Babeş-Bolyai University

Radii of starlikeness and convexity of some special functions

Abstract of the habilitation thesis

Árpád Baricz, PhD

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Introduction

The purpose of the present thesis is the presentation of some of the author's main contributions concerning the geometric properties of some special functions such as Bessel, q-Bessel, Struve and Lommel functions of the first kind. The common feature of the studied functions is that they are entire functions of which Taylor series coefficients can be expressed explicitly and all of their zeros are real for the corresponding values of the parameters. In order to study the geometric properties of the special functions mentioned above we use some basic techniques of complex analysis and of real entire functions.

The thesis is divided into eight sections. The first two sections contain the precise description of the radii of starlikeness and convexity of three kind or normalized Bessel functions of the first kind, while the third section contains the q-analogue of the results of the first two sections on Jackson and Hahn-Exton (or third Jackson) q-Bessel functions. The fourth section is devoted to the study of the radii of starlikeness of Struve and Lommel functions of the first kind. In the fifth section we deal with a special linear combination of Bessel functions of the first kind, called Dini function and prove some interesting close-to-convexity results. Sections six and seven are devoted to important auxiliary results concerning Bessel, q-Bessel, Struve and Lommel functions. Here the results on zeros of these functions are also of independent interest, although they were deduced to prove the main results of the second, third and fourth sections. Finally, the last section contains some concluding remarks and a short description of the impact of the obtained results. In this section we also present the recent research and mobility activity of the author.

The thesis is based on the published papers [BDY, 2016], [BDM, 2016], [BDOY, 2016], [BKS, 2014] and [BS, 2014].

The present abstract contains the main results (without proofs) of the first five sections of the habilitation thesis. The preliminary results contained in the sixth and seventh sections, as well as the content of the eight section of the thesis are not presented in this abstract.

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RADII OF STARLIKENESS AND CONVEXITY OF SOME SPECIAL FUNCTIONS

1. Radii of starlikeness of normalized Bessel functions

In this section our aim is to determine the radius of starlikeness of the normalized Bessel functions of the first kind for three different kinds of normalization. The key tool in the proof of our main result is the Mittag-Leffler expansion for Bessel functions of the first kind and the fact that, according to Ismail and Muldoon [IM, 1995], the smallest positive zeros of some Dini functions are less than the first positive zero of the Bessel function of the first kind.

Let $\mathbb{D}(0,r)$ be the open disk $\{z \in \mathbb{C} : |z| < r\}$, where $r > 0$, and set $\mathbb{D} = \mathbb{D}(0,1)$. By $\mathcal{A}$ we mean the class of analytic functions $f : \mathbb{D}(0,r) \to \mathbb{C}$ which satisfy the usual normalization conditions $f(0) = f'(0) - 1 = 0$. Denote by $\mathcal{S}$ the class of functions belonging to $\mathcal{A}$ which are univalent in $\mathbb{D}(0,r)$ and let $\mathcal{S}^*(\alpha)$ be the subclass of $\mathcal{S}$ consisting of functions which are starlike of order α in $\mathbb{D}(0,r)$, where $0 \leq \alpha < 1$. The analytic characterization of this class of functions is

$$\mathcal{S}^*(\alpha) = \left\{ f \in \mathcal{S} \; : \; \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \;\; \text{for all} \;\; z \in \mathbb{D}(0,r) \right\},$$

and we adopt the convention $\mathcal{S}^* = \mathcal{S}^*(0)$. The real number

$$r_\alpha^*(f) = \sup \left\{ r > 0 \; : \; \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \;\; \text{for all} \;\; z \in \mathbb{D}(0,r) \right\},$$

is called the radius of starlikeness of order α of the function f. Note that $r^*(f) = r_0^*(f)$ is the largest radius such that the image region $f(\mathbb{D}(0, r^*(f)))$ is a starlike domain with respect to the origin.

Recall that a function $g \in \mathcal{S}$ belongs to the class $\mathcal{K}$ of convex functions if maps the disk $\mathbb{D}(0,r)$ conformally onto $g(\mathbb{D}(0,r))$, which is a convex domain in $\mathbb{C}$. Moreover, for

$\alpha \in [0,1)$ we consider also the class of convex functions of order α defined by

$$\mathcal{K}(\alpha) = \left\{ g \in \mathcal{S} \ : \ \mathrm{Re}\left(1 + \frac{zg''(z)}{g'(z)}\right) > \alpha \ \text{ for all } \ z \in \mathbb{D}(0,r) \right\},$$

which for $\alpha = 0$ reduces to $\mathcal{K}$. We note that the convex functions does not need to be normalized, that is, the definition of $\mathcal{K}(\alpha)$ is also valid for non-normalized analytic function $g : \mathbb{D}(0,r) \to \mathbb{C}$ with the property $g'(0) \neq 0$. Now, let us consider the radius of convexity of order α of the analytic locally univalent function g

$$r_\alpha^c(g) = \sup\left\{ r > 0 \ : \ \mathrm{Re}\left(1 + \frac{zg''(z)}{g'(z)}\right) > \alpha \ \text{ for all } \ z \in \mathbb{D}(0,r) \right\}.$$

We note that $r^c(f) = r_0^c(g)$ is in fact the largest radius for which the image domain $g(\mathbb{D}(0, r^c(g)))$ is a convex domain in $\mathbb{C}$. For more details on starlike and convex functions we refer to the book [Du, 1983] and the references therein.

By definition the analytic function $h : \mathbb{D}(0,r) \to \mathbb{C}$ is close-to-convex if there exists a convex function $\phi : \mathbb{D}(0,r) \to \mathbb{C}$ such that $h'(z)/\phi'(z)$ has positive real part for all $z \in \mathbb{D}(0,r)$. We note that every close-to-convex function is univalent, and the class of close-to-convex functions clearly includes the convex functions themselves. Moreover, every starlike function is close-to-convex. For intrinsic characterization and geometric interpretation of close-to-convex functions we refer to the paper [Ka, 1952].

