*, Ann. of Math., 17(1915), 12-22.

[5] R.M. Ali, V. Ravichandran, N. Seenivasagan, *On subordination and superordination of the multiplier transformation for meromorphic functions*, Bull. Malays. Math. Sci. Soc. (2)33(2)(2010), 311-324.

[6] S.K. Bajpai, *A note on a class of meromorphic univalent functions*, Rev. Roum. Math. Pures Appl., 22(1997), 295-297.

[7] I.E. Bazilevič, *On a case of integrability in quadratures of the Loewner-Kufarev equation*, Math. Sb.,N.S.,37(79)(1955),471-476.

[8] L. Bieberbach, *Über die Koeffizienten derjenigen Potenzreihen, welche eine schlichte Abbildung des Einheitskreises vermitteln*, Preuss, Akad. Wiss. Sitzungsb., 1916, 940-955.

[9] L. Bieberbach, *Über einige Extremalprobleme im Gebiete der Konformen Abbildung*, Math. Ann., 77(1916), 153-172.

[10] L. Brickman, *Extreme points of the set of univalent functions,* Bull. Amer. Math. Soc. 76(1970), 372-376.

[11] L. Brickman, D.J. Hallenbeck, T.H. MacGregor, D.R. Wilken, *Convex hulls and extreme points of families of starlike and convex mappings,* Trans. Amer. Math. Soc. 185(1973), 413-428.

[12] T. Bulboacă, *On some classes of differential subordinations*, Studia Univ. Babeş-Bolyai, Math., 31, 1(1986),45-50.

[13] T. Bulboacă, *On some classes of differential subordinations II*, Studia Univ. Babeş-Bolyai, Math., 31, 2(1986),13-21.

[14] T. Bulboacă, *Classes of first-order differential subordinations* ,Mathematica (Cluj), 29(52), 1(1987), 11-17.

[15] T. Bulboacă, *Classes of first-order differential superordinations*, Demonstratio Mathematica, 352(2002), 287-292.

[16] T. Bulboacă, *Differential subordinations and superordinations. Recent results.*, Casa Cărţii de Ştiinţa, Cluj-Napoca, 2005.

[17] N.E. Cho, Ji A. Kim, *On certain classes of meromorphically starlike functions*, Internat. J. Math. & Math. Sci., vol. 18, no. 3 (1995), 463-468.

[18] H.B. Coonce,S.S. Miller, *Distortion properties of p-fold symmetric alpha-starlike functions*, Proc. Amer.Math.Soc.,44,2(1974),336-340.

[19] L. De Branges, *A proof of the Bieberbach conjecture,* Acta Math., 154(1985), 137-152.

[20] A. Derneck, *Subclasses of univalent meromorphic functions*, The Journal of Math.

[21] E. Drăghici, *On certain analytic functions with positive real part*, Bull. Korean Math. Soc.34(1997),no.1,pg.29-34.

[22] P.J. Eenigenburg, S.S. Miller, P.T. Mocanu, M.O. Reade, *On a Briot-Bouquet differential subordination*, General Inequalities 3, I.S.N.M., vol. 64, Birkhäuser Verlag, Basel (1983), 339-348.

[23] G.M. Goluzin, *Asupra unei metode variaţionale în reprezentarea conformă II,*(l. rusă), Mat. Sb. 21(1947), 83-117.

[24] G.M. Goluzin, *Teoria geometrică afuncţiilor de o variabilă complexă,* (l. rusă), Ed. de Stat pt. lit. tehnică-teoretică, Moscova-leningrad, 1952.

[25] R.M. Goel, N.S. Sohi, *On a class of meromorphic functions*, Glas. Mat. Ser. III, 17(37)(1981), 19-28.

[26] I. Graham, G. Kohr, *Geometric Function Theory in One and Higher Dimensions*, Marcel Dekker Inc., New York, 2003.

[27] T.H. Gronwall, *Some remarks on conformal representation,* Ann. of Math. 2 (1914/1915), 72-76.

