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Habilitation Thesis
ABSTRACT

Contributions to Brown representability problem

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2018

1 Preliminaries

Brown representability is replacement for the celebrated Freyd's Adjoint Functor Theorem, allowing us to construct adjoint functors, in the setting of triangulated categories. In the following $\mathcal{T}$ is a triangulated category and $\mathcal{A}$ is an additive category (often $\mathcal{A}$ is even abelian).

We say that $\mathcal{T}$ satisfies *Brown representability* if it has coproducts and every cohomological functor $F : \mathcal{T} \to \mathcal{A}b$ which sends coproducts into products is (contravariantly) representable, that is, it is naturally isomorphic to $\mathcal{T}(-, X)$ for some $X \in \mathcal{T}$.

2 Abelianization

The category $\mathrm{mod}(\mathcal{T})$ of all functors $F : \mathcal{T}^o \to \mathcal{A}b$ for which there is an exact sequence

$$\mathcal{T}(-, X) \to \mathcal{T}(-, Y) \to F \to 0,$$

is called the *abelianization* of $\mathcal{T}$.

2.1 A reformulation of Brown representability

Theorem 2.1.1. *The following are equivalent, for a triangulated category with arbitrary coproducts $\mathcal{T}$:*

(i) $\mathcal{T}$ satisfies the Brown representability theorem.

(ii) For every homological, coproducts preserving functor $f : \mathcal{T} \to \mathcal{A}$, into an abelian AB3 category with enough injectives $\mathcal{A}$, the induced functor

$$f_* : \mathrm{mod}(\mathcal{T}) \to \mathcal{A}$$

has a right adjoint.

(iii) Every exact, coproducts preserving functor $F : \mathrm{mod}(\mathcal{T}) \to \mathcal{A}$, into an abelian AB3 category with enough injectives $\mathcal{A}$, has a right adjoint.

(iv) Every exact, coproducts preserving functor $F : \mathrm{mod}(\mathcal{T}) \to \mathcal{A}b^o$ has a right adjoint.

2.2 Heller's criterion revisited

We say that $F : \mathcal{T} \to \mathcal{A}b$ has a *solution object* provided that there is an object $S \in \mathcal{T}$ and a functorial epimorphism

$$\mathcal{T}(S, -) \to F \to 0.$$

The next Theorem was shown by Heller in [28, Theorem 1.4], hence we call it *Heller's criterion* of representability.

Theorem 2.2.3. *If $\mathcal{T}$ is a triangulated category with products, then a homological product preserving functor $F : \mathcal{T} \to \mathcal{A}b$ is representable if and only if it has a solution object.*

3 Deconstructibility in triangulated categories

3.1 Deconstructibiliy

Consider a Σ-closed set of objects in $\mathcal{T}$ and denote it by $\mathcal{S}$. We define $\mathrm{Prod}(\mathcal{S})$ to be the full subcategory of $\mathcal{T}$ consisting of all direct factors of products of objects in $\mathcal{S}$. Next we define inductively $\mathrm{Prod}_0(\mathcal{S}) = \{0\}$ and $\mathrm{Prod}_n(\mathcal{S})$ is the full subcategory of $\mathcal{T}$ which consists of all objects Y lying in a triangle

$$X \to Y \to Z \to \Sigma X$$

with $X \in \mathrm{Prod}(\mathcal{S})$ and $Z \in \mathrm{Prod}_n(\mathcal{S})$. An object $X \in \mathcal{T}$ will be called $\mathcal{S}$-*cofiltered* if it may be written as a homotopy limit $X \cong \underleftarrow{\mathrm{holim}}\, X_n$ of an inverse tower

$$X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots$$

with $X_0 \in \mathrm{Prod}_0(\mathcal{S})$, and X_{n+1} lying in a triangle $P_n \to X_{n+1} \to X_n \to \Sigma P_n$, for some $P_n \in \mathrm{Prod}_1(\mathcal{S})$. Inductively we have $X_n \in \mathrm{Prod}_n(\mathcal{S})$, for all $n \in \mathbb{N}^*$.

