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Contributions to the study of nonlinear evolution equations

Ph.D. Thesis Summary

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Contents

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Introduction

This work is concerned with evolution equations and is structured in three parts as follows.

The first part (**Part I**) of this thesis deals with abstract evolution problems in generalized Banach (or metric) spaces endowed with vector-valued norms (or metrics). This is a nontrivial extension of usual normed spaces. The advantage of vector-valued norms in the study of semilinear (stationary) operator systems is explained in Precup [58], and we apply the ideas in [58] to semilinear evolution systems. Historically, the idea of using vector-valued norms goes back to Perov [56], and it provides a setting in which one can work with more general growth conditions than those expressed in terms of scalar norms.

The first problem that we study (Chapter 2) are systems of abstract semilinear evolution equations

$$\begin{cases} \dfrac{du_1}{dt}(t) + A_1 u_1(t) = F_1(t, u_1(t), u_2(t)) \\ \dfrac{du_2}{dt}(t) + A_2 u_2(t) = F_2(t, u_1(t), u_2(t)) \\ u_1(0) = u_1^0, \quad u_2(0) = u_2^0. \end{cases}$$

In Section 2.1 we establish the well-posedness of the evolution system under Lipschitz-like growth conditions by means of Perov's fixed point theorem and an abstract Gronwall lemma of I. A. Rus [66].

In the second section of the chapter we prove the existence of solutions under more general growth conditions which are compensated by supplementary compactness assumptions on the solutions of the linear part of the system. The proofs rely on the fixed point principles of Schauder and Leray-Schauder respectively.

The main original contributions are: **Theorem 14, 15, 16, 17** and **18**, some of which have already been published (see Precup and Viorel [60]).

In Chapter 3 we extend the ideas of Chapter 2 to a multivalued setting. Instead of abstract differential equations here we deal with a system of differential inclusions

$$\begin{cases} \dfrac{du_1}{dt}(t) + A_1 u_1(t) \in F_1(u_1(t), u_2(t)) \\ \dfrac{du_2}{dt}(t) + A_2 u_2(t) \in F_2(u_1(t), u_2(t)) \\ u_1(0) = u_1^0, \quad u_2(0) = u_2^0. \end{cases}$$

It proves that a similar analysis to the siglevalued case can be carried out.

When adapting the ideas of Chapter 2 to a multivalued setting the main observation is that the solution operator corresponding to the system of semilinear inclusions is a composition between a singlevalued operator (corresponding to the linear part of the system) and a multivalued operator (corresponding to the multivalued nonlinearity $F = (F_1, F_2)$). In this way it becomes clear how properties of F transfer to the solution operator.

The main results of this chapter are: a vector version of Nadler's fixed point theorem (**Theorem 20**) and the existence results **Theorem 21, 22** and **23**. All the mentioned personal results of Chapter 3 have been published in the paper R. Precup and A. Viorel [61].

The final chapter of Part I (Chapter 4), deals with semilinear evolution problems with nonlocal initial conditions

$$\begin{cases} \dfrac{du}{dt}(t) + Au(t) = F(t, u(t)), \quad 0 < t < 1 \\ u(0) = \int_0^1 u(t) d\alpha(t). \end{cases}$$

Evolution problems where the initial condition has been replaced by a nonlocal condition are motivated by applications in physics, since the nonlocal condition is more precise for physical measurement than the local condition. Pioneering works in this field are due to L. Byszewski, and his work [18] shows how semigroup methods apply to nonlocal evolution problems. Nonlocal conditions of Rieman-Stieltjes type appear in J. R. L. Webb's study [79] of nonlocal boundary value problems.

In their fixed point formulation, nonlocal evolution problems can be regarded as a systems of nonlinear equations, where one equation is of evolutionary type, while the second is of stationary type. Spaces endowed with vector-valued norms prove to be an appropriate setting for these nonstandard problems.

In Section 4.1 we prove a series of vector versions of Krasnoselskii's fixed point theorem for the sum of two operators. These original results are intended to be used in order to prove existence results for the nonlocal Cauchy problem, but they are also of theoretical interest as they extend the results of Avramescu and Vladimirescu [7] for example. Also, we note that the existence results **Theorem 28, 29** and **30** are just some of the possible applications of the abstract results in **Theorem 24, 25, 26** and **27**. All the mentioned results are part of [76].

Part II is centered around a nonlocal model in nonlinear elasticity

$$\begin{aligned} u_{tt} = \delta_1 (u_{xx} - p_x) - \delta_2 (u_{xx} - q_x) + \varepsilon u_{txx} + \sigma (u_x)_x \\ -\gamma_1^2 p_{xx} + p = u_x \\ -\gamma_2^2 q_{xx} + q = u_x \end{aligned}$$

proposed by X. Ren and L. Truskinovsky (Journal of Elasticity, 2000).

Our aim is to give a rigorous analysis of the Ren and Truskinovsky model, which up to our knowledge is new, and to clarify its relationship with standard models in nonlinear elasticity. This line of inquiry, as well as the model itself, have been suggested to the author by Prof. C. Rohde during a research stay at the University of Stuttgart, Institute of Applied Analysis and Numerical Simulation.

Obviously, not all of the five parameters $\delta_1, \delta_2, \gamma_1, \gamma_2$ and ε of the model are essential, and for simplicity we restrict ourselves to the study of the following two cases[1]

$$(\mathrm{A}_k) \qquad \begin{aligned} u_{tt} = k (u_x - p)_x + u_{xxt} + \sigma (u_x)_x \\ -\tfrac{1}{k} p_{xx} + p = u_x \end{aligned}$$

$$(\mathrm{B}_k) \qquad \begin{aligned} u_{tt} = (u_x - p)_x + u_{xxt} + \sigma (u_x)_x \\ -\tfrac{1}{k} p_{xx} + p = u_x. \end{aligned}$$

Energy arguments suggest that these two models are approximations of the srain-gradient model

$$(\mathrm{A}) \quad u_{tt} = -u_{xxxx} + u_{xxt} + \sigma (u_x)_x$$

and respectively of the nonlinear viscoelastic model

$$(\mathrm{B}) \quad u_{tt} = u_{txx} + \sigma (u_x)_x .$$

The key step in understanding the two $k \to \infty$ limit regimes is the observation that the auxiliary

[1] We have chosen $\delta_1 = k, \delta_2 = 0, \gamma_1^2 = \frac{1}{k}, \varepsilon = 1$ for (A_k) and $\delta_1 = 1, \delta_2 = 0, \gamma_1^2 = \frac{1}{k}, \varepsilon = 1$ for (B_k)

variable p is defined via a resolvent equation. This explains the approximation properties of Ren and Truskinovsky's model, which, from a functional analytical point of view, relies on the idea of replacing an unbounded (differential) operator by its bounded Yosida approximation.

In Chapters 6 and 7 we establish the global existence of classical solutions to (A_k) and (B_k) - **Theorem 35** and **Theorem 41**. Then we give rigorous proofs which confirm the conjectured behaviour

$$(\mathrm{A}_k) \to (\mathrm{A})$$

$$(\mathrm{B}_k) \to (\mathrm{B})$$

in the two $k \to \infty$ limits - **Theorem 38** and **Theorem 42**. These results are partially included in Engel, Rohde and Viorel [34].

The main interest in such approximations is related to the fact that they are computationally less expensive than their limit equations. Especially in real-time simulations or in large-scale, complex applications such properties become highly relevant.

In **Part III** we apply similar approximation ideas to those in **Part II** in a totally different model, namely the Allen-Cahn equation

$$u_t = \varepsilon u_{xx} - u^3 + u.$$

This is a Ginzburg-Landau type equation which describes the evolution of a non-conserved order parameter during dynamic phase transitions in binary alloys, and it has attracted much attention over the last decades due to its dynamic properties, a full account of which can be found in Chen [23].

In Chapter 9 we propose a "nonlocal" version of the Allen-Cahn equation

$$\begin{aligned} u_t &= \varepsilon p_{xx} - u^3 + u \\ -\varepsilon p_{xx} + p &= u \end{aligned}$$

where the regularizing diffusion term its replaced by its Yosida approximation. If the order of the Yosida approximation is chosen to be exactly the inverse diffusion coefficient, then one obtains a new model which shares many common properties with the standard Allen-Cahn model, but is much simpler form a numerical point of view since it is equivalent to

$$\begin{aligned} u_t &= -u^3 + p \\ -\varepsilon p_{xx} + p &= u \end{aligned}$$

which is a regular perturbation of

$$u_t = -u^3 + u.$$

The p-term in the evolution equation is actuaually a nonlocal interaction term as one can see by using the Green's function representation for the solution of the associated to the elliptic problem

$$p(t,x) = \int G(x,y)\, u(t,y)\, dy.$$

By this observation we can relate to the studies of Fife [36], Bates et. al. [11] or Cortazar et. al. [25] who all deal with models containing nonlocal diffusion terms.

In the first section of Chapter 9 we give a qualitative analysis of this new model, and show that it has important common properties with the Allen-Cahn model. More precisely we show the global existence of solutions (**Theorem 47**) and the fact that solutions are a priori bounded (**Theorem 48**) and that in the long time asymptotics $t \to \infty$ (**Theorem 49**) we have $\|u_t(t)\|_{L^2(0,1)} \to 0$. All these results are contained in A. Viorel [77].

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Chapter 1
Preliminaries

1.1 Semigroups and evolution equations

In this section we recall basic results from Semigroup Theory. The main references that we use are [22], [33], [37], [44], [54], [78] or [84].

1.2 Fixed point theorems

In this section we list some well known fixed point theorems that we will need throughout the text. Some standard references for this subject are [1], [29],[38], [53] and [68]. Besides the well-known fixed point principles of Banach and Schauder we will also use the following theorems.

