Ringel-Hall algebras in tame cases and applications

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Abstract

Classical Hall algebras associated with discrete valuation rings were introduced by Steinitz and Hall to provide an algebraic approach to the classical combinatorics of partitions. The multiplication is given by classical Hall polynomials which play an important role in the representation theory of symmetric groups and general linear groups (see [**42**]). In 1990 Ringel defined Hall algebras for a large class of rings, namely finitary rings, including in particular path algebras of quivers over finite fields. In general these Ringel-Hall algebras are not commutative, in contrast with the classical ones (which correspond to the one-loop quiver in this context). In case of Ringel-Hall algebras associated to quivers we know due to Ringel in [**57**] and Green in [**30**] that the so called composition subalgebra generated by the simple modules is up to a minor modification the positive part of the corresponding Drinfeld-Jimbo quantum group, the compatibility relation between the described multiplication and comultiplication being expressed by the famous Green's formula (see in [**30**]).

Ringel proved that in case of Ringel-Hall algebras corresponding to Dynkin quivers the structure constants of the multiplication are again polynomials in the number of elements of the base field. We call them Hall polynomials. The existence of Hall polynomials in this case uses Gabriel's theorem which states that indecomposables corresponds to positive roots via their dimension, so modules can be treated field independently via roots. In his famous paper [**51**] Ringel listed these Dynkin type Hall polynomials associated to indecomposable modules. It turns out that they can have degree up to 5, so they are not just 0 or 1 as the classical ones corresponding to indecomposables.

The first explicit results on tame Ringel-Hall algebras were formulated in the Kronecker case (which is the easiest tame case) by Zhang in [**80**] and Baumann-Kassel in [**5**]. Completing and expanding these results in his PhD thesis, the author described explicitly the structure of the Ringel-Hall algebra in the Kronecker case, obtaining all the formulas for the Hall numbers and constructing various Poincaré-Birkhoff-Witt (PBW) type of bases for the composition subalgebra. Using Kac's theorem (which is a generalization of Gabriel's theorem) we can see that all the tame indecomposable modules can be treated field independently (via their positive real root dimensions or quasi top and quasi-length) excepting the so called regular homogeneous indecomposables. The author in his PhD thesis showed a method in the Kronecker case how to group these regular homogeneous modules into classes such that up to these classes we can prove the existence of Hall polynomials.

Hubery in [**35**] generalized the author's previous results in the Kronecker case observing that these regular homogeneous classes are in fact Segre classes and proved (up to Segre classes) the existence of tame Hall polynomials. However we do not know them explicitly and as we will see it is very difficult to obtain them. Knowing these tame Hall polynomials would help us in various contexts: as mentioned above, they are the structure constants of quantum groups, they are used in the theory of cluster and quantum cluster algebras (via Grassmannian cardinalities) and they can also be used successfully to investigate the structure of the module category (via the Gabriel-Roiter measure).

If we drop the finiteness condition on the field k, we can use instead of the Ringel-Hall product the so called extension monoid product introduced by Reineke in [**47**]. In this way we lose the counting part for a specific extension, but we can still control the existence or non-existence of it. And this over an arbitrary field, not only over finite fields. As we will see in some applications (for example in matrix pencil theory) it is more important to be able to work over arbitrary fields.

The present habilitation thesis records the progresses made by the author in the last ten years regarding tame Hall polynomials and Kronecker extension monoid products and their various applications.

The first chapter is a preliminary one serving as a comprehensive survey of the main notions and tools used throughout the thesis. It covers combinatorics, representations of tame quivers, reflection functors, Schofield sequences, Ringel-Hall algebras, extension monoid products and some specializations of these notions.

The second chapter is dedicated to tame Hall polynomials. We will describe all the tame Ringel-Hall products involving indecomposables of absolute defect not higher than 1. More precisely we will present explicit formulas for three types of Ringel-Hall products. These results were published mainly in [**66**] and partially in [**75**]. Finally we will obtain some special Hall polynomials of the form $F^{\delta}_{\delta-aa}$ (where a is a positive real root of arbitrary negative defect and δ the minimal radical vector), which will be applied in the next chapter. These results appear in [**67**].