We say that $\mathcal{T}$ (respectively, $\mathcal{T}^o$) is *deconstructible* if $\mathcal{T}$ has coproducts (products) and there is a Σ-closed set $\mathcal{S} \subseteq \mathcal{T}$, which is not a proper class, such that every object $X \in \mathcal{T}$ is $\mathcal{S}$-filtered (cofiltered).

Theorem 3.1.3. *Let $\mathcal{T}$ be a triangulated category with products. If $\mathcal{T}^o$ is deconstructible, then $\mathcal{T}^o$ satisfies Brown representability.*

This Theorem is called the *deconstructibility criterion* for Brown representability.

3.3 Well–generation and deconstructibility

Theorem 3.3.3. *Let $\mathcal{T}$ be a triangulated category with coproducts which is $\aleph_1$-perfectly generated by a set. Then $\mathcal{T}$ is deconstructible and satisfies Brown representability.*

Corollary 3.3.5. *If $\mathcal{T}$ is a well–generated triangulated category then $\mathcal{T}$ is deconstructible, therefore it satisfies Brown representability.*

4 Quasi-locally presentable categories

4.1 Quasi-locally presentable abelian categories

Denote by $\mathfrak{R}$ the class of all regular cardinals.

We consider a cocomplete category $\mathcal{A}$ which is a union

$$\mathcal{A} = \bigcup_{\lambda \in \mathfrak{R}} \mathcal{A}_\lambda,$$

of a chain of subcategories $\{\mathcal{A}_\lambda \mid \lambda \in \mathfrak{R}\}$ such that $\mathcal{A}_\kappa \subseteq \mathcal{A}_\lambda$ for all $\kappa \leq \lambda$, the subcategory $\mathcal{A}_\lambda$ locally λ–presentable and the inclusion functor $I_\lambda : \mathcal{A}_\lambda \to \mathcal{A}$ has a right adjoint $R_\lambda : \mathcal{A} \to \mathcal{A}_\lambda$, for any $\lambda \in \mathfrak{R}$. closed under colimits in $\mathcal{A}$, for any $\lambda \in \mathfrak{R}$. We call *quasi-locally presentable* a category $\mathcal{A}$ as above satisfying the additional property that R_λ preserves colimits for all $\lambda \in \mathfrak{R}$.

Theorem 4.1.5. *Let $\mathcal{A}$ be a quasi-locally presentable, abelian category satisfying some additional technical conditions. Then every exact, contravariant functor $F : \mathcal{A} \to \mathcal{A}b$ which sends coproducts into products is representable (necessarily by an injective object).*

4.2 The abelianization of a well–generated triangulated category is quasi-locally presentable

Proposition 4.2.2. *Fix a regular cardinal $\kappa > \aleph_0$. If $\mathcal{T}$ is a a well-generated, namely compactly κ–generated triangulated category, then* $\mathrm{mod}(\mathcal{T})$ *is a quasi-locally presentable abelian category satisfying the additional propreties from Theorem 4.1.5.*

5 Homotopy category of complexes

5.1 Homtopy categories satisfying Brown representability

The category $\mathcal{T}$ is called *locally well–generated* if for any set $\mathcal{S}$ (not a proper class!) of objects of $\mathcal{T}$, $\mathrm{Loc}(\mathcal{S})$ is well–generated.

Theorem 5.1.3. *Let $\mathcal{T}$ be a locally well–generated triangulated category. Then $\mathcal{T}$ satisfies Brown representability if and only if $\mathcal{T}$ is well–generated. In particular, if R is a ring which is not right pure semisimple, for instance $R = \mathbb{Z}$, then* $\mathbf{K}(\mathrm{Mod}(R))$ *does not satisfy Brown representability.*

5.2 Brown representability for the dual of a homotopy category

We say that $\mathcal{A}$ has a *product generator* if there is an object $G \in \mathcal{A}$ such that $\mathcal{A} = \mathrm{Prod}(G)$.