Theorem 1 (Krasnoselskii) *Let X be a Banach space, C a nonempty closed bounded convex set and $N : C \to C$ such that:*

(i) $N = N_1 + N_2$ *with* $N_1 : C \to C$ *completely continuous and* $N_2 : C \to C$ *a contraction;*

(ii) $N_1(x) + N_2(y) \in C$ *for all* $x, y \in C$.

Then N has at least one fixed point in C.

Theorem 2 (Leray–Schauder) *Let $(X, ||\,.\,||)$ be a Banach space, $R > 0$ and $N : \overline{B}_R(0; X) \to X$ a completely continuous operator. If $||u|| < R$ for every solution u of the equation $u = \lambda N(u)$ and any $\lambda \in (0,1)$, then N has at least one fixed point.*

Theorem 3 (Granas' topologigal transversality theorem [38], [53]) *Let U be a nonempty bounded open set in a closed convex set K of a Banach space X and let $H : \overline{U} \times [0,1] \to K$ be compact. Assume*

(A) $H(x, \lambda) \neq x \quad \forall x \in \partial U \quad \lambda \in [0,1]$

(B) $H(\cdot, 0)$ *is essential in the set* $\mathbf{M}_C$ *of all compact maps from* $\overline{U}$ *into* K.

Then for each $\lambda \in [0,1]$ there exists a fixed point of $H(\cdot,\lambda)$ in U, moreover $H(\cdot,\lambda)$ is essential in $\mathbf{M}_C$ for every $\lambda \in [0,1]$.

Finally we recall two basic topological fixed point theorems for set-valued maps (see e.g. [29])

Theorem 4 (Bohnenblust-Karlin) *Let X be a Banach space, $D \subset X$ nonempty closed convex bounded and $N : D \to 2^X$ upper semicontinuous with $N(x)$ nonempty closed convex for all $x \in D$. If $N(D) \subset D$ and $N(D)$ is relatively compact, then N has at least one fixed point.*

Theorem 5 *Let X be a Banach space, $U \subset X$ open bounded and $N : \overline{U} \to 2^X$ upper semicontinuous with $N(x)$ nonempty closed convex for all $x \in \overline{U}$. If $N(\overline{U})$ is relatively compact and $x_0 + \lambda(x - x_0) \notin N(x)$ on ∂U for all $\lambda > 1$, then N has at least one fixed point.*

1.3 Convergent to zero matrices

This section introduces concepts that are intensively used in Part I. The key notions of a vector-valued metric (or normed) space, as well as convergent to zero matrices are presented here. banach's contraction mapping principle was generalized to spaces endowed with vector-valued metrics by Perov [56].

Definition 6 *Let X be a nonempty set. By a* vector-valued metric *on X we mean a mapping $d : X \times X \to \mathbb{R}^n_+$ that satisfies the axioms*

(i) $d(u,v) \geq 0$ for all $u, v \in X$ and if $d(u,v) = 0$ then $u = v$;

(ii) $d(u,v) = d(v,u)$ for all $u, v \in X$;

(iii) $d(u,v) \leq d(u,w) + d(w,v)$ for all $u, v, w \in X$

with respect to the natural order relation of $\mathbb{R}^n$. More precisely if $x, y \in \mathbb{R}^n$, $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_n)$, by $x \leq y$ we mean $x_i \leq y_i$ for $i = 1, 2, ..., n$.

For generalized metric spaces convergence is defined component wise and the notions of Cauchy sequence, completeness, open or closed set are similar to those for usual metric spaces.

The following notion plays a similar role to that of a contraction constant in usual metric spaces.

Definition 7 *Let $M \in M_{n \times n}(\mathbb{R}^n)$ be a square matrix with nonnegative elements is said to be* convergent to zero *if*

$$M^k \to 0 \qquad as\, k \to \infty.$$

Convergent to zero matrices can be characterized as follows (for details see [51], [57], [63]).

Lemma 8 *Let M be a square matrix with nonnegative elements. The following are equivalent*

(a) M converges to zero;

(b) $I - M$ is non-singular and $(I - M)^{-1} = I + M + M^2 + \ldots$

(c) the eigenvalues of M are located inside the unit disc of the complex plane;

(d) $I - M$ is nonsingular and $(I - M)^{-1}$ has nonnegative elements.

Definition 9 *An operator $N : X \to X$ is said to be* contractive *(with respect to the vector-valued metric d on X) if there exists a convergent to zero matrix M such that*

$$d(N(u), N(v)) \leq M d(u,v) \qquad \text{for all} \qquad u, v \in X.$$

The following result is due to Perov

Theorem 10 (Perov) *Let (X, d) be a complete generalized metric space and $N : X \to X$ a contractive operator with Lipschitz matrix M. Then N has a unique fixed point u^* and for each $u \in X$ we have*

$$d(N^k(u), u^*) \leq M^k(I - M)^{-1}d(u, N(u)) \qquad \text{for all} \qquad k \in \mathbb{N}.$$

A proof of Perov's fixed point theorem can be found for example in [57].

Corollary 11 *Under the same assumptions as in Theorem 10, $I - N$ is bijective and $(I - N)^{-1}$ is continuous.*

In the same manner as for vector-valued metrics one can intorduce vector-valued norms.

Definition 12 *Let X be a linear space. A mapping $\|\cdot\| : X \to \mathbb{R}^n$ is called a* vector-valued norm *if*

(i) $\|x\| \geq 0$ *for all* $x \in X$, *and if* $\|x\| = 0$ *then* $x = 0$;

(ii) $\|\lambda x\| = |\lambda| \, \|x\|$ *for all* $x \in X$ *and* $\lambda \in \mathbb{R}$;

(iii) $\|x + y\| \leq \|x\| + \|y\|$ *for all* $x, y \in X$.

Completeness of a linear space endowed with a vector-valued norm can be defined similarly to the case of standard normed spaces. And then, a similar notion to that of a Banach space can be defined.

Definition 13 *A linear space endowed with a vector-valued norm, which is complete with respect to the vector-valued norm is called* generalized Banach space.

I. SYSTEMS OF EVOLUTION EQUATIONS

Chapter 2
Systems of semilinear evolution equations

In his work [58] R. Precup has developed a technique for the investigation of systems of nonlinear operator equations which is based on vector-valued norms and convergent to zero matrices together with fundamental principles of nonlinear functional analysis. It is shown in [58] that the use of vector-valued metrics is more appropriate when treating systems of equations than the more familiar product space methods. In this chapter we are concerned with the existence (and the uniqueness) of solutions for the Cauchy problem associated to a semilinear system of abstract evolution equations:

$$\begin{cases} \dfrac{du_1}{dt}(t) + A_1u_1(t) = F_1(t, u_1(t), u_2(t)) \\ \dfrac{du_2}{dt}(t) + A_2u_2(t) = F_2(t, u_1(t), u_2(t)) \\ u_1(0) = u_1^0, \quad u_2(0) = u_2^0. \end{cases} \tag{2.1}$$

Here the linear operator $A_i : D(A_i) \subseteq X_i \to X_i$ is densely defined on the real Banach space X_i and $-A_i$ generates the strongly continuous semigroup of contractions $\{S_i(t), t \geq 0\}$, for $i = 1, 2$.

We shall look for *global mild solutions* on the interval $[0,T]$, i.e., $(u_1,u_2)\in C([0,T],X_1)\times C([0,T],X_1)$ satisfying

$$u_i(t)=S_i(t)u_i^0+\int_0^t S_i(t-\tau)F_i(\tau,u_1(\tau),u_2(\tau))d\tau \tag{2.2}$$

for all $t\in[0,T]$, $i=1,2$. The nonlinear operator defined by the right hand side of (2.2) will be denoted by $N_i(u)$, where $u=(u_1,u_2)\in C([0,T],X_1)\times C([0,T],X_2)$.

2.1 Well-posedness via Perov's fixed point theorem

Our first result is an existence and uniqueness theorem for the case of nonlinearities which satisfy a Lipschitz condition. Under the same basic assumptions on X_i and A_i, we have:

Theorem 14 (existence and uniqueness) *Suppose that $F_i:[0,T]\times X_1\times X_2\to X_i$ satisfies the Lipschitz condition*

$$||F_i(t,u)-F_i(t,v)||_{X_i}\le a_{i1}(t)||u_1-v_1||_{X_1}+a_{i2}(t)||u_2-v_2||_{X_2} \tag{2.3}$$

for all $u=(u_1,u_2)$, $v=(v_1,v_2)\in X_1\times X_2$, $t\in[0,T]$ and $i=1,2$, where $a_{ij}\in L^p([0,T],\mathbb{R}_+)$ for $i,j=1,2$. Then for any $(u_1^0,u_2^0)\in X_1\times X_2$ the Cauchy problem (2.1) *has a unique global mild solution.*

We can also show that the solution depends continuously on the initial data, the problem being thus wellposed.

Theorem 15 (data dependence) *Under the assumptions of Theorem 14 and if u and v are two mild solutions of (2.1) with different initial data u^0 and v^0 respectively, the following estimate holds for any $t\in[0,T]$*

$$\begin{pmatrix} ||u_1(t)-v_1(t)||_{X_1} \\ ||u_2(t)-v_2(t)||_{X_2}\end{pmatrix}\le U(t)\begin{pmatrix} ||u_1^0-v_1^0||_{X_1} \\ ||u_2^0-v_2^0||_{X_2}\end{pmatrix}.$$

Here $U(t)$ is a fundamental matrix of the system of ordinary differential equations

$$\begin{cases} \dfrac{dx_1}{dt}(t)=a_{11}(t)x_1(t)+a_{12}(t)x_2(t) \\ \dfrac{dx_2}{dt}(t)=a_{21}(t)x_1(t)+a_{22}(t)x_2(t)\end{cases} \tag{2.4}$$

where the coefficients $a_{ij}(t)$ are the same functions as in (2.3).