The Bessel function of the first kind of order ν is defined by [OLBC, 2010, p. 217]

$$J_\nu(z) = \sum_{n \geq 0} \frac{(-1)^n}{n!\Gamma(n+\nu+1)} \left(\frac{z}{2}\right)^{2n+\nu}, \quad z \in \mathbb{C},$$

and is a particular solution of the second-order linear homogeneous Bessel differential equation. Since the Bessel function J_ν does not belong to class $\mathcal{A}$, we focus on the following normalized forms

$$f_\nu(z) = \left(2^\nu \Gamma(\nu+1) J_\nu(z)\right)^{\frac{1}{\nu}} = z - \frac{1}{4\nu(\nu+1)} z^3 + \ldots, \ \nu \neq 0,,$$

$$g_\nu(z) = 2^\nu \Gamma(\nu+1) z^{1-\nu} J_\nu(z) = z - \frac{1}{4(\nu+1)} z^3 + \frac{1}{32(\nu+1)(\nu+2)} z^5 - \ldots,$$

$$h_\nu(z) = 2^\nu \Gamma(\nu+1) z^{1-\frac{\nu}{2}} J_\nu(\sqrt{z}) = z - \frac{1}{4(\nu+1)} z^2 + \ldots,$$

where $\nu > -1$. We note that in fact for $z \in \mathbb{C} \setminus \{0\}$ we have

$$f_\nu(z) = \exp\left(\frac{1}{\nu} \operatorname{Log}\left(2^\nu \Gamma(\nu+1) J_\nu(z)\right)\right),$$

where Log represents the principal branch of the logarithm, and in this thesis every multi-valued function is taken with the principal branch.

Now, let us recall some results on the geometric behavior of the functions f_ν, g_ν and h_ν. Brown [Br, 1960] determined the radius of starlikeness for f_ν in the case when $\nu > 0$. Namely, in [Br, 1960, Theorem 2] it was shown that the radius $r^*(f_\nu)$ is the smallest positive zero of the function $z \mapsto J'_\nu(z)$. Moreover, in [Br, 1960, Theorem 3] Brown proved that if $\nu > 0$, then the radius of starlikeness of the function g_ν is the smallest positive zero of the function $z \mapsto zJ'_\nu(z) + (1-\nu)J_\nu(z)$. Kreyszig and Todd [KT, 1960, Theorem 3] proved that when $\nu > -1$ the function g_ν is univalent in the circle $|z| \leq \rho_\nu$ but not in any concentric circle with larger radius, where ρ_ν is the point on the real axis in which the function g_ν takes its maximum. Brown [Br, 1960, p. 282] pointed out that when $\nu > 0$ the radius of starlikeness of the function g_ν, that is, $r^*(g_\nu)$ is exactly the radius of univalence ρ_ν obtained by Kreyszig and Todd [KT, 1960]. Furthermore, Brown [Br, 1962, Theorem 5.1] showed that the radius of starlikeness of the function g_ν is also ρ_ν when $\nu \in (-1/2, 0)$. On the other hand, Hayden and Merkes [HM, 1964, Theorem C] deduced that when $\mu = \operatorname{Re}\nu > -1$ the radius of starlikeness of g_ν is not less than the smallest positive zero of g'_μ. It is worth to mention that Brown used the methods of Nehari [Ne, 1949] and Robertson [Ro, 1954], and an important tool in the proofs was the fact that the Bessel function of the first kind is a particular solution of the Bessel differential equation. For related (more general) results the interested reader is referred to [Br, 1982, MRS, 1962, Ro, 1954, Wi, 1962] and to the references therein. Finally, let us mention that other geometric properties of the functions g_ν and h_ν were obtained in [Ba, 2008, Ba2, 2010, BP, 2010, Sz, 2010, SK, 2009]. See also the references therein.

Motivated by the above results in this section we make a contribution to the subject and we determine the radius of starlikeness of order β for the functions f_ν, g_ν and h_ν. We note that our approach is much simpler than the methods used in [Br, 1960, Br, 1962, HM, 1964, KT, 1960], and is based only on the Mittag-Leffler expansion for Bessel functions of the first kind and on the fact that the smallest positive zeros of certain Dini functions are less than the first positive zero of the Bessel function of the first kind, according to Ismail and Muldoon [IM1, 1988, IM, 1995].

Our main result of this section is the following theorem [BKS, 2014, Theorem 1]. Here I_ν denotes the modified Bessel function of the first kind, which in view of the relation $I_\nu(z) = \mathrm{i}^{-\nu} J_\nu(\mathrm{i}z)$ is also called sometimes as the Bessel function of the first kind with imaginary argument.

Theorem 1. [BKS, 2014, Theorem 1] *Let $1 > \beta \geq 0$. Then the following assertions are true:*

- **a.** *If $\nu \in (-1, 0)$, then $r_\beta^*(f_\nu) = x_{\nu,\beta}$, where $x_{\nu,\beta}$ denotes the unique positive root of the equation $zI_\nu'(z) - \beta\nu I_\nu(z) = 0$. Moreover, if $\nu > 0$, then we have $r_\beta^*(f_\nu) = x_{\nu,\beta,1}$, where $x_{\nu,\beta,1}$ is the smallest positive root of the equation $zJ_\nu'(z) - \beta\nu J_\nu(z) = 0$.*
- **b.** *If $\nu > -1$, then $r_\beta^*(g_\nu) = y_{\nu,\beta,1}$, where $y_{\nu,\beta,1}$ is the smallest positive root of the equation $zJ_\nu'(z) + (1 - \beta - \nu)J_\nu(z) = 0$.*
- **c.** *If $\nu > -1$, then $r_\beta^*(h_\nu) = z_{\nu,\beta,1}$, where $z_{\nu,\beta,1}$ is the smallest positive root of the equation $zJ_\nu'(z) + (2 - 2\beta - \nu)J_\nu(z) = 0$.*

In particular, when $\beta = 0$, we get the following result.

Corollary 1. [BKS, 2014, Corollary 1] *The following assertions are true:*

- **a.** *If $\nu \in (-1, 0)$, then the radius of starlikeness of f_ν is $x_{\nu,0}$, where $x_{\nu,0}$ is the unique positive root of the equation $I_\nu'(z) = 0$. If $\nu > 0$, then the radius of starlikeness of the function f_ν is $x_{\nu,0,1}$, which denotes the smallest positive root of the equation $J_\nu'(z) = 0$.*
- **b.** *If $\nu > -1$, then the radius of starlikeness of the function g_ν is $y_{\nu,0,1}$, which denotes the smallest positive root of the equation $zJ_\nu'(z) + (1 - \nu)J_\nu(z) = 0$.*
- **c.** *If $\nu > -1$, then the radius of starlikeness of the function h_ν is $z_{\nu,0,1}$, which denotes the smallest positive root of the equation $zJ_\nu'(z) + (2 - \nu)J_\nu(z) = 0$.*

