Second Order Differential Equations and Systems: Theory and Applications to Oscillators

Summary

In Part 1 of this work, I will present my research activity after the defense of my PhD dissertation in 2001. Since then, I have continued to investigate several aspects concerning qualitative properties of the solutions of certain ODEs or systems of ODEs and to work on further research subjects, publishing 35 papers either in mathematical journals or in proceedings of international conferences, a monograph, one paper is accepted, and another paper is submitted for possible publication. My interest has focused on the following two main research directions: qualitative properties of solutions of some classes of second order ODEs with applications to damped nonlinear oscillators and qualitative properties of solutions of some classes of systems of second order ODEs with applications to coupled damped nonlinear oscillators. The papers published in the first direction are presented in Chapter 1 and the results obtained in the second direction are described in Chapter 2.

The main problems I investigated concern the stability of equilibria of specific systems formed by either a single nonlinear oscillator or several coupled nonlinear oscillators, for the sake of a consistent investigation of the large-time behavior of the dynamics of these mechanical systems. After an Introduction, wherein the main elements that will be used in this thesis are briefly specified, I will present in Chapter 1 the results obtained regarding qualitative properties of solutions of second order ODEs. At first, I investigated the stability of the equilibrium of a nonlinear oscillator whose dynamics is described by the second order ODE (E) $\ddot{x} + 2f(t)\dot{x} + \beta(t)x + g(t,x) = 0$, $t \in \mathbb{R}_+$. This research originated in the paper of T.A. Burton and T. Furumochi [32], wherein the authors have introduced a new method to study the stability of the null solution $x = \dot{x} = 0$ of this equation with $\beta(t) = 1$, $\forall t \in \mathbb{R}_+$, based on the Schauder fixed point theorem. In [81] we provided stability results in the same case $\beta(t) = 1$, $\forall t \in \mathbb{R}_+$, by using relatively classical arguments, and in [82] we proved certain stability results for the null solution of this ODE under more general assumptions, which required more sophisticated arguments. I will also present a new stability result, which has not been published yet, by using a classical method based on a suitable Lyapunov function. The approach allows extensions to both the vector case and the case $t \in \mathbb{R}$. In fact, an ongoing problem that I have investigated over time concerns the existence of solutions of certain ODEs or systems of ODEs on the whole real line, publishing a series of results in this direction. The results I will present in Section 1.1 improve the results from [82] and [81]. In Section 1.2 I will provide a result about the stability of the null solution of (E) and will also show that for any solution x of the equation, we have $x(+\infty) = \dot{x}(+\infty) = 0$, for small initial data, in the case when the uniqueness of solutions is not guaranteed. The proofs are based on a generalized form of the Schauder–Tychonoff fixed point theorem. If the uniqueness of

solutions is ensured, I will present a stability result for the same equation, based on the Banach fixed point theorem on Fréchet spaces. Much work from this section is found in [113], but the present section contains improvements to the results from [113], and also extends the previous results from [19].

In [26] (wherein we considered the case $\beta(t) = 1$, $\forall t \in \mathbb{R}$) and [114], we proved the existence of homoclinic solutions of (E) under certain hypotheses on the coefficients. For this purpose, we used a method mainly based on differential inequalities and classical qualitative analysis of the solutions of ODEs. The material in Section 1.3 is mostly found in those papers, but contains significant improvements. In [115] I discussed the existence of solutions x of (E), which are not identically 0 and which have the limits $x(+\infty) = \dot{x}(+\infty) = 0$, using the Lyapunov's method and differential inequalities. That approach allowed extension to the case $t \in \mathbb{R}$, the existence of homoclinic solutions being thus deduced. Section 1.4 contains improvements to the results from [115] and also generalizes the result from [21]. In [22] we considered a general second order ODE on the real line and presented an existence result for solutions x satisfying the boundary conditions $x(-\infty) = x(+\infty)$ and $\dot{x}(-\infty) = \dot{x}(+\infty)$. Our proof is mainly based on the application of the Bohnenblust–Karlin fixed point theorem for multivalued mappings. The material in Section 1.5 is taken from that paper.

The second research direction was suggested by Gheorghe Moroşanu and in Chapter 2 I will present the results we obtained concerning the dynamics of coupled nonlinear oscillators.

In [84] we investigated the stability of the null solution of systems modelling the motion of two coupled nonlinear oscillators, both being under the action of some external forces. Under certain assumptions, we derived some stability results. The material in Section 2.1 is mostly found in [84], but it also contains improvements to our results from that article. In Subsection 2.1.2 we will consider the case of two coupled damped nonlinear oscillators. Subsection 2.1.3 is devoted to the case of two coupled nonlinear oscillators with partial lack of damping which we investigated in [83]. The hypotheses we will assume on the damping coefficients and the external forces are new compared to the ones in [83] and the results are more general. The case of two coupled undamped nonlinear oscillators is treated in Subsection 2.1.4. Each of these three cases will be studied by two approaches based on classical arguments, by using differential inequalities and the Lyapunov's method.

In [85] we reconsidered the mechanical system of two coupled damped nonlinear oscillators and investigated the stability of the null solution of the system of ODEs describing the motion. We proved that for any solution (x, y) of the system, $x(+\infty) = \dot{x}(+\infty) = y(+\infty) = \dot{y}(+\infty) = 0$, for small initial data, in the case when the uniqueness of solutions is not guaranteed. Our proofs are mainly based on a generalized form of the Schauder–Tychonoff fixed point theorem. The results provided in Section 2.2 are very similar to those from [85], but they also contain improvements.

In [86] we investigated nonlinear systems of second order ODEs describing the dynamics of two coupled nonlinear oscillators of a mechanical system of vibration reduction. We obtained, under certain assumptions, some stability results for the null solution. Also, we showed that in the presence of a time-dependent external force, every solution (x, y) starting from sufficiently small initial data and its derivative are bounded or have the limits $x(+\infty) = \dot{x}(+\infty) = y(+\infty) = \dot{y}(+\infty) = 0$, provided that suitable conditions are satisfied. The work from Section 2.3 is taken from that article.

Almost all the theoretical results presented in Chapters 1 and 2 are confirmed by numerical simulations obtained using `Matlab`.

I have also been working on research topics not directly related to the main research directions described in this thesis. At the and, in Part 4, I will give details regarding the results I obtained on the fixed point theory and its applications to the existence of solutions or asymptotically stable solutions of some classes of nonlinear integral equations and systems of nonlinear integral equations.

Part 2 contains plans for the evolution and development of my scientific and academic careers. Some open problems that could complete the results obtained until now and which I consider useful for my further scientific research are described. Finally, a research plan to approach these problems and also a plan for my academic career, with likely ways of practical implementation, are presented.

In Part 3 the most important bibliographical references studied for the preparation of this thesis are presented. All these references are cited in the first two parts of the work.