To prove this result we need the following vector-version of the Gronwall lemma. This is a special case of the abstract Gronwall introduced by I. A. Rus [66]

Theorem 16 *Let $a_{ij}\in L^p([0,T],\mathbb{R}_+)$, $p\ge 1$ and $b_i>0$ for $i,j=1,2$. If $x_i\in C[0,T]$, $i=1,2$, are two continuous functions such that*

$$\begin{pmatrix} x_1(t) \\ x_2(t)\end{pmatrix}\le\begin{pmatrix} b_1+\displaystyle\int_0^t(a_{11}(\tau)x_1(\tau)+a_{12}(\tau)x_2(\tau))\,d\tau \\ b_2+\displaystyle\int_0^t(a_{21}(\tau)x_1(\tau)+a_{22}(\tau)x_2(\tau))\,d\tau\end{pmatrix}, \tag{2.5}$$

then for any $t\in[0,T]$ we have $(x_1(t),x_2(t))^T\le U(t)(b_1,b_2)^T$ with $U(t)$ a fundamental matrix of (2.4).

2.2 Other existence results

Assuming that the operator N is completely continuous we can weaken condition (2.3). But now Schauder's fixed point theorem that we apply will only guarantee the existence not also the uniqueness of the solution. A sufficient condition for N to be completely continuous is that the semigroups $S_i(\cdot)$, $i = 1, 2$, are both compact. A typical example are analytical semigroups with compact resolvent such as the semigroup generated by the Laplacian with Dirichlet boundary conditions.

Theorem 17 *If the operator N is completely continuous and F_i satisfies*

$$||F_i(t, u)||_{X_i} \leq a_{i1}(t)||u_1||_{X_1} + a_{i2}(t)||u_2||_{X_2} + b_i(t) \tag{2.6}$$

for all $u = (u_1, u_2) \in X_1 \times X_2$, where $a_{ij} \in L^p([0, T], \mathbb{R}_+)$ and $b_i \in L^1([0, T], \mathbb{R}_+)$, for $i, j = 1, 2$, then problem (2.1) *has at least one global mild solution.*

Now in the case of Hilbert spaces, and if all mild solutions are classical solutions , i.e., $u \in C([0, T], D(A_i)) \cap C^1([0, T], X_i)$, we have the following result based on the Leray-Schauder fixed point theorem.

Theorem 18 *Let $(X_i, \langle ., . \rangle_{X_i})$, $i = 1, 2$ be real Hilbert spaces. If all mild solutions of the equations $u_i = \lambda N_i(u)$, $\lambda \in (0, 1)$, are classical solutions, the nonlinear operator N is completely continuous and F_i satisfies*

$$\langle F_i(t, u), u_i \rangle_{X_i} \leq a_{i1}(t)||u_1||^2_{X_1} + a_{i2}(t)||u_2||^2_{X_2} + b_i(t) \tag{2.7}$$

for all $u \in X_1 \times X_2$, where $a_{ij} \in L^p([0, T], \mathbb{R}_+)$ and $b_i \in L^1([0, T], \mathbb{R}_+)$ for $i, j = 1, 2$, then problem (2.1) *has at least one solution.*

Chapter 3

Systems of semilinear differential inclusions

The aim of this chapter is to extend the results of the previous chapter to a multi-valued setting. More exactly we are concerned with the Cauchy problem associated to the semilinear system of abstract evolution inclusions:

$$\begin{cases} \dfrac{du_1}{dt}(t) + A_1 u_1(t) \in F_1(u_1(t), u_2(t)) \\ \dfrac{du_2}{dt}(t) + A_2 u_2(t) \in F_2(u_1(t), u_2(t)) \\ u_1(0) = u_1^0, \quad u_2(0) = u_2^0. \end{cases} \tag{3.1}$$

Under the same assumptions as in Chapter 2 we shall look for *global mild solutions* to (3.1) on the interval $[0, T]$, i.e., $u = (u_1, u_2) \in C([0, T], X_1) \times C([0, T], X_2)$ such that

$$u_i(t) = S_i(t)u_i^0 + \int_0^t S_i(t - \tau)w_i(\tau)d\tau \quad t \in [0, T], \tag{3.2}$$

where $w_i \in L^1([0, T], X_i)$ is a selection for the multivalued function $t \mapsto F_i(u(t))$, i.e.,

$$w_i(t) \in F_i(u(t)) \quad \text{a.e. } t \in [0, T] \quad i = 1, 2. \tag{3.3}$$

3.1 Some preliminary remarks

Let (X, d) be a metric space. For two nonempty sets $A, B \subset X$ and $x \in X$ we use the following notations:

$$\begin{aligned} d(x, A) &= \inf \{d(x, a) : a \in A\}; \\ H(A, B) &= \max \left\{ \sup_{a \in A} d(a, B) : \sup_{b \in B} d(b, A) \right\}; \\ \delta(A, B) &= \sup \{d(a, b) : a \in A, b \in B\}. \end{aligned}$$

We recall that H is a metric (the Hausdorff-Pompeiu metric) on the set of all nonempty closed bounded subsets of (X, d). We will use the following property of the Hausdorff-Pompeiu metric.

Remark 19 *Let (X, d) be a metric space, $A, B \subset X$ nonempty closed bounded sets and $q > 1$. Then for each $a \in A$ there exists $b \in B$ such that $d(a, b) \leq qH(A, B)$.*

Our first result is a vector version of Nadler's fixed point theorem. For the original result of Nadler we refer for example to [38, page 28].

Theorem 20 *Let $(X_1, d_1), (X_2, d_2)$ be two complete metric spaces and $N : X_1 \times X_2 \to 2^{X_1 \times X_2}$ a multivalued operator with $N(x)$ nonempty closed bounded for each $x \in X_1 \times X_2$. Assume that there exists matrix M which is convergent to zero, such that*

$$\begin{pmatrix} H_1(N_1(u), N_1(v)) \\ H_2(N_2(u), N_2(v)) \end{pmatrix} \leq M \begin{pmatrix} d_1(u_1, v_1) \\ d_2(u_2, v_2) \end{pmatrix} \tag{3.4}$$

for all $u = (u_1, u_2), v = (v_1, v_2) \in X_1 \times X_2$, where $N_1 : X_1 \times X_2 \to 2^{X_1}$ and $N_2 : X_1 \times X_2 \to 2^{X_2}$ are the two components of N and H_1, H_2 stand for the Hausdorff-Pompeiu metrics associated to d_1 and d_2, respectively. Then N has a fixed point.

3.2 Existence results

We follow a fixed point approach based on the fact that any mild solution of (3.1) is a fixed point of the multivalued operator $N : C([0, T], X_1) \times C([0, T], X_2) \to 2^{C([0,T], X_1 \times X_2)}$, where

$$\begin{aligned} N_i(u) = \Big\{ S_i(t)u_i^0 + \int_0^t S_i(t-\tau)w_i(\tau)d\tau : \\ w_i \in L^1([0, T], X_i), w_i(t) \in F_i(u(t)) \text{ a.e. } t \in [0, T] \Big\}. \end{aligned} \tag{3.5}$$

The multivalued operator N can be written as a composition of a single-valued operator $\mathcal{N}$ with a multivalued operator W

$$N = \mathcal{N} \circ W,$$

where

$$\begin{aligned} &\mathcal{N} = (\mathcal{N}_1, \mathcal{N}_2) \\ &\mathcal{N}_i(f)(t) = S_i(t)u_i^0 + \textstyle\int_0^t S_i(t-\tau)f_i(\tau)d\tau \\ &W = (W_1, W_2) \\ &W_i(u) = \left\{ w_i \in L^1([0, T], X_i) : w_i(t) \in F_i(u(t)) \text{ a.e. } t \in [0, T] \right\}. \end{aligned}$$

In this way it becomes clear how the properties of F transfer to N. For example if F_i has bounded values and is upper semicontinuous for $i = 1, 2$ then the operator N is also upper semicontinuous.

Our first existence result is established by means of the vector version of Nadler's theorem proved in the previous section.

Theorem 21 *Let $F_i : X_1 \times X_2 \to 2^{X_i}$ and assume that $F_i(x)$ is nonempty closed bounded for each $x \in X_1 \times X_2$. In addition assume that there are constants $a_{ij} \geq 0$ for $i, j = 1, 2$ such that*

$$\delta_{X_i}\left(F_i(u), F_i(v)\right) \leq a_{i1}\|u_1 - v_1\|_{X_1} + a_{i2}\|u_2 - v_2\|_{X_2} \tag{3.6}$$

for all $u = (u_1, u_2)$, $v = (v_1, v_2) \in X_1 \times X_2$ and $i = 1, 2$. Then problem (3.1) has a mild solution.

The next existence result is an application of Theorem 4 and uses growth conditions on F_i which are more general than the Lipschitz condition (3.6).

Theorem 22 *Let $F_i : X_1 \times X_2 \to 2^{X_i}$ be upper semicontinuous with $F_i(x)$ nonempty closed bounded for each $x \in X_1 \times X_2$. Assume that there exist constants $a_{ij} \geq 0$ and $b_i \geq 0$ for $i, j = 1, 2$, such that*

$$\|w\|_{X_i} \leq a_{i1}\|u_1\|_{X_1} + a_{i2}\|u_2\|_{X_2} + b_i \tag{3.7}$$

for all $u = (u_1, u_2) \in X_1 \times X_2$ and $w \in F_i(u)$ $(i = 1, 2)$. If in addition operator N is completely continuous, then problem (3.1) has at least one mild solution.

In the case of Hilbert spaces the existence of solutions can be also derived based on Theorem 5.