Observe that part **a** and **b** of Corollary 1 complement the results of [Br, 1960, Theorem 2], [Br, 1960, Theorem 3] and [Br, 1962, Theorem 5.1], mentioned above. Part **c** complements the results from [Ba, 2008, BP, 2010, Sz, 2010, SK, 2009]. It is of interest to note here that very recently Szász [Sz, 2010] proved that the normalized Bessel function h_ν is starlike if and only if $\nu \geq \nu_0$, where $\nu_0 = -0.5623\ldots$ is the root of the equation $h'_\nu(1) = 0$, that is, $J'_\nu(1) + (2-\nu)J_\nu(1) = 0$. Finally, we mention that if we consider the function $z \mapsto \lambda_\nu(z) = h_\nu(z)/z$, then part **c** of Theorem 1 and Corollary 1 can be rewritten in terms of convex functions. The idea is to use the differentiation formula

$$\lambda'_\nu(z) = -\frac{1}{4(\nu+1)}\lambda_{\nu+1}(z)$$

together with the duality theorems of Alexander [Al, 1915] and Jack [Ja, 1971]. See also [Ba2, 2010, p. 25] for the results of Alexander and Jack, and also [Ba2, 2010, Ch. 2] for similar results on convex Bessel functions.

2. **Radii of convexity of normalized Bessel functions**

In this section we determine the radius of convexity for three kind of normalized Bessel functions of the first kind, which we studied in the previous section. In the mentioned cases the normalized Bessel functions are starlike-univalent and convex-univalent, respectively, on the determined disks. The key tools in the proofs of the main results are some new Mittag-Leffler expansions for quotients of Bessel functions of the first kind, special properties (like interlacing) of the zeros of Bessel functions of the first kind and their derivative, and the fact that the smallest positive zeros of some Dini functions are less than the first positive zero of the Bessel function of the first kind. Moreover, we find the optimal parameters for which these normalized Bessel functions are convex in the open unit disk. In addition, we disprove a conjecture of Baricz and Ponnusamy concerning the convexity of the Bessel function of the first kind. The results of this section naturally complement the starlikeness results of the previous section.

We start with the following new sharp results on the radii of convexity of normalized Bessel functions.

Theorem 2. [BS, 2014, Theorem 1.1] *If $\nu > 0$ and $\alpha \in [0,1)$, then the radius of convexity of order α of the function f_ν is the smallest positive root of the equation*

$$1 + \frac{rJ''_\nu(r)}{J'_\nu(r)} + \left(\frac{1}{\nu} - 1\right)\frac{rJ'_\nu(r)}{J_\nu(r)} = \alpha.$$

Moreover, $r^c_\alpha(f_\nu) < j'_{\nu,1} < j_{\nu,1}$, where $j_{\nu,1}$ and $j'_{\nu,1}$ denote the first positive zeros of J_ν and J'_ν, respectively.

Theorem 3. [BS, 2014, Theorem 1.2] *If $\nu > -1$ and $\alpha \in [0,1)$, then the radius of convexity of order α of the function g_ν is the smallest positive root of the equation*

$$1 + r\frac{rJ_{\nu+2}(r) - 3J_{\nu+1}(r)}{J_\nu(r) - rJ_{\nu+1}(r)} = \alpha.$$

Moreover, we have $r^c_\alpha(g_\nu) < \alpha_{\nu,1} < j_{\nu,1}$, where $\alpha_{\nu,1}$ is the first positive zero of the Dini function $z \mapsto (1-\nu)J_\nu(z) + zJ'_\nu(z)$.

Theorem 4. [BS, 2014, Theorem 1.3] *If $\nu > -1$ and $\alpha \in [0,1)$, then the radius of convexity of order α of the function h_ν is the smallest positive root of the equation*

$$1 + \frac{r^{\frac{1}{2}}}{2} \cdot \frac{r^{\frac{1}{2}}J_{\nu+2}(r^{\frac{1}{2}}) - 4J_{\nu+1}(r^{\frac{1}{2}})}{2J_\nu(r^{\frac{1}{2}}) - r^{\frac{1}{2}}J_{\nu+1}(r^{\frac{1}{2}})} = \alpha.$$

Moreover, we have $r^c_\alpha(h_\nu) < \beta_{\nu,1} < j_{\nu,1}$, where $\beta_{\nu,1}$ is the first positive zero of the Dini function $z \mapsto (2-\nu)J_\nu(z) + zJ'_\nu(z)$.

Now, we are going to present some other sharp results on the functions f_ν, g_ν and h_ν. We determine the best possible parameters such that the normalized Bessel functions map the open unit disk into a convex domain.

Theorem 5. [BS, 2014, Theorem 1.4] *The function f_ν is convex of order $\alpha \in [0,1)$ in $\mathbb{D}$ if and only if $\nu \geq \nu_\alpha(f_\nu)$, where $\nu_\alpha(f_\nu)$ is the unique root of the equation*

$$\nu(\nu^2-1)J^2_\nu(1) + (1-\nu)(J'_\nu(1))^2 = \alpha\nu J_\nu(1)J'_\nu(1),$$

situated in (ν^*, ∞), where $\nu^* \simeq 0.3901\ldots$ is the root of the equation $J'_\nu(1) = 0$. Moreover, f_ν is convex in $\mathbb{D}$ if and only if $\nu \geq 1$.

Theorem 6. [BS, 2014, Theorem 1.5] *The function g_ν is convex of order $\alpha \in [0,1)$ in $\mathbb{D}$ if and only if $\nu \geq \nu_\alpha(g_\nu)$, where $\nu_\alpha(g_\nu)$ is the unique root of the equation*

$$(2\nu + \alpha - 2)J_{\nu+1}(1) = \alpha J_\nu(1),$$

situated in $[0, \infty)$. In particular, g_ν is convex in $\mathbb{D}$ if and only if $\nu \geq 1$.

Theorem 7. [BS, 2014, Theorem 1.6] *The function h_ν is convex of order $\alpha \in [0,1)$ in $\mathbb{D}$ if and only if $\nu \geq \nu_\alpha(h_\nu)$, where $\nu_\alpha(h_\nu)$ is the unique root of the equation*

$$(2\nu + 2\alpha - 4)J_{\nu+1}(1) = (4\alpha - 3)J_\nu(1),$$

situated in $[0, \infty)$. In particular, h_ν is convex if and only if $\nu \geq \nu_0(h_\nu)$, where $\nu_0(h_\nu) \simeq -0.1438\ldots$ is the unique root of the equation

$$(2\nu - 4)J_{\nu+1}(1) + 3J_\nu(1) = 0.$$

Moreover, in particular, the function h_ν is convex of order $\frac{3}{4}$ if and only if $\nu \geq \frac{5}{4}$.