In the third chapter we will apply our knowledge on Hall polynomials in the theory of Gabriel-Roiter measures. The Gabriel-Roiter measure (GR measure for short) was introduced by Gabriel in order to give a combinatorial interpretation of the induction scheme used by Roiter in his proof of the first Brauer-Thrall conjecture. Ringel used it as a foundation tool for the representation theory of Artin algebras. First of all we will prove that the GR inclusions in preprojective indecomposables and homogeneous modules of dimension δ as well as their GR measures are field independent. A similar result for Dynkin quivers was obtained by Ringel in [**52**]. As an application of the theorems above we will prove using Hall polynomials a result by Bo Chen in [**13**] in a more general context: our result is valid also for the case $\tilde{E}_8$ (this case is missing from [**13**]) and it is field independent (in [**13**] k is algebraically closed). More precisely we prove that a GR submodule P of a homogeneous module R of dimension δ has defect -1. These results appear in [**67**].

In the fourth chapter we determine cardinalities of Kronecker quiver Grassmannians via Ringel-Hall numbers. We consider Grassmannian varieties of fixed dimensional submodules of indecomposable Kronecker modules. In [**12**] Caldero and Zelevinsky described (using Schubert cells) explicit combinatorial formulas for the Euler-Poincaré characteristics of these Grassmannians using them in cluster theory. Using the Ringel-Hall algebra approach and reflection functors we obtain specific recursions for the Grassmannian cardinalities. We also prove a q-analogue of a combinatorial identity due to Nanjundiah. Combining these two results we deduce explicit combinatorial formulas for the cardinalities of the Grassmannians above. We realize in this way a quantification of the formulas by Caldero and Zelevinsky, with applications in quantum cluster theory. All these results were published in [**68**].

The fifth chapter describes the extensions of Kronecker modules. Although extensions of arbitrary tame modules are field dependent, it turns out that in the Kronecker case the extensions are field independent up to Segre classes, so the extension monoid products can be described in a purely combinatorial way. We will describe explicitly these products in many cases using a generic version of Green's formula and partition combinatorics. We obtain in this way a generic, field independent version of the Kronecker type Ringel-Hall product list obtained in the author's PhD thesis. The final section of this chapter surveys the sometimes difficult combinatorics related with the extensions and embeddings of decomposable preinjective modules (or dually preprojective ones). As we will see in the final chapter, these results published in [**69, 70, 71**] can and will be applied in matrix pencil theory.

The main subject of the sixth chapter is the following problem, called Modular Challenge: characterize the embedding of Kronecker modules via their Kronecker invariants. This will lead us towards the description of the submodule category of the Kronecker algebra (see [**58**]) and also towards the solution of the following (unsolved) problem in matrix pencil theory (see the next chapter for details), called Pencil Challenge: find a necessary and sufficient condition (in terms of classical Kronecker invariants) for a pencil to be a subpencil of an another one; moreover construct the completion of the smaller pencil to the bigger one (see [**41**]). The chapter is dedicated to the solution of the Modular Challenge, splitting the modules into smaller components and using results on extensions of Kronecker modules listed in the previous chapter. As a consequence we can see that as the extensions the embeddings of Kronecker modules are also field independent up to Segre classes. The results of this chapter were published in [**72**].

In the seventh chapter the results presented in the previous chapters on extensions and embeddings of Kronecker modules are applied in the theory of matrix pencils. We will see that pencils correspond to Kronecker modules so in this way a modular approach is possible to all the problems in pencil theory. In particular we will show how to solve the Pencil Challenge via the solved Modular Challenge (see the previous chapter). As a first application we will give a short modular proof for the codimension formula of a matrix pencil (see Demmel, Edelman [**17**]). Finally we will present a short explicit solution to the Pencil Challenge involving pencils determined only by minimal indices for columns (respectively for rows). These results were published in [**73**] and [**74**].

The last chapter gives a brief account of the author's future research plans.

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