Theorem 5.2.6. *Let $\mathcal{A}$ be an additive category with products. If $\mathbf{K}(\mathcal{A})^o$ satisfies Brown representability, then $\mathcal{A}$ has a product generator. In particular $\mathbf{K}(\mathcal{A}b)^o$ does not satisfy Brown representability.*

Theorem 5.2.10. *Let $\mathcal{A}$ be an additive category with products and split idempotents, possessing also images or kernels. Then $\mathbf{K}(\mathcal{A})^o$ satisfies Brown representability if and only if $\mathcal{A}$ has a product generator. In particular, if R is a ring then $\mathbf{K}(\mathrm{Mod}(R))^o$ satisfies Brown representability if and only if* $\mathrm{Mod}(R)$ *has a product generator.*

5.3 Functors without adjoints

Theorem 5.3.1. *Let R be a countable ring and let $\mathcal{D}$ be the class of all right flat Mittag–Leffler R-modules in the sense of [71]. Then $\mathbf{K}(\mathcal{D})$ is always closed under coproducts in $\mathbf{K}(\mathrm{Mod}(R))$, but the inclusion functor $\mathbf{K}(\mathcal{D}) \to \mathbf{K}(\mathrm{Mod}(R))$ has a right adjoint if and only if R is a right perfect ring. In particular, a right adjoint does not exist for $R = \mathbb{Z}$.*

6 Brown representability for the dual

6.1 The dual of Brown representability for some derived categories

For a complex $X^\bullet$ consider the inverse tower

$$X^{\geq 0} \leftarrow X^{\geq -1} \leftarrow X^{\geq -2} \leftarrow \cdots .$$

obtained from the so called "clever" truncations of $X^\bullet$.

Following [65], the category $\mathbf{D}(\mathcal{A})$ is said to be *left–complete*, provided that it has products and with the notation above $X^\bullet \cong \underleftarrow{\operatorname{holim}} X^{\geq -n}$.

Theorem 6.1.1. *Let $\mathcal{A}$ be a complete abelian category possessing an injective cogenerator, and let $\mathbf{D}(\mathcal{A})$ be its derived category. If $\mathbf{D}(\mathcal{A})$ is left–complete, then $\mathbf{D}(\mathcal{A})$ has small hom–sets and $\mathbf{D}(\mathcal{A})^o$ satisfies Brown representability.*

Corollary 6.1.2. *Let $\mathcal{A}$ be an abelian complete category possessing an injective cogenerator. If $\mathcal{A}$ is AB4*-n, for some $n \in \mathbb{N}$ and $\mathbf{D}(\mathcal{A})$ has products, then $\mathbf{D}(\mathcal{A})$ has small hom–sets and $\mathbf{D}(\mathcal{A})^o$ satisfies Brown representability.*

Corollary 6.1.4. *Let $\mathcal{A}$ be an abelian complete category possessing an injective cogenerator. If $\mathcal{A}$ is of finite global injective dimension and $\mathbf{D}(\mathcal{A})$ has products, then $\mathbf{D}(\mathcal{A})$ has small hom–sets and $\mathbf{D}(\mathcal{A})^o$ satisfies Brown representability.*

Corollary 6.1.5. *If $\mathcal{A}$ is the category of quasi-coherent sheaves over a quasi-compact and separated scheme then $\mathbf{D}(\mathcal{A})$ has small hom–sets and $\mathbf{D}(\mathcal{A})^o$ satisfies Brown representability. In particular, the conclusion holds for the category $\mathcal{A}$ of quasi-coherent sheaves over $\mathbb{P}^d_R$, where $\mathbb{P}^d_R$ is the projective d-space, $d \in \mathbb{N}^*$, over an arbitrary commutative ring with one R.*

6.3 The dual of Brown representability for homotoy category of projectives

Theorem 6.3.2. *If R is a ring with several objects, then $\mathbf{K}(\operatorname{Proj}(R))^o$ satisfies Brown representability.*

Corollary 6.3.4. *If R is a ring, then the dual of the homotopy category of pure–projective modules satisfies Brown representability.*

Theorem 6.3.8. *For a quiver Q we denote by $\operatorname{Mod}(R, Q)$ the categorii of all representations of R-modules of Q and by $\operatorname{Proj}(R, Q)$ the subcategory of projective objects in $\operatorname{Mod}(R, Q)$. Then $\mathbf{K}(\operatorname{Proj}(R, Q))^o$ satisfies Brown representability.*

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