Selinger [Se, 1995] by using the method of differential subordinations proved that the function $\varphi_\nu : \mathbb{D} \to \mathbb{C}$, defined by

$$\varphi_\nu(z) = \frac{h_\nu(z)}{z} = 2^\nu \Gamma(\nu+1) z^{-\frac{\nu}{2}} J_\nu(\sqrt{z}) = 1 - \frac{1}{4(\nu+1)} z + \ldots, \ z \in \mathbb{C} \setminus \{0\},$$

is convex if $\nu \geq -\frac{1}{4}$. In 2009 Szász and Kupán [SK, 2009], by using a completely different approach, improved this result, and proved that φ_ν is convex in $\mathbb{D}$ if $\nu \geq \nu_1 \simeq -1.4069\ldots$, where ν_1 is the root of the equation $4\nu^2 + 17\nu + 16 = 0$. Recently, Baricz and Ponnusamy [BP, 2010] presented four improvements of the above result, and their best result was the following [BP, 2010, Theorem 3]: the function φ_ν is convex in $\mathbb{D}$ if $\nu \geq \nu_2 \simeq -1.4373\ldots$, where ν_2 is the unique root of the equation $2^\nu \Gamma(\nu+1)(I_{\nu+2}(1) + 2I_{\nu+1}(1)) = 2$. Moreover, Baricz and Ponnusamy [BP, 2010] conjectured that φ_ν is convex in $\mathbb{D}$ if and only if $\nu \geq -1.875$. Now, we are able to disprove this conjecture and to find the radius of convexity of the function φ_ν.

Theorem 8. [BS, 2014, Theorem 1.7] *If $\nu > -2$ and $\alpha \in [0,1)$, then the radius of convexity of order α of the function φ_ν is the smallest positive root of the equation*

$$\frac{r^{\frac{1}{2}} J_\nu(r^{\frac{1}{2}})}{2J_{\nu+1}(r^{\frac{1}{2}})} - \nu = \alpha.$$

Moreover, we have $r^c_\alpha(\varphi_\nu) < j_{\nu+1,1}$.

Theorem 9. [BS, 2014, Theorem 1.8] *The function φ_ν is convex of order $\alpha \in [0,1)$ in $\mathbb{D}$ if and only if $\nu \geq \nu_\alpha(\varphi_\nu)$, where $\nu_\alpha(\varphi_\nu)$ is the unique root of the equation*

$$(2\nu + 2\alpha) J_{\nu+1}(1) = J_\nu(1),$$

situated in $(\nu^\star, \infty)$, where $\nu^\star \simeq -1.7744\ldots$ is the root of the equation $J_{\nu+1}(1) = 0$. In particular, φ_ν is convex in $\mathbb{D}$ if and only if $\nu \geq \nu_0(\varphi_\nu)$, where $\nu_0(\varphi_\nu) \simeq -1.5623\ldots$ is the unique root of the equation $J_\nu(1) = 2\nu J_{\nu+1}(1)$, situated in $(\nu^\star, \infty)$.

3. Radii of starlikeness and convexity of normalized q-Bessel functions

In this section we consider the Jackson and Hahn-Exton (or third Jackson) q-Bessel functions. They are explicitly defined by

$$J_\nu^{(2)}(z;q) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} \sum_{n\geq 0} \frac{(-1)^n \left(\frac{z}{2}\right)^{2n+\nu}}{(q;q)_n \left(q^{\nu+1};q\right)_n} q^{n(n+\nu)}$$

and

$$J_\nu^{(3)}(z;q) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} \sum_{n\geq 0} \frac{(-1)^n z^{2n+\nu}}{(q;q)_n \left(q^{\nu+1};q\right)_n} q^{\frac{1}{2}n(n+1)},$$

where $z \in \mathbb{C}$, $\nu > -1$, $q \in (0,1)$ and

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{k=1}^{n} \left(1 - aq^{k-1}\right), \quad (a;q)_\infty = \prod_{k\geq 1} \left(1 - aq^{k-1}\right).$$

A common feature of these analytic functions is that they are q-extensions of the classical Bessel function of the first kind J_ν. Namely, for fixed z we have $J_\nu^{(2)}((1-z)q;q) \to J_\nu(z)$ and $J_\nu^{(3)}((1-z)q;q) \to J_\nu(2z)$ as $q \nearrow 1$. Watson's treatise [Wa, 1944] contains comprehensive information about the Bessel function of the first kind and properties of the above q-extensions of Bessel functions can be found in [Ab, 2005, AMA, 2010, Is, 1982, IM2, 1988, Ko, 1994, Ko, 1992] and in the references therein. Motivated by the results on classical Bessel functions, in this section we study the geometric properties of the Jackson and Hahn-Exton (or third Jackson) q-Bessel functions. For each of them, three different normalization are applied in such a way that the resulting functions are analytic in the unit disk of the complex plane. Using the Weierstrassian decomposition of $J_\nu^{(2)}$ and $J_\nu^{(3)}$ and combining the methods from [BKS, 2014, BS, 2014, BDOY, 2016] we determine precisely the radii of starlikeness and convexity for each of the six functions. These results are the q-generalizations of the corresponding results for Bessel functions of the first kind obtained in [BKS, 2014, BS, 2014]. A characterization of entire functions from the Laguerre-Pólya class involving hyperbolic polynomials and an interlacing property of the zeros of Jackson and Hahn-Exton q-Bessel functions and their derivatives play an important role in the proofs. We establish a necessary and sufficient condition for the close-to-convexity of a normalized Jackson q-Bessel function and its derivatives. The results obtained in the present section show that there is no essential difference between Jackson and Hahn-Exton q-Bessel functions when one threats the problem about the radii of starlikeness and convexity. Therefore one may expect that other geometric properties of these two q-extensions of Bessel's function are also similar.