Theorem 23 *Let $(X_i, \langle .,. \rangle_{X_i})$, $i = 1, 2$ be real Hilbert spaces, assume that all mild solutions of the system*

$$\begin{cases} \dfrac{du_1}{dt}(t) + A_1 u_1(t) \in \lambda F_1(u(t)) \\ \dfrac{du_2}{dt}(t) + A_2 u_2(t) \in \lambda F_2(u(t)) \\ u_1(0) = \lambda u_1^0, \quad u_2(0) = \lambda u_2^0 \end{cases} \tag{3.8}$$

for $\lambda \in (0, 1)$ are classical solutions and that the nonlinear operator N is completely continuous. In addition assume that there exist constants $a_{ij} \geq 0$ and $b_i \geq 0$ for $i, j = 1, 2$ such that

$$\sup_{w_i \in F_i(u)} \langle w_i, u_i \rangle_{X_i} \leq a_{i1}\|u_1\|_{X_1}^2 + a_{i2}\|u_2\|_{X_2}^2 + b_i \tag{3.9}$$

for all $u \in X_1 \times X_2$, $i = 1, 2$. Then problem (3.1) *has at least one solution.*

Chapter 4

Semilinear evolution equations with nonlocal initial conditions

This chapter deals with semilinear evolution equations with nonlocal initial conditions given in the form of a Riemann-Stieltjes integral. Such conditions have already been used in the case of two point boundary value problems on the real line by Webb and Infante [79] and [80].

More exactly, we consider the following problem

$$\begin{cases} \dfrac{du}{dt}(t) + Au(t) = F(t, u(t)), & 0 < t < 1 \\ u(0) = \int_0^1 u(t)d\alpha(t) \end{cases} \tag{4.1}$$

where $-A : D(A) \subseteq X \to X$ is the infinitesimal generator of the C_0-semigroup of contractions $\{S(t), t \geq 0\}$ defined on the real Banach space X, F is a given continuous nonlinear operator and $\alpha : [0,1] \to \mathbb{R}$ is a monotonic real function of bounded variation. Notice that the condition $u(0) = \sum_{k=1}^{m} a_k u(t_k)$ used for example in [14] or [18] is a special case of the nonlocal condition used in (4.1).

The aim of this chapter is to provide sufficient conditions on the nonlinearity F and on the function α in order to guarantee the existence (and uniqueness) of a solution $(u, u_0) \in C([0,1], X) \times X$ to

$$\begin{cases} u(t) = S(t)u_0 + \int_0^t S(t-\tau)F(\tau, u(\tau))d\tau \\ u_0 = \int_0^1 u(t)d\alpha(t). \end{cases} \tag{4.2}$$

Notice that, if (u, u_0) solves (4.2), then $u(0) = u_0$ and u is a mild solution of (4.1). The main difference to earlier works (for instance [14], [18], [19] or [30]) lies in the new method of treating (4.2) as a fixed point problem on the product space $C([0,1], X) \times X$ and using vector versions of classical fixed point theorems. A crucial role in this vector approach will be played by convergent to zero matrices and vector-valued metrics (or norms).

4.1 Vector versions of Krasnoselskii's fixed point theorem for the sum of two operators

The aim of this section is to present some fixed point theorems in generalized Banach spaces endowed with a vector-valued norm. We begin a with a vector version of Krasnoselskii's fixed point theorem for the sum of two operators. The classical result and other results of the same type can be found in [1], [7], [17] or [52].

Theorem 24 *Let $(X, \|\cdot\|)$ be a generalized Banach space, C a nonempty closed bounded convex set and $N : C \to C$ such that:*

(A) $N = N_1 + N_2$ with $N_1 : C \to C$ completely continuous and $N_2 : C \to C$ contractive, i.e. there exists a convergent to zero matrix M such that $\|N_2(u) - N_2(v)\| \leq M \|u - v\|$ for all $u, v \in C$;

(B) $N_1(x) + N_2(y) \in C$ for all $x, y \in C$.

Then N has at least one fixed point in C.

Now as a direct consequence of this theorem we can give a fixed point result for an operator defined on a product space, where one component satisfies a Lipschitz condition while the other component is completely continuous.

Theorem 25 *Let $(X, |\cdot|_X)$ and $(Y, |\cdot|_Y)$ be real Banach spaces, and C, D two nonempty closed bounded convex subsets of X, Y respectively, and let $N : C \times D \to C \times D$,*

$$N = \begin{pmatrix} P \\ Q \end{pmatrix}.$$

Assume that $P : C \times D \to C$ is completely continuous and for $Q : C \times D \to D$ there are $L_1 \geq 0$ and $L_2 < 1$ such that

$$|Q(x_1, y_1) - Q(x_2, y_2)|_Y \leq L_1 |x_1 - x_2|_X + L_2 |y_1 - y_2|_Y \tag{4.3}$$

for all $(x_1, y_1), (x_2, y_2) \in C \times D$. Then, N has at least one fixed point.

Using the topological transversality theorem instead of Schauder's fixed point principle we can replace the domain invariance condition with a Leray-Schauder condition to prove the following result.

Theorem 26 *Let $(X,\|\cdot\|)$ be a generalized Banach space,*

$$U=\left\{x\in X:\|x\|<u,u\in\mathbb{R}_{+}^{n}\right\}\subset X$$

and let $N:\overline{U}\to X$ given by $N=N_1+N_2$ where N_1 is completely continuous and N_2 is contractive. If

$$x\neq\lambda N(x)\quad\text{for all}\quad x\in\partial U,\lambda\in(0,1),\tag{4.4}$$

then there exists a fixed point for N in U.

The previous result can be applied to operators with two components in the following way.

Theorem 27 *Let X,Y be two Banach spaces, $\overline{B}_1:=\{x\in X:|x|_X\leq R_1\}$, $\overline{B}_2:=\{y\in Y:|y|_Y\leq R_2\}$ two closed balls in X, and Y respectively, and the operator $N:\overline{B}_1\times\overline{B}_2\to X\times Y$*

$$N=\begin{pmatrix}P\\Q\end{pmatrix}.$$

Assume that $P:\overline{B}_1\times\overline{B}_2\to X$ is completely continuous and for $Q:\overline{B}_1\times\overline{B}_2\to Y$ there are $L_1\geq 0$ and $L_2\in(0,1)$ such that for all $(x_1,y_1),(x_2,y_2)\in\overline{B}_1\times\overline{B}_2$

$$|Q(x_1,y_1)-Q(x_2,y_2)|_X\leq L_1|x_1-x_2|_X+L_2|y_1-y_2|_Y.\tag{4.5}$$

If for all solutions of

$$\begin{pmatrix}x\\y\end{pmatrix}=\lambda N\begin{pmatrix}x\\y\end{pmatrix},\ \ \lambda\in(0,1),\quad\text{we have}\quad\begin{array}{l}|x|_X<R_1\\|y|_Y<R_2\end{array}\tag{4.6}$$

then N has at least one fixed point.

4.2 Semilinear evolution equations with nonlocal initial conditions

In this section we apply the different fixed point results of the previous section to problem (4.2). First we give an existence and uniqueness theorem for the mild solution of (4.1) based on Perov's theorem.

In what follows X is a Banach space with norm $|\cdot|_X$.

Theorem 28 *Suppose that nonlinearity F satisfies the Lipschitz condition*

$$|F(t,u)-F(t,v)|_X\leq a(t)|u-v|_X\tag{4.7}$$

for all $u,v\in X$, $t\in[0,1]$, where $a\in L^p([0,1],\mathbb{R}_+)$. If there is $k\geq 0$ such that

$$\frac{\|a\|_{L^p}}{(qk)^{1/q}}<(1-e^kV_\alpha)\tag{4.8}$$

with $1/p+1/q=1$ and $V_\alpha=|\alpha(1)-\alpha(0)|$, then the problem (4.1) has a unique mild solution..

The next result is based on Theorem 25.

Theorem 29 *Assume that the semigroup $\{S(t),t\geq 0\}$ is compact, and that F satisfies the growth condition*

$$|F(t,u)|_X\leq a(t)|u|_X+b(t)\tag{4.9}$$

for all $u \in X$, $t \in [0,1]$, where $a \in L^p([0,1],\mathbb{R}_+)$, $b \in L^1([0,1],\mathbb{R}_+)$ and there is $k \geq 0$ such that (4.8) holds and

$$e^k V_\alpha < 1.$$

Then the problem (4.1) has at least one mild solution.

Finally in the case of Hilbert spaces, and if all mild solutions are classical solutions, i.e., they are in $C([0,T], D(A_i) \cap C^1([0,T], X_i)$ and satisfy (4.1), we have the following result based on Theorem :

Theorem 30 *Let $(X, \langle .,. \rangle_X)$, be a real Hilbert space. Assume that the semigroup $\{S(t), t \geq 0\}$ is compact, all mild solutions of*

$$\begin{cases} \dfrac{du}{dt}(t) + Au(t) = \lambda F(t, u(t)) \\ u(0) = \lambda u_0 \end{cases}, \qquad \lambda \in (0,1),$$

are classical solutions and that F satisfies the growth condition

$$\langle F(t,u), u \rangle_X \leq a(t)|u|_X^2 + b(t) \tag{4.10}$$

for all $u \in X$, where $a \in L^p([0,T],\mathbb{R}_+)$ and $b \in L^1([0,T],\mathbb{R}_+)$.

If

$$e^k V_\alpha < 1 \quad \text{and} \quad \frac{2\,\|a\|_{L^p}}{(2qk)^{1/q}} \leq (1 - e^{2k} V_\alpha^2), \tag{4.11}$$

for some $k > 0$, then the problem (4.1) has at least one solution.

Notice that the conditions in (4.11) can be interpreted in the following way: there is a tradeoff between the growth of the nonlinear term F and the total variation of α. We cannot control both simultanously, but if one of the two is small enough, then we can allow the other to be large.