[28] T.H. Gronwall, *Sur la déformation dans la représentation conforme*, C. R. Acad. Sci. Paris 162 (1916), 249-252.

[29] H. Grunsky, *Zwei Bemerkungen zur konformen Abbildung*, Jahresber. Deutsch. Math. Ver., 43(1933), 140-142.

[30] H. Grunsky, *Koeffizientenbedingungen für schlicht abbildende meromorphe Funktionen,* Math. Z. 45(1939), 29-61.

[31] D.J. Hallenbeck, S. Ruscheweyh, *Subordination by convex functions*, Proc. Amer. Math. Soc.,52(1975), 191-195.

[32] P. Hamburg, P. Mocanu, N. Negoescu, *Analiză Matematică (Funcţii complexe)*, Editura Didactică şi Pedagogică, Bucureşti-1982.

[33] W. Kaplan, *Close to convex schlicht functions*, Michig. Math. J., 1, 2(1952), 169-185.

[34] P. Koebe, *Über die Uniformisierung beliebiger analitischen Kurven*, Nachr. Ges. Wiss. Götingen, Math.-phys. 1(1907), 191-210.

[35] Y. Komatu, *Distorsion theorems in relation to linear integral transforms*, Kluwer Academic Publishers, Dordrecht, Boston and London, 1996.

[36] Z. Lewandowski, *Sur l'identité de certaines classesde fonctions univalentes I*, Ann. Univ. Mariae Curie-Sklodowska, Sect. A, 12(1958), 131-146.

[37] Z. Lewandowski, *Sur l'identité de certaines classesde fonctions univalentes II*, Ann. Univ. Mariae Curie-Sklodowska, Sect. A, 14(1960), 19-46.

[38] E. Lindelöf, *Memoire sur certaines inegalités dans la théory des fonctions monogènes et sur quelques propriétés nouvelles de ces fonctions dans le voisinage d'un point singulier essentiel*, Acta Soc. Fenn., 35(1908), No. 7.

[39] K. Löwner, *Untersuchungen über die Verzerrung bei konformen Abbidungen des Einheitskreises $|z| < 1$, die durch Funktionen mit nichtverschwindender Ableitung geliefert werden*, S.B. Sächs. Akad. Wiss. Leipzig Berichte, 69(1917), 89-106.

[40] T.H. MacGregor, *The radius of convexity for starlike functions of order 1/2*, Proc. Amer. Math. Soc., 14(1963), 71-76.

[41] T.H. MacGregor, *Applications of extreme point theory to univalent functions,* Michigan Math. J. 19(1972), 361-376.

[42] A. Marx, *Untersuchungen über schlichte Abbidungen*, Math. Ann., 107(1932-1933), 40-67.

[43] S.S. Miller, *Distortion properties of alpha-starlike functions*, Proc. Amer. Math. Soc., 38, 2(1973), 311-318.

[44] S.S. Miller, P.T. Mocanu, *Second order differential inequalities in the complex plane*, J. Math. Anal. Appl., 65(1978), 298-305.

[45] S.S. Miller, P.T. Mocanu, *Differential subordinations and univalent functions*, Michig. Math. J., 28(1981), 157-171.

[46] S.S. Miller, P.T. Mocanu, *Differential subordinations and inequalities in the complex plane*, J. Diff. Eqn., 67(1987), 199-211.

[47] S.S. Miller, P.T. Mocanu, *The theory and applicatins of second-order differential subordinations*, Studia Univ. Babeş-Bolyai, math.,34, 4(1989), 3-33.

[48] S.S. Miller, P.T. Mocanu, *Integral operators on certain classes of analytic functions*, Univalent Functions, Fractional Calculus and their Applications, Halstead Press, J. Wiley & Sons, New York (1989), 153-166.

[49] S.S. Miller, P.T. Mocanu, *Classes of univalent integral operators*, J. Math. Anal. Appl., 157, 1(1991), 147-165.

[50] S.S. Miller, P.T. Mocanu, *A class of nonlinear averaging integral operator*, J. Math. Anal. Appl., 197(1996), 318-323.