Observe that neither $J_\nu^{(2)}(\cdot;q)$, nor $J_\nu^{(3)}(\cdot;q)$ belongs to $\mathcal{A}$, and thus first we perform some natural normalization. For $\nu > -1$ we define three functions originating from $J_\nu^{(2)}(\cdot;q)$:

$$f_\nu^{(2)}(z;q) = \left(2^\nu c_\nu(q) J_\nu^{(2)}(z;q)\right)^{\frac{1}{\nu}},\ \nu \neq 0,$$

$$g_\nu^{(2)}(z;q) = 2^\nu c_\nu(q) z^{1-\nu} J_\nu^{(2)}(z;q),$$

$$h_\nu^{(2)}(z;q) = 2^\nu c_\nu(q) z^{1-\frac{\nu}{2}} J_\nu^{(2)}(\sqrt{z};q),$$

where $c_\nu(q) = (q;q)_\infty / \left(q^{\nu+1};q\right)_\infty$. Similarly, we associate with $J_\nu^{(3)}(\cdot;q)$ the functions

$$f_\nu^{(3)}(z;q) = \left(c_\nu(q) J_\nu^{(3)}(z;q)\right)^{\frac{1}{\nu}},\ \nu \neq 0,$$

$$g_\nu^{(3)}(z;q) = c_\nu(q) z^{1-\nu} J_\nu^{(3)}(z;q),$$

$$h_\nu^{(3)}(z;q) = c_\nu(q) z^{1-\frac{\nu}{2}} J_\nu^{(3)}(\sqrt{z};q).$$

Clearly the functions $f_\nu^{(s)}(\cdot;q)$, $g_\nu^{(s)}(\cdot;q)$, $h_\nu^{(s)}(\cdot;q)$, $s\in\{2,3\}$, belong to the class $\mathcal{A}$.

The first principal result we establish concerns the radii of starlikeness and reads as follows.

Theorem 10. [BDM, 2016, Theorem 1] *Let $\nu>-1$ and $s\in\{2,3\}$. The following statements hold:*

a. *If $\alpha\in[0,1)$ and $\nu>0$, then $r_\alpha^*\left(f_\nu^{(s)}\right)=x_{\nu,\alpha,1}$, where $x_{\nu,\alpha,1}$ is the smallest positive root of the equation*

$$r\cdot dJ_\nu^{(s)}(r;q)/dr-\alpha\nu J_\nu^{(s)}(r;q)=0.$$

Moreover, if $\alpha\in[0,1)$ and $\nu\in(-1,0)$, then $r_\alpha^*\left(f_\nu^{(s)}\right)=x_{\nu,\alpha}$, where $x_{\nu,\alpha}$ is the unique positive root of the equation

$$\mathrm{i}r\cdot dJ_\nu^{(s)}(\mathrm{i}r;q)/dr-\alpha\nu J_\nu^{(s)}(\mathrm{i}r;q)=0.$$

b. *If $\alpha\in[0,1)$, then $r_\alpha^*\left(g_\nu^{(s)}\right)=y_{\nu,\alpha,1}$, where $y_{\nu,\alpha,1}$ is the smallest positive root of the equation*

$$r\cdot dJ_\nu^{(s)}(r;q)/dr-(\alpha+\nu-1)J_\nu^{(s)}(r;q)=0.$$

c. *If $\alpha\in[0,1)$, then $r_\alpha^*\left(h_\nu^{(s)}\right)=z_{\nu,\alpha,1}$, where $z_{\nu,\alpha,1}$ is the smallest positive root of the equation*

$$\sqrt{r}\cdot dJ_\nu^{(s)}(\sqrt{r};q)/dr-(2\alpha+\nu-2)J_\nu^{(s)}(\sqrt{r};q)=0.$$

Our second result of this section concerns the radii of convexity.

Theorem 11. [BDM, 2016, Theorem 2] *Let $\nu>-1$ and $s\in\{2,3\}$. The following statements hold:*

a. *If $\nu>0$ and $\alpha\in[0,1)$, then the radius of convexity of order α of the function $f_\nu^{(s)}(\cdot;q)$ is the smallest positive root of the equation*

$$1+\frac{r\cdot d^2J_\nu^{(s)}(r;q)/dr^2}{dJ_\nu^{(s)}(r;q)/dr}+\left(\frac{1}{\nu}-1\right)\frac{r\cdot dJ_\nu^{(s)}(r;q)/dr}{J_\nu(r;q)}=\alpha.$$

Moreover, we have $r_\alpha^c(f_\nu^{(2)}) < j'_{\nu,1}(q) < j_{\nu,1}(q)$ *and* $r_\alpha^c(f_\nu^{(3)}) < l'_{\nu,1}(q) < l_{\nu,1}(q)$, *where* $j_{\nu,1}(q)$, $l_{\nu,1}(q)$, $j'_{\nu,1}(q)$ *and* $l'_{\nu,1}(q)$ *are the first positive zeros of the functions* $J_\nu^{(2)}(\cdot;q)$, $J_\nu^{(3)}(\cdot;q)$, $z \mapsto dJ_\nu^{(2)}(z;q)/dz$ *and* $z \mapsto dJ_\nu^{(3)}(z;q)/dz$.

b. *If* $\nu > -1$ *and* $\alpha \in [0,1)$, *then the radius of convexity of order* α *of the function* $g_\nu^{(s)}(\cdot;q)$ *is the smallest positive root of the equation*

$$1-\nu+r\frac{(2-\nu)\cdot dJ_\nu^{(s)}(r;q)/dr + r\cdot d^2J_\nu^{(s)}(r;q)/dr^2}{(1-\nu)J_\nu^{(s)}(r;q)+r\cdot dJ_\nu^{(s)}(r;q)/dr}=\alpha.$$

Moreover, we have $r_\alpha^c(g_\nu^{(2)}) < \alpha_{\nu,1}(q) < j_{\nu,1}(q)$ *and* $r_\alpha^c(g_\nu^{(3)}) < \gamma_{\nu,1}(q) < l_{\nu,1}(q)$, *where* $\alpha_{\nu,1}(q)$ *and* $\gamma_{\nu,1}(q)$ *are the first positive zeros of the functions* $z \mapsto z\cdot dJ_\nu^{(2)}(z;q)/dz + (1-\nu)J_\nu^{(2)}(z;q)$ *and* $z \mapsto z\cdot dJ_\nu^{(3)}(z;q)/dz + (1-\nu)J_\nu^{(3)}(z;q)$.

c. *If* $\nu > -1$ *and* $\alpha \in [0,1)$, *then the radius of convexity of order* α *of the function* $h_\nu^{(s)}(\cdot;q)$ *is the smallest positive root of the equation*

$$1-\frac{\nu}{2}+\frac{\sqrt{r}}{2}\frac{(3-\nu)\cdot dJ_\nu^{(s)}(\sqrt{r};q)/dr+\sqrt{r}\cdot d^2J_\nu^{(s)}(\sqrt{r};q)/dr^2}{(2-\nu)J_\nu^{(s)}(\sqrt{r};q)+\sqrt{r}\cdot dJ_\nu^{(s)}(\sqrt{r};q)/dr}=\alpha.$$

Moreover, we have $r_\alpha^c(h_\nu^{(2)}) < \beta_{\nu,1}(q) < j_{\nu,1}(q)$ *and* $r_\alpha^c(h_\nu^{(3)}) < \delta_{\nu,1}(q) < l_{\nu,1}(q)$, *where* $\beta_{\nu,1}(q)$ *and* $\delta_{\nu,1}(q)$ *are the first positive zeros of the functions* $z \mapsto z\cdot dJ_\nu^{(2)}(z;q)/dz + (2-\nu)J_\nu^{(2)}(z;q)$, *and* $z \mapsto z\cdot dJ_\nu^{(3)}(z;q)/dz + (2-\nu)J_\nu^{(3)}(z;q)$.