II. DYNAMICAL PHASE TRANSITIONS IN SOLIDS

Chapter 5
Modeling multi-phase material dynamics

5.1 Ericksen's nonlinear elasticity model.

A question that rises in te study of active materials is:

How to describe a material in which two phases coexist, still using the framework of continuum mechanics?

The first step leading towards an answer of the above question was done by J. Ericksen in a seminal paper [32] published in 1975. His idea is simple: since at equilibrium each of the two phases is stable, they should each correspond to a minimum of the potential energy of the system. This means that the potential energy should have **two minima**, in contrast to linear elasticity where the potential energy is a quadratic function with only one minimum.

The model proposed by Ericksen is based on a so called double well potential. This new type of potential should replace the potential of quadratic type encountered in linear elasticity. Smoothness (at least C^2) and polynomial growth at $\pm\infty$ come as natural (and sometimes implicit) assumptions for W. Although, for clear physical reasons, the two minima of double well potential ought not to be equal, the mathematically simpler symmetric double well potential is most commonly used in literature.

The evolution equation for the system with non-quadratic potential energy (Ericksen's bar) is a nonlinear wave equation

$$u_{tt} = \sigma\left(u_x\right)_x \tag{5.1}$$

with

$$\sigma : \mathbb{R} \to \mathbb{R}, \qquad \sigma\left(w\right) = W'\left(w\right),$$

and it can be derived from Hamilton's variational principle for the corresponding action integral

$$S\left(u\right) = \int_0^t \int \left(\frac{1}{2}u_t^2 + W\left(u_x\right)\right) dtdx.$$

In the double-well case the stress-strain relation σ is nonmonotone.

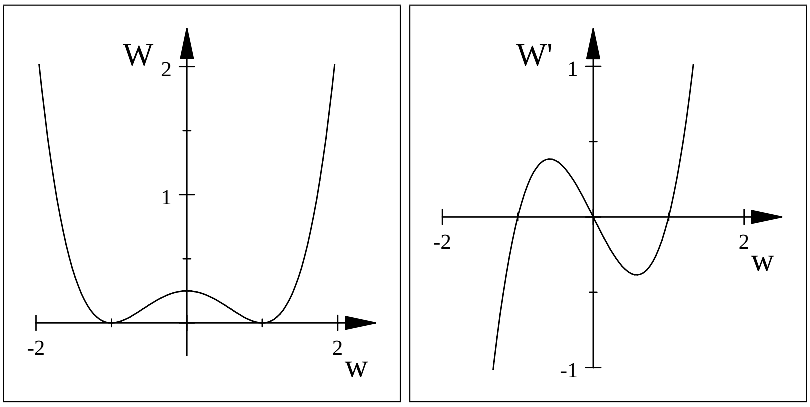

Figure 1. The standard double-well potential $W\left(w\right) = \frac{1}{4}\left(w^2-1\right)^2$ and its corresponding stress-strain relationship $\sigma\left(w\right) = w^3 - w$

Remark 31 *The equation (5.1) can be rewritten as a system of two evolution equations in two ways depending of the choice of state variables. We refer to the recent works [47] and [48] for a detailed discussion of alternative Hamiltonian formulations of physical systems. In our case, the two alternative formulations are either in terms of state variables displacement* u *and velocity* $v := u_t$

$$\begin{aligned} u_t &= v \\ v_t &= \sigma(u_x)_x, \end{aligned}$$

or in terms of strain $w := u_x$ *and velocity as state variables*

$$\begin{aligned} w_t &= v_x \\ v_t &= \sigma(w)_x. \end{aligned} \tag{5.2}$$

5.2 Extensions of Ericksen's model. Regularizing effects.

Extensive experimental observations done on materials which display dynamic phase boundaries showed phenomena that can not be explained by Erickson's model only. In the following we will discuss some extensions of the model related to regularizing effects.

A. Dissipative effects. The model of Ericksen is conservative, and it does not account for dissipative effects, which is unrealistic. This drawback is corrected by complementing the equation (5.1) with a diffusion term u_{txx}

$$u_{tt} = \varepsilon u_{txx} + \sigma\left(u_x\right)_x \tag{5.3}$$

A thorough study of the equation of nonlinear viscoelasticity (5.3) was done by Andrews and Ball [5], [6] in the 1980s. For a semigroup approach to linear viscoelasticity we refer to the book of Liu and Zheng [46] and the references therein.

When writing (5.3) as a system in terms of strain and velocity as state variables one obtains

$$\begin{aligned} w_t &= v_x \\ v_t &= \varepsilon v_{xx} + \sigma(w)_x, \end{aligned}$$

a viscous regularization of (5.2). The $\varepsilon \to 0$ vanishing viscosity limit plays a very important role in the theory of conservation laws since it is widely used to single-out certain weak solutions of (5.2) which satisfy supplementary, for example entropy, conditions [72], [40].

B. The finite scale of the microstructure. The classical nonlinear elasticity of Ericksen fails to predict the observed finite scale of the equilibrium domains. Indeed, any piecewise constant function taking values in the set of minimizers of the double-well potential energy W is an admissible equilibrium of the system described by (5.1).

These arbitrarily fine patterns are in contradiction with observations made on relevant materials which exhibit an intrinsic length scale related to domain dimensions, and which although considerably smaller compared to the length scale of the bulk material, is nevertheless finite. To deal with this issue so called capillarity, or strain gradient, models where introduced (see [41]). Here the dynamics of the system is described by

$$u_{tt} = -\delta u_{xxxx} + \varepsilon u_{txx} + \sigma\left(u_x\right)_x \tag{5.4}$$

and the total energy

$$E = \int \left(\frac{1}{2} u_t^2 + \frac{\delta}{2} u_{xx}^2 + W\left(u_x\right) \right) dx$$

contains a surface term proportional to $\left\|u_{xx}\right\|_{L^2}^2$ which penalizes sharp strain interfaces. Regularization terms of this kind were first studied by Slemrod [74] in connection with liquid-vapour phase transitions in van der Waals fluids.

In stress-velocity state variables the model reads

$$\begin{aligned} w_t &= v_x \\ v_t &= -\delta w_{xxx} + \varepsilon v_{xx} + \sigma(w)_x. \end{aligned}$$

The appropriate scaling between ε and δ is usually considered to be $\delta \sim \varepsilon^2$ as discussed in [45] or [65].

5.3 The Model of Ren and Truskinovsky.

In their work [62] Ren and Truskinovsky develop a new one-dimensional model which extends Ericksen's nonlinear elastic bar with non-convex energy. Their model accounts for the formation and growth of globally stable finite scale microstructures.

The main new ingredient of this model are nonlocal interaction terms, and the motivation for introducing them is twofold:

1. Although the observed patterns are mainly one-dimensional, a more realistic description of the actually three-dimensional material should also include possible higher-dimensional stabilizing effects. In connection with these modeling issues we refer to the work of Alberti et. al. [2] which motivates nonlocal interaction terms by arguments from statistical physics.
2. A common problem of strain gradient models like (5.4) is that they lead to over-regularized unrealistically diffuse interfaces, in contrast, observations show phase boundaries which are close to being atomically sharp.

The model allowing for sharp interfaces proposed by Ren and Truskinovsky has dynamics described by

$$u_{tt} = \delta_1 \left(u_{xx} - p_x\right) - \delta_2 \left(u_{xx} - q_x\right) + \varepsilon u_{txx} + \sigma \left(u_x\right)_x \tag{5.5}$$

coupled with

$$\begin{aligned} -\gamma_1^2 p_{xx} + p &= u_x \\ -\gamma_2^2 q_{xx} + q &= u_x. \end{aligned}$$

This is only apparently a local model since the additional two internal (artificial) variables p and q are used to introduce two nonlocal interaction terms. For example p has the following integral representation

$$p\left(x\right) = \int_0^1 G^{\gamma}\left(x,y\right) w\left(y\right) dy. \tag{5.6}$$

Here $G^{\gamma_1}\left(x,y\right)$ is the associated Green's function and its expression can be given explicitly for Neumann boundary conditions by

$$G^{\gamma}\left(x,y\right) = \frac{1}{\gamma\left(e^{1/\gamma} - e^{-1/\gamma}\right)}\left(\cosh\frac{x+y-1}{\gamma} + \cosh\frac{|x-y|-1}{\gamma}\right).$$

This type of nonlocal interaction was first introduced by Rogers and Truskinovsky [64].

Chapter 6
A model of Ren and Truskinovsky - the strain gradient limit regime

6.1 A low order visco-capillarity model

In the following we are interested in a special case of the Ren an Truskinovsky model (5.5) corresponding to the following choice of parameters[1]

$$\delta_1 = k, \quad \delta_2 = 0, \quad \gamma_1^2 = \frac{1}{k}, \quad \varepsilon = 1.$$

[1] One can notice that not all the parameters in (5.5) ought to be independent, and actually the interesting asymptotic regimes are obtained for an appropriate cupling.

We study well-posedness and the limit regime $k \to \infty$ for the following one-dimensional model

$$\begin{aligned} &u_{tt} = u_{xxt} + k\left(u_x - p\right)_x + \sigma\left(u_x\right)_x \\ &-\tfrac{1}{k}p_{xx} + p = u_x \end{aligned} \tag{P_k}$$

where

$$\sigma : \mathbb{R} \to \mathbb{R}, \qquad \sigma(w) = w^3 - w$$

with Dirichlet boundary conditions for u, Neumann boundary conditions for p and initial conditions.

This is a low-orde viso-capillarity model, and energy arguments suggest that the appropriate limit problem is the strain gradient model

$$u_{tt} = -u_{xxxx} + u_{xxt} + \sigma\left(u_x\right)_x \tag{P}$$

with boundary conditions and and initial conditions. Andrews and Ball [6] have studied the initial boundary value problem corresponding to (P) and we recall here their existence and regularity result.

