HABILITATION THESIS

Special Types of Fuzzy Spaces and Fuzzy Relations

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Abstract of the Thesis

The aim of this thesis is to highlight the author's main scientific results, obtained in the last few years and published in several high impact international journals, as well as the evolution and development plan of the scientific and professional career.

Considering the proposed aim, this thesis is structured into two parts. The first one, which presents the scientific results, is organized in four chapters. The first of these presents the terms and concepts as well as some preliminary results that are going to be used in the following chapters in order to give the thesis a unitary form.

After L.A. Zadeh introduced in its famous paper [106] the concept of fuzzy set, many mathematicians have tried to extend, within a fuzzy context, many of the classical mathematic results. An important problem was the obtaining of an adequate definition of the fuzzy metric space. I need to mention that the notion of fuzzy metric space was introduced by I. Kramosil and J. Michálek [53] in 1975. Their concept is equivalent, to a certain extent, to that of statistical metric space. I should highlight that the statistical metric spaces had been studied many years before and a survey had been made by B. Schweizer and A. Sklar in paper [91]. In 1994, A. George and P. Veeramani [35] modified I. Kramosil and J. Michálek's definition in order to obtain a Hausdorff topology on a fuzzy metric space. Lately many types of fuzzy metric spaces have been considered by different authors under different approaches. Thus V. Gregori and S. Romaguera [40] introduced the notions of fuzzy quasi-metric space and fuzzy quasi-pseudo-metric space. New concepts of generalized fuzzy metric spaces were introduced by A.D. Ray and P.K. Saha [81], G. Sun and K. Yang [98], T. Bag [5], R. Pleabaniak [79], B.C. Tripathy et al. [102]. The obtained results within this context, published in the journals Informatica(2016) and International Journal of Computers Communications & Control(2016) (see [75], [76]), are the subject of chapter two "Special types of fuzzy metric spaces".

In the first section I establish some properties of fuzzy quasi-pseudo-metric spaces. An important result of my research is that any partial ordering relation can be defined by a fuzzy quasi-metric, while any equivalence relation can be modeled through a fuzzy pseudo-metric. Moreover, I obtain decompositions theorems of a fuzzy quasi-pseudo metric into a right continuous and ascending family of quasi-pseudo metrics. I also need to highlight some important examples of fuzzy quasi-(pseudo)-metrics. Thus, I have shown that any quasi-(pseudo)-metric induces, in a natural way, a fuzzy quasi-(pseudo)-metric. Also, I have given an example of fuzzy quasi-metric space which is not a fuzzy metric space. Another example shows us

that there exists a fuzzy quasi-pseudo-metric space which is neither a fuzzy quasi-metric space nor a fuzzy pseudo-metric space. Finally, I have given an example of fuzzy pseudo-metric space which is not a fuzzy metric space.

In section two, I have introduced and studied the concept of fuzzy b-metric space, generalizing, in this way, both the notion of fuzzy metric space introduced by I. Kramosil and J. Michálek and the concept of b-metric space. On the other hand, I have introduced the concept of fuzzy quasi-b-metric space, extending the notion of fuzzy quasi-metric space recently introduced by V. Gregori and S. Romaguera. Finally, a decomposition theorem for a fuzzy quasi-pseudo-b-metric into an ascending family of quasi-pseudo-b-metrics was obtained.

The notion of fuzzy norm was introduced for the first time by A.K. Katsaras [48] in 1984. In 1992, C. Felbin [32] introduced another concept of fuzzy norm, by assigning a fuzzy real number to each element of the linear space. Following S.C. Cheng and J.N. Mordeson [22], in 2003, T. Bag and S.K. Samanta [7] introduced a new concept of fuzzy norm, which proved to be the most suitable and the easiest to apply in different and diverse developments. But, according to T. Bag and S.K. Samanta, a fuzzy norm is a fuzzy set which satisfies five axioms. In order to obtain different results, T. Bag and S.K. Samanta imposed two more axioms. Regarded together the seven axioms are very strong and they narrow very much the family of fuzzy normed linear spaces. Thus, I can say that a clear definition regarding the concept of fuzzy norm has not yet been given and many authors have tried to simplify and improve T. Bag and S.K. Samanta's definition (see[86], [60], [37], [1], [49]). Chapter 3 of the present thesis, "Fuzzy normed linear spaces", has its roots in the results I have obtained within this context and that were published in journals like: Fuzzy Sets and Systems(2016), Informatica(2014), International Journal of Computers Communications & Control(2015) (see [70, 72, 73, 74]).