We note that these theorems are natural q-extension to Jackson and Hahn-Exton (or third Jackson) q-Bessel functions of the results obtained in [BKS, 2014] and [BS, 2014].

Finally, we state a result, which is the q-extension of the first part of [BS, 2016, Theorem 1] for the Jackson q-Bessel function.

Theorem 12. [BDM, 2016, Theorem 3] *If* $\nu > -1$, *then* $h_\nu(\cdot;q) = h_\nu^{(2)}(\cdot;q)$ *is starlike and all of its derivatives are close-to-convex in* $\mathbb{D}$ *if and only if* $\nu \geq \max\{\nu_0(q), \nu^*(q)\}$, *where* $\nu_0(q)$ *is the unique root of the equation* $dh_\nu^{(2)}(z;q)/dz\Big|_{z=1} = h'_\nu(1;q) = 0$, *and* $\nu^*(q)$ *is the unique root of the equation* $j_{\nu,1}(q) = 1$.

4. Radii of starlikeness of some special functions

In this section geometric properties of the classical Lommel and Struve functions, both of the first kind, are studied. For each of them, there different normalization are applied in such a way that the resulting functions are analytic in the unit disc of the complex plane. For each of the six functions we determine the radius of starlikeness precisely. The reality and interlacing properties of the zeros of Struve and Lommel functions of the first play also an important role in the proofs of the main results. The results of the zeros are realized via properties of real entire functions belonging to the Laguerre-Pólya class. An old result of Pólya plays also an important role.

We consider two classical special functions, the Lommel function of the first kind $s_{\mu,\nu}$ and the Struve function of the first kind $\mathbf{H}_\nu$. They are explicitly defined in terms of the hypergeometric function ${}_1F_2$ by

$$s_{\mu,\nu}(z) = \frac{z^{\mu+1}}{(\mu-\nu+1)(\mu+\nu+1)}\, {}_1F_2\left(1; \frac{\mu-\nu+3}{2}, \frac{\mu+\nu+3}{2}; -\frac{z^2}{4}\right), \tag{4.1}$$

where $\frac{1}{2}(-\mu \pm \nu - 3) \notin \mathbb{N}$, and

$$\mathbf{H}_\nu(z) = \frac{\left(\frac{z}{2}\right)^{\nu+1}}{\sqrt{\frac{\pi}{4}}\,\Gamma\left(\nu+\frac{3}{2}\right)}\, {}_1F_2\left(1; \frac{3}{2}, \nu+\frac{3}{2}; -\frac{z^2}{4}\right), \quad -\nu-\frac{3}{2} \notin \mathbb{N}. \tag{4.2}$$

A common feature of these functions is that they are solutions of inhomogeneous Bessel differential equations [Wa, 1944]. Indeed, the Lommel function of the first kind $s_{\mu,\nu}$ is a solution of

$$z^2w''(z) + zw'(z) + (z^2-\nu^2)w(z) = z^{\mu+1}$$

while the Struve function $\mathbf{H}_\nu$ obeys

$$z^2w''(z) + zw'(z) + (z^2-\nu^2)w(z) = \frac{4\left(\frac{z}{2}\right)^{\nu+1}}{\sqrt{\pi}\Gamma\left(\nu+\frac{1}{2}\right)}.$$

We refer to Watson's treatise [Wa, 1944] for comprehensive information about these functions and recall some more recent contributions. In 1972 Steinig [St, 1972] examined the sign of $s_{\mu,\nu}(z)$ for real μ,ν and positive z. He showed, among other things, that for $\mu < \frac{1}{2}$ the function $s_{\mu,\nu}$ has infinitely many changes of sign on $(0,\infty)$. In 2012 Koumandos and Lamprecht [KL, 2012] obtained sharp estimates for the location of the zeros of $s_{\mu-\frac{1}{2},\frac{1}{2}}$ when $\mu \in (0,1)$. The Turán type inequalities for $s_{\mu-\frac{1}{2},\frac{1}{2}}$ were established in [BK, 2016] while those for the Struve function were proved in [BPS, 2017].

Geometric properties of $s_{\mu-\frac{1}{2},\frac{1}{2}}$ and of the Struve function were obtained in [BS, 2016] and in [YO, 2013], respectively. Motivated by those results we study the problem of starlikeness of certain analytic functions related to the classical special functions under discussion. Since neither $s_{\mu,\nu}$, nor $\mathbf{H}_\nu$ belongs to $\mathcal{A}$, first we perform some natural normalization. We define three functions originating from $s_{\mu,\nu}$:

$$f_{\mu,\nu}(z) = ((\mu-\nu+1)(\mu+\nu+1)s_{\mu,\nu}(z))^{\frac{1}{\mu+1}},$$

$$g_{\mu,\nu}(z) = (\mu-\nu+1)(\mu+\nu+1)z^{-\mu}s_{\mu,\nu}(z)$$

and

$$h_{\mu,\nu}(z) = (\mu-\nu+1)(\mu+\nu+1)z^{\frac{1-\mu}{2}}s_{\mu,\nu}(\sqrt{z}).$$

Similarly, we associate with $\mathbf{H}_\nu$ the functions

$$u_\nu(z) = \left(\sqrt{\pi}2^\nu\Gamma\left(\nu+\frac{3}{2}\right)\mathbf{H}_\nu(z)\right)^{\frac{1}{\nu+1}},$$

$$v_\nu(z) = \sqrt{\pi}2^\nu z^{-\nu}\Gamma\left(\nu+\frac{3}{2}\right)\mathbf{H}_\nu(z)$$

and

$$w_\nu(z) = \sqrt{\pi}2^\nu z^{\frac{1-\nu}{2}}\Gamma\left(\nu+\frac{3}{2}\right)\mathbf{H}_\nu(\sqrt{z}).$$

Clearly the functions $f_{\mu,\nu}$, $g_{\mu,\nu}$, $h_{\mu,\nu}$, u_ν, v_ν and w_ν belong to the class $\mathcal{A}$. The main results of the present section concern the exact values of the radii of starlikeness for these six function, for some ranges of the parameters.