Theorem 32 (Andrews and Ball, [6]) *For any $T > 0$, whenever*

$$u_0 \in H^2(0,1) \cap H_0^1(0,1) \quad \text{and} \quad v_0 \in L^2(0,1) \tag{6.1}$$

there is a unique global solution to the problem (P)

$$\begin{aligned} u &\in C\left([0,T], H^2(0,1) \cap H_0^1(0,1)\right) \\ u_t &\in C\left([0,T], L^2(0,1)\right) \end{aligned} \tag{6.2}$$

and furthermore for $t > 0$

$$u(t) \in H^4(0,1) \cap H_0^1(0,1), \quad \text{and} \quad u_t(t) \in H^2(0,1) \cap H_0^1(0,1).$$

We return now to the problem (P_k) and start by giving a functional analytic interpretation to the nonlocal term.

Remark 33 *For any positive integer $k = 1, 2, \ldots$ and any fixed $w \in L^2(0,1)$ the elliptic boundary value problem*

$$-\frac{1}{k}p_{xx} + p = w \tag{6.3}$$

with Neumann conditions

$$p_x(0) = p_x(1) = 0$$

has a unique solution $p \in H^2(0,1)$ (see [16]). Furthermore, if $w \in H^1(0,1)$ then also the solution has better regularity $p \in H^3(0,1)$ and

$$\left\|p_x\right\|^2_{L^2(0,1)} \le \left(p_x, w_x\right)_{L^2(0,1)} \le \left\|w_x\right\|^2_{L^2(0,1)}. \tag{6.4}$$

We note that (6.3) is a resolvent equation *for the Laplacian subject to Neumann boundary conditions and the previous facts imply also that the operator defined by the linear part of (P_k) approximates the linear part of (P) in the sense that*

$$A_k z \to Az \qquad \text{as} \qquad k \to \infty \tag{6.5}$$

for any z in the domain $D(A) := \left(H^4(0,1) \cap H_0^1(0,1)\right) \times \left(H^2(0,1) \cap H_0^1(0,1)\right)$ of A. Where A_k and A

are given by

$$A_k := \begin{pmatrix} 0 & I \\ -\partial_x \left(\partial_{xx} k R_k\right) \partial_x & \partial_{xx} \end{pmatrix} \quad \text{and} \quad A := \begin{pmatrix} 0 & I \\ -\partial_{xxxx} & \partial_{xx} \end{pmatrix}.$$

6.2 Existence of classical solutions

Remark 34 *For any classical solution of (P_k) one can show that the energy*

$$E\left[u\left(t\right), v\left(t\right)\right] = \frac{1}{2}\left\|v\left(t\right)\right\|^2_{L^2(0,1)} + \frac{1}{2}\left(p_x\left(t\right), u_{xx}\left(t\right)\right)_{L^2(0,1)} + \int_0^1 W\left(u_x\left(t\right)\right) dx$$

is bounded by the energy of the initial state $E\left[u\left(t\right), v\left(t\right)\right] \leq E\left[u_0, v_0\right]$, *and decreasing with time. On the other hand we have that*

$$E\left[u\left(t\right), v\left(t\right)\right] - E\left[u_0, v_0\right] = -\int_0^t \left\|v_x\left(s\right)\right\|^2_{L^2(0,1)} ds$$

Theorem 35 (global classical solutions) *The initial boundary value problem associated to (P_k), with $k \in \mathbb{N}$ fixed, has a unique global classical solution*

$$\begin{aligned} u &\in C\left(\left[0,T\right], H^2\left(0,1\right) \cap H^1_0\left(0,1\right)\right) \\ u_t &\in C\left(\left[0,T\right], L^2\right) \end{aligned}$$

for any initial conditions $u_0 \in H^2\left(0,1\right) \cap H^1_0\left(0,1\right)$ *and* $v_0 \in H^2\left(0,1\right) \cap H^1_0\left(0,1\right)$.

6.3 Convergence in the limit regime

In order to prove a convergence result in the limit $k \to \infty$ we need k-independent estimates on the family $\left(u^k, v^k\right)_{k\in\mathbb{N}}$ of solutions to (P_k) with $k = 1, 2, \ldots$ Such estimates are the object of the following Lemma.

Lemma 36 (k-independent energy estimates) *Let $\left(u^k, v^k\right)$ be the classical solutions of (P_k) for $k = 1, 2, \ldots$. Then the following k-independent energy estimates hold $\forall t \geq 0$*

$$\frac{1}{2}\left\|v^k\left(t\right)\right\|^2_{L^2(0,1)} + \frac{1}{2}\left(p^k_x\left(t\right), u^k_{xx}\left(t\right)\right)_{L^2(0,1)} + \int_0^1 W\left(u^k_x\left(t\right)\right) dx \leq E_0 \tag{6.6}$$

$$\int_0^T \left\|v^k_x\left(t\right)\right\|^2_{L^2(0,1)} dt \leq E_0 \tag{6.7}$$

where $E_0 := \frac{1}{2}\left\|v_0\right\|^2_{L^2(0,1)} + \frac{1}{2}\left\|u_{0xx}\right\|^2_{L^2(0,1)} + \int_0^1 W\left(u_{0x}\right) dx$.

Using the Aubin lemma [71] we can prove the weak convergence of solutions to (P_k) when $k \to \infty$.

Theorem 37 (weak convergence) *There exists a subsequence $\left(u^k, v^k\right)_{k\in\mathbb{N}}$ of the family of classical solutions of (P_k) all corresponding to the same initial conditions $u_0 \in H^2\left(0,1\right) \cap H^1_0\left(0,1\right)$ and $v_0 \in H^2\left(0,1\right) \cap H^1_0\left(0,1\right)$, and a pair $\left(\overline{w}, \overline{v}\right)^T \in L^2\left(0,T; L^2\left(0,1\right)\right) \times L^2\left(0,T; H^1_0\left(0,1\right)\right)$ such that*

(i) *we have*

$$\begin{aligned} u^k_x &\rightarrow \overline{w} && \text{in } L^2\left(0,T; L^2\left(0,1\right)\right) \\ v^k &\rightarrow \overline{v} && \text{in } L^2\left(0,T; H^1_0\left(0,1\right)\right) \\ \sigma\left(u^k_x\right) &\rightarrow \sigma\left(\overline{w}\right) && \text{in } L^2\left(0,T; L^2\left(0,1\right)\right) \end{aligned}$$

and $\overline{w} \in L^\infty\left(\left[0,T\right] \times \left[0,1\right]\right)$.

(ii) $(\overline{w},\overline{v})^T$ *is a weak solution of the problem (P) in conservation form, i.e.,*

$$\int_0^T \int_0^1 \overline{w}\psi_t - \overline{v}\psi_x dxdt = 0 \tag{6.8}$$

$$\int_0^T \int_0^1 \overline{v}\psi_t - \overline{v}\psi_{xx} - \overline{w}\psi_{xxx} + \sigma(\overline{w})\psi_x dxdt = 0 \tag{6.9}$$

for all $\psi \in C_0^\infty([0,T]\times[0,1])$.

Based on semigroup methods we can actually prove a result which is stronger than Theorem 37, namely that the classical solutions of (P_k) converge to the classical solution of (P).

Theorem 38 *Let* (u^k, v^k) *and* (u,v) *be the classical solutions of* (P_k) *and (P) with identical initial data*

$$u_0 \in H^2(0,1)\cap H_0^1(0,1) \quad \text{and} \quad v_0 \in H^2(0,1)\cap H_0^1(0,1).$$

Then, we have that for any $t\in[0,T]$

$$\left(u^k(t), v^k(t)\right) \to (u(t), v(t)) \qquad \text{as} \qquad k\to\infty$$

in the norm topology of $X = \left(H^2(0,1)\cap H_0^1(0,1)\right)\times L^2(0,1)$.

Chapter 7
A model of Ren and Truskinovsky - the nonlinear viscoelasticity limit regime

7.1 A special nonlinear elasticity model

This Chapter is devoted to a different limit regime of the Ren and Truskinovsky model, namely the regime in which solutions to (5.5) for a special choice of the parameters converge to the unique solution of the nonlinear viscoelasticity equation

$$u_{tt} = u_{txx} + \sigma(u_x)_x. \tag{7.1}$$

We are interested in case[1] when

$$\delta_1 = 1, \quad \delta_2 = 0, \quad \gamma_1^2 = \frac{1}{k}, \quad \varepsilon = 1.$$

The equations we will deal with are thus

$$u_{tt} = u_{xx} - p_x + u_{txx} + \sigma(u_x)_x \tag{7.2}$$

$$-\frac{1}{k}p_{xx} + p = u_x \tag{7.3}$$

[1] In previous chapter we have dealt with $\delta_1 = k, \delta_2 = 0, \gamma_1^2 = \frac{1}{k}, \varepsilon = 1$.

with Dirichlet boundary conditions for u and u_t, Neumann boundary conditions for p, and initial conditions.

The particular nonlinear stress-strain relationship we consider is the same as in the previous two chapters

$$\sigma(w) = w^3 - w.$$

which means we can reduce the two u_{xx} terms in (7.2) to obtain the equivalent form

$$u_{tt} = u_{txx} + \sigma^k(u_x)_x. \tag{7.4}$$

where $\sigma^k(u_x)_x = 3u_x^2 u_{xx} - p_x$.