[51] S.S. Miller, P.T. Mocanu, *Briot-Bouquet differential equations and differential subordinations*, Complex Variables, 33(1997), 217-237.

[52] S.S. Miller, P.T. Mocanu, *Differential subordinations. Theory and applications*, Marcel Dekker Inc. New York, Basel, 2000.

[53] S.S. Miller, P.T. Mocanu, *Subordinants of differential superordinations*, Complex Variables, 48(10), (2003), 815-826.

[54] S.S. Miller, P.T. Mocanu, *Briot-Boquet differential superordinations and sandwich theorems*, J. Math. Anal. Appl., 329(2007), 327-335.

[55] S.S. Miller, P.T. Mocanu, M.O. Reade, *On the radius of alpha-convexity*, Studia Univ. Babeş-Bolyai, Math., 22, 1(1977), 35-39.

[56] S.S. Miller, P.T. Mocanu, M.O. Reade, *A particular starlike integral operator*, Studia Univ. Babeş-Bolyai, Math., 22, 2(1977), 44-47.

[57] S.S. Miller, P.T. Mocanu, M.O. Reade, *Starlike integral operators*, Pacific J. Math., 79(1978), 157-168.

[58] S.S. Miller, P.T. Mocanu, M.O. Reade, *On some particular classes of starlike integral operators*, Babeş-Bolyai Univ., Fac. of Math., Res.Sem., Seminar of Geometric Function Theory, Preprint 4(1982), 159-165.

[59] P.T. Mocanu, *Une propriété de convexité généralisée dans la théorie de la représentation conforme*, Mathematica(Cluj), 11(34)(1969), 127-133.

[60] P.T. Mocanu, *Some integral operators and starlike functions*, Rev. Roum. Math. Pures Appl., 31,3(1986), 231-235.

[61] P.T. Mocanu, *Some simple criteria for starlikeness and convexity*, Libertas Mathematica, 13(1993), 27-40.

[62] P.T. Mocanu, *A simple convexity condition for meromorphic functions*, Bull. Math. Soc. Sc. Math. Roumanie,42(90),2(1999),115-119.

[63] P.T. Mocanu, *Some extensions of Aksentiev's univalence condition for meromorphic functions*, Complex Variables, 41(2001),29-38.

[64] P.T. Mocanu, T. Bulboacă, Gr. Şt. Sălăgean, *Teoria geometrică a funcţiilor univalente*, Casa Cărţii de Ştiinţă, Cluj-Napoca, 2006.

[65] P.T. Mocanu, D. Ripeanu, I. Şerb, *The order of starlikeness of certain integral operators*, Mathematica (Cluj), 23(46), 2(1981), 225-230.

[66] P.T. Mocanu, Gr.Şt. Sălăgean, *Integral operators and meromorphic starlike functions*, mathematica (Cluj), 32(55), 2(1990), 147-152.

[67] R. Nevanlinna, *Über die schlichten Abbildungen Einheitkreises*, Översikt av Finska Vet. Soc., Förh. (A), No. 7, 62(1920), 1-14.

[68] R. Nevanlinna, *Über die konforme Abbildung Sterngebieten*, Översikt av Finska Vet. Soc., Förh. (A), No. 6, 63(1921).

[69] M. Nunokawa, *On properties of Non-Carathéodory functions*, Proc. Japan Acad. 68(1992), 152-153.

[70] Gh. Oros, Georgia Irina Oros, *An application of Briot-Bouquet differential subordinations*, Buletinul Academiei de Ştiinţe a Republicii Moldova. Matematica, 50(2006), 101-104.

[71] S. Owa, J. Kang, *A property of certain analytic functions*, Bull. Korean Math.Soc.32(1995),201-204.

[72] S. Ozaki, *On the theory of multivalent functions*, Sci. Rep. Tokyo Bunrika Daigaku, A, 2, 40(1935), 167-188.

[73] Ch. Pommerenke, *Univalent Functions*, Vanderhoeck & Ruprecht, Göttingen, 1975.