Let us set $f_\mu(z) = f_{\mu-\frac{1}{2},\frac{1}{2}}(z)$, $g_\mu(z) = g_{\mu-\frac{1}{2},\frac{1}{2}}(z)$ and $h_\mu(z) = h_{\mu-\frac{1}{2},\frac{1}{2}}(z)$. The first principal result we establish reads as follows:

Theorem 13. [BDOY, 2016, Theorem 1] *Let $\mu \in (-1,1)$, $\mu \neq 0$. The following statements hold:*

a. *If $0 \le \alpha < 1$ and $\mu \in \left(-\frac{1}{2},0\right)$, then $r^*_\alpha(f_\mu) = x_{\mu,\alpha}$, where $x_{\mu,\alpha}$ is the smallest positive root of the equation*

$$z\,s'_{\mu-\frac{1}{2},\frac{1}{2}}(z) - \alpha\left(\mu+\frac{1}{2}\right)s_{\mu-\frac{1}{2},\frac{1}{2}}(z) = 0.$$

In particular, $r^*(f_\mu) = x_\mu$, where x_μ is the smallest positive root of the equation $z\, s'_{\mu-\frac{1}{2},\frac{1}{2}}(z) = 0$. Moreover, if $0 \le \alpha < 1$ and $\mu \in \left(-1, -\frac{1}{2}\right)$, then $r^*_\alpha(f_\mu) = q_{\mu,\alpha}$, where $q_{\mu,\alpha}$ is the unique positive root of the equation

$$\mathrm{i}z s'_{\mu-\frac{1}{2},\frac{1}{2}}(\mathrm{i}z) - \alpha\left(\mu + \frac{1}{2}\right) s_{\mu-\frac{1}{2},\frac{1}{2}}(\mathrm{i}z) = 0.$$

b. *If $0 \le \alpha < 1$, then $r^*_\alpha(g_\mu) = y_{\mu,\alpha}$, where $y_{\mu,\alpha}$ is the smallest positive root of the equation*

$$z\, s'_{\mu-\frac{1}{2},\frac{1}{2}}(z) - \left(\mu + \alpha - \frac{1}{2}\right) s_{\mu-\frac{1}{2},\frac{1}{2}}(z) = 0.$$

In particular, $r^*(g_\mu) = y_\mu$, where y_μ is the smallest positive root of the equation

$$z\, s'_{\mu-\frac{1}{2},\frac{1}{2}}(z) - \left(\mu - \frac{1}{2}\right) s_{\mu-\frac{1}{2},\frac{1}{2}}(z) = 0.$$

c. *If $0 \le \alpha < 1$, then $r^*_\alpha(h_\mu) = t_{\mu,\alpha}$, where $t_{\mu,\alpha}$ is the smallest positive root of the equation*

$$z s'_{\mu-\frac{1}{2},\frac{1}{2}}(z) - \left(\mu + 2\alpha - \frac{3}{2}\right) s_{\mu-\frac{1}{2},\frac{1}{2}}(z) = 0.$$

In particular, $r^*(h_\mu) = t_\mu$, where t_μ is the smallest positive root of the equation

$$z s'_{\mu-\frac{1}{2},\frac{1}{2}}(z) - \left(\mu - \frac{3}{2}\right) s_{\mu-\frac{1}{2},\frac{1}{2}}(z) = 0.$$

The corresponding result about the radii of starlikeness of the functions, related to Struve's one, is the following theorem.

Theorem 14. [BDOY, 2016, Theorem 2] *Let $|\nu| < \frac{1}{2}$. The following assertions are true:*

a. *If $0 \le \alpha < 1$, then $r^*_\alpha(u_\nu) = \delta_{\nu,\alpha}$, where $\delta_{\nu,\alpha}$ is the smallest positive root of the equation*

$$z\mathbf{H}'_\nu(z) - \alpha(\nu + 1)\mathbf{H}_\nu(z) = 0.$$

In particular, $r^*(u_\nu) = \delta_\nu$, where δ_ν is the smallest positive root of the equation

$$z\mathbf{H}'_\nu(z) = 0.$$

b. *If* $0 \leq \alpha < 1$, *then* $r^*(v_\nu) = \rho_{\nu,\alpha}$, *where* $\rho_{\nu,\alpha}$ *is the smallest positive root of the equation*

$$z\mathbf{H}'_\nu(z) - (\alpha + \nu)\mathbf{H}_\nu(z) = 0.$$

In particular, $r^*(v_\nu) = \rho_\nu$, *where* ρ_ν *is the smallest positive root of the equation*

$$z\mathbf{H}'_\nu(z) - \nu\mathbf{H}_\nu(z) = 0.$$

c. *If* $0 \leq \alpha < 1$, *then* $r^*_\alpha(w_\nu) = \sigma_{\nu,\alpha}$, *where* $\sigma_{\nu,\alpha}$ *is the smallest positive root of the equation*

$$z\mathbf{H}'_\nu(z) - (2\alpha + \nu - 1)\mathbf{H}_\nu(z) = 0.$$

In particular, $r^*(w_\nu) = \sigma_\nu$, *where* σ_ν *is the smallest positive root of the equation*

$$z\mathbf{H}'_\nu(z) - (\nu - 1)\mathbf{H}_\nu(z) = 0.$$

It is worth mentioning that the starlikeness of h_μ, when $\mu \in (-1, 1)$, $\mu \neq 0$, as well as of w_ν, under the restriction $|\nu| \leq \frac{1}{2}$, were established in [BS, 2016], and it was proved there that all the derivatives of these functions are close-to-convex in $\mathbb{D}$.

5. Close-to-convexity of normalized Dini functions

Recently, in [BCD, 2016] and [BS, 2016] the close-to-convexity of the derivatives of Bessel functions has been considered. In this section we make a contribution to the subject by deducing necessary and sufficient conditions for the close-to-convexity of some special combinations of Bessel functions of the first kind and their derivatives. In order to prove our sharp main results we use a result of Shah and Trimble [ST, 1971, Theorem 2] about transcendental entire functions with univalent derivatives and some newly discovered Mittag-Leffler expansions for Bessel functions of the first kind.