Remark 39 *Due to the properties of the elliptic equation we have that for a fixed* $u \in H^2(0,1)$

$$p_x^k \to u_{xx} \qquad as \qquad k \to \infty,$$

and hence

$$\sigma^k(u_x)_x \to \sigma(u_x)_x \qquad as \qquad k \to \infty.$$

7.2 Existence of classical solutions

The equation (7.1) has been studied among others by G. Andrews and J. Ball in the 1980s. They were able to prove a local existence and uniqueness result [5] and in a second article [6] they study the asymptotic behaviour of solutions for $t \to \infty$. The local existence proof of Andrews relies on Krasnoselskii's fixed point theorem for the sum of two operators, and on properties of Green's function for the one-dimensional heat equation. Using a semigroup argument we can prove the following result which is stronger than the result of Andrews. Actually, for smooth enough initial data (7.1) admits a global classical solution, not just an integral solution as in [5]. Our proof follows ideas in Engel-Nagel [33, VI.3].

Theorem 40 *The initial boundary value problem associated to the nonlinear viscoelastic model*

$$u_{tt} = u_{txx} + \sigma(u_x)_x$$

has a unique global solution

$$\begin{aligned} u &\in C\left([0,\infty), H^2(0,1) \cap H_0^1(0,1)\right) \\ u_t &\in C\left([0,\infty), L^2(0,1)\right) \end{aligned}$$

for any initial conditions

$$u_0 \in H^2(0,1) \cap H_0^1(0,1) \quad and \quad v_0 \in H^2(0,1) \cap H_0^1(0,1).$$

In a similar way we can prove that the particular case of the Ren and Truskinovsky model (7.4) also admits global classical solutions.

Theorem 41 *The initial boundary value problem associated to the modified nonlinear viscoelastic model*

$$u_{tt} = u_{txx} + \sigma^k(u_x)_x$$

with $\sigma^k (u_x)_x = 3u_x^2 u_{xx} - p_x$*, has a unique global solution*

$$\begin{aligned} u &\in C\left([0,\infty), H^2(0,1) \cap H_0^1(0,1)\right) \\ u_t &\in C\left([0,\infty), L^2(0,1)\right) \end{aligned}$$

for any initial conditions

$$u_0 \in H^2(0,1) \cap H_0^1(0,1) \quad \textit{and} \quad v_0 \in H^2(0,1) \cap H_0^1(0,1).$$

7.3 Convergence in the limit regime

In this section we study the behaviour of solutions to (7.4) in the limit $k \to \infty$. As already stated, we expect that this sequence of solutions converges to the unique classical solution of the nonlinear viscoelasticity model with the same initial data.

The main result of this section is based on k-independent estimates.

Theorem 42 *Let* (u, v) *and* $\left(u^k, v^k\right)$ *be the classical solutions of (7.1) and (7.2) with identical initial data*

$$u_0 \in H^2(0,1) \cap H_0^1(0,1) \quad \textit{and} \quad v_0 \in H^2(0,1) \cap H_0^1(0,1).$$

Then, we have that for any $t \in [0,T]$

$$\left(u^k(t), v^k(t)\right) \to (u(t), v(t)) \qquad \textit{as} \qquad k \to \infty$$

in the norm topology of $X = \left(H^2(0,1) \cap H_0^1(0,1)\right) \times L^2(0,1)$.

III. A NON-LOCAL ALLEN-CAHN EQUATION

Chapter 8
The Allen-Cahn equation

The Allen-Cahn equation

$$u_t = \varepsilon u_{xx} - f(u) \tag{8.1}$$

is a scalar Ginzburg-Landau equation first introduced by J. W. Cahn and S. M. Allen [4] to describe the evolution of a non-conserved order parameter during dynamic phase transitions in binary alloys. The equation is also related to pattern formation models (see [8], [27] or [49]) and to phase-field models (see [20]).

We will study the Allen-Cahn equation on a bounded domain $\Omega = [0,1]$, and the physically natural boundary conditions are Neumann conditions

$$u_x(t,0) = 0 = u_x(t,1), \qquad \text{for} \quad t \geq 0.$$

The diffusion coefficient ε is usually very small and the nonlinearity f is of bi-stabile, non-monotone type $f(u) = u^3 - u$.

For any smooth enough initial data the model has a unique classical solution which is bounded by the L^∞-norm of the initial state and which converges in the long-time limit to a solution of the stationary problem $\varepsilon u_{xx} = f(u)$, as we will see in more detail later.

We remark also that (8.1) is the $L^2(0,1)$ gradient flow (for details see [36] for example) of the free-energy functional

$$E = \int_0^1 \frac{\varepsilon}{2} u_x^2 - W(u)\, dx$$

where $W(u)$ is a double well potential with equal minima at ± 1. The two minima of W correspond to the two phases that coexist in the material.

The recent article [23] of X. Chen gives a detailed account of the dynamics described by the Allen-Cahn model. The four stages ideintified by X. Chen are:

(i) phase separation ($O(|\ln \sqrt{\varepsilon}|)$ long)

(ii) generation of the metastable pattern ($O(\varepsilon^{-1})$ long)

(iii) super-slow movement of the metastable pattern ($O(e^{1/\varepsilon})$ long), and

(iv) an annihilation of interfaces that are $O(\varepsilon)$ close, which interlaces with (iii).

The next three theorems establish characteristic properties of solutions to (8.1). We give a full account of these results as they will serve as a guidline in the study of a new nonlocal version of the Allen-Cahn equation later (Chapter 9).

Theorem 43 (global existence of classical solutions, [84]) *For any initial data*

$$u_0 \in D(A) \subset H^2(0,1),$$

where $D(A) = \{u \in H^2(0,1) : u_x(0) = u_x(1) = 0\}$ *is the domain of the Laplacian with Neumann boundary condition, the problem (8.1) admits a unique global classical solution*

$$u \in C^1([0,\infty), L^2(0,1)) \cap C([0,\infty), D(A)).$$

To prove the L^∞ boundedness of solutions we employ a technique due to Stampacchia cited in [16].

Theorem 44 (a priori boundedness) *Let* $u \in C^1([0,\infty), L^2(0,1)) \cap C([0,\infty), H^2(0,1))$ *be the classical solution of (8.1). Then*

$$\|u(t)\|_{L^\infty(0,1)} \le \max\left\{1, \|u_0\|_{L^\infty(0,1)}\right\}.$$

Theorem 45 (long-time behaviour, see [84]) *If* $u \in C^1([0,\infty), L^2(0,1)) \cap C([0,\infty), D(A))$ *is a classical solution of (8.1) then*

$$\|u_t(t)\|_{L^2(0,1)} \to 0, \qquad as \;\; t \to \infty.$$

Theorem 46 (convergence to stationary solutions, [84]) *For any given* $u_0 \in D(A)$, *where* A *is the Laplacian with Neumann conditions, there is an equilibrium* $\psi \in C^\infty(0,1)$ *satisfying*

$$\begin{aligned} -\varepsilon \psi_{xx} &= \psi - \psi^3 \qquad in \;\; [0,1] \\ \psi_x &= 0 \qquad on\; \partial[0,1], \end{aligned}$$

such that the classical solution of (8.1) converges to ψ in the following sense

$$\lim_{t\to\infty} \|u(t) - \psi\|_{H^2(0,1)} = 0.$$

For more details on the topic of convergence to stationary solutions we refer to [43].

Chapter 9
A special nonlocal Allen-Cahn Equation

The Allen-Cahn equation can be regarded as a singularly perturbed nonlinear evolution equation, the dissipative perturbation having a regularizing effect.

It is standard in semigroup theory to approximate unbounded operators like the Laplacian in the dissipative term of (8.1) by bounded operators called Yosida approximants.

The main idea of this chapter is to perform such an approximation in a "tuned" way, choosing the order n of the Yosida approximation to be exactly the inverse diffusion coefficient

$$n = \frac{1}{\varepsilon}.$$

Due to the bi-stabile form of the nonlinearity and the special choice of the approximation the resulting equation has a very simple form, as we will see .

As an alternative to the Allen-Cahn equation

$$u_t = \varepsilon u_{xx} - \left(u^3 - u\right)$$

we propose

$$u_t = \varepsilon p_{xx} - \left(u^3 - u\right) \tag{9.1}$$

$$-\varepsilon p_{xx} + p = u \tag{9.2}$$

where p is given by the auxiliary elliptic problem. The interesting fact is that if $\varepsilon = 1/n$, as announced above, then by substituting p_{xx} (9.1) becomes equivalent to

$$u_t = -u^3 + p \tag{9.3}$$

$$-\varepsilon p_{xx} + p = u \tag{9.4}$$

The equation (9.3) has the following remarkable properties:

(i) it is a regular perturbation of the nonlinear evolution equation

$$u_t = -u^3 + u;$$

(ii) it does not contain any spatial derivatives (in contrast to (9.1));

(iii) ε does not explicitly appear in the equation.

These properties are interesting from numerics point of view. Their meaning is that (9.3) can be solved numerically by a direct time step method and the only price that we have to pay is that we need to compute p at each step. But, since the elliptic problem relating p to u has constant coefficients, solving it amounts to performing one matrix multiplication at each time step. The matrix is the same for any time point, and can be constructed beforehand. For interesting results and numerical simulations of both local and nonlocal Allen-Cahn equations we refer to [12].

On the other hand, we can also use Green's function for the elliptic problem exactly like in (5.6) for it is the same elliptic problem, and thus

$$p(x) = \int_0^1 G^\varepsilon(x,y)\,u(y)\,dy$$

From this point of view, the above integral defines a nonlocal bounded operator. The interest in nonlocal versions of the Allen-Cahn

$$u_t = (J * u - u) - f(u)$$

is not new, and the class of nonlocal regularizations defined by means of convolution

$$(J * u)(x) = \int J(x-y)\,u(y)\,dy$$

has been intensively studied in recent years starting with the seminal work concerning travelling wave solutions [11]. In [25] and [26] the authors show that the solutions of convolutive nonlocal diffusion equations converge to the solution of the corresponding classical heat equation. For more details about travelling wave solutions, the slow-motion of phase-interfaces and the numerical analysis of nonlocal Allen-Cahn models we also refer to [12],[23],[24],[39].