[74] B.N. Rahmanov, *On the theory of univalent functions*, Dokl. Akad. Nauk. SSSR (N.S.), 78(1951), 209-211 (limba rusă).

[75] M.O. Reade, *Sur une classe de fonctions univalentes*, C. R. Acad. Sci. Paris, 239(1954), 1758-1759.

[76] M.O. Reade, *On close-to-convex functions*, Michig. Math. J., 3(1955), 59-62.

[77] T.R. Reddy, O.P. Juneja, *Integral operators on a class of meromorphic functions*, C. R. Acad. Bulgare Sci., 40(1987), 21-23.

[78] S. Ruscheweyh, V. Singh, *On a Briot-Bouquet equation related to univalent functions*, Rev. Roum. Math. Pures Appl.,24(1979), 285-290.

[79] P. Sahoo, S. Singh, *Starlikeness conditions for an integral operator*, J. Inequal. Pure and Appl. Math., **10**(3)(2009), Art.77, 6pp.

[80] K. Sakaguchi, *A note on p-valent functions*, J. Math. Soc. Japan, 14(1962), 312-321.

[81] K. Sakaguchi, *A variational method for functions with positive real part,* J. Math. Soc. Japan 16(1964), 287-297.

[82] Gr. Şt. Sălăgean, *Subclasses of univalent functions*, Lectures Notes in Math., 1013, 362-372, Springer-Verlag, Heideberg, 1983.

[83] Gr. Şt. Sălăgean, *Meromorphic starlike univalent functions*, Babeş-Bolyai Univ., Fac. Math. and Phys. Res. Sem., Itinerant Seminar on Functional Equations, Approximation and Convexity, Preprint 7(1986), 261-266.

[84] Gr. Şt. Sălăgean, *Integral operators and meromorphic functions*, Rev. Roum. Math. Pures Appl., 33, 1-2(1988), 135-140.

[85] M. Schiffer, *Variation of the Green function and theory of p-valued functions*, Amer. J. Math. 65(1943), 341-360.

[86] C. Selvaraj, K.R. Karthikeyan, *Some inclusion relationships for certain subclasses of meromorphic functions associated with a family of integral operators*, Acta Math. Univ. Comenianae, Vol. LXXVIII, 2(2009), 245-254.

[87] S. Shams, S.R. Kulkarni, J.M. Jahangiri, *Subordination properties of p-valent functions defined by integral operators*, Int. J. Math. Math. Sci., Article ID 94572 (2006), 1-3.

[88] E. Strohhäcker, *Beiträge zur Theorie der schlichten Functionen*, Math. Z., 37(1933), 356-380.

[89] A. Totoi, *On some subclasses of starlike and convex functions*, General Mathematics, Vol. 17, No. 1 (2009), 107-114.

[90] A. Totoi, *On starlikeness of a class of integral operators for meromorphic starlike functions*, Studia Univ."Babeş Bolyai", Mathematica, Vol. LV, No. 3(2010), 229-240.

[91] A. Totoi, *On integral operators of meromorphic functions*, General Mathematics, Vol. 18, No. 3(2010), 91-108.

[92] A. Totoi, *On Inverse-Convex Meromorphic Functions*, Studia Univ."Babeş Bolyai", Mathematica, Vol. LV, No. 4(2010), 167-174.

[93] A. Totoi, *On some classes of meromorphic functions defined by a multiplier transformation*, Acta Universitatis Apulensis, No. 25(2011), 41-53.

[94] A. Totoi, *On some classes of meromorphic functions defined by subordination and superordination*, Opuscula Mathematica (to apear).

[95] A. Totoi, *Integral operators applied on classes of meromorphic functions defined by subordination and superordination*, Proceedings of the 23rd Operator Theory Conference, Timişoara, 2010 (to appear).

[96] A. Totoi, *Integral operators on some classes of meromorphic close-to-convex multivalent functions*(preprint).

[97] B.A. Uralegaddi, C. Somanatha, *Certain differential operators for meromorphic functions*, Houston J. Math., 17 (1991), 279-284.