Now, consider the function $h_\nu : \mathbb{D} \to \mathbb{C}$, defined by

$$h_\nu(z) = 2^\nu \Gamma(\nu + 1) z^{1-\frac{\nu}{2}} J_\nu(\sqrt{z}) = \sum_{n \geq 0} \frac{(-1)^n \Gamma(\nu + 1) z^{n+1}}{4^n n! \Gamma(\nu + n + 1)},$$

where J_ν stands for the Bessel function of the first kind (see [OLBC, 2010, p. 217]). Very recently by using a result of Shah and Trimble [ST, 1971, Theorem 2] in [BS, 2016] the authors proved that the function h_ν and all of its derivatives are convex in $\mathbb{D}$ if and only if $\nu \geq \nu_\star$, where $\nu_\star \simeq -0.1438\ldots$ is the unique root of the equation

$$3J_\nu(1) + 2(\nu-2)J_{\nu+1}(1) = 0$$

on $(-1,\infty)$. We note that in view of the Alexander's duality theorem the first part of the above result is equivalent to the fact that the function

$$\begin{aligned} z \mapsto q_\nu(z) = zh_\nu'(z) &= 2^{\nu-1}\Gamma(\nu+1)z^{1-\frac{\nu}{2}}\left((2-\nu)J_\nu(\sqrt{z}) + \sqrt{z}J_\nu'(\sqrt{z})\right) \\ &= \sum_{n\geq 0} \frac{(-1)^n(n+1)\Gamma(\nu+1)z^{n+1}}{4^n n!\Gamma(\nu+n+1)} \end{aligned}$$

is starlike in $\mathbb{D}$ if and only if $\nu \geq \nu_\star$. In this section we would like to point out that actually we have the following stronger result.

Theorem 15. [BDY, 2016, Theorem 1.2] *The function q_ν is starlike and all of its derivatives are close-to-convex (and hence univalent) in $\mathbb{D}$ if and only if $\nu \geq \nu_\star$, where $\nu_\star \simeq -0.1438\ldots$ is the unique root of the transcendental equation $3J_\nu(1)+2(\nu-2)J_{\nu+1}(1) = 0$ on $(-1,\infty)$.*

Moreover, in this section we are interested on the normalized Dini function $r_\nu : \mathbb{D} \to \mathbb{C}$, which is another special combination of Bessel functions of the first kind and is defined as

$$\begin{aligned} r_\nu(z) &= 2^\nu\Gamma(\nu+1)z^{1-\frac{\nu}{2}}\left((1-\nu)J_\nu(\sqrt{z}) + \sqrt{z}J_\nu'(\sqrt{z})\right) \\ &= \sum_{n\geq 0} \frac{(-1)^n(2n+1)\Gamma(\nu+1)z^{n+1}}{4^n n!\Gamma(\nu+n+1)}. \end{aligned}$$

By using the idea of the proof of Theorem 15 our aim is to present the following interesting sharp result. We note that some similar results were proved for Bessel functions of the first kind in [BKS, 2014, BS, 2014, BS, 2016], but by using different approaches.

Theorem 16. [BDY, 2016, Theorem 1.3] *The function r_ν is starlike and all of its derivatives are close-to-convex (and hence univalent) in $\mathbb{D}$ if and only if $\nu \geq \nu^\star$, where $\nu^\star \simeq$*

$0.3062\ldots$ *is the unique root of the transcendental equation* $J_\nu(1) - (3-2\nu)J_{\nu+1}(1) = 0$ *on* $(0,\infty)$.

It is important to mention here that the first part of the above result is quite similar to the following sharp result (see Theorem 6 or [BS, 2014, Theorem 1.5]): the function $z \mapsto g_\nu(z) = 2^\nu \Gamma(\nu+1) z^{1-\nu} J_\nu(z)$ is convex in $\mathbb{D}$ if and only if $\nu \geq 1$. Note that according to the Alexander's duality theorem this result is equivalent to the following: the function

$$z \mapsto 2^\nu \Gamma(\nu+1) z^{1-\nu}\left((1-\nu)J_\nu(z) + zJ'_\nu(z)\right) = \frac{r_\nu(z^2)}{z}$$

is starlike in $\mathbb{D}$ if and only if $\nu \geq 1$.

Now, let us consider the function $w_{a,\nu} : \mathbb{D} \to \mathbb{C}$, defined by

$$\begin{aligned} w_{a,\nu}(z) &= \frac{2^\nu}{a}\Gamma(\nu+1) z^{1-\frac{\nu}{2}}\left((a-\nu)J_\nu(\sqrt{z}) + \sqrt{z}J'_\nu(\sqrt{z})\right) \\ &= \sum_{n\geq 0} \frac{(-1)^n(2n+a)\Gamma(\nu+1)z^{n+1}}{a\cdot 4^n n!\Gamma(n+\nu+1)}. \end{aligned}$$

The following sharp result is a common generalization of Theorems 15 and 16.

Theorem 17. [BDY, 2016, Theorem 1.4] *Let* $\nu > -\frac{3}{4}$ *and* $a \geq \frac{2}{4\nu+3}$. *The function* $w_{a,\nu}$ *is starlike and all of its derivatives are close-to-convex (and hence univalent) in* $\mathbb{D}$ *if and only if* $\nu \geq \nu_a$, *where* ν_a *is the unique root of the transcendental equation* $(2a-1)J_\nu(1) - (a-2\nu+2)J_{\nu+1}(1) = 0$ *on* $\left(-\frac{3}{4},\infty\right)$.

Finally, we mention that in particular Theorem 15 and Theorem 16 yield that

$$z \mapsto q_{\frac{1}{2}}(z) = \frac{3}{2}\sqrt{z}\left(\sin\sqrt{z} + \sqrt{z}\cos\sqrt{z}\right),$$

$$z \mapsto q_{\frac{3}{2}}(z) = \frac{3}{2\sqrt{z}}\left(\sqrt{z}\cos\sqrt{z} + (z-1)\sin\sqrt{z}\right),$$

$$z \mapsto r_{\frac{1}{2}}(z) = z\cos\sqrt{z}$$

and

$$z \mapsto r_{\frac{3}{2}}(z) = 3\cos\sqrt{z} - \frac{3(z-2)\sin\sqrt{z}}{2\sqrt{z}}$$

are starlike in $\mathbb{D}$ and all of their derivatives are close-to-convex (and hence univalent) there. Moreover, by using the above result we obtain that

$$z \mapsto \frac{r_{\frac{3}{2}}(z^2)}{z} = \frac{3\cos z}{z} - \frac{3(z^2-2)\sin z}{2z^2}$$

is starlike in $\mathbb{D}$. Here we used that [OLBC, 2010, p. 228]

$$J_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}}\sin z \quad \text{and} \quad J_{\frac{3}{2}}(z) = \sqrt{\frac{2}{\pi z}}\left(\frac{\sin z}{z} - \cos z\right).$$

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