Following the ideas of T. Bag and S.K. Samanta, in the first section of chapter 3, I have obtained decompositions theorems for fuzzy norms into a family of semi-norms, in more general settings. The results are for both fuzzy norms of Bag-Samanta type and for fuzzy norms of Katsaras type. As a consequence, I have obtained local convex topologies induced by these types of norms. Finally, I introduce the concept of atomic decomposition of a fuzzy normed linear space which has an important role in the development of a fuzzy wavelet theory.

The aim of section two is to introduce some special fuzzy norms on $\mathbb{K}^n$ and to obtain, in this way, fuzzy Euclidean normed spaces. Firstly, I prove that the cartesian product of a finite family of fuzzy normed linear spaces is a fuzzy normed linear space. Thus, any fuzzy norm on $\mathbb{K}$ generates a fuzzy norm on $\mathbb{K}^n$. Finally, I show that any fuzzy Euclidean normed space is complete.

Section three introduces, firstly, the notion of uniformly fuzzy continuous mapping and establishes the uniform continuity theorem in fuzzy settings. Moreover, the concept of of fuzzy Lipschitzian mapping is introduced and a fuzzy version of Banach's contraction principle is obtained. Finally, a special attention is given to various characterizations of fuzzy continuous linear operators and classical principles of functional analysis (such as the uniform boundedness principle, the open mapping theorem and the closed graph theorem) are extended in a more general fuzzy context.

In the last section of chapter three, I begin by introducing the notion of fuzzy pseudo-norm, and then I extend, improve and complete the results obtained by T.Bag and S.K. Samanta for fuzzy norms, in the fuzzy pseudo-norms context. Further on, I introduce and discuss the notions of fuzzy F-norm and fuzzy F-space. By means of several auxiliary results, I obtain a characterization of metrizable topological linear spaces in terms of fuzzy F-norm, by replacing fuzzy norms of type $(N, *_L)$ or $(N, \cdot)$ as stated in [1], with fuzzy F-norms $(F, \min)$.

In chapter 4 of the thesis, entitled "Special types of fuzzy relations" I present, in a unitary form, some special types of fuzzy relations: fuzzy affine relations, fuzzy linear relations, fuzzy convex relations, fuzzy M-convex relations. All these fuzzy relations are characterized and the inclusions between these classes of fuzzy relations are established. The sum of two fuzzy relations and the scalar multiplication are defined, in a natural way, and some properties of this operations are established. Section four investigates the fuzzy linear relations. Among the results obtained, there should be underlined a characterization of fuzzy linear relations and the fact that the inverse of a fuzzy linear relation is also a fuzzy linear relation. Moreover, I show that the composition of two fuzzy linear relations is a fuzzy linear relation as well. Finally, I prove that the family of all fuzzy linear relations is closed under addition and it is closed under scalar multiplication. This chapter is based on the results published in the journals Procedia Computer Science(2014) and Filomat(2016) (see [71, 77]).

In the second part of the thesis, namely chapter 5, I will describe my plans of scientific and professional career development. The evolution and development plan of my career maintains, in great lines, the directions already elaborated, but extends them as well, opening to new didactic and scientific horizons by identifying approaches to improve the quality of the teaching process as well as of to increase the level of the research I am involved in, both individually and collectively.

Firstly, there are presented the directions of research that I intend to pursue: fuzzy normed algebras, fixed point theorems in fuzzy normed linear spaces, fuzzy Hilbert spaces, bounded linear operators on fuzzy F-spaces, fuzzy wavelet, fixed point theorems in fuzzy b-metric spaces. A special attention will be shown to applications to real world problems.Within this context, my domains of interest can evolve towards an interdisciplinary research, through the use of the results obtained in applications in economics, computer science or engineering. Besides the scientific articles I intend to publish in the next years, I consider as a long term project a scientific monograph on fuzzy functional analysis.

In what the didactic activity is concerned, I intend to teach courses for the master degree programmes, with the desire to share my experience to the younger generation of mathematicians in order to attract them to work in the fuzzy functional analysis domain. Last but not least, I intend to extend my scientific and didactic cooperations both in Romania and abroad and to apply for research grants.