9.1 A qualitative analysis of the nonlocal model

In this section we prove that the nonlocal Allen-Cahn model (9.1) with its equivalent reformulations has the same "qualitative" properties as the classical Allen-Cahn equation. Here we are concerned with well-posedness, a priori L^∞ bounds and long-time asympthotics for (9.1), while in the next section we will give "quantitative" comparison.

Theorem 47 (existence of classical solutions) *For any initial data* $u_0 \in D(A) \subset H^2(0,1)$, *where* $D(A) = \{u \in H^2(0,1) : u_x(0) = u_x(1) = 0\}$ *is the domain of the Laplacian with Neumann boundary conditions, the problem (9.1) admits a unique global classical solution* $u \in C^1([0,\infty), L^2(0,1)) \cap C([0,\infty), H^1(0,1))$.

Theorem 48 (a priori boundedness of solutions) *Let* $u(t,x)$ *be a classical solution of (9.3) defined on* $[0,T] \times [0,1]$. *If* $u_0 = u(0,\cdot) \in L^\infty(0,1)$ *then*

$$\|u(t,\cdot)\|_{L^\infty(0,1)} \le \|u_0\|_{L^\infty(0,1)}$$

for any $t \in [0,T]$.

The long-time behaviour of the solution to (9.1) is similar to that of solutions of (8.1) as the following result states.

Theorem 49 (long-time behaviour of solutions) *Let* u *be a classical solution of (9.1), then*

$$\|u_t(t)\|_{L^2(0,1)} \to 0 \qquad as \qquad t \to \infty.$$

9.2 Error estimates

The aim of this section is to show that, at an appropriate time scale for the evolution process, the difference between the solution of the Allen-Cahn equation, and the solution of the proposed nonlocal Allen-Cahn equation can be controlled by ε-dependent terms, where ε is the diffusion coefficient.

We remark that the Allen-Cahn equation (8.1) and the nonlocal Allen-Cahn equation (9.1) differ only in the linear part, so estimates can be derived using representations of the corresponding semigroups in terms of the eigenvalues of their generators.

An important problem is to identify the time scale at which the nonlocal Allen-Cahn equation is a good approximation of the original equation. Our starting point lies in the detailed studies that have been carried out on the Allen-Cahn equation during the last two decades. We know about the solutions of this well studied equation that they develop steep transition layers, that is separated phase domains, in a time proportional to

$$\varepsilon \left|\ln \sqrt{\varepsilon}\right| .$$

The study of this phenomenon goes back to P. Fife and X. Chen [23], but there are also recent studies like those of M. Alfaro, D. Hilhorst and H. Matano [3].

We will show that at this very time scale the solutions of the local and nonlocal Allen-Cahn equations are close to each other, their difference depending on the diffusion coefficient ε.

Theorem 50 *Let u and u_n be the solutions of the local and nonlocal Allen-Cahn equations with identical initial data*

$$u_0 \in H^2(0,1), \qquad \|u_0\|_{L^\infty(0,1)} \leq 1.$$

Then at a time scale

$$t \sim \varepsilon \left|\ln \sqrt{\varepsilon}\right|$$

we have

$$\|u(t) - u_n(t)\|_{L^2(0,1)} \leq \left(\frac{1}{\left|\ln \sqrt{\varepsilon}\right|} + 1 \right) \sqrt{\varepsilon}.$$

Conclusions and further work

Conclusions

Part I has its starting point in recent works which study systems of nonlinear operator equations by means of vector-valued norms. The original idea goes back to Perov ([56]) and since the 1960s it has been extended and applied in many ways (for an abstract approach see [82]). In this context, our aim was to study systems of semilinear evolution problems in spaces endowed with vector-valued norms/metrics. Such extensions of the classical norm concept are more appropriate when dealing with systems of equations as they give you the freedom of treating each component separately.

First, in Chapter 2, we were able to show wellposedness for a system of semilinear evolution equations in a generalized Banach space. Our results have the same structure as those in the classical theory of ordinary differential equations. Existence and uniqueness of solutions are a consequence of a generalized contraction principle, while data dependence is obtained via a generalized Gronwall inequality. Subsequently, the growth conditions imposed on the nonlinear terms of the systems were weakened and existence results were proved using Schauder's or Leray-Schauder's fixed point principle.

In Chapter 3 we went a step further, from single-valued to multi-valued problems. An analysis similar to that in Chapter 2 has been carried out with success for systems of differential inclusions. Although the proofs in this chapter are technically more involved, the fundamental ideas are the same, and a key role is played by convergent to zero matrices.

Part I ends with a nonstandard problem, an abstract Cauchy problem with nonlocal initial data. Our idea was to treat this as a system of two coupled operator equations, one of the equations being of evolutionary type, while the second is stationary. Due to the contrasting properties of the two equations it was possible to prove existence of solutions only by means of vector versions of Krasnoselskii's fixed point theorem for the sum of two operators. These generalizations of the classical result of Krasnoselskii are new.

In **Part II** we have discussed a model in nonlinear elasticity suggested by X. Ren and L. Truskinovsky in 2000. The crucial observation was that the nonlocal regularizing terms proposed by these authors are actually Yosida approximations of unbounded differential operators, chosen such that they are compatible with the capillarity coefficient. Guided by this observation we have started to study the Ren and Truskinovsky model using semigroup methods.

Our main interest was the relationship between the new model and older, well studied models. The surprising fact that came out during this investigation is that, for special choices of the parameters, the model of Ren and Truskinovsky can serve as an approximation of both the strain gradient model (capillarity model) and the nonlinear viscoelasticity model. This surprising versatility is due to the unconventional nonlocal regularizing term(s) which are the main novelty of the model.

Two chapters (Chapter 6 and Chapter 7) deal with one of the limit regimes (the strain gradient limit and the nonlinear viscoelasticity limit) each. The structure of these two chapters is similar and in both, the main results concern the existence of global solutions and the convergence of these solutions in the limit regime to the solutions of the limit equations.

Once formulated in the terms of Yosida approximations and semigroups, the approximation idea of Truskinovsky can be applied to any nonlinear evolution model stemming from a double-well potential (free) energy. In **Part III**, our aim has been to show that a nonlocal "Truskinovsky" version of the Allen-Cahn model has similar properties to those of its classical counterpart. In perfect analogy to the well-established theory for the Allen-Cahn equation, we could show that also its nonlocal version admits global classical solutions which are a priori bounded. The the long time asymptotics ($t \to \infty$) are also similar for the two models.

Further work

A possible direction in which the results obtained in Part I of this work could be extended is considering an ℓ^2-valued (sequence-valued) norm instead of the $\mathbb{R}^n$-valued norm that we have used. This would be in many ways a natural extension since systems of infinitely many ordinary differential equations have been studied for quite some time, and they are strongly related to partial differential equations.

A different direction would be that of applying the abstract results in Part I to systems of coupled nonlinear viscoelastic equations. In a very recent paper, M. Mustafa [50, Nonlinear Analysis 2012] studies systems of the following type

$$\begin{aligned} u_{tt} - \Delta u + \int_0^t g_1(t-\tau)\,\Delta u(\tau)\,d\tau + f_1(u,v) &= 0 \\ v_{tt} - \Delta v + \int_0^t g_2(t-\tau)\,\Delta v(\tau)\,d\tau + f_2(u,v) &= 0. \end{aligned}$$

We believe that his results, which are obtained by Galerkin methods, can be recover and possibly extended by the vector-valued norm methods developed in this work.

The fixed point results obtained in Chapter 3 are constructed in such a way that they are suitable for treating systems of two coupled operator equations where one component has a compactness property but the other does not. Typically, dissipative systems (described for example diffusion equations) have very good, smoothing, properties, their solution operators being compact; whereas solutions to wave-type equations don't have such properties. A further direction of research can be exactly in the field of coupled parabolic-hyperbolic equations

$$\begin{aligned} u_t - \Delta u &= f_1(u, v) \\ v_{tt} - \Delta v &= f_2(u, v). \end{aligned}$$

In Part II we have considered two limit regimes for the Ren and Truskinowsky model in nonlinear elasticity, but one can see that neither of these limits is singular. A still open question is related to the behavior of solutions in the singular, also called sharp interface, limit when both the diffusion coefficient and also the capillarity coefficient go to zero. The study of such limits was initiated among others by Kruzkov [42] and since then became a classical approach to single out weak entropy solutions for hyperbolic systems of conservation laws. For the system of nonlinear elasticity the vanishing viscosity method was first applied by DiPerna [31] and then extended by Shearer [72], LeFloch [40], Bressan [13] and many others. Unfortunately, the compensated compactness technique of Murat and Tartar used by DiPerna, Shearer and LeFloch, as well as the centre manifold theorem technique of Bressan apply only to hyperbolic systems (with convex energy) and are therefore not suitable for the double-well potential energy of Ericksen. Up to some extent nonlinear semigroups could provide an appropriate setting in which to treat the above mentioned singular limits in the non-convex energy case, and the recent monograph [10] already contains existence results for related models.

A second interesting problem related to nonlinear elasticity models which allow multiple phases is **boundary control** of the microstructure. More precisely, the question is whether it is possible only by thermo-mechanical actions on the boundary of the body (bar) to produce a prescribed pattern of phase domains in the interior of the body. Obviously, to develop such a theory, one needs a very good understanding, which is still lacking, of the behaviour of solutions to these nonlinear problems.

As we have seen in Part III, the approximation idea of Truskinovsky does not apply only to nonlinear elasticity but to any evolution problem involving a double-well potential energy. The Allen-Cahn model that we have discussed in some detail is only one example in this class. Other models to which this approximation idea could be applied are: the **Cahn-Hilliard model, phase-field models** or **nonlinear diffusion models**.

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